Probabilistic Analysis of Aircraft Fuel Consumption Using Ensemble Weather Forecasts

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***Abstract*—The effects of wind uncertainty on aircraft fuel consumption are analyzed using a probabilistic trajectory predictor. The case of cruise flight subject to an average constant wind is considered. The average wind is modeled as a random variable; the wind uncertainty is obtained from ensemble weather forecasts. The probabilistic trajectory predictor is based on the Probability Transformation Method, which is a method that evolves the wind probability density function; the output of the probabilistic trajectory predictor is the probability density function of the fuel consumption. A general analysis is performed for arbitrary winds distributed uniformly, with a twofold objective: 1) present the capabilities of the probabilistic trajectory predictor, and 2) understand how the wind uncertainty affects the fuel consumption.**

Keywords- probabilistic trajectory prediction; wind uncertainty; ensemble weather forecasting

I. INTRODUCTION

The future Air Traffic Management (ATM) system must address the performance challenges posed by today's airspace: the capacity and the efficiency of the system must be increased while preserving or augmenting the safety levels. To accomplish these goals it is required a paradigm shift in operations through innovative technology and research. In this future system the trajectory becomes the fundamental element of a new set of operating procedures, collectively referred to as Trajectory-Based Operations (TBO), which aim at evolving from the current airspace-based ATM system to a trajectory-based system designed to accommodate airspace users' requests to the maximum extent possible [1].

One key factor that affects those challenges is uncertainty, which is an inherent property of real-world socio-technical complex systems, and ATM is clearly not an exception. Uncertainty is critical from different perspectives in air transport: safety, environmental and cost dimensions. Researchers must accept the fact that uncertainty is unavoidable and must be dealt with, rather than ignored. If the capacity of the ATM system is to be increased while maintaining high safety standards and improving the overall performance, uncertainty levels must be reduced and new strategies to deal with the remaining uncertainty must be found. In particular, procedures to integrate uncertainty information into the ATM planning process must be developed. In Rivas and Vazquez [2] one can find a review of all the uncertainty sources that affect the ATM system. Among those, weather has perhaps the greatest impact.

The analysis of weather uncertainty has been addressed by many authors, using different methods. For instance, Nilim et al. [3] consider a trajectory-based air traffic management scenario to minimize delays under weather uncertainty, where the weather processes are modeled as stationary Markov chains. Pepper et al. [4] present a method, based on Bayesian decision networks, for taking into account uncertain weather information in air traffic flow management. Clarke et al. [5] develop a methodology to study airspace capacity in the presence of weather uncertainty and formulate a stochastic dynamic programming algorithm for traffic flow management. Zheng and Zhao [6] develop a statistical model of wind uncertainties and apply it to stochastic trajectory prediction in the case of straight, level flight trajectories.

In this paper a probabilistic analysis of aircraft fuel consumption taking into account wind uncertainty is presented. The study is focused on the cruise phase and considers the wind uncertainty provided by Ensemble Prediction Systems (EPS), which have proved to be an effective way to quantify weather uncertainties. An analysis of wind-optimal cruise trajectories using ensemble probabilistic forecasts together with pseudospectral methods is performed in Gonzalez-Arribas et al. [7]. A conceptual vision of the integration of ensemble-based, probabilistic weather information with ATM decision support tools, focused on convective storms, is presented in Steiner et al. [8]. The importance of weather uncertainty information in probabilistic air traffic flow management is shown in Steiner et al. [9], where the translation of ensemble weather forecasts into probabilistic air traffic capacity impact is described. These papers clearly show the importance of making use of ensemble weather forecasts to generate probabilistic weather information for aviation needs.

The analysis presented in this work is based on a probabilistic trajectory predictor (pTP) which propagates the wind uncertainty along the aircraft trajectory. The method used for the uncertainty propagation is the Probabilistic Transformation Method (see Kadry [10] and Kadry and Smaily [11]) which is a non-parametric method according to the classification of Halder and Bhattacharya [12]; in this method the wind probability density function (pdf) is evolved. This method was presented in Vazquez and Rivas [13] where the propagation of uncertainty in the initial aircraft mass was studied, and some preliminary results applied to wind uncertainty are described in Vazquez and Rivas [14].

