Statistical solution and Liouville type theorem for the Klein-Gordon-Schrödinger equations*

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Abstract

In this article, the authors investigate the system of Schrödinger and Klein-Gordon equations with Yukawa coupling. They first prove the existence of pullback attractor and construct a family of invariant Borel probability measures. Then they establish that this family of probability measures satisfies a Liouville type theorem and is indeed a statistical solution for the coupling equations. Further, they reveal that the invariant property of the statistical solution is a particular situation of the Liouville type theorem.

Keywords: Klein-Gordon-Schödinger equations; Statistical solution; Pullback attractor; Invariant measure; Liouville type theorem

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1 Introduction

In this article, we investigate the following non-autonomous weakly dissipative Klein-Gordon-Schödinger (KGS for short) equations

$$u_{tt} + \nu u_t - \Delta u + \mu u - \beta |z|^2 = g(x,t), \ \ x \in \Omega, \ \ t > \tau, \tag{1.1}$$

$$iz_t + \Delta z + i\alpha z + zu = f(x,t), \ \ x \in \Omega, \ \ t > \tau, \tag{1.2}$$

with initial and boundary conditions

$$(u(x,t), u_t(x,t), z(x,t))\big|_{t=\tau} = (u_\tau, u_{0\tau}, z_\tau), \ \ x \in \Omega, \tag{1.3}$$

$$u(x,t)\big|_{\partial\Omega} = z(x,t)\big|_{\partial\Omega} = 0, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$.

Equations (1.1)-(1.2) describe the interaction of scalar nucleons with neutral mesons through Yukawa coupling (see [3]), where $u = u(x,t)$ and $z = z(x,t)$ denote a real meson field and a complex scalar nucleon field, respectively, the parameters $\alpha > 0$, $\nu > 0$ denote the dissipative mechanism of the system, $\mu > 0$, $\beta > 0$ are constants representing the damping coefficients, and the real-valued function $g(x,t)$ and complex-valued function $f(x,t)$ are the time-dependent external forces.

The autonomous KGS equations (1.1)-(1.2) and its related versions were extensively studied, one can see [1, 3, 17, 21, 23] and the references therein. For example, when $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain, Biler in [3] established the existence of global attractor in the weak topologies of

$$E = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \tag{1.5}$$

and

$$E_1 = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)). \tag{1.6}$$

Later, these results were improved by [29]. Also, Lange and Wang in [19] proved the regularity of the global attractor when $\Omega \subset \mathbb{R}$ is a bounded interval. The Cauchy problem associated to equations (1.1)-(1.2) were investigated in [21, 23]. For instance, Li and Guo in [21] used a Strichartz type inequality and some suitable decomposition to prove the asymptotic smoothing effect for the solutions. However, to the best of our knowledge, there are only some references concerning the asymptotic behavior of solutions for the non-autonomous KGS equations (1.1)-(1.2).

The motivation of the current article is to investigate the statistical solutions for the non-autonomous KGS equations (1.1)-(1.2). We are interested in the probability distribution of solutions within the phase space E. In turbulent flow regimes, the physical properties are universally recognized as randomly varying and characterized by some suitable probability distribution functions. In the theory of fluid mechanics, the invariant measures and statistical solutions have proven to be very useful in the understanding of turbulence (see Foias *et al.* [11]). The main reason is that the measurements of several important aspects (such as mass and velocity) of turbulent flows are actually measurements of time-average quantities. Statistical solutions have been introduced as a rigorous mathematical notion to formalize the object of ensemble average in the conventional statistical theory of turbulence. Nowadays, invariant measures and statistical solutions are widely used to describe certain characteristics of the fluids in the real world.

There are two prevalent notions of statistical solutions. The one is the so-called Foias-Prodi statistical solutions introduced by Foias and Prodi in [10] and the other is the so-called Vishik-Fursikov statistical solutions given by Vishik and Fursikov in [28]. The Foias-Prodi statistical solutions are a family of Borel measures parametrized by the time variable and defined on the phase space of the Navier-Stokes equations, representing the probability distribution of the velocity field of the flow at each time. The Vishik-Fursikov statistical solutions are a single Borel measure on the space of trajectories, representing the probability distribution of the space-time velocity field.

The invariant measures for well-posed dissipative systems were studied in a series of references (see [9,22,24–27,30]). For instance, Łukaszewicz, Real and Robinson [25] used the notion of Generalized Banach limit to construct the invariant measures for general continuous dynamical systems on metric spaces. Later, Chekroun and Glatt-Holtz [9] improved the results of [25] to construct invariant measures for a broad class of dissipative autonomous dynamical systems. Recently, Łukaszewicz and Robinson [26] extended the result of [9] to construct invariant measures for dissipative non-autonomous dynamical systems. The result of [26] was used to investigate the invariant measure for the three-dimensional (3D for short) globally modified Navier-Stokes equations and regularized MHD equations in [31, 37].

There also are some references investigating the statistical solutions and trajectory statistical solutions for some model evolution equations that possess global weak solutions but without a known result of global uniqueness. For instance, Foias, Rosa and Temam studied systematically the statistical solutions for the 3D Navier-Stokes equations in [12–16]. Bronzi, Mondaini and Rosa in [4, 6] proved an abstract framework for the theory of statistical solutions and trajectory statistical solutions for general evolution equations, including those with properties similar to the 3D Navier-Stokes equations. Bronzi and Rosa studied the convergence of statistical solutions of the 3D Navier-Stokes-α model as α vanishes in [5]. Caraballo, Kloeden and Real investigated the invariant measure and statistical solution for the 3D globally modified Navier-Stokes equations in [7]. Kloeden, Rubio and Real studied the equivalence of invariant measure and stationary statistical solutions for the autonomous globally modified Navier-Stokes equations in [18]. Zhao and Caraballo in [32] constructed the trajectory statistical solutions for the 3D globally modified Navier-Stokes equations. Zhao, Song and Caraballo in [34] constructed the strong trajectory statistical solutions for the 2D dissipative Euler equations. Zhao, Li and Caraballo in [33] proved sufficient conditions ensuring the existence of trajectory statistical solutions for autonomous evolution equations. In addition, Zhao, Li and Song in [35] constructed the trajectory statistical solutions for the 3D Navier-Stokes equations via the trajectory attractor approach. Also, Zhao, Jiang and Caraballo constructed in [36] the trajectory statistical solutions for the nonlinear wave equations with polynomial growth

The main result of the current article is to prove the existence of the statistical solution for the non-autonomous KGS equations (1.1)-(1.2). This statistical solution describes the probability distribution of the meson field and nucleon field in the phase space. We will first use the abstract theory for dissipative non-autonomous system in [26, Theorem 3.1] to obtain the existence of a family of invariant Borel probability measures $\{m_t\}_{t\in\mathbb{R}}$. Then we establish that $\{m_t\}_{t\in\mathbb{R}}$ satisfies a Liouville type theorem and is indeed a statistical solution for equations (1.1)-(1.2). Finally, we reveal that the invariant property of the statistical solution is a particular situation of the Liouville type theorem.

To apply the abstract theory of [26, Theorem 3.1] to obtain the existence of a family of invariant Borel probability measures, we shall prove that the solution operators associated to problem (1.1)-(1.4) generate a continuous process $\{U(t,\tau)\}_{t\geqslant\tau}$ in the phase space E and

(1) the process $\{U(t,\tau)\}_{t\geqslant\tau}$ is pullback strongly bounded in E;

(2) the process $\{U(t,\tau)\}_{t\geqslant\tau}$ is pullback asymptotically compact in E;

(3) for each given $t\in\mathbb{R}$ and given $\psi_*\in E$, the E-valued function $\tau\mapsto U(t,\tau)\psi_*$ is continuous and bounded on $(-\infty,t]$.

By definition, a continuous process $\{U(t,\tau)\}_{t\geqslant\tau}$ in the phase space E means $\{U(t,\tau)\}_{t\geqslant\tau}$ is a two-parameter family of mappings in E satisfying:

(a) $U(t,s)U(s,\tau)=U(t,\tau)$, $\forall\, t\geqslant s\geqslant\tau,\tau\in\mathbb{R}$;

(b) $U(\tau,\tau)=\mathrm{Id}$ (identity operator), $\tau\in\mathbb{R}$;

(c) For given t and τ with $t\geqslant\tau$, the mapping $U(t,\tau)$ is continuous from E to E.

It is not a standard fact to prove above assertions (1)-(3) for the process $\{U(t,\tau)\}_{t\geqslant\tau}$.

Firstly, on the one hand, it is not a direct generalization of the dynamics from autonomous system to non-autonomous system. On the other hand, in [3], the authors established the existence of global attractor with the conditions $f,g\in\mathcal{C}_b(\mathbb{R}_+;L^2(\Omega))$ or $f,g\in L^2(\Omega)$, which implies that f and g are uniformly bounded in $L^2(\Omega)$ with respect to time t. In this article, the conditions imposed on f and g are weaker than those in [3]. In fact, we allow that the external forces are unbounded and actually even exponentially growing time-dependent functions. At the same time, the nonlinear terms $|z|^2$ and zu produce some difficulties when we estimate the solutions and prove the pullback strongly boundedness of $\{U(t,\tau)\}_{t\geqslant\tau}$ in E.

Secondly, it is not easy to prove directly the pullback asymptotically compactness of $\{U(t,\tau)\}_{t\geqslant\tau}$ in E because of the special coupling of a hyperbolic equation with a parabolic one. Here we will employ some delicate decomposition of the process $\{U(t,\tau)\}_{t\geqslant\tau}$. Precisely, we decompose the addressed system into two equations, and simultaneously split the nonlinear term $|z|^2$ into $\mathbf{Re}(\overline{z}z_1)$ and $\mathbf{Re}(\overline{z}z_2)$ (here $z=z_1+z_2$). By this way we can prove that the solutions of the first decomposed equations pullback decay exponentially, whereas the solutions of the second decomposed equations are pullback bounded in E_1. Note that the embedding $E\hookrightarrow E_1$ is compact. We then obtain the pullback asymptotically compactness of $\{U(t,\tau)\}_{t\geqslant\tau}$ in E by the abstract theory of [8, Theorem 3.2].

Thirdly, to prove assertion (3) the key step is to establish the continuous dependence of the solutions on the initial data in E. This continuous dependence has been proved by Wang and Lange in [29, Theorem 3.4] via the method of energy equation. Here we will present a simple and direct proof to this continuous dependence. The main technique we used is to construct a suitable space E_μ which is equivalent to the usual phase space E. Then we establish that the abstract operator corresponding to the linear part of equations (1.1)-(1.2) is coercive on E_μ. This coerciveness allows us to prove directly the continuous dependence of the solutions on the initial data in the norm of E_μ, which is equivalent to the continuous dependence of the solutions on the initial data in the norm of E.

To establish that $\{m_t\}_{t\in\mathbb{R}}$ is a statistical solution of the KGS equations, the important step is to prove that $\{m_t\}_{t\in\mathbb{R}}$ satisfies a Liouville type theorem similar to that from Statistical Mechanics. Fortunately, the form of the construction of the Borel probability measures $\{m_t\}_{t\in\mathbb{R}}$ plays essential role in our proof. We also want to point out an interesting relation between the invariant property and the Liouville type theorem for the statistical solution. We all know that Liouville theorem from Statistical Mechanics indicates that the distribution of a set in the phase space could change with the evolution of time, but its Liouville measures is conserved. We will discover in this article that the invariant property of the statistical solution describes exactly that the shape of the pullback attractor $\mathcal{A}_{\mathcal{D}_\delta}(\tau)$ could change with the evolution of time from τ to t, but the measure of $\mathcal{A}_{\mathcal{D}_\delta}(\tau)$ and $\mathcal{A}_{\mathcal{D}_\delta}(t)$ coincides with each other.

The rest of the article is arranged as follows. In the next section, we estimate the solutions and then show the global well-posedness of problem (1.1)-(1.4). In Section 3, we establish the existence of the pullback attractor for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ associated to problem (1.1)-(1.4). In Section 4, we first construct a family of invariant Borel probability measures for the process $\{U(t,\tau)\}_{t\geqslant\tau}$. Then we establish that this family of probability measures satisfies a Liouville type theorem and is indeed a statistical solution for the KGS equations. Further, we reveal that the invariant property of the statistical solution is a particular situation of the Liouville type theorem.

2 Estimates and global well-posedness of solutions

We first introduce some notations. Let $L^p(\Omega)$, $H_0^1(\Omega)$ and $W^{m,p}(\Omega)$ denote the usual Lebesgue and Sobolev spaces with norms $\|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_{H_0^1(\Omega)}$ and $\|\cdot\|_{W^{m,p}(\Omega)}$, respectively. Especially, $H^m(\Omega)=W^{m,2}(\Omega)$ and $\|\cdot\|_{L^2(\Omega)}=\|\cdot\|$. Throughout this article, we will use the function spaces E (see (1.5)) and E_1 (see (1.6)) and the norms are defined respectively as

$$\|\psi\|_E=(\|\nabla u\|^2+\|v\|^2+\|\nabla z\|^2)^{1/2},\ \text{ for }\ \psi=(u,v,z)^T\in E, \tag{2.1}$$
$$\|\psi\|_{E_1}=(\|\Delta u\|^2+\|\nabla v\|^2+\|\Delta z\|^2)^{1/2},\ \text{ for }\ \psi=(u,v,z)^T\in E_1. \tag{2.2}$$

In addition, we will employ the notation $a\lesssim b$ (also $a\gtrsim b$) to mean that $a\leqslant cb$ (also $a\geqslant cb$) for a universal constant $c>0$ that only depends on the parameters coming from the problem.

