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PULLBACK EXPONENTIAL ATTRACTORS FOR DIFFERENTIAL EQUATIONS WITH DELAY

Sana Netchaoui, Mohamed Ali Hammami

Department of Mathematics, Faculty of Sciences of Sfax,
University of Sfax, Route de la Soukra km 4 Sfax 3038, Tunisia

and

Tomás Caraballo

Departamento de Ecuaciones Diferenciales y Análisis Numérico
Facultad de Matemáticas, Universidad de Sevilla
c/ Tarfia s/n, 41012-Sevilla, Spain

Abstract. We show the existence of an exponential attractor for non-autonomous dynamical system with bounded delay. We considered the case of strong dissipativity then prove that the result remains for the weak dissipativity. We conclude then the existence of the global attractor and ensure the boundedness of its fractal dimension.

1. **Introduction.** The theory of attractors for retarded dynamical systems was developed in the global sense by Hale [11]. Then, the notion of pullback attractor was introduced to generalize the autonomous case to the general non autonomous framework and, recently, the theory of exponential attractors was introduced for evolution processes considering their robustness under perturbation. However, the existence of an exponential attractor implies the existence of the global attractor, which means that one way of proving the existence of the global attractor is to find an exponential attractor which also means that its fractal dimension is finite. Technical theorems were established in [5, 4] to prove the existence of attractors for asymptotically compact evolution processes. The existence of pullback attractors for differential equation with one variable delay was studied in [2]. Then Caraballo et al. [1] proved a more general result for the existence of pullback attractors for autonomous and non-autonomous retarded systems with one varying and distributed delay, where the variable delay was differentiable and its derivative is bounded by 1. In paper [16] the authors improve that result by showing the existence of exponential attractors for differential an equation with one varying delay which, in particular, means the existence of a pullback attractor with bounded fractal dimension. With the analysis we carry out in our current paper, in particular we extend the previous results obtained for variable delay to a much wider class of delay differential equations (which, in particular, includes the cases of variable and distributed delay).

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The main feature for our research is to impose a Lipschitz-like integral condition on the term containing the delays.
Consider the following delay differential equation

$$\begin{cases} x'(t) &= f(t,x_t), \qquad x_t \in \mathcal{C}, \\ x_s &= u, \quad u \in \mathcal{C}, \end{cases} \tag{1}$$

where $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is continuous and maps bounded sets into bounded sets, where $\mathcal{C} = C([-h,0],\mathbb{R}^n)$ denotes the Banach space of continuous functions with the sup-norm, which is the usual space when we deal with delay equations, and for a given continuous function $x(\cdot) : \mathbb{R} \to \mathbb{R}^n$ and $t \in \mathbb{R}$, we denote by $x_t(\cdot)$ an element in $\mathcal{C}$ given by

$$x_t(\theta) = x(t+\theta), \qquad \theta \in [-h,0]$$

When $x(\cdot)$ is a solution of (1), then $x_t(\cdot)$ is said to be the solution segment at time t. It is known that for any $(s,u) \in \mathbb{R} \times \mathcal{C}$, there exists a solution $x(t,s,u)$ for (1) defined on $[s-h,\alpha_{s,u})$. We assume that $\alpha_{s,u} = +\infty$, for all $s \in \mathbb{R}$ since we are interested in the long-time behavior of solutions.

Remark 1.1. Uniqueness holds straightforwardly if, for instance f is locally Lipschitz with respect to its second variable, see [11] for more details.

If f is such that uniqueness of solution holds, then we define the associated evolution process by:

$$U(t,s)u = x_t(\cdot,s,u), \qquad \forall t \geq s \in \mathbb{R} \qquad u \in \mathcal{C},$$

where $x(\cdot,s,u)$ denotes the solution of (1).

The article is organized as follows. In Section 2 we recall main definitions and notations for the theory of non-autonomous dynamical systems that will be needed. Next, in Section 3, we state two main results for the existence of exponential attractors when the absorbing set is uniform and when it is time-dependent. Our main result is proved in Section 4 where we develop an abstract concept for the existence of an exponential attractor for a class of differential equation with finite delay and, finally in Section 5, we prove first the existence of an exponential attractor when the dependence in time is uniform, and then prove that the result remains true when the dissipativity is time dependent.

2. Preliminaries.

We recall some basic notions from the theory of dynamical systems that will be needed in the subsequent sections.

Definition 2.1. A two-parameter family $\{U(t,s)|\ t,s \in \mathbb{R},\ t \geq s\}$ of continuous operators from $\mathcal{C}$ into itself is called an evolution process in $\mathcal{C}$ if it satisfies the following properties

(i) $U(t,s) \circ U(s,r) = U(t,r), \qquad t \geq s \geq r,$
(ii) $U(t,t) = Id, \qquad t \in \mathbb{R},$
(iii) $U(t,s) : \mathcal{C} \to \mathcal{C}$ is a continuous map for all $t \geq s$.