This study is relevant because wind is one of the main sources of uncertainty in trajectory prediction, and because cruise uncertainties have a large impact on the overall flight since the cruise phase is the largest portion of the flight (at least for long-haul routes). In particular it is expected that this study be relevant for the determination of the contingency fuel, and, hence, for allowing a more effective decision making, as concluded by SESAR WP-E IMET project (http://www.sesarju.eu/print/2352).

II. Ensemble Weather Forecasting

To model weather for strategic planning horizons, a probabilistic approach is the appropriate one, so that the inherent weather uncertainty can be taken into account. Today's trend is to use Ensemble Prediction Systems (EPS), which attempt to characterize and quantify the inherent prediction uncertainty based on ensemble modeling. Ensemble forecasting is a prediction technique that consists in running an Ensemble of Weather Forecasts (EWF) by slightly altering the initial conditions and/or the parameters that model the atmospheric physical processes, and/or by considering time-lagged or multi-model approaches (Arribas et al. [15]; Lu et al. [16]). Thus, this technique generates a representative sample of the possible (deterministic) realizations of the potential weather outcome [8].

An ensemble forecast is a collection of typically 10 to 50 weather forecasts (referred to as members). Cheung et al. [17] review various EPSs: PEARP (form Météo France), consisting of 35 members; MOGREPS (form the UK Met Office), with 12 members; the European ECMWF, with 51 members; and a multi-model ensemble (SUPER) constructed by combining the previous three forming a 98-member ensemble. Some examples of EPSs from the US are MEPS (form the Air Force Weather Agency) with 10 members, and SREF (form the National Centers for Environmental Prediction) comprised of 21 members.

Ensemble forecasting has proved to be an effective way to quantify weather prediction uncertainty. The uncertainty information is on the spread of the solutions in the ensemble, and the hope is that this spread bracket the true weather outcome [8]. It is important to notice that for strategic planning the analysis of all the individual ensemble members must be included (rather than an ensemble mean) [9].

III. Trajectory Prediction Considering Ensemble Weather Uncertainty

As described in the IMET project (http://sesarinnovationdays.eu/files/2013/Posters/SID%202013%20poster%20IMET.pdf) there are two approaches for trajectory prediction subject to uncertainty provided by ensemble weather forecasts.

1) Ensemble trajectory prediction (see Fig. 1), where, for each member of the ensemble, a deterministic trajectory predictor (TP) is used, leading to an ensemble of trajectories from which probability distributions can be derived. This approach generates a large volume of data; some type of post processing is required. This is the approach used in the IMET project.

2) Probabilistic trajectory prediction (see Fig. 2), where probability distributions of meteorological parameters of interest (such as wind) are evolved along the aircraft trajectory using a probabilistic trajectory predictor (pTP), leading to probability distributions of trajectory parameters of interest (such as fuel consumption). This approach, as compared to the previous one, saves computation time. This is the approach followed in this paper.

The required input from the EWF to the trajectory predictors will depend on the ATM problem under consideration. In this paper the fuel consumption in cruise flight is studied, subject to wind uncertainty; therefore, w_1, w_2, ... w_n represent the wind fields defined by the ensemble members.

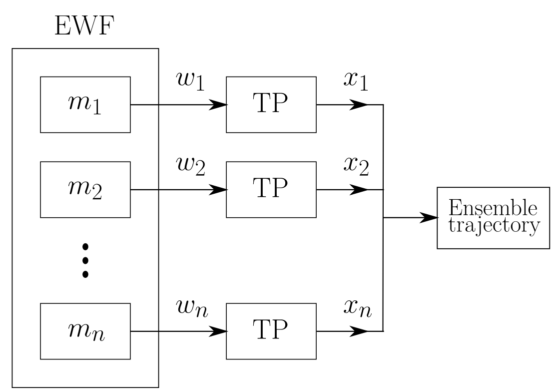


Figure 1. Ensemble trajectory prediction. Legend: m - member, w - weather, x - trajectory.