Put

$$v=v(t)=u_t+\delta u, \tag{2.3}$$

where $\delta>0$ is some constant that will be specified later. Then problem (1.1)-(1.4) is equivalent to the following problem

$$u_t+\delta u-v=0,\ \ t>\tau, \tag{2.4}$$
$$v_t-\Delta u-\delta(\nu-\delta)u+\mu u+(\nu-\delta)v=\beta|z|^2+g(x,t),\ \ t>\tau, \tag{2.5}$$
$$z_t-i\Delta z+\alpha z=izu-if(x,t),\ \ t>\tau, \tag{2.6}$$
$$(u(x,t),v(x,t),z(x,t))\big|_{t=\tau}=(u_\tau,v_\tau,z_\tau),\ \ x\in\Omega, \tag{2.7}$$
$$(u(x,t),v(x,t),z(x,t))\big|_{\partial\Omega}=(0,0,0), \tag{2.8}$$

hereinafter $v_\tau=u_{0\tau}+\delta u_\tau$ and δ is the constant from (2.3). Denote

$$\psi=\psi(x,t)=(u(x,t),v(x,t),z(x,t))^T$$

and

$$\Theta=\begin{pmatrix}\delta I & -I & 0\\ -\Delta-\delta(\nu-\delta)I+\mu I & (\nu-\delta)I & 0\\ 0 & 0 & -i\Delta+\alpha I\end{pmatrix}, \tag{2.9}$$
$$F(\psi,t)=(0,\beta|z|^2+g(x,t),izu-if(x,t))^T, \tag{2.10}$$

where I in the matrix Θ is the identity operator. Then problem (2.4)-(2.8) can be written as

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}+\Theta\psi=F(\psi,t),\ \ t>\tau, \tag{2.11}$$
$$\psi(\tau)=\psi_\tau=(u_\tau,v_\tau,z_\tau)^T. \tag{2.12}$$

We next estimate the solutions of problem (2.11)-(2.12).

Lemma 2.1. *Let $f(x,t)$, $f_t(x,t)$, $g(x,t)$ belong to $L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$. Then for any $\psi_\tau = (u_\tau, v_\tau, z_\tau)^T \in E$, every solution $\psi(x,t) = (u(x,t), v(x,t), z(x,t))^T$ of problem (2.11)-(2.12) corresponding to ψ_τ satisfies*

$$\|z(t)\|^2 \lesssim \|z_\tau\|^2 e^{-\alpha(t-\tau)} + e^{-\alpha t}\int_\tau^t e^{\alpha s}\|f(s)\|^2 \mathrm{d}s, \ \ \forall\, t \geqslant \tau, \tag{2.13}$$

$$\begin{aligned}\|\psi(t)\|^2_E \lesssim &\Upsilon(\tau)e^{-\delta(t-\tau)} + e^{-\delta t}\int_\tau^t e^{\delta s}\|G(s)\|^2 \mathrm{d}s \\ &+ e^{-\delta t}\int_\tau^t e^{\delta s}\|z(s)\|^6 \mathrm{d}s + \|z(t)\|^6 + \|f(t)\|^2, \ \ \forall\, t \geqslant \tau,\end{aligned} \tag{2.14}$$

where

$$\Upsilon(\tau) = \|\nabla z_\tau\|^2 + \frac{1}{2}\|\nabla u_\tau\|^2 + \frac{1}{2}\|v_\tau\|^2 + 2\boldsymbol{Re}\int_\Omega \overline{z_\tau} f(\tau)\mathrm{d}x - \int_\Omega |z_\tau|^2 u_\tau \mathrm{d}x + \frac{\mu}{2}\|u_\tau\|^2, \tag{2.15}$$

$$\|G(s)\|^2 = \|f(s)\|^2 + \|\frac{\partial f(x,s)}{\partial t}\|^2 + \|g(s)\|^2. \tag{2.16}$$

Proof. Let the assumption of this lemma hold. Then the existence and uniqueness of the solution $\psi(x,t)$ corresponding to the initial data $\psi_\tau = (u_\tau, v_\tau, z_\tau)^T \in E$ can be proved in a standard way, using the Galerkin approximations as in [3]. The estimate (2.13) is easily obtained by taking the scalar product of (2.6) with $z(t)$ and the real part of the resulting equality. These details are omitted and here we prove (2.14).

In fact, in [3, (2.10)], it is proved that

$$\begin{aligned}&\frac{\mathrm{d}\Upsilon(t)}{\mathrm{d}t} + \delta\Upsilon(t) + (2\alpha - 3\delta)\|\nabla z(t)\|^2 + (\nu - \frac{3\delta}{2})\|v(t)\|^2 + \frac{\delta}{2}\|\nabla u(t)\|^2 + \frac{\delta\mu}{2}\|u(t)\|^2 \\ =&2\alpha\int_\Omega |z(t)|^2 u(t)\mathrm{d}x - 2(\alpha-\delta)\mathbf{Re}\int_\Omega \overline{z}(t) f(x,t)\mathrm{d}x + \delta(\nu-\delta)\int_\Omega u(t)v(t)\mathrm{d}x \\ &+ \int_\Omega v(t)g(x,t)\mathrm{d}x + 2\mathbf{Re}\int_\Omega \overline{z}(t)\frac{\partial f(x,t)}{\partial t}\mathrm{d}x, \ \ \forall\, t \geqslant \tau,\end{aligned} \tag{2.17}$$

where

$$\Upsilon(t) = \|\nabla z(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\|v(t)\|^2 + 2\mathbf{Re}\int_\Omega \overline{z(t)} f(x,t)\mathrm{d}x - \int_\Omega |z(t)|^2 u(t)\mathrm{d}x + \frac{\mu}{2}\|u(t)\|^2.$$

We next estimate the terms on the right-hand side of (2.17). By Hölder's inequality, Gagliardo-Nirenberg's inequality and the embedding $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$, we obtain

$$|\int_\Omega |z(t)|^2 u(t)\mathrm{d}x| \lesssim \|z(t)\|^2_{L^{12/5}(\Omega)}\|u(t)\|_{L^6(\Omega)} \lesssim \|z(t)\|^{3/2}\|\nabla z(t)\|^{1/2}\|\nabla u(t)\|,$$

and thus

$$2\alpha\int_\Omega |z(t)|^2 u(t)\mathrm{d}x \lesssim \frac{\alpha}{2}\|\nabla z(t)\|^2 + \frac{\delta}{4}\|\nabla u(t)\|^2 + \|z(t)\|^6. \tag{2.18}$$

The other terms on the right-hand side of (2.17) are simpler to deal with:

$$\begin{cases} 2(\alpha-\delta)\big|\mathbf{Re}\displaystyle\int_\Omega \overline{z}(t)f(x,t)\mathrm{d}x\big| \leqslant (\alpha-\delta)\|\nabla z(t)\|^2+\frac{4(\alpha-\delta)}{\lambda_1^2}\|f(t)\|^2, \\ 2\big|\mathbf{Re}\displaystyle\int_\Omega \overline{z}(t)\frac{\partial f(x,t)}{\partial t}\mathrm{d}x\big| \leqslant \frac{\alpha}{2}\|\nabla z(t)\|^2+\frac{2}{\alpha\lambda_1^2}\|\frac{\partial f(x,t)}{\partial t}\|^2, \\ \big|\displaystyle\int_\Omega v(t)g(x,t)\mathrm{d}x\big| \leqslant \frac{\nu}{4}\|v(t)\|^2+\frac{1}{\nu}\|g(t)\|^2, \\ \delta(\nu-\delta)\big|\displaystyle\int_\Omega u(t)v(t)\mathrm{d}x\big| \leqslant \frac{\nu-\delta}{2}\|v(t)\|^2+\frac{\delta^2(\nu-\delta)}{2}\|u(t)\|^2, \end{cases} \tag{2.19}$$

where we have used the following Poincaré inequality

$$\|u\|^2 \leqslant \lambda_1^{-1}\|\nabla u\|^2, \ \ \forall\, u\in H_0^1(\Omega).$$

Inserting (2.18)-(2.19) into (2.17) yields

$$\frac{\mathrm{d}\Upsilon(t)}{\mathrm{d}t}+\delta\Upsilon(t)+(\frac{\alpha}{2}-2\delta)\|\nabla z(t)\|^2+(\frac{\nu}{4}-\delta)\|v(t)\|^2+\frac{\delta}{4}\|\nabla u(t)\|^2+\frac{\delta}{2}(\mu-\delta(\nu-\delta))\|u(t)\|^2$$
$$\lesssim \|z(t)\|^6+\|G(t)\|^2, \ \ \forall\, t\geqslant\tau, \tag{2.20}$$

where $G(\cdot)$ is defined by (2.16). We now choose δ such that

$$0<\delta<\min\{\frac{\alpha}{4},\frac{\nu}{4},\frac{\mu}{\nu}\}.$$

Then (2.20) implies

$$\frac{\mathrm{d}\Upsilon(t)}{\mathrm{d}t}+\delta\Upsilon(t)\lesssim \|z(t)\|^6+\|G(t)\|^2, \ \ \forall\, t\geqslant\tau,$$

and applying Gronwall's inequality we deduce

$$\Upsilon(t)\lesssim \Upsilon(\tau)e^{-\delta(t-\tau)}+e^{-\delta t}\int_\tau^t e^{\delta s}\|G(s)\|^2\mathrm{d}s+e^{-\delta t}\int_\tau^t e^{\delta s}\|z(s)\|^6\mathrm{d}s, \ \ \forall\, t\geqslant\tau. \tag{2.21}$$

Combining (2.21), the estimates similar to (2.18) and the first inequality in (2.19), we obtain (2.14). The proof of Lemma 2.1 is completed. □

According to the estimates in Lemma 2.1, we next analyze under what assumptions on the data there exists a pullback absorbing set for the process associated to problem (2.11)-(2.12).

From (2.15) we see that

$$\Upsilon(\tau)\lesssim \|\psi_\tau\|_E^2+\|z_\tau\|^6+\|f(\tau)\|^2. \tag{2.22}$$

From (2.14) and (2.22) it follows that the following assumptions

$$\lim_{\tau\to-\infty}\|\psi_\tau\|_E^2 e^{\frac{\delta\tau}{3}}=0, \tag{2.23}$$

$$\lim_{\tau\to-\infty}\|f(\tau)\|^2 e^{\delta\tau}=0, \tag{2.24}$$

$$\int_{-\infty}^t e^{\delta s}\|G(s)\|^2\mathrm{d}s<+\infty, \ \ \text{for each}\, t\in\mathbb{R}, \tag{2.25}$$

are needed. We next analyze the third term on the right-hand side of (2.14). In fact, it follows from (2.13) that

$$e^{-\delta t}\int_\tau^t e^{\delta s}\|z(s)\|^6 \mathrm{d}s \lesssim \rho_1(t,\tau)+\rho_2(t,\tau)+\rho_3(t,\tau)+\rho_4(t,\tau),$$

where

$$\begin{cases} \rho_1(t,\tau)=e^{-\delta t}\displaystyle\int_\tau^t e^{\delta s}\|z_\tau\|^6 e^{-3\alpha(s-\tau)}\mathrm{d}s,\\ \rho_2(t,\tau)=e^{-\delta t}\displaystyle\int_\tau^t e^{\delta s}\|z_\tau\|^4 e^{-2\alpha(s-\tau)}e^{-\alpha s}\int_\tau^s e^{\alpha\theta}\|f(\theta)\|^2\mathrm{d}\theta\mathrm{d}s,\\ \rho_3(t,\tau)=e^{-\delta t}\displaystyle\int_\tau^t e^{\delta s}\|z_\tau\|^2 e^{-\alpha(s-\tau)}\big(e^{-\alpha s}\int_\tau^s e^{\alpha\theta}\|f(\theta)\|^2\mathrm{d}\theta\big)^2\mathrm{d}s,\\ \rho_4(t,\tau)=e^{-\delta t}\displaystyle\int_\tau^t e^{\delta s}\big(e^{-\alpha s}\int_\tau^s e^{\alpha\theta}\|f(\theta)\|^2\mathrm{d}\theta\big)^3\mathrm{d}s.\end{cases}$$

By (2.23),

$$\lim_{\tau\to-\infty}\|z_\tau\|^6 e^{\delta\tau}\lesssim\lim_{\tau\to-\infty}(\|\psi_\tau\|^2 e^{\frac{\delta\tau}{3}})^3=0,$$

and thus when $\tau\to-\infty$,

$$\rho_1(t,\tau)=(\|z_\tau\|^2 e^{\frac{\delta\tau}{3}})^3 e^{-\delta t}\int_\tau^t e^{\delta s}e^{-3\alpha(s-\tau)}e^{-\delta\tau}\mathrm{d}s\lesssim(\|z_\tau\|^2 e^{\frac{\delta\tau}{3}})^3 e^{-\delta t}\longrightarrow 0. \tag{2.26}$$

We write $\rho_2(t,\tau)$ in the form

$$(\|z_\tau\|^2 e^{\frac{\delta\tau}{3}})^2 e^{-\frac{2\delta\tau}{3}}e^{-\delta t}\int_\tau^t e^{\delta s}e^{-2\alpha(s-\tau)}e^{-\alpha s}\int_\tau^s e^{\alpha\theta}\|f(\theta)\|^2\mathrm{d}\theta\mathrm{d}s. \tag{2.27}$$

Thus, $\rho_2(t,\tau)\longrightarrow 0$ as $\tau\to-\infty$ if the integral in (2.27) stays bounded as $\tau\to-\infty$. For this purpose, we assume

$$e^{(\frac{2\delta}{3}-2\alpha)s}\int_{-\infty}^s e^{\alpha\theta}\|f(\theta)\|^2\mathrm{d}\theta\leqslant K(s), \tag{2.28}$$

where $K(s)$ is a continuous function on the real line which is bounded on every interval of the form $(-\infty,t)$. If (2.28) holds, then

$$\begin{aligned}\rho_2(t,\tau)\lesssim&(\|z_\tau\|^2 e^{\frac{\delta\tau}{3}})^2 e^{(2\alpha-\frac{2\delta}{3})\tau}e^{-\delta t}\int_\tau^t e^{-(3\alpha-\delta)s}e^{(2\alpha-\frac{2\delta}{3})s}K(s)\mathrm{d}s\\ \lesssim&(\|z_\tau\|^2 e^{\frac{\delta\tau}{3}})^2 e^{(\alpha-\frac{\delta}{3})\tau}\widetilde{K}(t)\longrightarrow 0,\ \ \tau\to-\infty,\end{aligned} \tag{2.29}$$

where $\widetilde{K}(t)$ is a bounded quantity depending only on t and the function $K(\cdot)$.

