On some $\mathcal{D}$-modules in dimension 2

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Abstract.- We prove a duality formula for two $\mathcal{D}$-modules arising from logarithmic derivations w.r.t. a plane curve. As an application we give a differential proof of a logarithmic comparison theorem in [4].

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1 Introduction

Let $\mathcal{O} = \mathbf{C}\{x, y\}$ be the ring of convergent power series in two variables and $\mathcal{D}$ the ring of linear differential operators with coefficients in $\mathcal{O}$. For each reduced power series $f \in \mathcal{O}$, with $f(0,0) = 0$, we will denote by I^{log} the left ideal of $\mathcal{D}$ generated by the logarithmic derivations (see [11]) with respect to f. We denote by $\mathrm{Der}_{\mathbf{C}}(\mathcal{O})$ the Lie algebra of $\mathbf{C}$-derivations on $\mathcal{O}$. Recall that a derivation $\delta \in \mathrm{Der}_{\mathbf{C}}(\mathcal{O})$ is logarithmic if there exists $a \in \mathcal{O}$ such that $\delta(f) = af$. We denote by $\widetilde{I}^{log}$ the left ideal of $\mathcal{D}$ generated by the operators of the form $\delta + a$ where $\delta(f) = af$.

We first prove that the $\mathcal{D}$-modules $M^{log} = \mathcal{D}/I^{log}$ and $\widetilde{M}^{log} = \mathcal{D}/\widetilde{I}^{log}$ are dual each to the other and then that both $\mathcal{D}$-modules are regular holonomic 2.3.

Let $\mathcal{O}[1/f]$ be the $\mathcal{D}$-module of (the germs of) the meromorphic functions in two variables with poles along f. There exists a natural surjective

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morphism $\psi : \widetilde{M}^{log} \to \mathcal{O}[1/f]$. Using [4] we prove that $\mathrm{Ext}^2_{\mathcal{D}}(\widetilde{M}^{log}, \mathcal{O}) = 0$ if and only if f is quasi-homogeneous and then we obtain that the morphism ψ is an isomorphism if and only if f is quasi-homogeneous (see 2.5). As a consequence we give a new "differential" proof of the logarithmic comparison theorem of [4].

These results are susceptible to be generalized to the case of higher dimensions but no general results are known up to now. See [6] for a proof of the duality formula in higher dimension. Nevertheless we give a complete example showing that some results of the present work are true in dimension 3.

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2 The module $\widetilde{M}^{log}$ in the general case.

Let us consider any reduced $f \in \mathcal{O} = \mathbf{C}\{x, y\}$ with a singular point at the origin. It is possible to obtain, from the logarithmic derivations, an ideal inside $Ann_{\mathcal{D}}(1/f)$: if $\delta(f) = af$ then $\delta + a \in Ann_{\mathcal{D}}(1/f)$. This fact suggested us a general way to present the annihilating ideal of $1/f$ for a constructive proof of the equality $\mathrm{Ext}^2_{\mathcal{D}}(\mathcal{O}[1/f], \mathcal{O}) = 0$ for any "polynomial" curve (see [14]).

We have $\widetilde{I}^{\log} \subset Ann_{\mathcal{D}}(1/f)$, where $\widetilde{I}^{\log}$ is the left ideal in $\mathcal{D}$ generated by the operators $\delta + a$ for $\delta \in Der_{\mathbf{C}}(\mathcal{O})$ and $\delta(f) = af$. Then we have a surjective morphism $\psi : \widetilde{M}^{log} = \frac{\mathcal{D}}{\widetilde{I}^{\log}} \longrightarrow \mathcal{D}/\mathcal{D}Ann_{\mathcal{D}}(1/f) \simeq \mathcal{O}[1/f]$ (for the last isomorphism we use that the Bernstein polynomial of f has no integer roots smaller than -1 (see [15])). It is well known that around each smooth point of $f = 0$ the morphism ψ is in fact an isomorphism. So, the kernel K of ψ is a $\mathcal{D}$-module concentrated at the origin. Then K is a direct sum of "couches-multiples" modules [8], and this type of modules are regular holonomic [9]. In particular $\widetilde{M}^{\log}$ is regular holonomic because $\mathcal{O}[\frac{1}{f}]$ and K are.