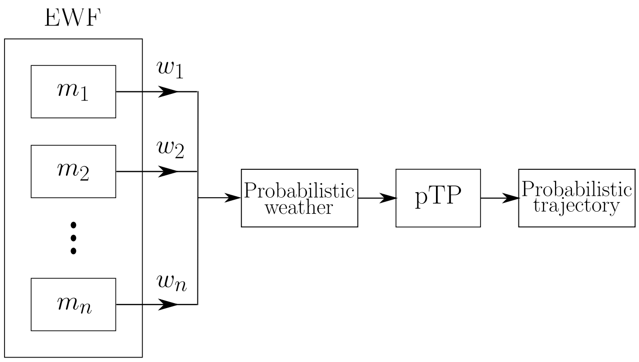


Figure 2. Probabilistic trajectory prediction. Legend: m - member, w - weather.

IV. Fuel Consumption in Cruise Flight

As already indicated, in this paper the fuel consumption in cruise flight is studied. The cruise is supposed to be formed by a given number of cruise segments, each one of them defined by a constant heading, and flown at constant speed and constant altitude, as required by Air Traffic Control (ATC) procedures.

In each cruise segment the flight is supposed to be subject to a constant average wind, which can be different for the different segments, thus modeling the along-track wind variation. This condition of having a constant average wind will be key to decide the number of segments that compose the cruise.

In this paper, as a first step in this research, the case of a cruise defined by only one segment is considered. The case of several cruise segments is left for future work since it involves more than one random variable.

Assuming symmetric flight and the flat Earth model, the equations of motion for a cruise segment are (see Vinh [18])

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= V + w, \quad \frac{\mathrm{d}m}{\mathrm{d}t} = -cT \\ T &= D, \quad L = mg \end{aligned} , \qquad (1)$$

where x is the horizontal distance, t is the time, V is the airspeed, w is the average wind speed, considered constant, T is the thrust, D and L are the aerodynamic drag and the lift, m is the aircraft mass, $g = 9.8\ m/s^2$ is the acceleration of gravity, and c is the specific fuel consumption, which can be taken as a function of altitude and speed, and it is therefore constant under the given cruise condition.

The drag can be written as $D = \rho V^2 S C_D / 2$, where ρ is the air density, S is the wing surface area, and the drag coefficient C_D is modeled by a parabolic polar $C_D = C_{D_0} + C_{D_2} C_L^2$, where C_L is the lift coefficient given by $C_L = 2L/(\rho V^2 S)$, and the coefficients C_{D_0} and C_{D_2} are constant under the given cruise condition.

Using these definitions and (1), the following equation is obtained for the aircraft mass

$$\frac{\mathrm{d}m}{\mathrm{d}x} = -\frac{A + Bm^2}{V + w}, \qquad (2)$$

where the constants A and B are defined as

$$A = \frac{c}{2} \rho V^2 S C_{D_0}, \quad B = \frac{2cC_{D_2} g^2}{\rho V^2 S}. \qquad (3)$$

Note that $A, B > 0$. Equation (2) is a nonlinear equation describing the evolution of the aircraft mass as a function of distance. Even though this model is quite simple, it is adequate to describe the cruise flight of commercial transport aircraft, since they usually fly segments of constant Mach number (M) and constant altitude (h) following ATC procedures, and it is assumed that the constant values of the parameters of the aircraft model (C_{D_0}, C_{D_2}, and c) correspond to the values of M and h set for the flight.

In this paper, the cruise range x_f and the final aircraft mass m_f are given. Fixing m_f (instead of the initial aircraft mass) is consistent with having a fixed landing weight. It also allows for a fair comparison for different values of the wind, which lead to different fuel loads and therefore to different values of the initial aircraft mass. Hence, (2) is to be solved *backwards* with the boundary condition

$$m(x_f) = m_f. \qquad (4)$$

To emphasize the dependence of the aircraft mass $m(x)$ on the wind, it is written as $m(x; w)$. If the average wind w is uncertain, then the evolution of mass with distance is uncertain as well. Note that, in such a case, the solution of (2) and (4) is still valid but in a probabilistic sense, i.e., $m(x; w)$ is a random process.

