Quasi-free divisors and duality

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Abstract

We prove a duality theorem for some logarithmic $\mathcal{D}$-modules associated with a class of divisors. We also give some results for the locally quasi-homogeneous case.

Diviseurs quasi-libres et dualité. On montre un théorème de dualité pour certains $\mathcal{D}$-modules logarithmiques associés à une classe de diviseurs. On donne aussi quelques résultats dans le cas localement quasi-homogène.

Version française abrégée

Dans [13] Saito a développé la théorie des diviseurs libres dans $X = \mathbf{C}^n$. Rappelons qu'un diviseur $D \subset X$ est libre si le faisceau des champs de vecteurs logarithmiques relativement à D, noté $\mathrm{Der}(-\log D)$, est localement libre en tant que faisceau de modules sur $\mathcal{O}$, le faisceau structural de X. D'autre part, une forme méromorphe $\omega \in \Omega^p(\star D)$ à poles le long de D est dite logarithmique si $f\omega$ et $\mathrm{d}f \wedge \omega$ sont formes holomorphes pour une (ou pour toute) équation locale réduite f de D. On définit comme cela un complexe $\Omega^\bullet(\log D)$ des formes logarithmiques par rapport à D et une inclusion $i_D : \Omega^\bullet(\log D) \to \Omega^\bullet(\star D)$. La classification des diviseurs D tels que i_D est un quasi-isomorphisme est un problème ouvert. D'après le théorème de comparaison de Grothendieck les complexes $\Omega^\bullet(\star D)$ et $\mathbf{R}j_*(\mathbf{C})$ sont quasi-isomorphes, ici $j : X \setminus D \to X$ est l'inclusion ; ainsi, si i_D est un quasi-isomorphisme on dira simplement que D vérifie le théorème de comparaison logarithmique (ou que D vérifie le TCL). Le résultat central de [6] est que si $D \subset X$ est un diviseur libre et localement quasi-homogène alors D vérifie le TCL. Une réciproque de ce résultat est démontrée dans [4] pour la dimension 2.

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On écrit $\mathcal{D}$ pour le faisceau d'anneaux d'opérateurs différentiels linéaires à coefficients dans $\mathcal{O}$. Alors le $\mathcal{O}$-module $\mathcal{O}[\star D]$ des fonctions méromorphes à poles le long de D est un $\mathcal{D}$-module cohérent et holonome. Le $\mathcal{O}$-module des champs de vecteurs logarithmiques $\mathrm{Der}(-\log D)$ engendre un idéal $I^{\log D} := \mathcal{D}\,\mathrm{Der}(-\log D)$ et on note $M^{\log D}$ le $\mathcal{D}$-module quotient $\mathcal{D}/I^{\log D}$. Ce $\mathcal{D}$-module a été introduit dans [3] où on démontre que pour la classe de diviseurs libres de Koszul [3, Définition 4.1.1] le complexe $\mathrm{Sol}(M^{\log D})$ de solutions holomorphes de $M^{\log D}$ est quasi-isomorphe au complexe $\Omega^\bullet(\log D)$. En plus, comme on précise dans [8], ce dernier résultat est aussi vrai pour les diviseurs de Spencer. Rappelons que un diviseur libre D est dit de Spencer si $M^{\log D}$ est holonome et si en plus le complexe de Spencer logarithmique est une résolution localement libre de $M^{\log D}$. Si f est une équation locale réduite de D on peut aussi considerer le $\mathcal{D}$-module $\widetilde{M}^{\log f}$ défini localement comme le quotient de $\mathcal{D}$ par l'idéal $\tilde{I}^{\log D}$ engendré par les opérateurs $\delta + \frac{\delta(f)}{f}$ où $\delta \in \mathrm{Der}(-\log D)$. L'inclusion d'idéaux $\tilde{I}^{\log D} \subset \mathrm{Ann}_{\mathcal{D}}(1/f)$ permet de définir un morphisme naturel de $\mathcal{D}$-modules $\phi_f : \widetilde{M}^{\log D} \to \mathcal{O}[\star D]$ dont l'image est $\mathcal{D}\frac{1}{f}$.