As we look for attracting sets, we introduce first the Hausdorff semi-distance between subsets A and B in a metric space (X,d) as

$$dist(A,B) = \sup_{a\in A} \inf_{b\in B} d(a,b)$$

Definition 2.2. Let U be a process on a complete metric space X. A family of compact sets $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is said to be the global pullback attractor for U if it satisfies the following properties:

(i) It is invariant: $U(t,s)\mathcal{A}(s) = \mathcal{A}(t)$ for all $t \geq s$.
(ii) It is pullback attracting i.e

$$\lim_{s\to\infty} dist(U(t,t-s)D, \mathcal{A}(t)) = 0, \quad \text{for all bounded subsets} D \quad \text{of} \quad X$$

.

(iii) It is minimal, that is, for any closed attracting set Y at time t, we have $\mathcal{A}(t) \subset Y$.

We recall now the definition of an absorbing set which is crucial when we deal with global and exponential attractors.

Definition 2.3. $\{\mathcal{B}(t)\}_{t\in\mathbb{R}}$ is said to be absorbing with respect to the process U if, for all $t \in \mathbb{R}$ and all $D \subset X$ bounded, there exists $T_D(t) > 0$ such that for all $\tau \geq T_D(t)$

$$U(t, t-\tau)D \subset \mathcal{B}(t).$$

Theorem 2.4. *Suppose that $U(t,s)$ maps bounded sets into bounded sets and there exists a family $\{\mathcal{B}(t)\}_{t\in\mathbb{R}}$ of bounded absorbing sets for U. Then there exists a pullback attractor for problem (1).*

Proof. See [2]. □

Definition 2.5. Let $U(t,s), t \geq s$, be an evolution process in X. The family of non-empty compact subsets $\mathcal{M} = \{\mathcal{M}(t)|t \in \mathbb{R}\}$ is called a **pullback exponential attractor** for the evolution process U if

- $\mathcal{M}$ is positively invariant, i.e.

$$U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t) \qquad \forall t \geq s,$$

- the fractal dimension of the sections $\mathcal{M}(t), t \in \mathbb{R}$, is uniformly bounded,

$$\sup_{t\in\mathbb{R}}\{\dim_f(\mathcal{M}(t))\} < \infty,$$

- and $\mathcal{M}$ exponentially pullback attracts all bounded sets, i.e. there exists a constant $\omega > 0$ such that for every bounded subset $D \subset X$ and every $t \in \mathbb{R}$

$$\lim_{s\to\infty} e^{\omega s}\text{dist}_H\big(U(t,t-s)D, \mathcal{M}(t)\big) = 0.$$

Where the fractal dimension of a precompact subset $A \subset X$ is defined as

$$\dim_f(A) = \lim_{\varepsilon\to 0} \frac{\ln(N_\varepsilon^X(A))}{\ln(\frac{1}{\varepsilon})},$$

such that $N_\varepsilon^X(A)$ denotes the minimal number of ε-balls in X with centers in A needed to cover A.

3. **General existence Theorems.** For the reader convenience, we will include now some results which are crucial for our analysis and we have already been published, e.g., in [16].

3.1. **Strongly bounded evolution processes.** We first recall an existence theorem for evolution process that are strongly bounded.

Theorem 3.1. *Let $U(t,s), t \geq s$, be an evolution process in a Banach space W. Moreover, we assume that the following properties are satisfied for some $t_0 \in \mathbb{R}$:*

(A_0) *Let V be another Banach space such that the embedding $V \hookrightarrow\hookrightarrow W$ is dense and compact.*

(A_1) *There exists a bounded set $B \subset W$ that pullback absorbs all bounded sets at times $t \leq t_0$, i.e. for every bounded set $D \subset W$ there exists $T_D \geq 0$ such that*

$$\bigcup_{t \leq t_0} U(t, t-s)D \subset B \qquad \forall s \geq T_D.$$

(A_2) *The evolution process $U(t,s), t \geq s$, satisfies the smoothing property in B, i.e. there exist positive constants $\tilde{t} > 0$ and κ such that*

$$\|U(t, t-\tilde{t})u - U(t, t-\tilde{t})v\|_V \leq \kappa \|u - v\|_W \qquad \forall u, v \in B, t \leq t_0.$$

(A_3) *The evolution process $U(t,s), t \geq s$, is Lipschitz continuous in B, i.e. for all $t \leq t_0$, $t - \tilde{t} \leq s \leq t$, there exists $L_{t,s} \geq 0$ such that*

$$\|U(t,s)u - U(t,s)v\|_W \leq L_{t,s}\|u - v\|_W \qquad \forall u, v \in B.$$

Then, for every $\nu \in (0, \frac{1}{2})$ there exists a pullback exponential attractor $\mathcal{M}^\nu$ in W, and its fractal dimension is bounded by

$$\dim_f^W(\mathcal{M}^\nu(t)) \leq \log_{\frac{1}{2\nu}} \left(N^W_{\frac{\nu}{\kappa}}(B_1^V(0)) \right) \qquad \forall t \in \mathbb{R}.$$