Once the aircraft mass is obtained, the cruise fuel consumption follows from

$$m_F(w) = m(0;w) - m_f\,, \tag{5}$$

which is uncertain as well. This problem has the following explicit solution

$$m_F = \frac{(m_f^2 + A/B)\tan\left(\dfrac{\sqrt{AB}x_f}{V+w}\right)}{\sqrt{A/B} - m_f \tan\left(\dfrac{\sqrt{AB}x_f}{V+w}\right)}, \tag{6}$$

which defines the transformation $m_F = g(w)$. Notice that in the case of several cruise segments m_f would be uncertain after the first segment, and hence one would have a transformation with two random variables.

V. PROBABILISTIC TRAJECTORY PREDICTOR

The pTP is described in this section. The input is the pdf of the wind and the output is the pdf of the fuel consumption, see sketch in Fig. 3. The pTP is based on the Probability Transformation Method (PTM), see [10, 11]. The basis of this method is the following theorem (see Canavos [19]): Given a random variable y with probability density function $f_y(y)$ if one defines another random variable z using a transformation g such that $z = g(y)$, then the probability density function of z is given by

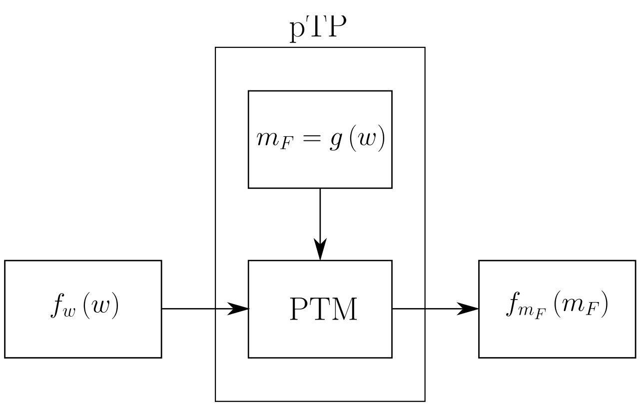


Figure 3. Probabilistic trajectory predictor (pTP).

$$f_z(z) = \frac{f_y(g^{-1}(z))}{|\,g'(g^{-1}(z))\,|}, \tag{7}$$

expression that is valid only if the function $g(y)$ is invertible on the domain of y.

Let $f_w(w)$ be the pdf of the wind (to be defined in Section VI). Then, the pdf of the fuel consumption follows from

$$f_{m_F}(m_F) = \frac{f_w(g^{-1}(m_F))}{|\,g'(g^{-1}(m_F))\,|}. \tag{8}$$

This analysis is valid only if the function $m_F = g(w)$ is invertible on the domain of *w*, that is, only if for two different values of wind w_1 and w_2, the aircraft fuel masses $m_{F,1}$ and $m_{F,2}$ are different, which in this problem is obvious.

Once the pdf is known, one can compute the mean and the typical deviation, as follows

$$\begin{aligned} E[m_F(w)] &= \int_{-\infty}^{\infty} m_F f_{m_F}(m_F)dm_F \\ \sigma[m_F(w)] &= \left[\int_{-\infty}^{\infty} m_F^2 f_{m_F}(m_F)dm_F - \left(E[m_F(w)]\right)^2\right]^{1/2}, \end{aligned} \tag{9}$$

VI. PROBABILISTIC WIND MODEL

In this section the input to the pTP is defined (see Fig. 3), that is, the probabilistic wind that affects the aircraft trajectory. In the following the approach to obtain the pdf of the average constant wind in the cruise segment is described.

Suppose that the ensemble has n members, then, the first step is to determine, for each member of the ensemble, the average wind along the segment, say w_i. Next, once the sample values $\{w_1,\ldots,w_n\}$ are obtained, one must assume that they follow a particular distribution. This is not a minor point, and in fact is one of the open challenges in this problem.

Taking into account that the uncertainty information is on the spread of the solutions in the ensemble, and that all members (including the outliers) must be considered, in this paper, to obtain the pdf of the wind, it is assumed that the wind is distributed as a uniform continuous variable in the interval $[w_m, w_M]$, where $w_m = \min\{w_1,\ldots,w_n\}$ and $w_M = \max\{w_1,\ldots,w_n\}$.