Also, if (2.28) holds, then for $K_1(s)=K(s)e^{(\alpha-\frac{\delta}{3})s}$ we have

$$e^{(\frac{2\delta}{3}-2\alpha)s}\big(\int_{-\infty}^s e^{\alpha\theta}\|f(\theta)\|^2\mathrm{d}\theta\big)^2\lesssim K_1^2(s),$$

and thus

$$e^{(\frac{\delta}{3}-\alpha)s}\int_{-\infty}^s e^{\alpha\theta}\|f(\theta)\|^2\mathrm{d}\theta\lesssim K_1(s).$$

Obviously, $K_1(s)$ possesses the same property of the function $K(s)$ in (2.28). Hence

$$\begin{aligned}\rho_3(t,\tau) \lesssim& \|z_\tau\|^2 e^{\frac{\delta\tau}{3}} e^{(\alpha-\frac{\delta}{3})\tau} e^{-\delta t}\int_\tau^t e^{-(3\alpha-\delta)s} e^{(2\alpha-\frac{2\delta}{3})s} K_1^2(s)\mathrm{d}s\\ \lesssim& \|z_\tau\|^2 e^{\frac{\delta\tau}{3}} e^{-\delta t}\widetilde{K_1}(t) \longrightarrow 0,\ \ \tau\to-\infty, \end{aligned} \tag{2.30}$$

where $\widetilde{K}_1(t)$ is a bounded quantity depending only on t and the function $K_1(\cdot)$.

At the same time, if (2.28) holds then

$$e^{(\frac{\delta}{3}-\alpha-(\alpha-\frac{\delta}{3}))s}\int_{-\infty}^s e^{\alpha\theta}\|f(\theta)\|^2\mathrm{d}\theta \leqslant K(s),$$

and so we can choose some $\gamma>0$ such that $\alpha-\frac{\delta}{3}>\frac{\delta}{3}>\gamma$ and

$$e^{(\frac{\delta}{3}-\alpha-\gamma)s}\int_{-\infty}^s e^{\alpha\theta}\|f(\theta)\|^2\mathrm{d}\theta \leqslant K_2(s)=K(s)e^{(\alpha-\frac{\delta}{3}-\gamma)s}.$$

Clearly, the function $K_2(s)$ also possesses the property of the function $K(s)$. Thus

$$\begin{aligned}\rho_4(t,\tau) \lesssim& e^{-\delta t}\int_\tau^t e^{-(3\alpha-\delta)s} e^{(\delta-3\alpha-3\gamma)s}\Big(\int_\tau^t e^{\alpha\theta}\|f(\theta)\|^2\mathrm{d}\theta\Big)^3 e^{(3\alpha+3\gamma-\delta)s}\mathrm{d}s\\ \lesssim& e^{-\delta t}\int_\tau^t e^{-(3\alpha-\delta)s}\Big(e^{(\frac{\delta}{3}-\alpha-\gamma)s}\int_\tau^t e^{\alpha\theta}\|f(\theta)\|^2\mathrm{d}\theta\Big)^3 e^{(3\alpha+3\gamma-\delta)s}\mathrm{d}s\\ \lesssim& e^{-\delta t}\int_\tau^t K_2^3(s)e^{3\gamma s}\mathrm{d}s \lesssim e^{-(\delta-3\gamma)t}\widetilde{K}_2(t),\ \ \forall\,\tau<t, \end{aligned} \tag{2.31}$$

where $\widetilde{K}_2(t)$ is a bounded quantity depending only on t and the function $K_2(\cdot)$.

We now summarize our assumptions on the external forces $f(x,t)$ and $g(x,t)$ leading to the existence of a bounded pullback absorbing set.

(H) *Assume* $f(x,t)$, $\dfrac{\partial f(x,t)}{\partial t}$, $g(x,t)\in L^2_{\mathrm{loc}}(\mathbb{R};L^2(\Omega))$ *and let*

$$\int_{-\infty}^t e^{\delta s}\|G(s)\|^2\mathrm{d}s<+\infty,\ \ for\ \ each\ \ t\in\mathbb{R}, \tag{2.32}$$

$$\lim_{\tau\to-\infty}\|f(\tau)\|^2 e^{\delta\tau}=0. \tag{2.33}$$

Moreover, let there exist some continuous function $K(\cdot)$ on the real line, bounded on intervals of the form $(-\infty,t)$, such that

$$\int_{-\infty}^t e^{\delta s}\big(\|f(s)\|^2+\|\frac{\partial f(x,s)}{\partial t}\|^2\big)\mathrm{d}s \leqslant K(t)e^{(\alpha+\frac{\delta}{3})t},\ \ for\ \ each\ \ t\in\mathbb{R}. \tag{2.34}$$

Remark 2.1. (1) *From (2.34) we see that for each $t\in\mathbb{R}$,*

$$\int_{-\infty}^t e^{\alpha s}\|f(s)\|^2\mathrm{d}s=\int_{-\infty}^t e^{(\alpha-\delta)s}e^{\delta s}\|f(s)\|^2\mathrm{d}s\leqslant e^{(\alpha-\delta)t}K(t)e^{(\alpha+\frac{\delta}{3})t},$$

so that (3.28) holds. The condition $\int_{-\infty}^t e^{\delta s}\|\frac{\partial f(x,s)}{\partial t}\|^2\mathrm{d}s\leqslant K(t)e^{(\alpha+\frac{\delta}{3})t}$ *will be used when deriving the pullback asymptotic compactness of the process in E.*

(2) *Let* $\|f(s)\|^2 + \|\dfrac{\partial f(x,s)}{\partial t}\|^2 \leqslant Me^{\kappa s}$ *for all* $s \in \mathbb{R}$*, where* $M > 0$ *and* $\kappa \geqslant \alpha - \dfrac{2\delta}{3}$. *Then*

$$\int_{-\infty}^{t} e^{\delta s}(\|f(s)\|^2 + \|\frac{\partial f(x,s)}{\partial t}\|^2)\mathrm{d}s \leqslant K(t)e^{(\alpha+\frac{\delta}{3})t}$$

with

$$K(t) = \frac{M}{\delta+\kappa}e^{(\frac{2\delta}{3}-\alpha+\kappa)t}.$$

This example shows that we allow the external forces to be unbounded and actually even exponentially growing time-dependent functions.

From now on, by $\mathcal{P}(E)$ we denote the family of all nonempty subsets of E. According to (2.23), we denote by $\mathcal{D}_\delta$ the class of family of nonempty subsets $\widehat{D} = \{D(s)|s \in \mathbb{R}\} \subset \mathcal{P}(E)$ satisfying

$$\lim_{s\to-\infty}\big(e^{\frac{\delta s}{3}} \sup_{\psi\in D(s)} \|\psi\|_E^2\big) = 0. \tag{2.35}$$

The class $\mathcal{D}_\delta$ will be called a universe in $\mathcal{P}(E)$. Clearly, all fixed bounded subsets of E lie in $\mathcal{D}_\delta$.

By above analyses and the Galerkin approximations as in [3], we can prove in a standard way the following result (cf. [3]).

Lemma 2.2. *Let assumption* **(H)** *hold. Then for any* $\widehat{D} = \{D(s)|s \in \mathbb{R}\} \in \mathcal{D}_\delta$ *and* $\psi_\tau = (u_\tau, v_\tau, z_\tau)^T \in D(\tau)$*, problem* (2.11)-(2.12) *possesses a unique global solution*

$$\psi(x,t) = (u(x,t), v(x,t), z(x,t))^T \in \mathcal{C}([\tau,+\infty); E). \tag{2.36}$$

Moreover, $\psi(x,t)$ *satisfies* (2.13) *and* (2.14).

The continuous dependence of the solution $\psi(x,t)$ on the initial data ψ_τ will be proved in Section 4. From this continuous dependence and Lemma 2.2 we see that the maps

$$U(t,\tau) : \psi_\tau = (u_\tau, v_\tau, z_\tau)^T \in E \longmapsto \psi(x,t) = (u(x,t), v(x,t), z(x,t))^T \in E, \ \ \forall\, t \geqslant \tau,$$

generate a continuous process $\{U(t,\tau)\}_{t\geqslant\tau}$ in E.

3 Existence of the pullback-$\mathcal{D}_\delta$ attractor

The aim of this section is to prove the existence of the pullback-$\mathcal{D}_\delta$ attractor for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in the phase space E. To this end, we will first show that $\{U(t,\tau)\}_{t\geqslant\tau}$ possesses a bounded pullback-$\mathcal{D}_\delta$ absorbing set and is pullback-$\mathcal{D}_\delta$ asymptotically compact in E.

Definition 3.1. (1) *It is said that a family of subsets $\widehat{D}_0 = \{D_0(s)|s \in \mathbb{R}\} \subset \mathcal{P}(E)$ is bounded pullback-$\mathcal{D}_\delta$ absorbing for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in E if for each $t \in \mathbb{R}$ and any $\widehat{D} = \{D(s)|s \in \mathbb{R}\} \in \mathcal{D}_\delta$, with $D(s) \subset E$ bounded for every $s \in \mathbb{R}$, there exists a $\tau_0(t, \widehat{D}) \leqslant t$ such that $U(t,\tau)D(\tau) \subset D_0(t)$ for all $\tau \leqslant \tau_0(t, \widehat{D})$.*

(2) *The process $\{U(t,\tau)\}_{t\geqslant\tau}$ is said to be pullback-$\mathcal{D}_\delta$ asymptotically compact in E if, for each given $t \in \mathbb{R}$, each $\widehat{D} = \{D(s)|s \in \mathbb{R}\} \in \mathcal{D}_\delta$ and any sequence $\{\tau_n\}_{n\geqslant1}$ in $(-\infty, t]$ such that $\tau_n \to -\infty$ as $n \to \infty$, the sequence $\{U(t,\tau_n)\psi(\tau_n)\}$ with any $\psi(\tau_n) \in D(\tau_n)$ possesses a convergent subsequence.*

Lemma 3.1. *Let assumption* **(H)** *hold. Then the process $\{U(t,\tau)\}_{t\geqslant\tau}$ possesses a bounded pullback-$\mathcal{D}_\delta$ absorbing set $\widehat{\mathcal{B}}_0 = \{\mathcal{B}_0(s)|s \in \mathbb{R}\} \subset \mathcal{P}(E)$, where $\mathcal{B}_0(s) = \mathcal{B}_0(0; r_\delta(s))$ is the ball of radius $r_\delta(s)$ and centered at zero in E.*

Proof. Let $r_\delta(t) = R_\delta^{\frac{1}{2}}(t)$, where

$$\begin{aligned} R_\delta(t) \lesssim & 1 + e^{-\delta t}\int_{-\infty}^t e^{\delta s}\|G(s)\|^2 \mathrm{d}s + e^{-\delta t}\int_{-\infty}^t e^{\delta s}\big(e^{-\alpha s}\int_{-\infty}^s e^{\alpha\theta}\|f(\theta)\|^2 \mathrm{d}\theta\big)^3 \mathrm{d}s \\ & + (1 + e^{-\alpha t}\int_{-\infty}^t e^{\alpha s}\|f(s)\|^2 \mathrm{d}s)^3 + \|f(t)\|^2, \ \ t \in \mathbb{R}. \end{aligned} \tag{3.1}$$

Then from (2.13)-(2.14), (2.22)-(2.26), (2.29)-(2.31) and the analyses in previous section, we conclude that $\widehat{\mathcal{B}}_0 = \{\mathcal{B}_0(s)|s \in \mathbb{R}\}$ constitutes the desired bounded pullback-$\mathcal{D}_\delta$ absorbing set for $\{U(t,\tau)\}_{t\geqslant\tau}$ in E. □

We next use the decomposition of equations (2.4)-(2.8) to prove that $\{U(t,\tau)\}_{t\geqslant\tau}$ is pullback-$\mathcal{D}_\delta$ asymptotically compact in E. Let $\psi(x,t) = (u(x,t), v(x,t), z(x,t))^T$ be a solution of problem (2.4)-(2.8) with initial data (u_τ, v_τ, z_τ). We decompose problem (2.4)-(2.8) into two problems as

$$u_{1t} + \delta u_1 - v_1 = 0, \ \ t > \tau, \tag{3.2}$$
$$v_{1t} - \Delta u_1 - \delta(\nu - \delta)u_1 + \mu u_1 + (\nu - \delta)v_1 = \beta \mathbf{Re}(\overline{z} z_1), \ \ t > \tau, \tag{3.3}$$
$$iz_{1t} + \Delta z_1 + i\alpha z_1 + z_1 u = 0, \ \ t > \tau, \tag{3.4}$$
$$\big(u_1(x,t), v_1(x,t), z_1(x,t)\big)\big|_{t=\tau} = (u_\tau, v_\tau, z_\tau), \ \ x \in \Omega, \tag{3.5}$$
$$\big(u_1(x,t), v_1(x,t), z_1(x,t)\big)\big|_{\partial\Omega} = (0,0,0), \tag{3.6}$$

and

$$u_{2t} + \delta u_2 - v_2 = 0, \ \ t > \tau, \tag{3.7}$$
$$v_{2t} - \Delta u_2 - \delta(\nu - \delta)u_2 + \mu u_2 + (\nu - \delta)v_2 = \beta \mathbf{Re}(\overline{z} z_2) + g(x,t), \ \ t > \tau, \tag{3.8}$$
$$iz_{2t} + \Delta z_2 + i\alpha z_2 + z_2 u = f(x,t), \ \ t > \tau, \tag{3.9}$$
$$\big(u_2(x,t), v_2(x,t), z_2(x,t)\big)\big|_{t=\tau} = (0,0,0), \ \ x \in \Omega, \tag{3.10}$$
$$\big(u_2(x,t), v_2(x,t), z_2(x,t)\big)\big|_{\partial\Omega} = (0,0,0). \tag{3.11}$$

Obviously, if $\psi_1(x,t) = (u_1(x,t), v_1(x,t), z_1(x,t))^T$ is a solution of problem (3.2)-(3.6), then

$$\begin{aligned}\psi_2(x,t) =&(u_2(x,t), v_2(x,t), z_2(x,t))^T\\ =&(u(x,t)-u_1(x,t), v(x,t)-v_1(x,t), z(x,t)-z_1(x,t))^T\end{aligned} \tag{3.12}$$

is a solution of problem (3.7)-(3.11).