Le complexe de Spencer logarithmique (voir Section 3.1) associé à un diviseur libre

$$\Bigl(\mathcal{D} \otimes_{\mathcal{O}} \bigwedge^{\bullet}\bigl(\mathrm{Der}(-\log D)\bigr), \nabla\Bigr)$$

a été défini dans [3]. Dans [8] on démontre que pour les diviseurs (libres) de Spencer on a une formule de dualité $(M^{\log D})^\star \simeq \widetilde{M}^{\log f}$. Ceci démontre que si D est un diviseur libre de Spencer on a une suite de quasi-isomorphismes

$$\Omega^\bullet(\log D) \simeq \mathrm{Sol}\bigl(M^{\log D}\bigr) \simeq DR\bigl(\bigl(M^{\log D}\bigr)^*\bigr) \simeq DR\bigl(\widetilde{M}^{\log f}\bigr),$$

où $DR(\cdot)$ dénote le complexe de de Rham correspondant. Comme on a toujours un morphisme, déduit de ϕ_f, entre $DR(\widetilde{M}^{\log f})$ et $DR(\mathcal{O}[\star D]) = \Omega^\bullet(\star D)$ ceci montre que le problème de classification des diviseurs $D \equiv (f = 0)$ vérifiant le TCL est étroitement lié à celui de la classification des diviseurs pour lesquels le morhisme ϕ_f est un isomorphisme.

Dans cette Note on annonce un théorème de dualité (4.1) pour certains $\mathcal{D}$-modules associés à un diviseur dit quasi-libre de Spencer. Ce théorème généralise [8, Théorème 4.3]. En plus le diviseur quasi-libre D est localement quasi-homogène alors il est de Spencer et pour une certaine équation f^α non nécessairement réduite on a un isomorphisme de $\mathcal{D}$-modules entre $\widetilde{M}^{\log f^\alpha}$ et $\mathcal{D} \cdot \frac{1}{f^\alpha}$. En particulier, dans ce cas les $\mathcal{D}$-modules logarithmiques associés à D sont holonomes réguliers.

1. Introduction

Let us denote $X = \mathbf{C}^n$ and $\mathcal{O} = \mathcal{O}_X$ the sheaf of holomorphic functions on X. Let $D \subset X$ be a divisor (i.e. a hypersurface) and $p \in X$. Denote by $\mathrm{Der}(\mathcal{O}_p)$ the $\mathcal{O}_p$-module of $\mathbf{C}$-derivations of $\mathcal{O}_p$ (the elements in $\mathrm{Der}(\mathcal{O}_p)$ are called *vector fields*).

According to Saito [13] a vector field $\delta \in \mathrm{Der}(\mathcal{O}_p)$ is said to be *logarithmic* with respect to D if $\delta(f) = af$ for some $a \in \mathcal{O}_p$, where f is a local equation of the germ $(D, p) \subset (X, p)$. The $\mathcal{O}_p$-module of logarithmic vector fields (or logarithmic derivations) is denoted by $\mathrm{Der}(-\log D)_p$ and it is a Lie algebra under the bracket product $[-,-]$. This yields a $\mathcal{O}$-module coherent sheaf denoted by $\mathrm{Der}(-\log D)$, which is a sub-module of the sheaf of vector fields over X. The $-\log D$ in parentheses is justified, according Saito (see [10, Section 0.1]) because the dual $\mathcal{O}$-module of $\mathrm{Der}(-\log D)$ is just $\Omega^1(\log D)$ the $\mathcal{O}$-module of 1-forms with logarithmic poles along D (see [13]).

Definition 1.1 [13]. The divisor D is said to be *free at the point* $p \in D$ if the $\mathcal{O}_p$-module $\mathrm{Der}(-\log D)_p$ is free (and, in this case, of rank n). We also say in this case that the germ (D, p) is free. The divisor D is called *free* if it is free at each point $p \in D$.

Example 1. Smooth divisors and normal crossing divisors are free. By [13] any germ of plane curve $D \subset \mathbf{C}^2$ is a free divisor. The divisor defined in $\mathbf{C}^3$ by $x^3 + y^3 + z^3 = 0$ is not free. More generally, the singular locus of a free divisor $D \subset \mathbf{C}^n$ has codimension 1 in D, so if $n \geqslant 3$ and D has only isolated singularities then D is not free.