Therefore, the wind has the following pdf

$$f_w(w) = \begin{cases} 1/(w_M - w_m), & w \in [w_m, w_M] \\ 0, & w \notin [w_m, w_M] \end{cases}. \quad (10)$$

The mean and the typical deviation of w are given by

$$\begin{aligned} \mathrm{E}[w] &= \int_{-\infty}^{\infty} w f_w(w) dw = (w_M + w_m)/2 \\ \sigma[w] &= \left[\int_{-\infty}^{\infty} w^2 f_w(w) dw - (\mathrm{E}[w])^2 \right]^{1/2} = \frac{w_M - w_m}{2\sqrt{3}} \end{aligned}, \quad (11)$$

VII. ANALYSIS OF FUEL CONSUMPTION UNCERTAINTY

Once the input to the pTP is defined (given by (10)), the pdf of the fuel consumption $f_{m_F}(m_F)$ is easily computed from (8), where the function $m_F = g(w)$ is defined by (6). One has

$$f_{m_F}(m_F) = \begin{cases} \dfrac{G(m_F)}{w_M - w_m}, & m_F \in [m_{F,1}, m_{F,2}] \\ 0, & m_F \notin [m_{F,1}, m_{F,2}] \end{cases}, \quad (12)$$

where

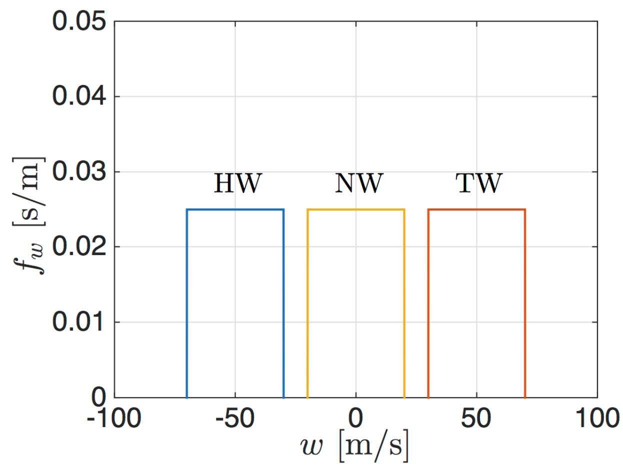
$$G(m_F) = \frac{A x_f}{(m_f + m_F)^2 + A/B} \times \left[\arctan\left(\frac{m_F \sqrt{A/B}}{m_f(m_f + m_F) + A/B} \right) \right]^{-2}, \quad (13)$$

and $m_{F,1} = g(w_m)$, $m_{F,2} = g(w_M)$. Note that the G function depends only on the transformation g, being independent of the probabilistic wind model considered.

In the following some initial results are presented corresponding to a general analysis for arbitrary winds distributed uniformly along one cruise segment. The objective is twofold: 1) present the capabilities of the pTP, and 2) understand how the wind uncertainty affects the fuel consumption. In this controlled experiment, for the wind distribution the mean value $\overline{w} = (w_M + w_m)/2$ varies between -50 m/s and 50 m/s, and the width $\delta_w = (w_M - w_m)/2$ varies between 0 and 25 m/s.

Results are presented for a given aircraft and a given cruise flight defined by the following parameters: $V = 240\,\mathrm{m/s}$, $\rho = 0.4127\,\mathrm{kg/m^3}$ ($h \approx 10000\,\mathrm{m}$), $C_{D_0} = 0.01744$, $C_{D_2} = 0.04823$, $c = 1.49 \cdot 10^{-5}\,\mathrm{s/m}$, $S = 283.5\,\mathrm{m^2}$, $m_f = 150000\,\mathrm{kg}$, and $x_f = 3000\,\mathrm{km}$.

The pdf of the fuel mass is shown in Fig. 4, for the following wind distributions: headwind (HW) $\overline{w} = -50$ m/s, no wind (NW) $\overline{w} = 0$, and tailwind (TW) $\overline{w} = 50$ m/s, with $\delta_w = 20$ m/s in all cases, and in Fig. 5 for different values of δ_w ($\delta_w = 10,\ 15,\ 20,\ 25$ m/s) in two different cases: HW $\overline{w} = -50$ m/s and TW $\overline{w} = 50$ m/s.