Lemma 3.2. *Let assumption* **(H)** *hold. Then for any given $\widehat{D} = \{D(s)|s \in \mathbb{R}\} \in \mathcal{D}_\delta$ and $\psi_\tau = (u_\tau, v_\tau, z_\tau)^T \in D(\tau)$, problem (3.2)-(3.6) possesses a unique solution $\psi_1(x,t) = (u_1(x,t), v_1(x,t), z_1(x,t))^T$ corresponding to ψ_τ. Moreover, for each given $t \in \mathbb{R}$ there exists a time $\tau_0 = \tau_0(t, \widehat{D})$ such that*

$$\|\psi_1(x,t)\|_E^2 \lesssim \|\psi_\tau\|_E^2(1+\|\psi_\tau\|_E^{\frac{3}{4}})e^{-\delta(t-\tau)}, \ \ \forall \tau \leqslant \tau_0. \tag{3.13}$$

Proof. We first estimate $\|\nabla z_1(t)\|^2$. Taking the imaginary part of the inner product of (3.4) with $z_1(t)$ in $L^2(\Omega)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z_1(t)\|^2 + 2\alpha\|z_1(t)\|^2 = 0.$$

By Gronwall's inequality,

$$\|z_1(t)\|^2 \leqslant \|z_\tau\|^2 e^{-2\alpha(t-\tau)}, \ \ \forall \tau \leqslant t. \tag{3.14}$$

At the same time, multiplying (3.4) by $(-\overline{z}_{1t} - \alpha\overline{z}_1)$, integrating over Ω and then taking the real part of the resulting equality yield

$$\frac{\mathrm{d}H_1(t)}{\mathrm{d}t} + \alpha H_1(t) = J_1(t), \tag{3.15}$$

where

$$\begin{aligned}H_1(t) &= \|\nabla z_1(t)\|^2 - \int_\Omega u(t)|z_1(t)|^2\mathrm{d}x,\\ J_1(t) &= -\alpha\|\nabla z_1(t)\|^2 + \alpha\int_\Omega u(t)|z_1(t)|^2\mathrm{d}x - \int_\Omega u_t(t)|z_1(t)|^2\mathrm{d}x.\end{aligned}$$

By Hölder's inequality, Gagliardo-Nirenberg's inequality and Lemma 3.1, we have

$$\begin{aligned}\Big|\alpha\int_\Omega u(t)|z_1(t)|^2\mathrm{d}x\Big| \lesssim&\|u(t)\|_{L^6(\Omega)}\|z_1(t)\|^2_{L^{\frac{12}{5}}(\Omega)} \lesssim \|\nabla u(t)\|\|z_1(t)\|^{\frac{3}{2}}\|\nabla z_1(t)\|^{\frac{1}{2}}\\ \lesssim&\frac{\alpha}{2}\|\nabla z_1(t)\|^2 + \|\nabla u(t)\|^{\frac{4}{3}}\|z_1(t)\|^2\\ \lesssim&\frac{\alpha}{2}\|\nabla z_1(t)\|^2 + \|z_1(t)\|^2, \ \ \forall \tau \leqslant \tau_0,\end{aligned} \tag{3.16}$$

where in the last inequality of (3.16) we have used the inequality $\|\nabla u(t)\| \lesssim \|\psi(t)\|_E$, and the pullback-$\mathcal{D}_\delta$ absorbing property $\|\psi(t)\|_E \lesssim 1$ for any $\tau \leqslant \tau_0$ since $t \in \mathbb{R}$ is given and $r_\delta(t)$ is a positive constant. Similarly,

$$\begin{aligned}\Big|\alpha\int_\Omega u_t(t)|z_1(t)|^2\mathrm{d}x\Big| \lesssim&\|u_t(t)\|\|z_1(t)\|^2_{L^4(\Omega)} \lesssim \|u_t(t)\|\|z_1(t)\|^{\frac{1}{2}}\|\nabla z_1(t)\|^{\frac{3}{2}}\\ \lesssim&\frac{\alpha}{4}\|\nabla z_1(t)\|^2 + \|z_1(t)\|^2, \ \ \forall \tau \leqslant \tau_0,\end{aligned} \tag{3.17}$$

It follows from (2.13) and (3.16)-(3.17) that

$$J_1(t) \lesssim \|z_\tau\|^2 e^{-2\alpha(t-\tau)} \lesssim \|\psi_\tau\|_E^2 e^{-2\alpha(t-\tau)}, \ \ \forall\, \tau \leqslant \tau_0. \tag{3.18}$$

Note that

$$H_1(\tau) = \|\nabla z_\tau\|^2 - \int_\Omega u_\tau |z_\tau|^2 \mathrm{d}x \lesssim \|\nabla z_\tau\|^2 + \|\nabla u_\tau\|^{\frac{4}{3}} \|z_\tau\|^2 \lesssim \|\psi_\tau\|_E^2 (1 + \|\psi_\tau\|_E^{\frac{4}{3}}). \tag{3.19}$$

Applying Gronwall's inequality to (3.15) and using (3.18)-(3.19) yield

$$\begin{aligned} H_1(t) \lesssim & H_1(\tau) e^{-\alpha(t-\tau)} + \int_\tau^t J_1(s) e^{-\alpha(s-\tau)} \mathrm{d}s \\ \lesssim & \|\psi_\tau\|_E^2 (1 + \|\psi_\tau\|_E^{\frac{4}{3}}) e^{-\alpha(t-\tau)} + \|\psi_\tau\|_E^2 \int_\tau^t e^{-2\alpha(s-\tau)} e^{-\alpha(t-s)} \mathrm{d}s \\ \lesssim & \|\psi_\tau\|_E^2 (1 + \|\psi_\tau\|_E^{\frac{4}{3}}) e^{-\alpha(t-\tau)}, \ \ \forall\, \tau \leqslant \tau_0. \end{aligned} \tag{3.20}$$

Similar with (3.16), we have

$$\Big| \int_\Omega u(t) |z_1(t)|^2 \mathrm{d}x \Big| \lesssim \frac{1}{2} \|\nabla z_1(t)\|^2 + \|z_1(t)\|^2, \ \ \forall\, \tau \leqslant \tau_0.$$

From above estimation and the expression of $H_1(t)$ we get

$$\|\nabla z_1(t)\|^2 \lesssim H_1(t) + \Big| \int_\Omega u(t) |z_1(t)|^2 \mathrm{d}x \Big| \lesssim H_1(t) + \frac{1}{2} \|\nabla z_1(t)\|^2 + \|z_1(t)\|^2, \ \ \forall\, \tau \leqslant \tau_0,$$

which, together with (3.14) and (3.20) gives

$$\|\nabla z_1(t)\|^2 \lesssim H_1(t) + \|z_1(t)\|^2 \lesssim \|\psi_\tau\|_E^2 (1 + \|\psi_\tau\|_E^{\frac{4}{3}}) e^{-\alpha(t-\tau)}, \ \ \forall\, \tau \leqslant \tau_0. \tag{3.21}$$

Secondly, we estimate $\|\nabla u_1(t)\|^2 + \|v_1(t)\|^2$. Using $v_1(t)$ to take inner product with (3.3) in $L^2(\Omega)$ yields

$$\begin{aligned} & \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|v_1(t)\|^2 + (\nu - \delta) \|v_1(t)\|^2 - \delta(\nu - \delta) \big(u_1(t), v_1(t)\big) \\ & \quad + \mu \big(u_1(t), v_1(t)\big) - \big(\Delta u_1(t), v_1(t)\big) = \int_\Omega v_1(t) \mathbf{Re}(\overline{z}(t) z_1(t)) \mathrm{d}x. \end{aligned} \tag{3.22}$$

Direct computations give

$$\begin{cases} \delta(\nu - \delta) \big(u_1(t), v_1(t)\big) = \dfrac{\delta(\nu - \delta)}{2} \dfrac{\mathrm{d}}{\mathrm{d}t} \|u_1(t)\|^2 + \delta^2 (\nu - \delta) \|u_1(t)\|^2, \\ \mu \big(u_1(t), v_1(t)\big) = \dfrac{\mu}{2} \dfrac{\mathrm{d}}{\mathrm{d}t} \|u_1(t)\|^2 + \mu\delta \|u_1(t)\|^2, \\ -\big(\Delta u_1(t), v_1(t)\big) = \dfrac{1}{2} \dfrac{\mathrm{d}}{\mathrm{d}t} \|\nabla u_1(t)\|^2 + \delta \|\nabla u_1(t)\|^2. \end{cases} \tag{3.23}$$

Put

$$\begin{aligned} H_2(t) &= \|\nabla u_1(t)\|^2 + \mu \|u_1(t)\|^2 + \|v_1(t)\|^2 - \delta(\nu - \delta) \|u_1(t)\|^2, \\ J_2(t) &= -2(\nu - 2\delta) \|v_1(t)\|^2 + 2\beta \int_\Omega v_1(t) \mathbf{Re}\big(\overline{z}(t) z_1(t)\big) \mathrm{d}x. \end{aligned}$$

It then follows from (3.22) and (3.23) that

$$\frac{\mathrm{d}H_2(t)}{\mathrm{d}t}+2\delta H_2(t)=J_2(t).$$

By Gronwall's inequality,

$$H_2(t)\lesssim H_2(\tau)e^{-2\delta(t-\tau)}+\int_\tau^t e^{-2\delta(t-s)}J_2(s)\mathrm{d}s,\ \ \forall\,\tau\leqslant t. \tag{3.24}$$

Obviously,

$$H_2(\tau)=\|\nabla u_\tau\|^2+\mu\|u_\tau\|^2+\|v_\tau\|^2-\delta(\nu-\delta)\|u_\tau\|^2\lesssim\|\psi_\tau\|_E^2. \tag{3.25}$$

Also, using Hölder's inequality and Cauchy's inequality,

$$\begin{aligned}J_2(t)\lesssim&-(\nu-2\delta)\|v_1(t)\|^2+\|v_1(t)\|\|z(t)\|_{L^6(\Omega)}\|z_1(t)\|_{L^3(\Omega)}\\ \lesssim&(\nu-2\delta)\|v_1(t)\|^2+\|z(t)\|^2_{L^6(\Omega)}\|z_1(t)\|^2_{L^3(\Omega)}-(\nu-2\delta)\|v_1(t)\|^2\\ \lesssim&\|\nabla z(t)\|^2\|\nabla z_1(t)\|^2\lesssim\|\psi(t)\|_E^2\|\nabla z_1(t)\|^2\lesssim\|\nabla z_1(t)\|^2\\ \lesssim&\|\psi_\tau\|_E^2(1+\|\psi_\tau\|_E^{\frac{4}{3}})e^{-\alpha(t-\tau)},\ \ \forall\,\tau\leqslant\tau_0.\end{aligned} \tag{3.26}$$

Inserting (3.25)-(3.26) into (3.24) yields

$$\begin{aligned}H_2(t)\lesssim&\|\psi_\tau\|^2e^{-2\delta(t-\tau)}+\|\psi_\tau\|_E^2(1+\|\psi_\tau\|_E^{\frac{4}{3}})\int_\tau^t e^{-2\delta(t-s)}e^{-2\alpha(s-\tau)}\mathrm{d}s\\ \lesssim&\|\psi_\tau\|_E^2(1+\|\psi_\tau\|_E^{\frac{4}{3}})e^{-2\delta(t-\tau)},\ \ \forall\,\tau\leqslant\tau_0.\end{aligned} \tag{3.27}$$

Now we choose $\delta>0$ satisfying

$$\delta(\nu-\delta)<\nu\delta<\mu.$$

Notice that $\alpha>2\delta$. Then (3.21) and (3.27) give

$$\|\psi_1(t)\|_E^2\lesssim H_2(t)+\|\nabla z_1(t)\|^2\lesssim\|\psi_\tau\|_E^2(1+\|\psi_\tau\|_E^{\frac{4}{3}})e^{-2\delta(t-\tau)},\ \ \forall\,\tau\leqslant\tau_0.$$

The proof of Lemma 3.2 is completed. □

Form (3.12), Lemma 3.1 and Lemma 3.2, we have

Lemma 3.3. *Let assumption* **(H)** *hold. Then for any given* $\widehat{D}=\{D(s)|s\in\mathbb{R}\}\in\mathcal{D}_\delta$ *and* $\psi_\tau=(u_\tau,v_\tau,z_\tau)^T\in D(\tau)$*, problem* (3.7)-(3.11) *possesses a unique solution* $\psi_2(x,t)=(u_2(x,t),v_2(x,t),z_2(x,t))^T$ *corresponding to* ψ_τ*. Moreover, for each given* $t\in\mathbb{R}$ *there exists a time* $\tau_0=\tau_0(t,\widehat{D})$ *such that*

$$\|\psi_2(x,t)\|_E^2\lesssim 1,\ \ \forall\,\tau\leqslant\tau_0. \tag{3.28}$$

We next prove that the solution of problem (3.7)-(3.11) is bounded in E_1.

Lemma 3.4. *Let assumption* **(H)** *hold. Then for any given* $\widehat{D} = \{D(s)|s \in \mathbb{R}\} \in \mathcal{D}_\delta$ *and* $\psi_\tau = (u_\tau, v_\tau, z_\tau)^T \in D(\tau)$, *there exists a time* $\tau_2 = \tau_2(t, \widehat{D})$ *such that the solution* $\psi_2(x,t) = \big(u_2(x,t), v_2(x,t), z_2(x,t)\big)^T$ *of problem* (3.7)-(3.11) *corresponding to* ψ_τ *satisfies*

$$\begin{aligned}
&\|\Delta u_2(x,t)\|^2 + \|\nabla v_2(x,t)\|^2 + \|\Delta z_2(x,t)\|^2 \\
\lesssim & 1 + \|f(t)\|^2 + \|g(t)\|^2 + \int_{-\infty}^{t} e^{-\delta(t-s)}\big(\|f(s)\|^2 + \|\frac{\partial f(s)}{\partial t}\|^2 + +\|\frac{\partial g(s)}{\partial t}\|^2\big)\mathrm{d}s \\
&+ \int_{-\infty}^{t} e^{-\delta(t-s)}\Big(\int_{-\infty}^{s} e^{-\alpha(s-\theta)}\big(\|f(\theta)\|^2 + \|\frac{\partial f(\theta)}{\partial t}\|^2\big)\mathrm{d}\theta\Big)\mathrm{d}s, \ \ \forall\, \tau \leqslant \tau_2. \qquad (3.29)
\end{aligned}$$

Proof. Let $\widehat{D} = \{D(s)|s \in \mathbb{R}\} \in \mathcal{D}_\delta$ and $\psi_\tau = (u_\tau, v_\tau, z_\tau)^T \in D(\tau)$ be given.