Proof. See Theorem 3.4 and Remark 3 in [4]. □

Corollary 3.2. *Let $U(t,s), t \geq s$, be an evolution process in a Banach space W. If the hypotheses $(A_0), (A_1)$ and (A_2) are satisfied, then the evolution process $U(t,s), t \geq s$, possesses a global pullback attractor, and its fractal dimension is bounded by*

$$\dim_f^W(\mathcal{A}(t)) \leq \inf_{\nu \in (0, \frac{1}{2})} \left\{ \log_{\frac{1}{2\nu}} \left(N^W_{\frac{\nu}{\kappa}}(B_1^V(0)) \right) \right\} \qquad \forall t \in \mathbb{R}.$$

Proof. See Theorem 3.4 in [4] and the proof of Theorem 3.1 in [5]. □

3.2. **Time-dependent families of pullback absorbing sets.** In this section the fixed bounded pullback absorbing set B is replaced by a time-dependent family of absorbing sets $B = \{B(t)\}_{t \in \mathbb{R}}$ with certain properties. The following result generalizes Theorem 3.1.

Theorem 3.3. *Let $U(t,s), t \geq s$, be an evolution process in a Banach space W and (A_0) be satisfied. Moreover, we assume that the following properties hold for some $t_0 \in \mathbb{R}$:*

$(\widetilde{A}_1)$ *There exists a family of bounded pullback absorbing sets $B = \{B(t)\}_{t \in \mathbb{R}}$ in W, i.e. for every bounded set $D \subset W$ and $t \leq t_0$, there exists $T_{D,t} \geq 0$ such that*

$$U(s, s-r)D \subset B(s) \qquad \forall r \geq T_{D,t},\ s \leq t.$$

Moreover, there exists $\tilde{t} > 0$ such that

$$U(t, t-\tilde{t})B(t-\tilde{t}) \subset B(t) \qquad \forall t \leq t_0,$$

and the diameter of the family of absorbing sets $B = \{B(t)\}_{t\in\mathbb{R}}$ grows at most sub-exponentially in the past, i.e.

$$\lim_{t\to-\infty} \operatorname{diam}(B(t))e^{\gamma t} = 0 \qquad \forall \gamma > 0.$$

$(\widetilde{A}_2)$ *The evolution process $U(t,s), t \geq s$, satisfies the smoothing property in B, i.e. there exist a positive constant κ such that*

$$\|U(t,t-\tilde{t})u - U(t,t-\tilde{t})v\|_V \leq \kappa \|u-v\|_W \qquad \forall u,v \in B(t-\tilde{t}),\ t \leq t_0.$$

$(\widetilde{A}_3)$ *The evolution process $U(t,s), t \geq s$, is Lipschitz continuous in B, i.e. for all $t \in \mathbb{R}$, $t \leq s \leq t+\tilde{t}$, there exists $L_{s,t} \geq 0$ such that*

$$\|U(s,t)u - U(s,t)v\|_W \leq L_{s,t}\|u-v\|_W \qquad \forall u,v \in B(t).$$

Then, for every $\nu \in (0, \frac{1}{2})$ there exists a pullback exponential attractor $\mathcal{M}^\nu$ in W, and its fractal dimension is bounded by

$$\dim_f^W(\mathcal{M}^\nu(t)) \leq \log_{\frac{1}{2\nu}}\left(N^W_{\frac{\nu}{\kappa}}(B_1^V(0))\right) \qquad \forall t \in \mathbb{R}.$$

Proof. This is a particular case of Theorem 3.4 in [4] □

Corollary 3.4. *Let $U(t,s), t \geq s$, be an evolution process in a Banach space W. If the hypotheses $(A_0), (\widetilde{A}_1)$ and $(\widetilde{A}_2)$ are satisfied, then the evolution process $U(t,s), t \geq s$, possesses a global pullback attractor, and its fractal dimension is bounded by*

$$\dim_f^W(\mathcal{A}(t)) \leq \inf_{\nu\in(0,\frac{1}{2})}\left\{\log_{\frac{1}{2\nu}}\left(N^W_{\frac{\nu}{\kappa}}(B_1^V(0))\right)\right\} \qquad \forall t \in \mathbb{R}.$$

Proof. See Theorem 3.4 in [4] and the proof of Theorem 3.1 in [5]. □

4. **Strong dissipativity.** Consider the delay differential equation in the general framework

$$\begin{aligned} &x'(t) = f(t,x_t), \qquad t > s,\\ &x_s = u, \qquad u \in \mathcal{C}, \end{aligned} \tag{2}$$

Assume that $f(\cdot,\cdot) : \mathbb{R}\times\mathcal{C} \to \mathbb{R}^n$ satisfies the following conditions