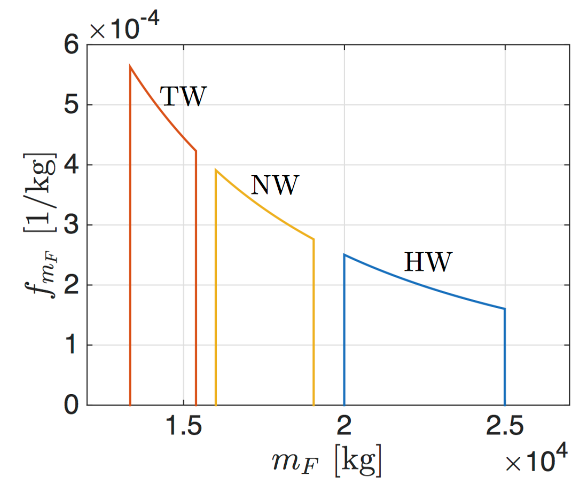


Figure 4. Fuel mass pdf for different wind distributions: HW $\overline{w} = -50$ m/s, NW $\overline{w} = 0$, and TW $\overline{w} = 50$ m/s; $\delta_w = 20$ m/s.

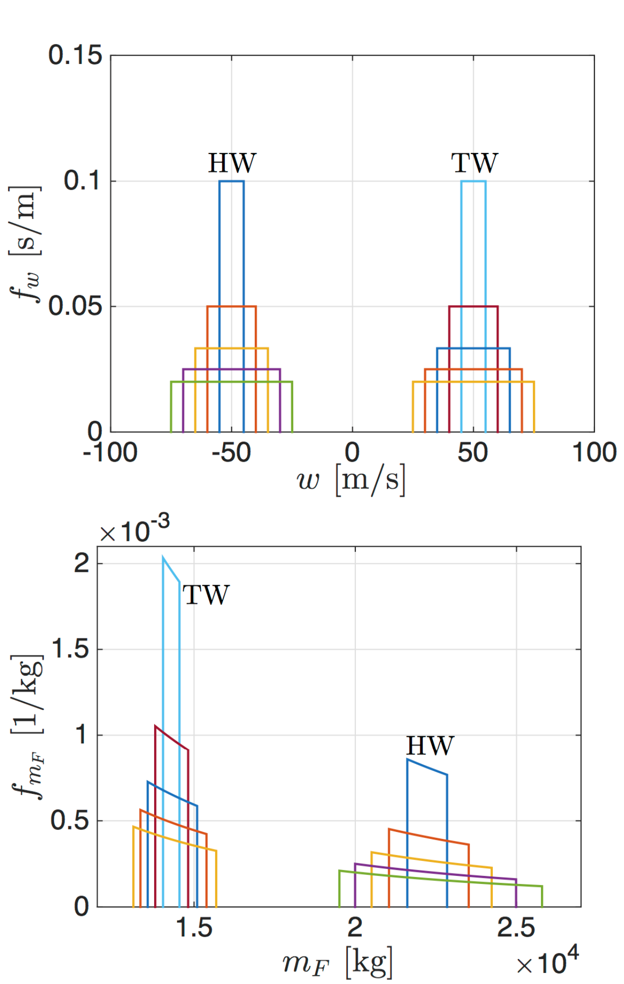


Figure 5. Fuel mass pdf for different wind distributions: HW $\overline{w} = -50$ m/s and TW $\overline{w} = 50$ m/s; $\delta_w = 10, 15, 20, 25$ m/s.