We first estimate $\|\Delta z_2(t)\|^2$. Lemma 3.3 shows that the solution

$$\psi_2(x,t) = (u_2(x,t), v_2(x,t), z_2(x,t))^T$$

of problem (3.7)-(3.11) corresponding to ψ_τ satisfies

$$\|\psi_2(x,t)\|_E^2 = \|\nabla u_2(x,t)\|^2 + \|v_2(x,t)\|^2 + \|\nabla z_2(x,t)\|^2 \lesssim 1, \ \ \forall\, \tau \leqslant \tau_0. \qquad (3.30)$$

Note that $z_2(x,\tau) = 0$ for $x \in \Omega$ due to (3.10). Thus $\Delta z_2(x,\tau) = 0$. We now differentiate equation (3.9) with respect to time t and find that z_{2t} is a solution of the following problem

$$\begin{aligned}
& i z_{2tt} + \Delta z_{2t} + i\alpha z_{2t} + z_{2t} u + u_t z_2 = \frac{\partial f(x,t)}{\partial t}, \ \ t > \tau, & (3.31)\\
& z_{2t}(x,\tau) = -i f(x,\tau). & (3.32)
\end{aligned}$$

Multiplying (3.31) by $2\overline{z}_{2t}$, integrating over Ω and then taking the imaginary part of the resulting equality, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|z_{2t}\|^2 + 2\alpha\|z_{2t}\|^2 =& -2\mathbf{Im}\int_\Omega u_t z_2 \overline{z}_{2t}\mathrm{d}x + 2\mathbf{Im}\int_\Omega \frac{\partial f(x,t)}{\partial t}\overline{z}_{2t}\mathrm{d}x \\
\lesssim& \|u_t\|\|z_2\|_{L^\infty(\Omega)}\|z_{2t}\| + \frac{\alpha}{3}\|z_{2t}\|^2 + \|\frac{\partial f(x,t)}{\partial t}\|^2. \qquad (3.33)
\end{aligned}$$

From (3.9) we see

$$\|\Delta z_2\| \lesssim \|z_{2t}\| + \|z_2\| + \|u\|_{L^4(\Omega)}\|z_2\|_{L^4(\Omega)} + \|f(t)\|. \qquad (3.34)$$

Using Gagliardo-Nirenberg's inequality, (3.30) and (3.34), we have

$$\begin{aligned}
\|z_2\|_{L^\infty(\Omega)} \lesssim& \|z_2\|^{\frac{1}{4}}\|\Delta z_2\|^{\frac{3}{4}} \lesssim \big(\|z_{2t}\| + 1 + \|f(t)\|\big)^{\frac{3}{4}} \\
\lesssim& \|z_{2t}\|^{\frac{3}{4}} + (1 + \|f(t)\|)^{\frac{3}{4}}, \ \ \forall\, \tau \leqslant \tau_0. \qquad (3.35)
\end{aligned}$$

It then follows from (3.33) and (3.35) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|z_{2t}\|^2+2\alpha\|z_{2t}\|^2 \lesssim&\|z_{2t}\|^{\frac{7}{4}}+(1+\|f(t)\|)^{\frac{3}{4}}\|z_{2t}\|+\frac{\alpha}{3}\|z_{2t}\|^2+\|\frac{\partial f(x,t)}{\partial t}\|^2\\ \lesssim&\alpha\|z_{2t}\|^2+(1+\|f(t)\|)^{\frac{3}{2}}+\|\frac{\partial f(x,t)}{\partial t}\|^2.\end{aligned}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z_{2t}(t)\|^2+\alpha\|z_{2t}\|^2\lesssim 1+\|f(t)\|^2+\|\frac{\partial f(x,t)}{\partial t}\|^2,\ \ \forall\,\tau\leqslant\tau_0. \tag{3.36}$$

Using again Gronwall's inequality to (3.36) and then using (3.32) yield

$$\begin{aligned}\|z_{2t}(t)\|^2\lesssim&\|z_{2t}(\tau)\|^2e^{-\alpha(t-\tau)}+\int_\tau^t e^{-\alpha(t-s)}\big(1+\|f(s)\|^2+\|\frac{\partial f(x,s)}{\partial t}\|^2\big)\mathrm{d}s\\ \lesssim&1+\|f(\tau)\|^2e^{-\alpha(t-\tau)}+\int_\tau^t e^{-\alpha(t-s)}\big(\|f(s)\|^2+\|\frac{\partial f(x,s)}{\partial t}\|^2\big)\mathrm{d}s,\ \ \forall\,\tau\leqslant\tau_0.\end{aligned}$$

Combining above estimate and (3.34) we see that there is a $\tau_1=\tau_1(t,\widehat{D})$ such that

$$\|\Delta z_2(t)\|^2\lesssim 1+\|f(t)\|^2+\int_\tau^t e^{-\alpha(t-s)}\big(\|f(s)\|^2+\|\frac{\partial f(x,s)}{\partial t}\|^2\big)\mathrm{d}s,\ \ \forall\,\tau\leqslant\tau_1, \tag{3.37}$$

since, by (2.33), $\lim\limits_{\tau\to-\infty}\|f(\tau)\|^2e^{\delta\tau}=0$.

Secondly, we estimate $\|\Delta u_2(t)\|^2+\|\nabla v_2(t)\|^2$. Multiplying (3.8) with $-\Delta v_2$ and then integrating the resulting equality over Ω yield

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v_2\|^2+(\nu-\delta)\|\nabla v_2\|^2+\delta(\nu-\delta)(u_2,\Delta v_2)-\mu(u_2,\Delta v_2)\\ &\quad+(\Delta u_2,\Delta v_2)=\int_\Omega-\Delta v_2\mathbf{Re}(\bar{z}z_2)\mathrm{d}x-(g,\Delta v_2).\end{aligned} \tag{3.38}$$

Direct computations give

$$\begin{cases}(u_2,\Delta v_2)=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_2\|^2-\delta\|\nabla u_2\|^2,\\ (\Delta u_2,\Delta v_2)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u_2\|^2+\delta\|\Delta u_2\|^2,\\ (g,-\Delta v_2)=-\delta\int_\Omega\Delta u_2g\mathrm{d}x-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Delta u_2g\mathrm{d}x+\int_\Omega\frac{\partial g(x,t)}{\partial t}\Delta u_2\mathrm{d}x.\end{cases} \tag{3.39}$$

Inserting (3.39) into (3.38) yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla v_2\|^2+\|\Delta u_2\|^2+(\mu-\delta(\nu-\delta)))\|\nabla u_2\|^2+2\int_\Omega\Delta u_2g\mathrm{d}x\big)\\ &\quad+(\nu-\delta)\|\nabla v_2\|^2+\delta\|\Delta u_2\|^2+\delta(\mu-\delta(\nu-\delta))\|\nabla u_2\|^2+\delta\int_\Omega\Delta u_2g\mathrm{d}x\\ =&\int_\Omega\nabla v_2\nabla\big(\mathbf{Re}(\bar{z}z_2)\big)\mathrm{d}x+\int_\Omega\frac{\partial g(x,t)}{\partial t}\Delta u_2\mathrm{d}x.\end{aligned} \tag{3.40}$$

Set

$$\begin{aligned}H_3(t) =&\|\nabla v_2\|^2 + (\mu - \delta(\nu - \delta))\|\nabla u_2\|^2 + \|\Delta u_2\|^2 + \int_\Omega \Delta u_2 g \mathrm{d}x,\\ J_3(t) =&2\int_\Omega \nabla v_2 \nabla\big(\mathbf{Re}(\overline{z}z_2)\big)\mathrm{d}x + 2\int_\Omega \frac{\partial g(x,t)}{\partial t}\Delta u_2 \mathrm{d}x\\ &- (2\nu - 3\delta)\|\nabla v_2\|^2 - \delta\|\Delta u_2\|^2 - \delta(\mu - \delta(\nu - \delta))\|\nabla u_2\|^2.\end{aligned}$$

Then from (3.40) it follows that

$$\frac{\mathrm{d}H_3(t)}{\mathrm{d}t} + \delta H_3(t) = J_3(t). \tag{3.41}$$

By Hölder's inequality, Sobolev's embedding theorem and (3.28),

$$\begin{aligned}\Big|2\int_\Omega \nabla v_2 \nabla(\mathbf{Re}(\overline{z}z_2))\mathrm{d}x\Big| \lesssim&\|\nabla v_2\|\big(\|\nabla z\|\|z_2\|_{L^\infty(\Omega)} + \|z\|_{L^6(\Omega)}\|\nabla z_2\|_{L^3(\Omega)}\big)\\ \lesssim&\|\nabla v_2\|\|\Delta z_2\| \lesssim (2\nu - 3\delta)\|\nabla v_2\|^2 + \|\Delta z_2\|^2, \ \ \forall\, \tau \leqslant \tau_0.\end{aligned}$$

Thus by (3.30), (3.37) and Cauchy's inequality,

$$\begin{aligned}J_3(t) \lesssim&(2\nu - 3\delta)\|\nabla v_2\|^2 + \|\Delta z_2\|^2 + 2\int_\Omega \frac{\partial g(x,t)}{\partial t}\Delta u_2 \mathrm{d}x\\ &- (2\nu - 3\delta)\|\nabla v_2\|^2 - \delta\|\Delta u_2\|^2 - \big(\mu - \delta(\nu - \delta)\big)\|\nabla u_2\|^2\\ \lesssim&1 + \|\frac{\partial g(x,t)}{\partial t}\|^2 + \|\Delta z_2\|^2\\ \lesssim&1 + \|f(t)\|^2 + \|\frac{\partial g(x,t)}{\partial t}\|^2\\ &+ \int_\tau^t e^{-\alpha(t-s)}\big(\|f(s)\|^2 + \|\frac{\partial f(x,s)}{\partial t}\|^2\big)\mathrm{d}s, \ \ \forall\, \tau \leqslant \tau_2,\end{aligned} \tag{3.42}$$

where $\tau_2 = \min\{\tau_0(t,\widehat{D}), \tau_1(t,\widehat{D})\}$. Note that for $x \in \Omega$ we have $(u_2(x,\tau), v_2(x,\tau)) \equiv (0,0)$ due to (3.10). Thus $\Delta u_2(x,\tau) = 0$ and $\|\nabla u_2(x,\tau)\| = \|\Delta u_2(x,\tau)\| = \|\nabla v_2(x,\tau)\| = 0$. These gives the fact that

$$\begin{aligned}H_3(\tau) =&\|\nabla v_2(x,\tau)\|^2 + (\mu - \delta(\nu - \delta))\|\nabla u_2(x,\tau)\|^2 + \|\Delta u_2(x,\tau)\|^2 + \int_\Omega \Delta u_2(x,\tau)g(x,\tau)\mathrm{d}x\\ =&0.\end{aligned}$$

Applying Gronwall's inequality to (3.41), then using (3.42) and the fact that $H_3(\tau) = 0$, we obtain

$$
\begin{aligned}
H_3(t) \lesssim & H_3(\tau)e^{-\delta(t-\tau)} + \int_\tau^t e^{-\delta(t-s)} J_3(s)\mathrm{d}s \\
\lesssim & 1 + \int_\tau^t e^{-\delta(t-s)} \big(\|f(s)\|^2 + \|\frac{\partial g(s)}{\partial t}\|^2\big)\mathrm{d}s \\
& + \int_\tau^t e^{-\delta(t-s)} \Big(\int_\tau^s e^{-\alpha(s-\theta)}(\|f(\theta)\|^2 + \|\frac{\partial f(\theta)}{\partial t}\|^2)\mathrm{d}\theta\Big)\mathrm{d}s \\
\lesssim & 1 + \int_{-\infty}^t e^{-\delta(t-s)} \big(\|f(s)\|^2 + \|\frac{\partial g(s)}{\partial t}\|^2\big)\mathrm{d}s \\
& + \int_{-\infty}^t e^{-\delta(t-s)} \Big(\int_{-\infty}^s e^{-\alpha(s-\theta)}(\|f(\theta)\|^2 + \|\frac{\partial f(\theta)}{\partial t}\|^2)\mathrm{d}\theta\Big)\mathrm{d}s, \ \ \forall \tau \leqslant \tau_2.
\end{aligned} \tag{3.43}
$$

From assumption **(H)** we see that the right-hand side of inequality (3.43) is a bounded quantity which is independent of τ. Hence

$$
\|\Delta u_2\|^2 + \|\nabla v_2\|^2 \lesssim H_3(t) + \|\nabla u_2(t)\|^2 + \big|\int_\Omega \Delta u_2 g \mathrm{d}x\big| \lesssim H_3(t) + 1 + \frac{1}{2}\|\Delta u_2\|^2 + \|g\|^2,
$$

that is

$$
\|\Delta u_2\|^2 + \|\nabla v_2\|^2 \lesssim H_3(t) + 1 + \|g(t)\|^2, \ \ \forall \tau \leqslant \tau_2. \tag{3.44}
$$

We obtain (3.29) from (3.37) and (3.43)-(3.44). The proof of Lemma 3.4 is completed. □

From Lemma 3.2 and Lemma 3.4 we see that the solutions operators of problem (3.2)-(3.6) and problem (3.7)-(3.11) generate in the space E continuous processes, $\{S(t,\tau)\}_{t\geqslant\tau}$ and $\{T(t,\tau)\}_{t\geqslant\tau}$, respectively, that is the process $\{U(t,\tau)\}_{t\geqslant\tau}$ can be decomposed as

$$
U(t,\tau) = S(t,\tau) + T(t,\tau).
$$

Moreover, Lemma 3.2 shows that $\{S(t,\tau)\}_{t\geqslant\tau}$ pullback decays exponentially in the manner

$$
\|S(t,\tau)\psi_\tau\|_E^2 \lesssim \|\psi_\tau\|_E^2(1 + \|\psi_\tau\|_E^{\frac{3}{4}})e^{-\delta(t-\tau)}, \ \ \forall \tau \leqslant \tau_0.
$$

Lemma 3.4 indicates that $\{T(t,\tau)\}_{t\geqslant\tau}$ is pullback strongly bounded in E_1. Notice that the embedding $E_1 \hookrightarrow E$ is compact. Thus $\{T(t,\tau)\}_{t\geqslant\tau}$ is pullback-$\mathcal{D}_\delta$ compact in E in the sense that for each given $t \in \mathbb{R}$, each $\widehat{D} = \{D(s)|s \in \mathbb{R}\} \in \mathcal{D}_\delta$ and any sequence $\{\tau_n\}_{n\geqslant 1}$ in $(-\infty, t]$ such that $\tau_n \to -\infty$ as $n \to \infty$, the sequence $\{T(t,\tau_n)\psi(\tau_n)\}$ with any $\psi(\tau_n) \in D(\tau_n)$ possesses a convergent subsequence in E. Combining these analyses and [8, Theorem 3.2], we obtain the following result.