- *i)* A Global Lipschitz condition

$$|f(t,\psi_1) - f(t,\psi_2)| \leq L_f\|\psi_1-\psi_2\|_{\mathcal{C}}, \qquad \forall \psi_1,\psi_2 \in \mathcal{C}. \tag{3}$$

- *ii)* For all $u,v \in C^0([\tau-h,t];\mathbb{R}^n)$,

$$\exists C_h > 0 : \int_\tau^t |f(s,u_s) - f(s,v_s)|ds \leq C_h \int_{\tau-h}^t |u(s)-v(s)|ds, \tag{4}$$

- *iii)* A dissipative condition:
  The function $f : \mathbb{R}\times\mathcal{C} \to \mathbb{R}^n$ in (1) is strongly dissipative, i.e., there exists positive constants α and β such that

$$\langle f(t,\phi), \phi(0)\rangle \leq -\alpha|\phi(0)|^2 + \beta \qquad \forall \phi \in \Phi(h)\mathcal{C}, \tag{5}$$

  where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^n$ and

$$\Phi(h)\mathcal{C} = \{\phi \in \mathcal{C} | \phi = U(s+h,s)\psi \text{ for some } s \in \mathbb{R}, \psi \in \mathcal{C}\}$$

  is the set of functions in $\mathcal{C}$ that are realisable as solutions after time h.

Theorem 4.1. *Let $U(t,s), t \geq s$, be the evolution process generated by problem (2). We assume that hypotheses (3), (4) and (5) are satisfied.*

Then, there exists a pullback exponential attractor for $U(t,s), t \geq s$, in $\mathcal{C}$. Moreover, the global pullback attractor exists, it is contained in the pullback exponential attractor, and its fractal dimension is finite.

Proof. We verify the hypotheses of Theorem 3.3 for the spaces $W = \mathcal{C}$ and $V = \mathcal{C}^1$. Certainly, (A_0) is satisfied due to the compact embedding $\mathcal{C}^1 \hookrightarrow\hookrightarrow \mathcal{C}$.

(i) Existence of a family of pullback absorbing sets:
We proved in [16] the existence of a family of bounded absorbing set when the delay was time varying, following the same computation we obtain the same result under condition (5).

(ii) Smoothing property $(\widetilde{A}_2)$: Let $s \in \mathbb{R}$, $u, v \in B(s)$ and $x(t,u,s)$ and $y(t,u,s)$, $t \geq s$, denote the corresponding solutions of (2). In order to shorten notations, in the sequel we write $x(t) = x(t,u,s)$ and $y(t) = y(t,v,s), t \geq s$. For the difference of x and y we obtain

$$\begin{aligned}\|U(s+h,s)u - U(s+h,s)v\|_{\mathcal{C}^1} &= \|x(s+h+\cdot) - y(s+h+\cdot)\|_{\mathcal{C}^1}\\ =&\|x(s+h+\cdot) - y(s+h+\cdot)\|_{\mathcal{C}} + \|x'(s+h+\cdot) - y'(s+h+\cdot)\|_{\mathcal{C}}.\end{aligned}$$

Using (4) we have:

$$\begin{aligned}|x(t) - y(t)| \leq &\|u - v| + \int_s^t |f(\tau, x_\tau) - f(\tau, y_\tau)| d\tau\\ &\leq \|u - v\|_{\mathcal{C}} + C_h \int_{s-h}^t |x(\tau) - y(\tau)| d\tau\\ &\leq \|u - v\|_{\mathcal{C}} + C_h \int_{s-h}^s |x(\tau) - y(\tau)| d\tau + C_h \int_s^t |x(\tau) - y(\tau)| d\tau\\ =&(1 + hC_h)\|u - v\|_{\mathcal{C}}.\end{aligned}$$

Hence Gronwall's Lemma implies

$$|x(t) - y(t)| \leq (1 + hC_h) \exp C_h(t - s) \|u - v\|_{\mathcal{C}}. \tag{6}$$

Let now $\theta \in [-h, 0]$ and $t = s + h + \theta$, then using (6)

$$|x(s+h+\theta) - y(s+h+\theta)| \leq (1 + hC_h) \exp C_h(h + \theta) \|u - v\|_{\mathcal{C}}. \tag{7}$$

After taking the supremum over $\theta \in [-h, 0]$

$$\|x(s+h+.) - y(s+h+.)\|_{\mathcal{C}} \leq (1 + hC_h) \exp C_h h \|u - v\|_{\mathcal{C}}. \tag{8}$$

On the other hand, we observe that

$$\begin{aligned}&|x'(s+h+\theta) - y'(s+h+\theta)|\\ &\leq |f(s+h+\theta, x_{s+h+\theta}) - f(s+h+\theta, y_{s+h+\theta})|\\ &\leq L_f |x(s+h+\theta) - y_(s+h+\theta)|\\ &\leq L_f \|x(s+h+\cdot) - y(s+h+\cdot)\|_{\mathcal{C}}\\ &\leq L_f (1 + hC_h) \exp C_h h \|u - v\|_{\mathcal{C}},\end{aligned}$$

where we used (8) in the last estimate. Now taking the supremum over $t \in [-h, 0]$, it follows that

$$\|x'(s+h+\cdot) - y'(s+h+\cdot)\|_{\mathcal{C}} \leq L_f (1 + hC_h) \exp hC_h \|u - v\|_{\mathcal{C}}. \tag{9}$$