We will denote by $\mathrm{Der}(\log f)$ the Lie algebra of logarithmic derivations with respect to f. By [11] $\mathrm{Der}(\log f)$ is a free $\mathcal{O}$-module of rank two. Let $\{\delta_1, \delta_2\}$ be a basis of $\mathrm{Der}(\log f)$,

$$\left\{ \begin{array}{lcl} \delta_1 & = & b_1\partial_x + c_1\partial_y, \\ \delta_2 & = & b_2\partial_x + c_2\partial_y \end{array} \right. .$$

We can suppose that

$$\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = f$$

We will take into account the following results for any (reduced) curve f:

- Every basis δ_1, δ_2 of Der(log f) verifies that

  $$\langle \sigma(\delta_1), \sigma(\delta_2) \rangle = \mathrm{gr}^F(I^{log}) = \mathrm{gr}^F(\widetilde{I}^{log}),$$

  because $\{\sigma(\delta_1), \sigma(\delta_2)\}$ is a regular sequence (see [2] and ([3], Corollary 4.2.2)). Here $\sigma(\cdot)$ denotes the principal symbol of the corresponding operator and $\mathrm{gr}^F(I^{log})$ is the graded ideal associated to the order filtration on $\mathcal{D}$. Therefore,

  $$CCh(\widetilde{M}^{log}) = CCh(M^{log}),$$

  where $CCh(\ )$ represents the *characteristic cycle* of the $\mathcal{D}$-module (see, for example, [7]). Of course both M^{log} and $\widetilde{M}^{log}$ define coherent $\mathcal{D}$-modules in some neighborhood of the origin and then we can properly speak of characteristic varieties and characteristic cycles. Since $\widetilde{M}^{log}$ is holonomic then M^{log} is holonomic.

- For any curve,

  $$Sol(M^{log}) \overset{q.i.}{\simeq} \Omega^\bullet(log f) \overset{\varphi}{\longrightarrow} \Omega^\bullet[1/f] \simeq DR(\mathcal{O}[1/f]),$$

  where $Sol(\ )$ and $DR(\ )$ are the *solutions complex* and the *De Rham complex* (see, for example, [9]) and where $\Omega^\bullet(\log f)$ (resp. $\Omega^\bullet([1/f])$) is the complex of logarithmic differential forms (resp. meromorphic differential forms). The first quasi-isomorphism appears in [3] and φ is the natural morphism.

Proposition 2.1 *Let f be a (reduced) curve and let $\{\delta_1, \delta_2\}$ be a basis of* Der (log f) *with $[\delta_1, \delta_2] = \alpha_1\delta_1 + \alpha_2\delta_2$ and $\delta_i(f) = a_i f,\ i = 1, 2$. Then*

$$\mathcal{D}\{\delta_2^t + \alpha_1, \delta_1^t - \alpha_2\} = \mathcal{D}\{\delta_1 + a_1, \delta_2 + a_2\}$$

where δ_i^t is the transposed *of δ_i.*

Proof: First we find an expression of the α_i from the a_j, b_k, c_l:

$$[\delta_1, \delta_2] = \alpha_1(b_1\partial_x + c_1\partial_y) + \alpha_2(b_2\partial_x + c_2\partial_y) =$$

$$= (\alpha_1 b_1 + \alpha_2 b_2)\partial_x + (\alpha_1 c_1 + \alpha_2 c_2)\partial_y =$$

$$= b_1\partial_x(b_2)\partial_x - b_2\partial_x(b_1)\partial_x + b_1\partial_x(c_2)\partial_y - c_2\partial_y(b_1)\partial_x +$$

$$+ c_1\partial_y(b_2)\partial_x - b_2\partial_x(c_1)\partial_y + c_1\partial_y(c_2)\partial_y - c_2\partial_y(c_1)\partial_y =$$

$$= (c_1\partial_y(b_2) - b_2\partial_x(b_1) - c_2\partial_y(b_1) + c_1\partial_y(b_2))\partial_x +$$

$$+ (b_1\partial_x(c_2) - b_2\partial_x(c_1) - c_2\partial_y(c_1) + c_1\partial_y(c_2))\partial_y.$$