Lemma 3.5. *Let assumption* **(H)** *hold. Then the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *is pullback-*$\mathcal{D}_\delta$ *asymptotically compact in* E.

At this stage, we use the results of Lemma 2.2, Lemma 3.1, Lemma 3.5 and [8, Theorem 3.1] to obtain the main results of this section.

Theorem 3.1. *Let assumption **(H)** hold. Then the process $\{U(t,\tau)\}_{t\geqslant\tau}$ possesses a pullback-$\mathcal{D}_\delta$ attractor $\widehat{\mathcal{A}}_{\mathcal{D}_\delta}=\{\mathcal{A}_{\mathcal{D}_\delta}(t)|t\in\mathbb{R}\}$ in E satisfying*

(a) *Compactness: for every $t\in\mathbb{R}$, $\mathcal{A}_{\mathcal{D}_\delta}(t)$ is a nonempty compact subset of E;*

(b) *Invariance: $U(t,\tau)\mathcal{A}_{\mathcal{D}_\delta}(\tau)=\mathcal{A}_{\mathcal{D}_\delta}(t),\quad\forall\,\tau\leqslant t$;*

(c) *Pullback attraction: $\mathcal{A}_{\mathcal{D}_\delta}(t)$ is pullback-$\mathcal{D}_\delta$ attracting in the following sense,*

$$\lim_{\tau\to-\infty}\mathrm{Dist}_E\big(U(t,\tau)D(\tau),\mathcal{A}_{\mathcal{D}_\delta}(t)\big)=0,\ \ \forall\,\widehat{D}=\{D(s)|s\in\mathbb{R}\}\in\mathcal{D}_\delta,t\in\mathbb{R}.$$

4 Invariant measure, Liouville type theorem and statistical solution

The goal of this section is to construct the statistical solution for the KGS equations. To this end, we will first prove the existence of the family of invariant Borel probability measures for the process $\{U(t,\tau)\}_{t\geqslant\tau}$. Then we establish that this family of probability measures satisfies a Liouville type theorem and is indeed a statistical solution for the KGS equations.

We first prove that the solutions of problem (2.11)-(2.12) depend continuously on the initial data. For any $u,v\in H_0^1(\Omega)$, define

$$(u,v)_\mu=\mu(u,v)+(\nabla u,\nabla v),$$

where μ is the positive constant from equation (1.1) and $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$. Obviously, we have

$$\|\nabla u\|^2\leqslant\|u\|_\mu^2\lesssim(1+\mu)\|\nabla u\|^2,\ \ \forall\,u\in H_0^1(\Omega),$$

which means that $(\cdot,\cdot)_\mu$ is an inner product in $H_0^1(\Omega)$ and $H_\mu=(H_0^1(\Omega),(\cdot,\cdot)_\mu)$ is a Hilbert space equivalent to $H_0^1(\Omega)$ with the usual inner product. Set

$$E_\mu=H_\mu\times L^2(\Omega)\times H_0^1(\Omega)$$

and equip it with the inner product and norm as

$$(\phi,\varphi)_{E_\mu}=(\phi_1,\varphi_1)_\mu+(\phi_2,\varphi_2)+(\nabla\phi_3,\nabla\varphi_3),\ \ \phi=(\phi_1,\phi_2,\phi_3)^T,\varphi=(\varphi_1,\varphi_2,\varphi_3)^T\in E_\mu,$$
$$\|\phi\|_{E_\mu}^2=(\phi,\phi)_{E_\mu},\ \ \phi=(\phi_1,\phi_2,\phi_3)^T\in E_\mu.$$

Obviously, $(E_\mu,(\cdot,\cdot)_{E_\mu})$ is a Hilbert space equivalent to E with the usual inner product.

Lemma 4.1. *For any $\psi = (u, v, z)^T \in E_\mu$, there holds*

$$\mathbf{Re}(\Theta\psi, \psi)_{E_\mu} \geqslant \vartheta(\|u\|_\mu^2 + \|v\|^2) + \frac{\nu}{2}\|v\|^2 + \alpha\|\nabla z\|^2, \tag{4.1}$$

where Θ is the operator defined by (2.9) *and*

$$0 < \vartheta = \frac{\mu\nu}{\sqrt{\nu^2 + 4\mu}(\sqrt{\nu^2 + 4\mu} + \nu)} \in (0, \frac{\nu}{4}). \tag{4.2}$$

Proof. Direct computations imply

$$\mathbf{Re}(\Theta\psi, \psi)_{E_\mu} = \mu\delta\|u\|^2 + \delta\|\nabla u\|^2 + \delta(\delta - \nu)(u, v) + \alpha\|\nabla z\|^2 + (\nu - \delta)\|v\|^2.$$

We now choose ϑ as in (4.2) and

$$0 < \delta = \frac{\mu\nu}{\nu^2 + 4\mu} < \frac{\nu}{4}. \tag{4.3}$$

Then $\delta > \vartheta$ and

$$\begin{aligned}
&\mathbf{Re}(\Theta\psi, \psi)_{E_\mu} - \vartheta(\|u\|_\mu^2 + \|v\|^2) - \frac{\nu}{2}\|v\|^2 - \alpha\|\nabla z\|^2 \\
=&(\delta - \vartheta)\|u\|_\mu^2 + (\frac{\nu}{2} - \delta - \vartheta)\|v\|^2 + \delta(\delta - \nu)(u, v) \\
\geqslant&(\delta - \vartheta)\|u\|_\mu^2 + (\frac{\nu}{2} - \delta - \vartheta)\|v\|^2 - \frac{\delta\nu}{\sqrt{\mu}}\|u\|_\mu\|v\| \geqslant 0,
\end{aligned}$$

since

$$4(\delta - \vartheta)(\frac{\nu}{2} - \delta - \vartheta) = \frac{\delta^2\nu^2}{\mu}.$$

This ends the proof. □

With the above coercivity of the operator Θ in E_μ, we can prove directly the continuous dependence of the solutions of problem (2.11)-(2.12) on the initial data.

Lemma 4.2. *Let $\psi^{(1)}(t) = \psi^{(1)}(x, t)$ and $\psi^{(2)}(t) = \psi^{(2)}(x, t)$ be two solutions of problem* (2.11)-(2.12) *corresponding to the initial data $\psi_\tau^{(1)}$ and $\psi_\tau^{(2)}$, respectively. Then*

$$\|\psi^{(1)}(t) - \psi^{(2)}(t)\|_{E_\mu}^2 \lesssim \|\psi_\tau^{(1)} - \psi_\tau^{(2)}\|_{E_\mu}^2 \exp\Big\{\int_\tau^t \big(\|\psi^{(1)}(s)\|_E + \|\psi^{(2)}(s)\|_E\big)\mathrm{d}s\Big\}. \tag{4.4}$$

Proof. Let

$$\psi^{(k)}(t) = \psi^{(k)}(x, t; \tau, \psi_\tau^{(k)}) = \big(u^{(k)}(x, t), v^{(k)}(x, t), z^{(k)}(x, t)\big)^T, \ \ k = 1, 2,$$

be two solutions of problem (2.11)-(2.12) corresponding to the initial data $\psi_\tau^{(1)}$ and $\psi_\tau^{(2)}$, respectively. Put

$$\begin{cases}
\widetilde{u}(t) = \widetilde{u}(x, t) = u^{(1)}(x, t) - u^{(2)}(x, t), \\
\widetilde{v}(t) = \widetilde{v}(x, t) = v^{(1)}(x, t) - v^{(2)}(x, t), \\
\widetilde{z}(t) = \widetilde{z}(x, t) = z^{(1)}(x, t) - z^{(2)}(x, t), \\
\widetilde{\psi}(t) = \psi^{(1)}(t) - \psi^{(2)}(t).
\end{cases}$$

Then $\widetilde{\psi}(t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\psi}(t)+\Theta\widetilde{\psi}(t)=F(\psi^{(1)}(t),t)-F(\psi^{(2)}(t),t),\quad t>\tau, \tag{4.5}$$
$$\widetilde{\psi}(\tau)=\widetilde{\psi}_\tau=\psi_\tau^{(1)}-\psi_\tau^{(2)}. \tag{4.6}$$

By Lemma 4.1,

$$\mathbf{Re}(\Theta\widetilde{\psi},\widetilde{\psi})_{E_\mu}\geqslant\vartheta(\|\widetilde{u}\|_\mu^2+\|\widetilde{v}\|^2)+\frac{\nu}{2}\|\widetilde{v}\|^2+\alpha\|\nabla\widetilde{z}\|^2,\ \forall\, t\geqslant\tau. \tag{4.7}$$

At the same time, direct computations imply

$$\begin{aligned}
&\|F(\psi^{(1)}(t),t)-F(\psi^{(2)}(t),t)\|_{E_\mu}^2\\
=&\beta^2\big\||z^{(1)}|^2-|z^{(2)}|^2\big\|^2+\||\nabla(z^{(1)}u^{(1)}-z^{(2)}u^{(2)})\|^2\\
\lesssim&\|z^{(1)}-z^{(2)}\|^2\big\||z^{(1)}|+|z^{(2)}|\big\|^2+\|\nabla z^{(1)}\|^2\|u^{(1)}-u^{(2)}\|^2+\|\nabla u^{(2)}\|^2\|z^{(1)}-z^{(2)}\|^2\\
&+\|z^{(1)}\|^2\|\nabla u^{(1)}-\nabla u^{(2)}\|^2+\|u^{(2)}\|^2\|\nabla z^{(1)}-\nabla z^{(2)}\|^2\\
\lesssim&\|\psi^{(1)}-\psi^{(2)}\|_{E_\mu}^2(\|\psi^{(1)}(t)\|_E^2+\|\psi^{(2)}(t)\|_E^2),\ \forall\, t\geqslant\tau.
\end{aligned} \tag{4.8}$$

Taking the real part of the inner product $(\cdot,\cdot)_{E_\mu}$ of equation (4.5) with $\widetilde{\psi}(t)$ first and then using (4.7)-(4.8), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\|\widetilde{\psi}(s)\|_{E_\mu}^2+\sigma\|\widetilde{\psi}(s)\|_{E_\mu}^2\lesssim\|\widetilde{\psi}(s)\|_{E_\mu}^2(\|\psi^{(1)}(s)\|_E+\|\psi^{(2)}(s)\|_E),\ \forall\, s>\tau, \tag{4.9}$$

where $\sigma=\min\{2\vartheta,2\alpha\}>0$. Integrating (4.9) over $[\tau,t]$ yields

$$\|\widetilde{\psi}(t)\|_{E_\mu}^2\lesssim\|\widetilde{\psi}(\tau)\|_{E_\mu}^2+\int_\tau^t\|\widetilde{\psi}(s)\|_{E_\mu}^2\big(\|\psi^{(1)}(s)\|_E+\|\psi^{(2)}(s)\|_E\big)\mathrm{d}s,\ \forall\, t>\tau. \tag{4.10}$$

Applying Gronwall's inequality to (4.10) gives (4.4). The proof of Lemma 4.2 is completed. □

Lemma 4.3. *Let assumption* **(H)** *hold. Then for every $\psi_*\in E$ and every $t\in\mathbb{R}$, the E-valued function $\tau\longmapsto U(t,\tau)\psi_*$ is continuous and bounded on $(-\infty,t]$.*

Proof. Let $\psi_*=(u_*,v_*,z_*)^T\in E$ and $t\in\mathbb{R}$ be given. For any $s_*\in(-\infty,t]$ we next prove that $U(t,\tau)\psi_*$ is continuous at $\tau=s_*$. To this end, we shall establish that for any $\epsilon>0$ there exists some $\eta=\eta(\epsilon)>0$, such that if $r<t$ with $|r-s_*|<\eta$ then

$$\|U(t,r)\psi_*-U(t,s_*)\psi_*\|_E<\epsilon. \tag{4.11}$$

We assume $r<s_*$ without loss of generality. Notice that the norm $\|\cdot\|_{E_\mu}$ is equivalent to $\|\cdot\|_E$. Employing (4.4) and the continuity property of the process, we have

$$\begin{aligned}
&\|U(t,r)\psi_*-U(t,s_*)\psi_*\|_E^2\\
=&\|U(t,s_*)U(s_*,r)\psi_*-U(t,s_*)U(r,r)\psi_*\|_E^2\\
\lesssim&\|U(s_*,r)\psi_*-U(r,r)\psi_*\|_E^2\exp\Big\{\int_{s_*}^t\big(\|U(\theta,r)\psi_*\|_E+\|U(\theta,s_*)\psi_*\|_E\big)\mathrm{d}\theta\Big\}.
\end{aligned} \tag{4.12}$$

Remember that (2.36) shows that $U(\cdot,r)\psi_*$ and $U(\cdot,s_*)\psi_*$ belong to $\mathcal{C}([s_*,t],E)$. Hence

$$\int_{s_*}^{t}\big(\|U(\theta,r)\psi_*\|_E+\|U(\theta,s_*)\psi_*\|_E\big)\mathrm{d}\theta<+\infty. \tag{4.13}$$

So from (2.36) and (4.13) we conclude that the right hand side of (4.12) is as small as needed if $|r-s_*|$ is small enough, that is (4.11) holds true. Therefore, the E-valued function $\tau\longmapsto U(t,\tau)\psi_*$ is continuous on $(-\infty,t]$.