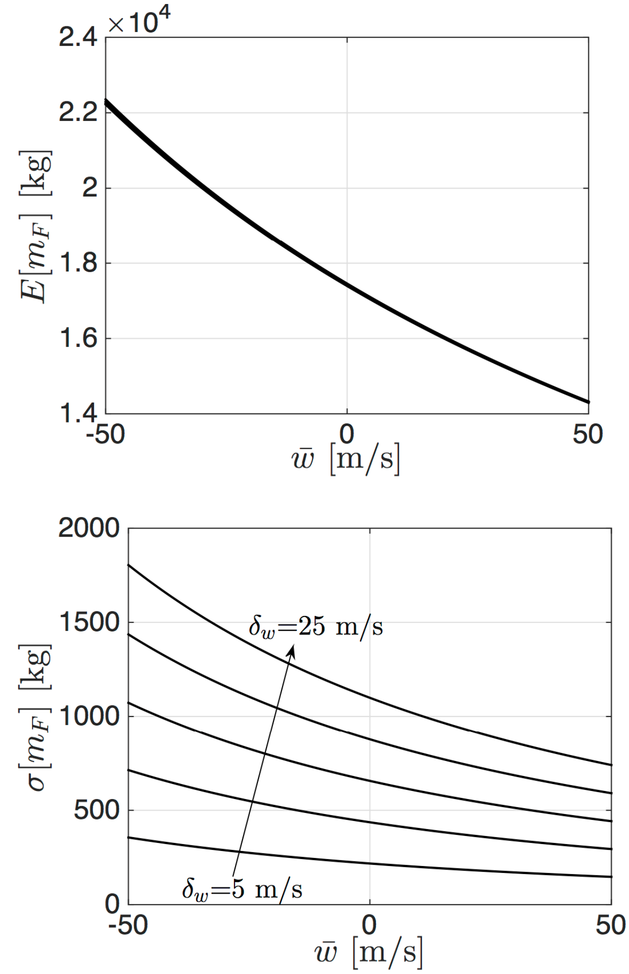


Figure 6. $E[m_F]$ and $\sigma[m_F]$ as a function of $\overline{w}$; $\delta_w = 5, 10, 15, 20, 25$ m/s.

The mean and the typical deviation of the fuel mass are shown in Fig. 6 as a function of the mean wind value $\overline{w}$ for different values of the wind distribution width δ_w ($\delta_w = 5, 10, 15, 20, 25$ m/s), and in Fig. 7 as a function of δ_w for different values of $\overline{w}$ ($\overline{w} = -50, -40, \ldots 40, 50$ m/s). Some results, for $\overline{w} = -50, 0, 50$ m/s and $\delta_w = 10, 20$ m/s, are given in Table I.

TABLE I. VALUES OF THE MEAN AND THE TYPICAL DEVIATION OF THE FUEL CONSUMPTION

	$E[m_F]$ (kg)		$\sigma[m_F]$ (kg)	
$\overline{w}$ (m/s)	δ_w = 10 m/s	δ_w = 20 m/s	δ_w = 10 m/s	δ_w = 20 m/s
−50	22235.5	22304.7	713.2	1436.2
0	17400.8	17433.8	436.5	876.6
50	14294.8	14313.1	294.5	590.6

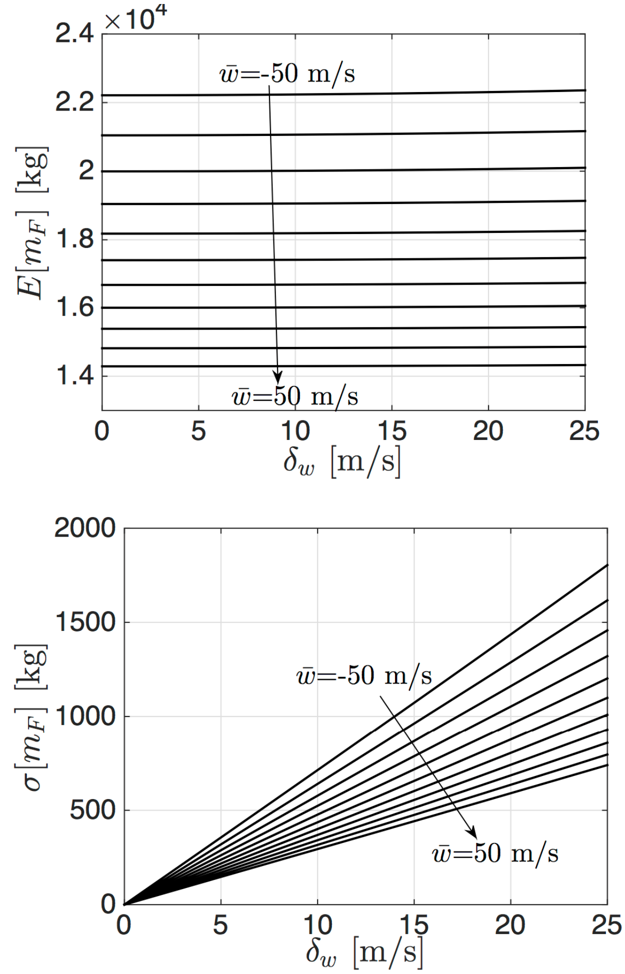


Figure 7. $E[m_F]$ and $\sigma[m_F]$ as a function of δ_w; $\overline{w} = -50, \ -40, \ \ldots \ 40, \ 50$ m/s.