By Saito's criterion [13, (1.8), (1.9)] D is free at $p \in D$ if and only if there exists a system $\delta_i = \sum_{j=1}^{n} a_{ij}\partial_j$, $i = 1, \ldots, n$ of vector fields in $\mathrm{Der}(-\log D)_p$ such that $\det(a_{ij}) = f$ defines a reduced equation of the germ (D, p).

For each divisor $D \subset X$ we have an inclusion $i_D : \Omega^\bullet(\log D) \to \Omega^\bullet(\star D)$ where $\Omega^\bullet(\star D)$ is the meromorphic de Rham complex and $\Omega^\bullet(\log D)$ is the de Rham logarithmic complex, both with respect to D. A meromorphic form $\omega \in \Omega^p(\star D)$ is said to be *logarithmic* if $fw \in \Omega^p$ and $\mathrm{d}f \wedge \omega \in \Omega^{p+1}$ for each local equation f of D.

A natural problem is to find the class of divisors $D \subset X$ for which $i_D : \Omega^\bullet(\log D) \to \Omega^\bullet(\star D)$ is a quasi-isomorphism (i.e., i_D induces an isomorphism on cohomology). By Grothendieck's comparison theorem we know that the complexes $\Omega^\bullet(\star D)$ and $\mathbf{R}j_*(\mathbf{C})$ are naturally quasi-isomorphic, where $j : U = X \setminus D \to X$ is the inclusion. So, if i_D is a quasi-isomorphism we say that the *logarithmic comparison theorem holds for* D (or simply LCT holds for D). In [6] the following theorem is proven

Theorem 1.2. *Suppose $D \subset X$ is a locally quasi-homogeneous free divisor. Then LCT holds for D.*

Following [6] a divisor $D \subset X$ is said to be *locally quasi-homogeneous* (or simply LQH) if for all $q \in D$ there exists a system of local coordinates $(V; x_1, \ldots, x_n)$ centered at q such that $D \cap V$ has a (strictly) weighted homogeneous defining equation with respect to $(x_1, \ldots, x_n)$. Smooth divisors and normal crossing divisors are LQH. A weighted homogeneous polynomial $f \in \mathbf{C}[x, y]$ defines a LQH divisor $D \equiv (f = 0) \subset \mathbf{C}^2$. The reciprocal of 1.2 is proved for dimension 2 in [4].

Let us denote by $\mathcal{D} = \mathcal{D}_X$ the sheaf of rings of linear differential operators with holomorphic coefficients on X. For any divisor $D \subset \mathbf{C}^n$ the sheaf $\mathcal{O}[\star D]$ of meromorphic functions with poles along D is naturally a left holonomic $\mathcal{D}$-module; that follows from the results of Bernstein, Björk and Kashiwara [1,2,11].

In [3] the author considers the left ideal $I^{\log D} \subset \mathcal{D}$ generated by the set of logarithmic vector fields $\mathrm{Der}(-\log D)$. It is a coherent sheaf of ideals in $\mathcal{D}$. The module $M^{\log D}$ is defined as the quotient $\mathcal{D}/I^{\log D}$.

On the other hand, in [15] (see also [7,8]) it is considered the $\mathcal{D}_p$-module $\widetilde{M}^{\log f} := \mathcal{D}_p/\tilde{I}^{\log f}$ where $\tilde{I}^{\log f}$ is the left ideal of $\mathcal{D}_p$ generated by the set $\{\delta + \frac{\delta(f)}{f} \mid \delta \in \mathrm{Der}(-\log D)_p\}$, for each local equation f of (D, p) (the equation f does not need to be reduced). In fact $\tilde{I}^{\log f}$ is generated by the set of linear differential operators of order 1 that annihilate the meromorphic function $1/f$. There exists a natural morphism $\phi_f : \widetilde{M}^{\log f} \to \mathcal{O}[\star D]_p$ defined by $\phi_D(\overline{P}) = P(1/f)$ where $\overline{P}$ denotes the class of the operator $P \in \mathcal{D}_p$ modulo $\tilde{I}^{\log f}$. The image of ϕ_f is $\mathcal{D}_p \frac{1}{f}$. As a natural question we ask for the class of D such that the morphism ϕ_f is an isomorphism (see Section 5).