We next prove that the E_μ-valued function $\tau\longmapsto U(t,\tau)\psi_*$ is bounded on $(-\infty,t]$. In fact, for above $\psi_*\in E$ and $t\in\mathbb{R}$, we see from Lemma 2.1 that

$$\begin{aligned}&\lim_{\tau\to-\infty}\|U(t,\tau)\psi_*\|_E^2\\ \lesssim\,& e^{-\delta t}\int_{-\infty}^{t}e^{\delta s}\|G(s)\|^2\mathrm{d}s+e^{-\delta t}\int_{-\infty}^{t}e^{\delta s}\big(e^{-\alpha s}\int_{-\infty}^{s}e^{\alpha\theta}\|f(\theta)\|^2\mathrm{d}\theta\big)^3\mathrm{d}s\\ &+\big(e^{-\alpha t}\int_{-\infty}^{t}e^{\alpha s}\|f(s)\|^2\mathrm{d}s\big)^3+\|f(t)\|^2,\;\; t\in\mathbb{R}.\end{aligned} \tag{4.14}$$

The right-hand side of (4.14) is a bounded quantity which is independent of τ. From this fact and the continuity of $\tau\longmapsto U(t,\tau)\psi_*$ on $(-\infty,t]$ we obtain the desired result. The proof of Lemma 4.3 is completed. □

We next recall the definition of generalized Banach limit and a useful property.

Definition 4.1. ([11,26]) *A generalized Banach limit is any linear functional, which we denote by* $\mathrm{LIM}_{t\to+\infty}$*, defined on the space of all bounded real-valued functions on* $[0,+\infty)$ *that satisfies*

(1) $\mathrm{LIM}_{t\to+\infty}h(t)\geqslant 0$ *for nonnegative functions* $h(\cdot)$ *on* $[0,+\infty)$;

(2) $\mathrm{LIM}_{t\to+\infty}h(t)=\lim\limits_{t\to+\infty}h(t)$ *if the usual limit* $\lim\limits_{t\to+\infty}h(t)$ *exists.*

Let B_+ be the collection of all bounded real-valued functions on $[0,+\infty)$. For any generalized Banach limit $\mathrm{LIM}_{t\to+\infty}$, the following useful property

$$|\mathrm{LIM}_{t\to+\infty}h(t)|\leqslant\limsup_{t\to+\infty}|h(t)|,\;\;\forall\, h(\cdot)\in B_+, \tag{4.15}$$

is presented in [11, (1.38)] and in [9, (2.3)].

Remark 4.1. *Notice that we consider the "pullback" asymptotic behavior and we require generalized limits as* $\tau\to-\infty$*. For a given real-valued function* φ *defined on* $(-\infty,0]$ *and a given Banach limit* $\mathrm{LIM}_{T\to+\infty}$*, we define*

$$\mathrm{LIM}_{t\to-\infty}\varphi(t)=\mathrm{LIM}_{t\to+\infty}\varphi(-t). \tag{4.16}$$

Combining Lemma 2.2, Theorem 3.1, Lemma 4.3 and [26, Theorem 3.1], we obtain the following result.

Theorem 4.1. *Let assumption* **(H)** *hold. Let* $\{U(t,\tau)\}_{t\geqslant\tau}$ *be the process associated to problem* (2.11)-(2.12) *and* $\widehat{\mathcal{A}}_{\mathcal{D}_\delta}=\{\mathcal{A}_{\mathcal{D}_\delta}(t)|t\in\mathbb{R}\}$ *the pullback-*$\mathcal{D}_\delta$ *attractor obtained in Theorem 3.1. Then for a given generalized Banach limit* $\mathrm{LIM}_{t\to+\infty}$ *and a continuous map* $\xi:\mathbb{R}\longmapsto E$ *with* $\xi(\cdot)\in\mathcal{D}_\delta$, *there exists a unique family of Borel probability measures* $\{m_t\}_{t\in\mathbb{R}}$ *in* E *such that the support of the measure* m_t *is contained in* $\mathcal{A}_{\mathcal{D}_\delta}(t)$ *and*

$$
\begin{aligned}
&\mathrm{LIM}_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t\Psi(U(t,s)\xi(s))\mathrm{d}s\\
=&\int_{\mathcal{A}_{\mathcal{D}_\delta}(t)}\Psi(\psi)\mathrm{d}m_t(\psi)=\int_E\Psi(\psi)\mathrm{d}m_t(\psi) \qquad (4.17)\\
=&\mathrm{LIM}_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t\int_E\Psi(U(t,s)\xi(s))\mathrm{d}m_s(\psi)\mathrm{d}s, \qquad (4.18)
\end{aligned}
$$

for any real-valued continuous functional Ψ *on* E. *Moreover,* m_t *is invariant in the sense that*

$$\int_{\mathcal{A}_{\mathcal{D}_\delta}(t)}\Psi(\psi)\mathrm{d}m_t(\psi)=\int_{\mathcal{A}_{\mathcal{D}_\delta}(\tau)}\Psi(U(t,\tau)\psi)\mathrm{d}m_\tau(\psi),\ \ t\geqslant\tau. \tag{4.19}$$

We next introduce the class $\mathcal{T}$ of test function associated to the definition of statistical solutions for equation (2.11). We write (2.11) as

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}=\mathcal{F}(\psi,t), \tag{4.20}$$

where $\mathcal{F}(\psi,t)=-\Theta\psi+F(\psi,t)$. Then $\mathcal{F}(\psi,t):E\times\mathbb{R}\longmapsto E^*$, here E^* is the dual space of E. We expect that the function $\Phi\in\mathcal{T}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\psi(t))=\langle\Phi'(\psi(t)),\mathcal{F}(\psi(t),t)\rangle, \tag{4.21}$$

for every global solution $\psi(t)$ of equation (2.11), where $\langle\cdot,\cdot\rangle$ is the dual pairing between E and E^*.

Definition 4.2. (cf. [11, page 178, Definition 1.2]) *We define the class* $\mathcal{T}$ *of test functions to be the set of real-valued functionals* $\Phi=\Phi(\psi)$ *on* E *that are bounded on bounded subset of* E *and satisfy*

(a) *for any* $\psi\in E$, *the Frechét derivative* $\Phi'(\psi)$ *exists: for each* $\psi\in E$ *there exists an element* $\Phi'(\psi)$ *such that*

$$\frac{|\Phi(\psi+\varphi)-\Phi(\psi)-\langle\Phi'(\psi),\varphi\rangle|}{\|\varphi\|_E}\longrightarrow 0\ \ as\ \ \|\varphi\|_E\to 0,\ \ \varphi\in E;$$

(b) $\Phi'(\psi)\in E$ *for all* $\psi\in E$, *and the mapping* $\psi\longmapsto\Phi'(\psi)$ *is continuous and bounded as a function from* E *to* E;

(c) *for every global solution $\psi(t)$ of equation* (2.11), (4.21) *holds true.*

For example, we can consider the cylindrical test function defined on E. Let φ_1, φ_2 and φ_3 belong to E and γ be a continuously differentiable real-valued function on $\mathbb{R}^3$ with compact support. For each $\psi \in E$, define $\Phi(\psi)$ via

$$\Phi(\psi) = \gamma(\langle\psi, \varphi_1\rangle, \langle\psi, \varphi_2\rangle, \langle\psi, \varphi_3\rangle),$$

where $\langle\psi, \varphi_j\rangle$ is the dual pairing between $\psi \in E \subset E^*$ and $\varphi_j \in E$. Then the function $\Phi(\cdot)$ is obviously continuous from E to $\mathbb{R}$ and in fact is differentiable in E, with differential $\Phi'(\cdot)$ at $\psi \in E$ given by

$$\Phi'(\psi) = \sum_{j=1}^{3} \partial_j \gamma(\langle\psi, \varphi_1\rangle, \langle\psi, \varphi_2\rangle, \langle\psi, \varphi_3\rangle)\varphi_j. \tag{4.22}$$

where $\partial_j \gamma$ denotes the derivative of γ with respect to its j-th coordinate. (4.22) shows that $\Phi'(\cdot) \in E$. Above analyses show that the cylindrical test functions of above form satisfy Definition 4.2.

We now introduce the definition of statistical solution for equation (4.20) and prove its existence.

Definition 4.3. *A family $\{\rho_t\}_{t\in\mathbb{R}}$ of Borel probability measures in E is said to be a statistical solution in the phase space E (or simply a statistical solution) of equation* (4.20) *if the following conditions are satisfied:*

(a) *the function $t \mapsto \int_E \Gamma(\psi)\mathrm{d}\rho_t(\psi)$ is continuous for every $\Gamma \in \mathcal{C}(E)$ (the collection of continuous and bounded functions on E);*

(b) *for almost $t \in \mathbb{R}$, the function $\psi \mapsto \langle\mathcal{F}(\psi(t), t), \phi\rangle$ is ρ_t-integrable for every $\phi \in E$. Moreover, the map*

$$t \mapsto \int_E \langle\mathcal{F}(\psi(t), t), \phi\rangle \mathrm{d}\rho_t(\psi)$$

belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$ for every $\phi \in E$;

(c) *for any cylindrical test function $\Phi \in \mathcal{T}$, it follows that*

$$\int_E \Phi(\psi)\mathrm{d}\rho_t(\psi) - \int_E \Phi(\psi)\mathrm{d}\rho_\tau(\psi) = \int_\tau^t \int_E \langle\mathcal{F}(\psi(s), s), \Phi'(\psi)\rangle \mathrm{d}\rho_s(\psi)\mathrm{d}s,$$

for all $t, \tau \in \mathbb{R}$.

Theorem 4.2. *Let assumption* **(H)** *hold. Then the family of Borel probability measures $\{\mathsf{m}_t\}_{t\in\mathbb{R}}$ obtained in Theorem 4.1 is a statistical solution of equation* (4.20).

Proof. We prove that the family of Borel probability measures $\{m_t\}_{t\in\mathbb{R}}$ satisfies conditions (a), (b) and (c) of Definition 4.3 item by item.

Firstly, for any given $t_* \in \mathbb{R}$, we establish that for every $\Gamma \in \mathcal{C}(E)$ there holds

$$\lim_{t\to t^*} \int_E \Gamma(\psi)\mathrm{d}m_t(\psi) = \int_E \Gamma(\psi)\mathrm{d}m_{t_*}(\psi). \tag{4.23}$$

In fact, from (4.17) and (4.19) we see for $t > t_*$ that

$$\int_E \Gamma(\psi)\mathrm{d}m_t(\psi) - \int_E \Gamma(\psi)\mathrm{d}m_{t_*}(\psi) = \int_{\mathcal{A}_{\mathcal{D}_\delta}(t_*)} \big(\Gamma(U(t,t_*)\psi) - \Gamma(\psi)\big)\mathrm{d}m_{t_*}(\psi). \tag{4.24}$$

Since $U(t,t_*)\psi \longrightarrow \psi$ strongly in E as $t \to t_*^+$, $\Gamma \in \mathcal{C}(E)$ and $\mathcal{A}_{\mathcal{D}_\delta}(t_*)$ is compact in E, (4.24) implies

$$\lim_{t\to t_*^+} \int_E \Gamma(\psi)\mathrm{d}m_t(\psi) = \int_E \Gamma(\psi)\mathrm{d}m_{t_*}(\psi).$$

Similarly,

$$\lim_{t\to t_*^-} \int_E \Gamma(\psi)\mathrm{d}m_t(\psi) = \int_E \Gamma(\psi)\mathrm{d}m_{t_*}(\psi).$$

Thus (4.23) is proved.

Secondly, for every $t \in \mathbb{R}$ we have proved that m_t is carried by $\mathcal{A}_{\mathcal{D}_\delta}(t) \subset E$. Now for every $\phi = (\phi_1,\phi_2,\phi_3)^T \in E$ we define for $\psi = (u,v,z)^T \in E$ that

$$\Psi(\psi) = \langle \mathcal{F}(\psi,s), \phi\rangle. \tag{4.25}$$

Then $\Psi(\cdot) : E \longmapsto \mathbb{R}$. We next establish $\Psi(\cdot) \in \mathcal{C}(E)$. Let $\psi_* = (u_*,v_*,z_*)^T \in E$ be fixed and consider $\psi = (u,v,z)^T \in E$ with $\|\psi_* - \psi\|_E \leqslant 1$. Then

$$|\Psi(\psi_*) - \Psi(\psi)| = |\langle \mathcal{F}(\psi_*,s) - \mathcal{F}(\psi,s), \phi\rangle| \leqslant |\langle \Theta(\psi_* - \psi), \phi\rangle| + |\langle F(\psi_*,s) - F(\psi,s), \phi\rangle|.$$

The term $|\langle \Theta(\psi_* - \psi), \phi\rangle|$ can be bounded as

$$\begin{aligned}
|\langle \Theta(\psi_* - \psi), \phi\rangle| \leqslant& \big|\big(\delta(u_* - u) - (v_* - v), \phi_1\big)\big| + \big|\big((-i\Delta + \alpha)(z_* - z), \phi_3\big)\big| \\
&+ \big|\big((-\Delta - \delta(\nu - \delta) + \mu)(u_* - u) + (\nu - \delta)(v_* - v), \phi_2\big)\big| \\
\lesssim& (\|u_* - u\| + \|v_* - v\|)(\|\phi_1\| + \|\phi_2\|) + \|z_* - z\|\|\phi_3\| \\
&+ \|\nabla(z_* - z)\|\|\nabla\phi_3\| + \|\nabla(u_* - u)\|\|\nabla\phi_2\| \\
\lesssim& \|\psi_* - \psi\|_E\|\phi\|_E.
\end{aligned}$$

At the same time, note that $E \hookrightarrow E^*$ and the norm between E_μ and E is equivalent. Hence, by (4.8) the term $|\langle F(\psi_*,s) - F(\psi,s), \phi\rangle|$ can be bounded as

$$\begin{aligned}
|\langle F(\psi_*,s) - F(\psi,s), \phi\rangle| \lesssim& \|F(\psi_*,s) - F(\psi,s)\|_{E^*}\|\phi\|_E \lesssim \|F(\psi_*,s) - F(\psi,s)\|_E\|\phi\|_E \\
\lesssim& \|F(\psi_*,s) - F(\psi,s)\|_{E_\mu}\|\phi\|_E \lesssim \|\psi_* - \psi\|_{E_\mu}(2\|\psi_*\|_E + 1)\|\phi\|_E \\
\lesssim& \|\psi_* - \psi\|_E(\|\psi_*\|_E + 1)\|\phi\|_E.
\end{aligned}$$

Therefore, we have

$$|\Psi(\psi_*) - \Psi(\psi)| \lesssim \|\psi_* - \psi\|_E(\|\psi_*\|_E + 1)\|\phi\|_E. \tag{4.26}$$

Inequality (4.26) implies that the real-valued function $\Psi(\cdot)$ defined by (4.25) is continuous on E. From (4.17) and (4.25) we conclude that the function $\psi \mapsto \langle \mathcal{F}(\psi(t), t), \phi\rangle = \Psi(\psi)$ is m_t-integrable for every $\phi \in E$. At the same time, we have proved in the previous step that the function

$$t \mapsto \int_E \langle \mathcal{F}(\psi(t), t), \phi\rangle \mathrm{d}m_t(\psi) = \int_E \Psi(\psi)\mathrm{d}m_t(\psi)$$

is continuous on $\mathbb{R}$. Hence it belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$ for every $\phi \in E$.