Besides,

$$-\delta_1^t + \alpha_2 = \partial_x b_1 + \partial_y c_1 + \alpha_2 = \delta_1 + \alpha_2 + \partial_x(b_1) + \partial_y(c_1).$$

To prove that $\alpha_2 + \partial_x(b_1) + \partial_y(c_1) = a_1$, we will establish that

$$\alpha_2 f = a_1 f - \partial_x(b_1) f - \partial_y(c_1) f$$

We have

$$a_1 f - \partial_x(b_1) f - \partial_y(c_1) f =$$

$$= (b_1\partial_x + c_1\partial_y) - \partial_x(b_1) - \partial_y(c_1))(b_1 c_2 - b_2 c_1) =$$

$$= b_1(b_1\partial_x(c_2) - c_1\partial_x(b_2) - b_2\partial_x(c_1)) +$$

$$+ c_1(c_2\partial_y(b_1) + b_1\partial_y(c_2) - c_1\partial_y(b_2)) +$$

$$+ b_2 c_1\partial_x(b_1) - b_1 c_2\partial_y(c_1).$$

Therefore

$$(\alpha_1, \alpha_2)\begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix} = (\gamma_1, \gamma_2),$$

where

$$\begin{array}{rcl} \gamma_1 & = & c_1\partial_y(b_2) - b_2\partial_x(b_1) - c_2\partial_y(b_1) + c_1\partial_y(b_2), \\ \gamma_2 & = & b_1\partial_x(c_2) - b_2\partial_x(c_1) - c_2\partial_y(c_1) + c_1\partial_y(c_2)). \end{array}$$

Multiplying by the transposed adjoint matrix and by f we obtain

$$(\alpha_1 f, \alpha_2 f) = (\gamma_1, \gamma_2)\begin{pmatrix} c_2 & -c_1 \\ -b_2 & b_1 \end{pmatrix}.$$

and hence the equality follows. In a similar way $\delta_2^t + \alpha_1 = -\delta_2 - a_2$. Then both ideals are equal. □

Prof. Narváez pointed us to consider, instead of the Lie algebra $\mathrm{Der}(\log f)$, the Lie algebra

$$L = \{\delta + a|\ \delta(f) = af\},$$

and try to construct of a free resolution (of "Spencer type") of $\widetilde{M}^{log}$ ([2], [3]). In fact, we have

Proposition 2.2 *A free resolution of $\widetilde{M}^{log}$ is*

$$0 \longrightarrow \mathcal{D} \xrightarrow{\phi_2} \mathcal{D}^2 \xrightarrow{\phi_1} \mathcal{D} \longrightarrow \widetilde{M}^{log} \longrightarrow 0,$$

where ϕ_2 is defined by the matrix

$$(-\delta_2 - a_2 - \alpha_1, \delta_1 + a_1 - \alpha_2),$$

and ϕ_1 by $\begin{pmatrix} \delta_1 + a_1 \\ \delta_2 + a_2 \end{pmatrix}$.

Proof: To check the exactness of the resolution above, it is enough to consider a discrete filtration on that complex and to verify the exactness of the resulting resolution (see [1], chapter 2, lemma 3.13). The same argument is used in [2], [3](proposition 4.1.3) to prove that the complex $\mathcal{D} \otimes_{\mathcal{V}_0^f(\mathcal{D})} Sp^\bullet(logf)$ is a free resolution of M^{log} (as a left $\mathcal{D}$-module)[1]. But, for $n = 2$, the exact graded complex in the proof of [3] is precisely

$$0 \longrightarrow \mathrm{gr}^F(\mathcal{D}) \xrightarrow{M_1} \mathrm{gr}^F(\mathcal{D})^2[-2] \xrightarrow{M_2} \mathrm{gr}^F(\mathcal{D})[-1] \longrightarrow \mathrm{gr}^F(M^{log}) \longrightarrow 0,$$

where the matrices are

$$M_1 = (-\sigma^F(\delta_2), \sigma^F(\delta_1)), \quad M_2 = \begin{pmatrix} \sigma^F(\delta_1) \\ \sigma^F(\delta_2) \end{pmatrix}.$$