2. Quasi-free divisors

Definition 2.1. A germ of divisor $(D, 0)$ in $(\mathbf{C}^n, 0)$ is called *quasi-free* if there exists a $\mathcal{O}$-submodule $\Theta(D) \subset \mathrm{Der}(-\log D)$ of rank n verifying:

(a) $\Theta(D)$ is a Lie subalgebra of $\mathrm{Der}(-\log D)$.
(b) There exists a basis $\delta_1, \ldots, \delta_n$ of $\Theta(D)$ such that if $\delta_i = \sum_j a_{ij}\partial_j$ then $\det((a_{ij}))$ is an (nonnecessarily reduced) equation of $(D, 0)$.
(c) $\mathcal{D}\Theta(D) = \mathcal{D}\,\mathrm{Der}(-\log D)$.

Condition (b) in Definition 2.1 is independent of the chosen basis of $\Theta(D)$.

Remark 1. Of course for any divisor $(D,0)$ the $\mathcal{O}$-module $\mathrm{Der}(-\log D)$ has a free $\mathcal{O}$-submodule of rank n verifying conditions (a) and (b) of Definition 2.1 just by considering $f\,\mathrm{Der}(\mathcal{O})$ where f is an equation of the germ D. Nevertheless, it is not obvious when a germ $(D,0)$ is quasi-free. In [9] Damon introduced the notion of *free* divisor structure* on a divisor $(D,0)$ defined by some free $\mathcal{O}$-submodule (of rank n') $\mathcal{L}\subset \mathrm{Der}(-\log D')$ where $D' = D\times \mathbf{C}^{n'-n}$ and $n'\geqslant n$. The $\mathcal{O}$-module $\mathcal{L}$ need not be a Lie algebra. Our notion of quasi-free divisor should be related in future works to Damon's notion of free* structure.

Example 2. (1) Any free divisor $(D,0)$ is quasi-free just by defining $\Theta(D)=\mathrm{Der}(-\log D)$. In particular any germ of plane curve is quasi-free.

(2) The germ of arrangement of hyperplanes $(D,0)\subset(\mathbf{C}^n,0)$ defined by $x_1x_2\cdots x_n(x_1+x_2+\cdots+x_n)=0$ is quasi-free choosing $\Theta(D)$ generated by

$$\begin{aligned}
\delta_1 &= \left(x_1^2+x_1x_n\right)\partial_1 + x_1x_2\partial_2+\cdots+x_1x_{n-1}\partial_{n-1},\\
\delta_2 &= x_1x_2\partial_1 + \left(x_2^2+x_2x_n\right)\partial_2+\cdots+x_2x_{n-1}\partial_{n-1},\\
&\ \ \vdots\\
\delta_{n-1} &= x_1x_{n-1}\partial_1 + x_2x_{n-1}\partial_2+\cdots+\left(x_{n-1}^2+x_{n-1}x_n\right)\partial_{n-1},\\
\delta_n &= x_1\partial_1+\cdots+x_n\partial_n.
\end{aligned}$$

It is not free (see [12]).

(3) Orlik–Terao's arrangement (see [12]) defined in dimension 4 by $xyzw(x+y)(x+z)(x+w)(y+z)(y+w)(z+w)(x+y+z)(x+y+w)(y+z+w)(x+y+z+w)=0$ is quasi-free but not free.

(4) If $(D,0)\subset(\mathbf{C}^n,0)$ is quasi-free then $(D\times\mathbf{C},0)\subset(\mathbf{C}^{n+1},0)$ is also quasi-free. We do not know if the converse of this last result is true.

Remark 2. The free module $\Theta(D)$ in Definition 2.1 is not unique as we can see in Example 2 just by considering the $\mathcal{O}$-module generated by the vector fields δ_i' obtained from δ_i by the substitutions x_n by x_{n-1} and ∂_n by ∂_{n-1}.