Finally, summing up inequalities (8) and (9) we obtain

$$\begin{aligned}
&\|U(s+h,s)u - U(s+h,s)v\|_{\mathcal{C}^1} \\
&= \|x(s+h+\cdot) - y(s+h+\cdot)\|_{\mathcal{C}^1} \\
&\leq (1+L_f)(1+hC_h)\exp hC_h \|u-v\|_{\mathcal{C}} \\
&= \kappa\|u-v\|_{\mathcal{C}},
\end{aligned}$$

which proves the smoothing property for $\tilde{t} = h$.

(iii) Lipschitz continuity $(\widetilde{A}_3)$**:** Using (6)

$$\|U(t,s)u - U(t,s)v\|_{\mathcal{C}} \leq (1+hC_h)\exp C_h(t-s)\|u-v\|_{\mathcal{C}}. \qquad \forall t \geq s,$$

which proves $(\widetilde{A}_3)$.

(iv) Existence of the pullback exponential attractor: The existence of a family of bounded absorbing sets and the smoothing property was proved in (i). The Lipschitz continuity $(\widetilde{A}_3)$ was shown in (ii). Consequently, all assumptions of Theorem 3.3 are fulfilled, which implies the existence of a pullback exponential attractor. □

4.1. **Application: (i)Differential Equation With One Variable Delay.** Consider a delay differential equation where

$$f(t,x_t) = F(x(t-\rho(t))), \tag{10}$$

with $F : \mathbb{R}^N \to \mathbb{R}^N$ a function such that there exists $L_F > 0$ for which

- F is Globally lipschitz in time

$$|F(x) - F(y)|_{\mathbb{R}^N} \leq L_F|x-y|_{\mathbb{R}^N}, \forall x,y \in \mathbb{R}^N, \tag{11}$$

  and $\rho \in \mathcal{C}^1(\mathbb{R},[0,h])$ such that $\sup_{t\in\mathbb{R}} = \rho^\star < 1$.
- The function F is dissipative in a similar sense to (5), so that there exists some $\alpha_0 > 0$ and $\beta_0 \geq 0$ such that

$$\langle F(x), x\rangle \leq -\alpha_0^2|x|^2 + \beta_0 \tag{12}$$

Remark 4.2. In this case we assumed in a more general way that $f(t,\psi) = F(\psi(-\rho(t)))$ for all $\psi \in \mathcal{C}, t \in \mathbb{R}$. The existence of pullback exponential attractor was proved in [16] by direct computation.

Here we ensure that, in this case, condition (4) holds true by setting $C_h = \frac{L_F}{1-\rho^\star}$. Indeed

$$\begin{aligned}
\int_\tau^t |f(s,x_s) - f(s,y_s)|ds &\leq \int_\tau^t |F(x(s-\rho(s))) - F(y(s-\rho(s)))|ds \\
&\leq L_F \int_\tau^t |x(s-\rho(s)) - y(s-\rho(s))|ds \\
&\leq \frac{L_F}{1-\rho^\star}\int_{\tau-\rho(\tau)}^{t-\rho(t)} |x(u)-y(u)|du \\
&\leq \frac{L_F}{1-\rho^\star}\int_{\tau-h}^{t} |x(u)-y(u)|du.
\end{aligned}$$

Hence all conditions of Theorem 4.1 are fulfilled and the existence of exponential attractor under these general conditions holds straightforwardly.

Next we prove that Theorem 4.1 can be applied even for equations containing not only variable delay but also distributed delay like the following integro–differential equation with finite delay.

(ii)Integro differential equation:

To simplify the computations we will consider only the term containing the distributed delay, the rest was already done in the previous example. We consider the integro–differential equation introduced below.

$$x'(t) = F_0(t, x(t)) + F_1(t, x(t-\rho(t))) + \int_{-h}^{0} b(t, s, x(t+s))ds \tag{13}$$

where the functions F_0, F_1, and b are uniformly Lipschitz, i.e., there exist positive constants L_0, L_1 and L_b such that
$(H1)$

$$\begin{aligned}
|F_0(t,x) - F_0(t,y)| &\leq L_0|x-y| \qquad \forall x, y \in \mathbb{R}^n, \quad \forall t \in \mathbb{R} \\
|F_1(t,x) - F_1(t,y)| &\leq L_1|x-y| \qquad \forall x, y \in \mathbb{R}^n, \quad \forall t \in \mathbb{R} \\
|b(t,s,x) - b(t,s,y)| &\leq L_b|x-y| \qquad \forall x, y \in \mathbb{R}^n, \forall \quad (t,s) \in \mathbb{R} \times [-h, 0].
\end{aligned}$$