And the last complex is the result of applying the same graduation to the resolution of $\widetilde{M}^{log}$ too, because

$$\sigma^F(\delta_i) = \sigma^F(\delta_i + a_i).$$

▯

[1]Here $Sp^\bullet(logf)$ is the Logarithmic Spencer complex and $\mathcal{V}_0^f(\mathcal{D})$ is the ring of degree zero differential operators w.r.t. $\mathcal{V}$-filtration relative to f. See [2], [3], section 1.2.

Proposition 2.3 *Given* $f \in \mathbf{C}\{x,y\}$, $\widetilde{M}^{log} \simeq (M^{log})^\star$ *where* $()^\star$ *is the dual in the sense of* $\mathcal{D}$*-modules. In particular* $\widetilde{M}^{log}$ *and* M^{log} *are regular* $\mathcal{D}$*-modules.*

Proof: We take the free resolution of M^{log} (see [2], ([3, Th. 3.1.2])

$$0 \longrightarrow \mathcal{D} \xrightarrow{\psi_2} \mathcal{D}^2 \xrightarrow{\psi_1} \mathcal{D} \longrightarrow M^{log} \longrightarrow 0,$$

where $\{\delta_1, \delta_2\}$ is a basis of the $\mathcal{O}$-module Der(log$f$), where

$$[\delta_1, \delta_2] = \alpha_1\delta_1 + \alpha_2\delta_2,$$

$$\psi_1 = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$$

and, on the other hand, ψ_2 is the syzygy matrix

$$\psi_2 = (-\delta_2 - \alpha_1, \delta_1 - \alpha_2).$$

Applying the $Hom_{\mathcal{D}}(-,\mathcal{D})$ functor to calculate the dual module, we obtain the sequence

$$0 \longrightarrow \mathcal{D} \xrightarrow{\psi_1^*} \mathcal{D}^2 \xrightarrow{\psi_2^*} \mathcal{D} \longrightarrow 0,$$

where ψ_2^* is the right product by $\begin{pmatrix} -\delta_2 - \alpha_1 \\ \delta_1 - \alpha_2 \end{pmatrix}$. Hence, $(M^{log})^\star$ is the left $\mathcal{D}$-module associated to the right $\mathcal{D}$-module $\mathcal{D}/(\delta_2 + \alpha_1, \delta_1 - \alpha_2)\mathcal{D}$, that is to say,

$$(M^{log})^\star \simeq \mathcal{D}/\mathcal{D}(\delta_2^t + \alpha_1, \delta_1^t - \alpha_2).$$

Using the proposition 2.1, we deduce that $(M^{log})^\star \simeq \widetilde{M}^{log}$. The regularity of M^{log} follows from the regularity of $\widetilde{M}^{log}$ (c.f. [9]). $\square$

Proposition 2.4 *If* f *is a non quasi homogeneous (reduced) curve, then*

$$\mathrm{Ext}^2_{\mathcal{D}}(\widetilde{M}^{log}, \mathcal{O}) \neq 0.$$

Proof: The proof of this result contains, as an essential ingredient, a re-reading of the demonstration of Theorem 3.7 of [4]. As a matter of fact, we include some tricks of this demonstration.

By proposition 2.2, a free resolution of $\widetilde{M}^{log}$ is

$$0 \longrightarrow \mathcal{D} \xrightarrow{\phi_2} \mathcal{D}^2 \xrightarrow{\phi_1} \mathcal{D} \longrightarrow \widetilde{M}^{log} \longrightarrow 0,$$

where ϕ_2 is the matrix

$$(-\delta_2 - a_2 - \alpha_1, \delta_1 + a_1 - \alpha_2).$$

Hence, $\mathrm{Ext}^2_{\mathcal{D}}(\widetilde{M}^{log}, \mathcal{O}) \simeq \mathcal{O}/Img\phi_2^*$. To guarantee that this vector space has dimension greater than zero, it is enough to show that a pair of functions $h_1, h_2 \in \mathcal{O}$ such that

$$(-\delta_2 - a_2 - \alpha_1, \delta_1 + a_1 - \alpha_2) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 1,$$

does not exist, that is to say, that $1 \notin Img\phi_2^*$.