3. Spencer divisors

Let $(D,0)$ be a divisor. Suppose there exists a free $\mathcal{O}$-submodule $\Theta(D)$ of $\mathrm{Der}(-\log D)$ verifying conditions (a) and (b) of Definition 2.1, but not necessarily condition (c). As $\Theta(D)$ is also a Lie algebra one has a complex (of Spencer type) of $\mathcal{D}$-modules denoted $(Sp^\bullet(\Theta(D)),\nabla)$ (see [3, 3.1]. The free $\mathcal{D}$-modules of such a complex are

$$\mathcal{D}\otimes_{\mathcal{O}}\bigwedge^{p}\Theta(D),$$

and the differential of the complex are the analogous to the case of $\mathrm{Der}(-\log D)$. We call this complex $(\mathrm{Sp}^\bullet(\Theta(D)),\nabla)$ the Spencer complex associated to $\Theta(D)$.

We denote by $M^{\Theta(D)}$ the quotient $\mathcal{D}$-module $M^{\Theta(D)}=\mathcal{D}/(\mathcal{D}\Theta(D))$.

Definition 3.1. Let $(D,0)$ be a germ of divisor. Suppose there exists a free $\mathcal{O}$-submodule $\Theta(D)$ of $\mathrm{Der}(-\log D)$ verifying conditions (a) and (b) of Definition 2.1 (but not necessarily condition (c)). The divisor $(D,0)$ is said to be *of Spencer type* (*or just Spencer divisor*) *with respect to* $\Theta(D)$ if the following conditions hold:

(i) The $\mathcal{D}$-module $M^{\Theta(D)}$ is holonomic;
(ii) The Spencer complex $(Sp^\bullet(\Theta(D)),\nabla)$ is a free resolution of $M^{\Theta(D)}$.

Example 3. (1) Any Spencer free divisor (see [8, 3.3]) is a Spencer divisor.

(2) The germ of arrangement of hyperplanes $(D,0) \subset (\mathbf{C}^n,0)$ defined by $x_1 \cdots x_n(x_1 + \cdots + x_n) = 0$ is a quasi-free divisor of Spencer type with respect to the module $\Theta(D)$ defined in Section 2 because the principal symbols of the δ_i form a regular sequence in $\mathcal{O}[\xi_1, \ldots, \xi_n]$.

(3) Orlik–Terao's arrangement is a quasi-free Spencer divisor.

4. Duality

In [8] it is proved that for any Spencer free divisor one has a *duality formula* (in the sense of $\mathcal{D}$-modules), namely $(M^{\log D})^\star \simeq \widetilde{M}^{\log f}$. We will prove here a duality theorem for logarithmic $\mathcal{D}$-modules associated with Spencer quasi-free divisors.

Let $(D,0) \subset (\mathbf{C}^n,0)$ be a germ of Spencer divisor with respect to a free $\mathcal{O}$-submodule $\Theta(D)$ of $\mathrm{Der}(-\log D)$ verifying conditions (a), (b) of Definition 2.1 but not necessarily condition (c). Suppose $f = f_1 \cdots f_r$ is the decomposition of a reduced equation f of $(D,0)$ in irreducible factors. Let $\{\delta_1, \ldots, \delta_n\}$ be a basis of $\Theta(D)$ where

- $\delta_i = \sum_{k=1}^n a_{ik}\partial_k$ for some $a_{ik} \in \mathcal{O}$. We denote $(a_{ij}) = A$.
- $\det(A) = f^\alpha$, for some $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbf{N}^r$, $\alpha_i \geqslant 1$.
- $\delta_i(f^\alpha) = m_i f^\alpha$, $m_i \in \mathcal{O}$.
- $[\delta_i, \delta_j] = \sum_{k=1}^n c_k^{ij}\delta_k$, with $c_k^{ij} \in \mathcal{O}$.