The previous figures show that the mean of the fuel mass distribution decreases as $\overline{w}$ increases (as expected, HWs lead to larger fuel consumption than TWs), and it is practically independent of δ_w. On the other hand, one has that the typical deviation of the fuel mass also decreases as $\overline{w}$ increases, and it increases as δ_w increases; the increase with δ_w is almost linear (for the range of values of δ_w considered), with a slope that decreases as $\overline{w}$ increases. Thus, one obtains the result that the uncertainty in the fuel consumption is larger in the case of HWs (for a given value of the wind uncertainty) than in the case of TWs; as a numerical reference, for $\delta_w = 20$ m/s, $\sigma[m_F]$ increases from 590.6 kg for TW $\overline{w} = 50$ m/s to 1436.2 kg for HW $\overline{w} = -50$ m/s.

It is interesting to note that, for a given wind distribution (defined by $\overline{w}$ and δ_w), the previous results satisfy the following relationship

$$E[m_F(w)] > m_F(E[w]) \,, \tag{17}$$

that is, the mean of the fuel mass distribution is always larger than the mass of fuel that it would be required for the wind mean value $E[w]$ (taken as a deterministic value).

The difference $\varepsilon = E[m_F(w)] - m_F(E[w])$ decreases as $\overline{w}$ increases, and increases as δ_w increases; thus, it is largest for the most uncertain headwinds. Some results, for $\overline{w} = -50, \ 0, \ 50$ m/s and $\delta_w = 15, \ 25$ m/s, are given in Table II.

Hence, the presence of uncertain winds leads one to expect (in a statistical sense) larger values of fuel loading.

VIII. Final Remarks

The general framework for this paper is the development of a methodology to manage weather uncertainty suitable to be integrated into the trajectory planning process. This work is a first step focused on the assessment of the impact of wind uncertainty on aircraft trajectory, and in particular on the cruise fuel load. It is expected that by considering the weather uncertainty in the trajectory prediction process, one could adjust the contingency fuel depending on the uncertainty obtained for the fuel consumption.

In this paper the fuel consumption in cruise flight has been obtained as an explicit function of the average wind, and hence the pdf $f_{m_F}(m_F)$ has been obtained explicitly as well. In problems where explicit solutions cannot be obtained, the pdf given by (12-13) must be obtained numerically. A numerical approach for this type of problems can be found in [13, 14].

TABLE II. Values of the Difference $\varepsilon = E[m_F(w)] - m_F(E[w])$

	ε (kg)	
$\overline{w}$ (m/s)	δ_w = 15 m/s	δ_w = 25 m/s
−50	51.6	144.3
0	24.7	68.8
50	13.7	38.1

Even though the analysis presented has not taken crosswinds into account, they can be considered in a simple manner by defining the ground speed as $V_g = \sqrt{V^2 - w_c^2} + w$, where w_c is the average crosswind for the given cruise segment. Then, for each member of the ensemble, one can define the average values w_i and $w_{c,i}$, and therefore $V_{g,i}$. The analysis now can be carried out straightforwardly just considering V_g as the random variable (instead of w). Another approach would be to consider a two-variable random process, defining $m_F = g(w, w_c)$. The extension of this problem to more than one random variable is left for future work.

The application of the probabilistic approach presented in this paper to real trajectories, composed of several cruise segments, taking into account the wind distributions obtained from real EWFs is to be carried out as a next step in this research.

Also for future work is left the task of including both wind uncertainty and the presence of convective regions in the problem.

The probabilistic trajectory predictor presented in this paper is capable of taking as input any type of wind distribution. In this work, a simple uniform distribution has been considered, although other types of distributions could be considered as well. It is clear that the determination of the wind pdf from the uncertainty information contained in the EWFs is an open challenge in this problem. This issue poses a multidisciplinary task to be addressed jointly by meteorologists, statisticians and ATM experts.

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