We suppose that F_0 is strongly dissipative, i.e. there exists $\alpha, \beta > 0$ such that

$$\langle x, F_0(x) \rangle \leq -\alpha|x|^2 + \beta. \tag{14}$$

F_1 and b are sublinear

$$|F_1(t,x)|^2 \leq k_1^2 + k_2^2|x|^2, \tag{15}$$

and

$$|b(t,s,x)| \leq m_0 + m_1(s)|x|. \tag{16}$$

We prove that this case fulfils also condition (4). Indeed

$$\begin{aligned}
\int_{\tau}^{t} |f(s, x_s) - f(s, y_s)|ds &\leq \int_{\tau}^{t} \int_{-h}^{0} |b(u, x(u+s)) - b(u, y(u+s))|duds \\
&\leq L_b \int_{\tau}^{t} \int_{-h}^{0} |x(u+s) - y(u+s)|duds \\
&\leq L_b \int_{-h}^{0} \int_{\tau}^{t} |x(u+s) - y(u+s)|dsdu \\
&\leq L_b \int_{-h}^{0} \int_{\tau+s}^{t+s} |x(r) - y(r)|drdu \\
&\leq L_b \int_{-h}^{0} \int_{\tau-h}^{t} |x(r) - y(r)|drdu \\
&\leq L_b h \int_{\tau-h}^{t} |x(s) - y(s)|ds.
\end{aligned}$$

The previous two examples prove how the assumptions of our main theorem hold for equations with variable delay and also with distributed delay.

5. **Weak dissipativity.** The function $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ in (1) is now assumed to be weakly dissipative, i.e., there exists a positive constant α and two continuous function $m_1, m_2 : \mathbb{R} \to (0, \infty)$ such that

$$\langle f(t,\phi), \phi(0)\rangle \leq (-\alpha + m_1(t))|\phi(0)|^2 + m_2(t) \qquad \forall \phi \in \Phi(h)\mathcal{C}, t \in \mathbb{R}. \tag{17}$$

We consider now a more general case when the Lipschitz condition and the integral Lipschitz condition are time dependent, i.e there exists two non-negative continuous functions $m_1(\cdot)$ and $\gamma(\cdot)$,

$$|f(t,\phi) - f(t,\psi)| \leq m_1(t)\|\phi - \psi\|_{\mathcal{C}} \qquad \forall t \in \mathbb{R}, \phi, \psi \in \mathcal{C}. \tag{18}$$

$$\int_\tau^t |f(s,x_s) - f(s,y_s)|ds \leq \int_{\tau-h}^t \gamma(s)|x(s) - y(s)|ds. \tag{19}$$

where $m_2(\cdot)$ is non-decreasing and $m_1(\cdot), m_2(\cdot)$ satisfy the integrability conditions

$$\int_{-\infty}^t m_1(s)ds < \infty, \qquad \int_{-\infty}^t e^{\varepsilon s} m_2(s)ds < \infty \qquad \forall \varepsilon > 0,\ t \in \mathbb{R}. \tag{20}$$

Theorem 5.1. *Let $U(t,s), t \geq s$, be the evolution process generated by problem (1). We assume that hypotheses (18), (17) and (19) are satisfied.*

Then, there exists a pullback exponential attractor for $U(t,s), t \geq s$, in $\mathcal{C}$. Moreover, the global pullback attractor exists, it is contained in the pullback exponential attractor, and its fractal dimension is finite.

Proof. **Smoothing Property:** Let $s \in \mathbb{R}$, $u, v \in B(s)$ and $x(t;u,s)$ and $y(t;u,s)$, $t \geq s$, denote the corresponding solutions of (1). In order to shorten notations, in the sequel we write $x(t) = x(t;u,s)$ and $xy(t) = y(t;v,s), t \geq s$. For the difference of x and y we obtain

$$\begin{aligned}&\|U(s+h,s)u - U(s+h,s)v\|_{\mathcal{C}^1} = \|x(s+h+\cdot) - y(s+h+\cdot)\|_{\mathcal{C}^1}\\ =&\|x(s+h+\cdot) - y(s+h+\cdot)\|_{\mathcal{C}} + \|x'(s+h+\cdot) - y'(s+h+\cdot)\|_{\mathcal{C}}.\end{aligned}$$

First part: Integrating the equation (1) from s to t we obtain

$$\begin{aligned}&|x(t) - y(t)|\\ \leq&\|u-v\|_{\mathcal{C}} + \int_s^t |f(\tau,x_\tau) - f(\tau,x_\tau)|d\tau\\ \leq&\|u-v\|_{\mathcal{C}} + \int_{s-h}^t \gamma(\tau)|x(\tau) - y(\tau)|d\tau\\ \leq&\|u-v\|_{\mathcal{C}} + \int_{s-h}^s \gamma(\tau)|x(\tau) - y(\tau)|d\tau + \int_s^t \gamma(\tau)|x(\tau) - y(\tau)|d\tau\\ \leq&\big(1 + \int_{s-h}^s \gamma(\tau)d\tau\big)\|u-v\|_{\mathcal{C}} + \int_s^t \gamma(\tau)|x(\tau) - y(\tau)|d\tau.\end{aligned}$$