Thirdly, for any $\Phi \in \mathcal{T}$, we have (4.21). Hence,

$$\Phi(\psi(t)) - \Phi(\psi(\tau)) = \int_\tau^t \langle \mathcal{F}(\psi(\theta), \theta), \Phi'(\psi(\theta))\rangle \mathrm{d}\theta. \tag{4.27}$$

Now for any $s < \tau$, let $\psi_* \in E$ and $\psi(\theta) = U(\theta, s)\psi_*$ for $\theta \geqslant s$. We use (4.27) to deduce

$$\Phi(U(t, s)\psi_*) - \Phi(U(\tau, s)\psi_*) = \int_\tau^t \langle \mathcal{F}(U(\theta, s)\psi_*, \theta), \Phi'(U(\theta, s)\psi_*)\rangle \mathrm{d}\theta. \tag{4.28}$$

By (4.21) and (4.28), we obtain after some calculations,

$$\begin{aligned}
&\int_E \Phi(\psi)\mathrm{d}m_t(\psi) - \int_E \Phi(\psi)\mathrm{d}m_\tau(\psi) \\
=&\int_{\mathcal{A}_{\mathcal{D}_\delta}(t)} \Phi(\psi)\mathrm{d}m_t(\psi) - \int_{\mathcal{A}_{\mathcal{D}_\delta}(\tau)} \Phi(\psi)\mathrm{d}m_\tau(\psi) \\
=&\mathrm{LIM}_{M\to-\infty}\frac{1}{\tau - M}\int_M^\tau \int_E \big(\Phi(U(t, s)\psi_*) - \Phi(U(\tau, s)\psi_*)\big)\mathrm{d}m_s(\psi_*)\mathrm{d}s \\
=&\mathrm{LIM}_{M\to-\infty}\frac{1}{\tau - M}\int_M^\tau \int_E \int_\tau^t \langle \mathcal{F}(U(\theta, s)\psi_*, \theta), \Phi'(U(\theta, s)\psi_*)\rangle \mathrm{d}\theta\mathrm{d}m_s(\psi_*)\mathrm{d}s \\
=&\mathrm{LIM}_{M\to-\infty}\frac{1}{\tau - M}\int_M^\tau \int_\tau^t \int_E \langle \mathcal{F}(U(\theta, s)\psi_*, \theta), \Phi'(U(\theta, s)\psi_*)\rangle \mathrm{d}m_s(\psi_*)\mathrm{d}\theta\mathrm{d}s,
\end{aligned} \tag{4.29}$$

where we have used Fubini's Theorem to swap the order of integration. Now by the property of the process $U(\theta, s) = U(\theta, \tau)U(\tau, s)$ and (4.19),

$$\begin{aligned}
&\int_E \langle \mathcal{F}(U(\theta, s)\psi_*, \theta), \Phi'(U(\theta, s)\psi_*)\rangle \mathrm{d}m_s(\psi_*) \\
=& \int_E \langle \mathcal{F}(U(\theta, \tau)U(\tau, s)\psi_*, \theta), \Phi'(U(\theta, \tau)U(\tau, s)\psi_*)\rangle \mathrm{d}m_s(\psi_*) \\
=& \int_E \langle \mathcal{F}(U(\theta, \tau)\psi_*, \theta), \Phi'(U(\theta, \tau)\psi_*)\rangle \mathrm{d}m_\tau(\psi_*),
\end{aligned}$$

the right-hand side of which is independent to s. Therefore,

$$\begin{aligned}
&\int_{\mathcal{A}_{\mathcal{D}_\delta}(t)} \Phi(\psi)\mathrm{d}m_t(\psi) - \int_{\mathcal{A}_{\mathcal{D}_\delta}(\tau)} \Phi(\psi)\mathrm{d}m_\tau(\psi) \\
&= \int_\tau^t \int_E \langle \mathcal{F}(U(\theta,\tau)\psi_*, \theta), \Phi'(U(\theta,\tau)\psi_*)\rangle \mathrm{d}m_\tau(\psi_*)\mathrm{d}\theta \\
&= \int_\tau^t \int_E \langle \mathcal{F}(\psi(s), s), \Phi'(\psi)\rangle \mathrm{d}m_s(\psi)\mathrm{d}s. \qquad (4.30)
\end{aligned}$$

The proof of Theorem 4.2 is completed. □

We point out that, if statistical equilibrium has been reached by the KGS system, then the statistical informations do not change with time, that is $\Phi'(\psi(t)) = 0$. In this situation, (4.30) implies

$$\int_{\mathcal{A}_{\mathcal{D}_\delta}(t)} \Phi(\psi)\mathrm{d}m_t(\psi) = \int_{\mathcal{A}_{\mathcal{D}_\delta}(\tau)} \Phi(\psi)\mathrm{d}m_\tau(\psi), \ \ t \geqslant \tau, \qquad (4.31)$$

and (4.31) is exactly (4.19), which describes the invariant property of the statistical solution under the action of the process $\{U(t,\tau)\}_{t\geqslant\tau}$. The invariant property of the statistical solution indicates that the shape of the pullback attractor $\mathcal{A}_{\mathcal{D}_\delta}(\tau)$ could change with the evolution of time from τ to t, but the measures of $\mathcal{A}_{\mathcal{D}_\delta}(\tau)$ and $\mathcal{A}_{\mathcal{D}_\delta}(t)$ coincide with each other. This is the result of Liouville Theorem from Statistical Mechanics. Thus we say the statistical solution $\{m_t\}_{t\in\mathbb{R}}$ of the KGS equations satisfies a Liouville type theorem.

Corollary 4.1. *The invariant property of the statistical solution $\{m_t\}_{t\in\mathbb{R}}$ of the KGS equations is a particular situation of the Liouville type theorem.*

References

[1] M. Abounouh, O. Goubet, A. Hakin, Regulartity of the attractor for a couped Klein-Gordon-Schrödinger system, *Diff. Integ. Equations*, **16**(2003), 573-581.

[2] C. D. Aliprentis, K. C. Border, *Infinite Dimensional Analysis*, *A Hithhiker's Guide*, third editon, Springer-Verlag, 2006.

[3] P. Biler, Attractors for the system of Schrödinger and Klein-Gordon equations with Yukawa coupling, *SIAM J. Math. Anal.*, **21**(1990), 1190-1212.

[4] A. Bronzi, C. F. Mondaini, R. Rosa, Trajectory statistical solutions for three-dimensional Navier-Stokes-like systems, *SIAM J. Math. Anal.*, **46**(2014), 1893-1921.

[5] A. Bronzi, R. Rosa, On the convergence of statistical solutions of the 3D Navier-Stokes-α model as α vanishes, *Discrete Cont. Dyn. Syst.*, **34**(2014), 19-49.

[6] A. Bronzi, C. F. Mondaini, R. Rosa, Abstract framework for the theory of statistical solutions, *J. Differential Equations*, **260**(2016), 8428-8484.

[7] T. Caraballo, P. E. Kloeden, J. Real, Invariant measures and statistical solutions of the globally modified Navier-Stokes equations, *Discrete Cont. Dyn. Syst.-B*, **10**(2008), 761-781.

[8] T. Caraballo, A. N. Carvalho, J. A. Langa, F. Rivero, A non-autonomous strongly damped wave equation: Existence and continuity of the pullback attractor, *Nonlinear Anal.*, **74**(2011), 2272-2283.

[9] M. Chekroun, N. E. Glatt-Holtz, Invariant measures for dissipative dynamical systems: Abstract results and applications, *Comm. Math. Phys.*, **316**(2012), 723-761.

[10] C. Foias, G. Prodi, Sur les solutions statistiques equations de Navier-Stokes, *Ann. Mat. Pura Appl.*, **111**(1976), 307-330.

[11] C. Foias, O. Manley, R. Rosa, R. Temam, *Navier-Stokes Equations and Turbulence*, Cambridge University Press, Cambridge, 2001.

[12] C. Foias, R. Rosa, R. Temam, A note on statistical solutions of the three-dimensional Navier-Stokes equations: the stationary case, *C. R. Math.*, **348**(2010), 235-240.

[13] C. Foias, R. Rosa, R. Temam, A note on statistical solutions of the three-dimensional Navier-Stokes equations: the time-dependent case, *C. R. Math.*, **348**(2010), 347-353.

[14] C. Foias, R. Rosa, R. Temam, Properties of time-dependent statistical solutions of the three-dimensional Navier-Stokes equations, *Annales de L'Institut Fourier*, **63**(2013), 2515-2573.

[15] C. Foias, R. Rosa, R. Temam, Convergence of time averages of weak solutions of the three-dimensional Navier-Stokes equations, *J. Stat. Phys.*, **160**(2015), 519-531.

[16] C. Foias, R. Rosa, R. Temam, Properties of stationary statistical solutions of the three-dimensional Navier-Stokes equations, *J. Dyn. Differential Equations*, **31**(2019), 1689-1741.

[17] B. Guo, Y. Li, Attractors for dissipative Klein-Gordon-Schrödinger equations in $\mathbb{R}^3$, *J. Differential Equations*, **136**(1997), 356-377.

[18] P. E. Kloeden, P. Marín-Rubio, J. Real, Equivalence of invariant measures and stationary statistical solutions for the autonomous globally modified Navier-Stokes equations, *Comm. Pure Appl. Anal.*, **8**(2009), 785-802.

[19] H. Lange, B. Wang, Regularity of global attractor for the Klein-Gordon-Schrödinger equations, *Math. Meth. Appl. Sci.*, **22**(1999), 535-1594.

[20] B. Levant, F. Ramos, E. S. Titi, On the statistical properties of the 3D incompressible Navier-Stokes-Voigt model, *Comm. Math. Sci.*, **8**(2010), 277-293.

[21] Y. Li, B. Guo, Asymptotic smoothing effect for weakly dissipative Klein-Gordon-Schrödinger equations, *J. Math. Anal. Appl.*, **282**(2003), 256-265.

[22] X. Li, W. Shen, C. Sun, Invariant measures for complex-valued dissipative

dynamical systems and applications, *Discrete Cont. Dyn. Syst.-B*, **22**(2017), 2427-2446.

[23] K. Lu, B. Wang, Global attractor for the dissipative Klein-Gordon-Schrödinger equations in unbounded domains, *J. Differential Equations*, **170**(2001), 281-316.

[24] G. Łukaszewicz, Pullback attractors and statistical solutions for 2-D Navier-Stokes equations, *Discrete Cont. Dyn. Syst.-B*, **9**(2008), 643-659.

[25] G. Łukaszewicz, J. Real, J. C. Robinson, Invariant measures for dissipative dynamical systems and generalised Banach limits, *J. Dyn. Differential Equations*, **23**(2011), 225-250.

[26] G. Łukaszewicz, J. C. Robinson, Invariant measures for non-autonomous dissipative dynamical systems, *Discrete Cont. Dyn. Syst.*, **34**(2014), 4211-4222.

[27] Cláudia B. Gentile Moussa, Invariant measures for multivalued semigroups, *J. Math. Anal. Appl.*, **455**(2017), 1234-1248.

[28] M. I. Vishik, A. V. Fursikov, Translationally homogeneous statistical solutions and individual solutions with infinite energy of a system of Navier-Stokes equations, *Siberian Mathematical Journal*, **19**(1978), 710-729.

[29] B. Wang, H. Lange, Attractors for the Klein-Gordon-Schrödinger equations, *J. Math. Phys.*, **40**(1999), 2445-2457.

[30] X. Wang, Upper-semicontinuity of stationary statistical properties of dissipative systems, *Discrete Cont. Dyn. Syst.*, **23**(2009), 521-540.

[31] C. Zhao, L. Yang, Pullback attractor and invariant measure for the globally modified Navier-Stokes equations, *Comm. Math. Sci.*, **15**(2017), 1565-1580.

[32] C. Zhao, T. Caraballo, Asymptotic regularity of trajectory attractor and trajectory statistical solution for the 3D globally modified Navier-Stokes equations, *J. Differential Equations*, **266**(2019), 7205-7229.

[33] C. Zhao, Y. Li, T. Caraballo, Trajectory statistical solutions and Liouville type equations for evolution equations: Abstract results and applications, *J. Differential Equations*, **269**(2020), 467-494.

[34] C. Zhao, Z. Song, T. Caraballo, Strong trajectory statistical solutions and Liouville type equations for dissipative Euler equations, *Appl. Math. Lett.*, **99**(2020), 105981.

[35] C. Zhao, Y. Li, Z. Song, Trajectory statistical solutions for the 3D Navier-Stokes equations: the trajectory attractor approach, *Nonlinear Anal.-RWA*, **53** (2020) 103077

[36] C. Zhao, H. Jiang, T. Caraballo, Trajectory statistical solutions and Liouville type theorem for nonlinear wave equations with polynomial growth, *Adv. Differential Equations*, Accepted for publication.

[37] Z. Zhu, C. Zhao, Pullback attractor and invariant measures for the three-dimensional regularized MHD equations, *Discrete Cont. Dyn. Syst.*, **38**(2018), 1461-1477.