Let us take $\delta_1 = b_1\partial_x + c_1\partial_y$. As $a_1 - \alpha_2 = \partial_x(b_1) + \partial_y(c_1)$, (proposition 2.1) we will prove that, or b_1 and c_1 have no lineal parts, or that after derivation those lineal parts become 0.

Of course f has no quadratic part: in that case, because of the classification of the singularities in two variables, f would be equivalent to a quasi homogeneous curve $x^2 + y^{k+1}$, for some k. Then we can suppose that

$$f = f_n + f_{n+1} + \cdots = \sum_{k \geq n} h_k = \sum_{k \geq n} \sum_{i+j=k} a_{ij} x^i y^j,$$

where $n \geq 3$ and $f_n \neq 0$.

We will write

$$\delta_1 = b_1\partial_x + c_1\partial_y = \delta_0^1 + \delta_1^1 + \cdots = \sum_{k \geq 0} \sum_{i+j=k+1} (\beta_{ij}^1 x^i y^j \partial_x + \gamma_{ij}^1 x^i y^j \partial_y),$$

where the linear part δ_0^1 is $(xy)A_0(\partial_x\partial_y)^t$, and A_0 is a matrix 2×2 with complex coefficients.

If $A_0 = 0$, we have finished. Otherwise, the possibilities of the Jordan form of A_0 are

$$A_0 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad A_0 = \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}.$$

As δ_1 is not an Euler vector (because f is not quasi homogeneous), we deduce:

- If we take the first Jordan form, then (see the cited demonstration of [4]) $f_n = x^p y^q$ y $\delta_0 = qx\partial_x - py\partial_y$. After a sequence of changes of coordinates we have that $f = x^p y^q$ with $p + q = n \geq 3$, that contradicts that f is reduced.

- For the second Jordan form with $\lambda_1 \neq 0$, it has to be $f_n = 0$, that contradicts that f has its initial part of grade n.
- For the second option with $\lambda_1 = 0$ we have $\delta_0^1 = y\partial_x$ and, in this situation, the linear of b_1 is y. If we precisely apply ∂_x, we obtain 0.

In a similar way, you prove the same for $a_2 + \alpha_1$. $\square$

Theorem 2.5 *The natural morphism* $\widetilde{M}^{log} \xrightarrow{\psi} \mathcal{O}[\frac{1}{f}]$ *is an isomorphism if and only if f is a quasi homogeneous (reduced) curve.*

Proof: As we pointed, if f is quasi homogeneous then $\widetilde{I}^{log} = Ann_{\mathcal{D}}(1/f)$ and therefore ψ is an isomorphism. Reciprocally, if ψ is an isomorphism, then $\mathrm{Ext}^2_{\mathcal{D}}(\mathcal{O}[1/f], \mathcal{O}) \simeq \mathrm{Ext}^2_{\mathcal{D}}(\widetilde{M}^{log}, \mathcal{O})$. Because of a result of [9], we have $\mathrm{Ext}^2_{\mathcal{D}}(\mathcal{O}[1/f], \mathcal{O}) = 0$ and, if we take into account proposition 2.4, we obtain that f has to be quasi homogeneous. $\square$

Remark.- The following result can be obtain using [13]: if f is not quasi-homogeneous curve then $Ann_{\mathcal{D}}(1/f)$ could not be generated by elements of degree one in ∂ and then $Ann_{\mathcal{D}}(1/f) \neq \widetilde{I}^{log}$.

Let us give a new "differential" proof of a version of the *Logarithmic Comparison Theorem* [4].