Now Gronwall's Lemma implies that:

$$|x(t) - y(t)| \leq \|u-v\|_{\mathcal{C}}\big(1 + \int_{s-h}^s \gamma(\tau)d\tau\big)\exp\int_s^t \gamma(\tau)d\tau. \tag{21}$$

Let now $\theta \in [-h, 0]$ and $t = s + h + \theta$. If we replace t in (21) and take the supremum over $\theta \in [-h, 0]$, we obtain:

$$\|x_1(s+h+\cdot) - x_2(s+h+\cdot)\|_{\mathcal{C}} \leq \|u-v\|_{\mathcal{C}}(1+\Gamma(s))\exp\Gamma(s), \tag{22}$$

where

$$\Gamma(s) = \int_{s-h}^{s+h} \gamma(\tau)d\tau.$$

Second part: We observe that:

$$\begin{aligned}
&\|x'(s+h+\theta) - y'(s+h+\theta)| \\
&\leq |f(s+h+\theta, x_{s+h+\theta}) - f(s+h+\theta, y_{s+h+\theta})| \\
&\leq m_1(s+h+\theta)\|x(s+h+\theta+\cdot) - y(s+h+\theta+\cdot)\|_{\mathcal{C}} \\
&\leq m_1(s+h+\theta)\|u-v\|_{\mathcal{C}}(1+\Gamma(s+\theta))\exp\Gamma(s+\theta) \\
&\leq m_1(s+h+\theta)\|u-v\|_{\mathcal{C}}(1+\Gamma(s))\exp\Gamma(s) \\
&\leq M_1(s)\|u-v\|_{\mathcal{C}}(1+\Gamma(s))\exp\Gamma(s),
\end{aligned}$$

where

$$M(s) = \sup\big\{m_1(\tau) : \tau \in (-\infty, s+h]\big\}.$$

Taking the supremum over $\theta \in [-h, 0]$,

$$\|x_1'(s+h+\cdot) - x_2'(s+h+\cdot)\|_{\mathcal{C}} \leq M_1(s)\|u-v\|_{\mathcal{C}}(1+\Gamma(s))\exp\Gamma(s). \tag{23}$$

Summing up the inequalities (22) and (23),

$$\|U(s+h,s)u - U(s+h,s)v\|_{\mathcal{C}^1} \tag{24}$$
$$= \|x_1(s+h+\cdot) - x_2(s+h+\cdot)\|_{\mathcal{C}^1} \tag{25}$$
$$\leq \|u-v\|_{\mathcal{C}}(1+\Gamma(s))\exp\Gamma(s)(1+M(s)) \tag{26}$$
$$= \kappa(s)\|u-v\|_{\mathcal{C}}. \tag{27}$$

(ii) Lipschitz continuity: Using (21), the Lipschitz continuity holds obviously.

Existence of Exponential Pullback attractor Let $r \in \mathbb{R}$ be arbitrary. We observe that the function in the smoothing property in part (ii) satisfies $\kappa(s) \leq \kappa(r)$ for all $s \leq r$. Hence, hypothesis $(\widetilde{A}_2)$ is satisfied with $\tilde{t} = h$ and $\kappa = \kappa(r)$. The Lipschitz continuity $(\widetilde{A}_3)$ was shown in (ii), and the existence of a family of pullback absorbing sets satisfying $(\widetilde{A}_1)$ was already proved in [16]. Consequently, all assumptions of Theorem 3.3 are fulfilled, which implies the existence of a pullback exponential attractor. □

5.1. **Application:** Consider the case where

$$f(t,\phi) = F_0(\phi(0)) + F_1(\phi(-\rho(t))).$$

In other words

$$f(t,x_t) = F_0(x(t)) + F_1(x(t-\rho(t)) \tag{28}$$

The delay function ρ is continuously differentiable and satisfies

$$\rho'(t) \leq \rho^* < 1 \qquad \forall t \in \mathbb{R}, \tag{29}$$

for some constant $\rho^* < 1$. We suppose that $F_0 : \mathbb{R}^n \to \mathbb{R}^n$ in (28) is uniformly Lipschitz i.e., there exists $L > 0$ such that

$$|F_0(x) - F_0(y)| \leq L|x-y| \qquad \forall x, y \in \mathbb{R}^n. \tag{30}$$

The function $F_0 : \mathbb{R}^n \to \mathbb{R}^n$ in (28) is dissipative, i.e., there exists some positive constants $\alpha > 0$ and $\beta \geq 0$ such that

$$\langle F_0(x), x\rangle \leq -\alpha|x|^2 + \beta \qquad \forall x \in \mathbb{R}^n. \tag{31}$$

We suppose that $F_1 : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$F_1(x) = \frac{m(t)x}{1+|x|^2}, \tag{32}$$

where the function $m(\cdot) : \mathbb{R} \to \mathbb{R}_+$ satisfies $\int_{-\infty}^{t} e^{\epsilon s} m(s)ds < \infty$, for all $t \in \mathbb{R}$ and $\epsilon \geq 0$.