In the following theorem we compare two logarithmic $\mathcal{D}$ modules: $M^{\Theta(D)} = \mathcal{D}/\mathcal{D}\Theta(D)$ and $M^{\widetilde{\Theta}(f^\alpha)} := \mathcal{D}/\mathcal{D}\widetilde{\Theta}(f^\alpha)$ where

$$\widetilde{\Theta}(f^\alpha) = \left\{\delta + \frac{\delta(f^\alpha)}{f^\alpha} \text{ for all } \delta \in \Theta(D)\right\}.$$

Recall that the dual $M^\star$ of a holonomic $\mathcal{D}$-module M is the left $\mathcal{D}$-module associated to the right $\mathcal{D}$-module $\mathrm{Ext}^n_{\mathcal{D}}(M, \mathcal{D})$.

Theorem 4.1. *With D as above, we have $(M^{\Theta(D)})^\star \simeq M^{\widetilde{\Theta}(f^\alpha)}$.*

Remark 3. If D is a quasi-free and Spencer divisor then $\mathcal{D}\Theta(D) = \mathcal{D}\,\mathrm{Der}(-\log D)$, so we obtain $(M^{\log D})^\star = \widetilde{M}^{\log f^\alpha}$. This result is useful (see [7] and [14]) to study which divisors verify the Logarithmic Comparison Theorem.

5. The locally quasi-homogeneous case

Following the ideas of [5], it can be shown that, if D is a LQH quasi-free divisor with respect to a corresponding $\Theta(D)$ (see 2.1), the symbols of the elements of any basis of $\Theta(D)$ form a regular sequence in $\mathcal{O}[\xi_1, \ldots, \xi_n]$.

Theorem 5.1. *If D is a LQH quasi-free divisor, then it is Spencer.*

There exists a natural morphism $\phi_{f^\alpha} : \widetilde{M}^{\log f^\alpha} \to \mathcal{O}[\star D]_p$ defined by $\phi_{f^\alpha}(\overline{P}) = P(1/f^\alpha)$ where $\overline{P}$ denotes the class of the operator $P \in \mathcal{D}_p$ modulo $\widetilde{I}^{\log f^\alpha}$. The image of ϕ_{f^α} is $\mathcal{D}_p \frac{1}{f^\alpha}$. As a natural question we ask for the class of D such that the morphism ϕ_{f^α} is an isomorphism. We proved in [8] that for any locally quasi-homogeneous free germ of divisor (D,p) defined by a local reduced equation f the morphism ϕ_f is an isomorphism and in particular $\mathrm{Ann}_{\mathcal{D}}(1/f) = \widetilde{I}^{\log f}$ is generated by differential operators of order 1. We have an analogous result for LQH Spencer quasi-free divisors:

Theorem 5.2. *If $D \equiv (f=0) \subset \mathbf{C}^n$ is a LQH quasi-free divisor, then the natural morphism $\phi_{f^\alpha} : \widetilde{M}^{\log f^\alpha} \to \mathcal{D}\frac{1}{f^\alpha}$ is an isomorphism. In particular, $M^{\log D}$ and $\widetilde{M}^{\log f^\alpha}$ are regular holonomic.*

The last theorem is based in the next lemma. It can be proved as in [8, Lemma 5.3.]. The key is that one can assume an Euler vector field in the basis of $\Theta(D)$ in the LQH case.

Lemma 5.3. *Under the hypotheses of* 5.2, $\mathrm{Ext}^n_{\mathcal{D}}(M^{\widetilde{\Theta}(f^\alpha)}, \mathcal{O})_0 = 0$.

Remark 4. If LCT holds for a divisor verifying all the hypotheses of 5.2 and in addition $\mathcal{D}\frac{1}{f^\alpha} = \mathcal{O}[\star D]$ then we have

$$\Omega^\bullet(\log D) \simeq DR\big(\mathcal{O}[\star D]\big) = DR\big(\widetilde{M}^{\log f^\alpha}\big) = DR\big(\big(M^{\log D}\big)^\star\big) = \mathrm{Sol}\big(M^{\log D}\big),$$

that is, the complex of logarithmic differential forms turns out to be quasi-isomorphic to a complex of solutions of a certain $\mathcal{D}$-module.

It is a natural question if this fact holds – independently of LCT – for all Spencer divisors, as in the free case (see [3]).

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