Theorem 2.6 *The complexes $\Omega^\bullet(log f)$ and $\Omega^\bullet[1/f]$ are isomorphic in the correspondent derived category if and only if f is quasi homogeneous.*

Proof: If f is quasi homogeneous we have pointed yet that $\widetilde{M}^{log}$ is isomorphic to $\mathcal{O}[\frac{1}{f}]$. By the proposition 2.3 $(M^{log})^\star \simeq \widetilde{M}^{log}$ and then we have

$$\Omega^\bullet(log f) \simeq Sol(M^{log}) \simeq DR((M^{log})^\star) \simeq DR(\widetilde{M}^{log}) \simeq \Omega^\bullet[1/f],$$

where the first isomorphism is obtained in [2] (see also [3]) and the second one could be found in [9]. Reciprocally, if f is not quasi homogeneous then

$\widetilde{M}^{log} \not\simeq \mathcal{O}[1/f]$ and, as both are regular holonomic, neither their De Rham complexes are isomorphic, that is

$$DR(\widetilde{M}^{log}) \not\simeq \Omega^\bullet[1/f],$$

using the Riemann-Hilbert correspondence of Mebkhout-Kashiwara. $\square$

3 Example in a constructive way of logarithmic comparison in surfaces.

We illustrate in this section an interesting example in dimension 3 of the situation presented in theorem 2.6 for curves. We consider the surface (see [2]) $h = 0$ with

$$h = xy(x+y)(xz+y),$$

which is not locally quasi-homogeneous. We prove that:

- $Ann_{\mathcal{D}}(1/h) = \widetilde{I}^{log}$.
- $\widetilde{M}^{log} \simeq (M^{log})^\star$

and we conclude the logarithmic comparison theorem holds in this case. Although this example appears in [4], here the treatment is under an effective point of view.

We can compute a basis of Der $(\log h)$ with a set of generators of the syzygies among $h, \frac{\partial h}{\partial_x}, \frac{\partial h}{\partial_y}, \frac{\partial h}{\partial_z}$. We obtain

- $\delta_1 = x\partial_x + y\partial_y$,
- $\delta_2 = xz\partial_z + y\partial_z$,
- $\delta_3 = x^2\partial_x - y^2\partial_y - xz\partial_z - yz\partial_z$,

with

$$\delta_1(h) = 4h, \quad \delta_2(h) = xh, \quad \delta_3(h) = (2x-3y)h,$$

and

$$\begin{vmatrix} x & y & 0 \\ 0 & 0 & xz+y \\ x^2 & -y^2 & -xz-yz \end{vmatrix} = h.$$

As a multiple of the b-function of h in $\mathcal{D}$ is

$$b(s) = (4s+5)(2s+1)(4s+3)(s+1)^3,$$

and this polynomial has no integer roots smaller than -1, we can assure that

$$\mathcal{O}[*h] \simeq \mathcal{D}\frac{1}{h}\ .$$

It is easy to check that $Ann_{\mathcal{D}}(1/h)$ is equal to $\widetilde{I}^{log}$. The computations of the b-function and the annihilating ideal of h^s have been made using the algorithms of [10], implemented in [12].

We calculate (using Gröbner bases) a free resolution of the module $\mathcal{D}/I^{\log}$ where $I^{\log} = (\delta_1, \delta_2, \delta_3)$ (see [5]). The first module of syzygies is generated (in this case) by the relations deduced from the expressions of the $[\delta_i, \delta_j]$ with $i \neq j$:

- $[\delta_1, \delta_2] = \delta_2$,
- $[\delta_1, \delta_3] = \delta_3$,
- $[\delta_2, \delta_3] = -x\delta_2$.

The second module of syzygies is generated by only one element $\mathbf{s} = (s_1, s_2, s_3)$:

- $s_1 = -y^2\partial_y + x^2\partial_x - zy\partial_z - zx\partial_z - x$,
- $s_2 = -y\partial_z - xz\partial_z$,
- $s_3 = y\partial_y + x\partial_x - 2$.