This example is within our framework and satisfies all the assumptions of Theorem 5.1. Indeed

1. The weak dissipativity is satisfied as follows

$$\begin{aligned}
&\langle f(t,\phi), \phi(0)\rangle \\
&= \langle F_0(\phi(0)), \phi(0)\rangle + \langle F_1(\phi(-\rho(t))), \phi(0)\rangle \\
&\leq -\alpha|\phi(0)|^2 + \beta + \frac{1}{2}\frac{m(t)}{1+|\phi(-\rho(t))|^2}\langle \phi(-\rho(t)), \phi(0)\rangle \\
&\leq -\alpha|\phi(0)|^2 + \beta + \frac{1}{2}m(t)\big(\frac{|\phi(-\rho(t))|^2}{1+|\phi(-\rho(t))|^2} + |\phi(0)|^2\big) \\
&\leq (-\alpha + \frac{1}{2}m(t))|\phi(0)|^2 + \frac{1}{2}m(t) + \beta \\
&\leq (-\alpha + m_1(t))|\phi(0)|^2 + m_2(t),
\end{aligned}$$

2. The Lipschitz condition is also fulfilled thanks to the mean value theorem:

$$\begin{aligned}
&|f(t,\phi) - f(t,\psi)| \\
&\leq |F_0(\phi(0)) - F_0(\psi(0))| + |F_1(\phi(-\rho(t))) - F_1(\psi(-\rho(t)))| \\
&\leq L|\phi(0) - \psi(0)| + n \cdot \frac{m(t)}{(1+|\phi(-\rho(t))|^2)(1+|\psi(-\rho(t))|^2)}|\phi(-\rho(t)) - \psi(-\rho(t))| \\
&\leq L\|\phi - \psi\|_{\mathcal{C}} + n \cdot m(t)\|\phi - \psi\|_{\mathcal{C}} \\
&\leq (L + n \cdot m(t))\|\phi - \psi\|_{\mathcal{C}}.
\end{aligned}$$

3. The integral Lipschitz condition is also fulfilled, hence for all $x, y \in \mathbb{R}^n$ and $\tau \leq t$ we have

$$\begin{aligned}
&\int_{\tau}^{t} |f(s,x_s) - f(s,y_s)|ds \\
&\leq \int_{\tau}^{t} |F_0(x(s)) - F_0(y(s))|ds + \int_{\tau}^{t} |F_1(s, x(s-\rho(s))) - F_1(s, y(s-\rho(s)))|ds \\
&\leq \int_{\tau}^{t} L|x(s) - y(s)|ds + \int_{\tau}^{t} n \cdot m(s)|x(s-\rho(s)) - y(s-\rho(s))|ds \\
&\leq L\int_{\tau-h}^{t} |x(s) - y(s)| \int_{\tau}^{t} n \cdot m(s)|x(s-\rho(s)) - y(s-\rho(s))|ds.
\end{aligned}$$

Now using the change of variable $\eta(s) = s - \rho(s)$

$$\begin{aligned}
&\int_{\tau}^{t} |f(s,x_s) - f(s,y_s)|ds \\
&\leq L\int_{\tau-h}^{t} |x(s)-y(s)|ds + \int_{\tau-\rho(\tau)}^{t-\rho(t)} \frac{1}{1-\rho^*} n\cdot m(\eta^{-1}(s))|x(s)-y(s)|ds \\
&\leq \int_{\tau-h}^{t} (L + n\cdot m(\eta^{-1}(s)))|x(s)-y(s)|ds.
\end{aligned}$$

Here

$$\gamma(s) = L + n\cdot m(\eta^{-1}(s)), \forall s \in \mathbb{R}.$$

Remark 5.2. We can see that using the same computation of the previous example we conclude that the previous result remains true in a general case where

$$f(t,x_t) = F_0(x(t)) + F_1(t, x(t-\rho(t))$$

such that (30), (31) are satisfied and F_1 is locally Lipschitz in time i.e there exists a positive function $m : \mathbb{R} \to (0,\infty)$ such that

$$|F_1(t,x) - F_1(t,y)| \leq m(t)|x-y| \qquad t \in \mathbb{R}, \qquad x,y \in \mathbb{R}^n.$$

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E-mail address: caraball@us.es
E-mail address: sananatchaoui@gmail.com
E-mail address: MohamedAli.Hammami@fss.rnu.tn