The above calculations provide a free resolution of M^{log}. With a procedure similar to the used in 2.3 we obtain that $(M^{log})^\star$ is the left $\mathcal{D}$-module associated to the right $\mathcal{D}$-module $\mathcal{D}/(s_1, s_2, s_3)\mathcal{D}$. Then

$$(M^{log})^\star \simeq \mathcal{D}/(s_1^t, s_2^t, s_3^t).$$

It is enough to compute s_1^t, s_2^t, s_3^t and check (using Gröbner basis) that they span $\widetilde{I}^{\log}$. Hence

$$(M^{log})^\star = (\mathcal{D}/\mathrm{Der}(\log(h))^\star \simeq \mathcal{D}/\widetilde{I}^{log} = \widetilde{M}^{log}.$$

At this point we have obtained that

$$Sol(M^{log}) \simeq DR((M^{log})^{\star}) \simeq DR(\widetilde{M}^{log}) \simeq \Omega^{\bullet}[1/h]$$

where the last two isomorphism are due to our computations (the first was used in the proof of 2.6). Taking into account that $\Omega^{\bullet}(\log h) \simeq Sol(M^{log})$ (it was showed for this example in [2],[3]) we can deduce that the logarithmic comparison theorem (i.e. $\Omega^{\bullet}(\log h) \simeq \Omega^{\bullet}[1/h]$, [4]) holds without the "locally quasi homogeneous" hypothesis. It is interesting to remark too (see [2]) that $\{\sigma^F(\delta_1), \sigma^F(\delta_2), \sigma^F(\delta_3)\}$ do not form a regular sequence in $\mathrm{gr}^F(\mathcal{D})$. We have $z\eta\zeta - \xi\zeta \notin \langle \sigma^F(\delta_1), \sigma^F(\delta_2) \rangle$ and

$$(z\eta\zeta - \xi\zeta)\sigma^F(\delta_3) \in \langle \sigma^F(\delta_1), \sigma^F(\delta_2) \rangle.$$

References

[1] Björk, J-E. *Rings of Differential Operators.* North-Holland, Amsterdam 1979.

[2] Calderón-Moreno, F.J. *Operadores diferenciales logarítmicos con respecto a un divisor libre.* Ph.D. Thesis. June 1997.

[3] Calderón-Moreno, F.J. *Logarithmic Differential Operators and Logarithmic De Rham Complexes relative to a Free Divisor.* Ann. Sci. E.N.S., 4^e série, t. 32, 1999, p. 701-714.

[4] Calderón-Moreno F.J., D. Mond, L. Narváez-Macarro and F.J. Castro-Jiménez. *Logarithmic Cohomology of the Complement of a Plane Curve.* Preprint, University of Warwick, 3/1999.

[5] Castro , F. *Thèse de 3ème cycle.* Université Paris VII, 1984.

[6] Castro-Jiménez, F.J., Ucha-Enríquez, J.M. *A duality formula for $\mathcal{D}$-modules.* Preprint

[7] Granger, J.M. and P. Maisonobe. *A basic course on differential modules.* Travaux en Cours 45, Hermann. Paris 1993.

[8] Kashiwara, M. On the Maximally Overdetermined Systems of Linear Differential Equations, I. Publ. RIMS, Kyoto Univ. 10:536-579, (1997).

[9] Mebkhout, Z. *Le formalisme des six opérations de Grothendieck pour les $\mathcal{D}_X$-modules cohérents.* Travaux en cours, volume 35. Hermann, Paris 1989.

[10] Oaku, T. *Algorithms for the b-function and $\mathcal{D}$-modules associated with a polynomial.* Journal of Pure and Applied Algebra, 117 & 118:495-518, 1997.

[11] Saito, K. *Theory of logarithmic differential forms and logarithmic vector fields.* J. Fac Sci. Univ. Tokyo 27:256-291, 1980.

[12] Takayama, N. **Kan:** *a system for computation in Algebraic Analysis.* Source code available for Unix computers from 'ftp.math.kobe-u.ac.jp', 1991.

[13] Torrelli, T. *Équations fonctionelles pour une fontion sur une intersection complète quasi homogène à singularité isolée.* C.R. Acad. Sci. Paris T. 330, Série I, pp:577-580 (2000).

[14] Ucha-Enríquez, J.M. Métodos constructivos en álgebras de operadores diferenciales Ph.D. Thesis, Sept.1999.

[15] Varchenko, A.N. *Asymptotic mixed Hodge structure in vanishing cohomologies.* Math. USSR Izvestija 18:3 (1982), 469-512.