

## A STOCHASTIC THRESHOLD FOR AN EPIDEMIC MODEL WITH ISOLATION AND A NON LINEAR INCIDENCE

TOMÁS CARABALLO\*

Dpto. Ecuaciones Diferenciales y Análisis Numérico  
Facultad de Matemáticas, Universidad de Sevilla  
c/ Tarfia s/n, 41012-Sevilla, Spain

MOHAMED EL FATINI AND IDRIS SEKKAK

Département de Mathématiques, Faculté des Sciences, Université Ibn Tofail  
BP 133, Kénitra, Morocco

REGRAGUI TAKI

Université Chouaib Doukkali, EST Sidi Benour, El Jadida, Morocco

AZIZ LAARIBI

Université Sultan Moulay Slimane, Faculté Polydisciplinaire, Beni Mellal, Morocco

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**ABSTRACT.** In this paper, we study a stochastic epidemic model with isolation and nonlinear incidence. In particular, we propose a stochastic threshold for the model without any sharp sufficient assumptions on model parameters as compared to existing works. Firstly, we establish the uniqueness of the global positive solution according to Lyapunov function method. Secondly, we prove stochastic permanence of the solutions. Then, we establish sufficient condition for the extinction. Thirdly, we investigate necessary and sufficient conditions for persistence in mean of the disease. Finally, we provide some numerical simulations to illustrate our theoretical results.

**1. Introduction.** Diseases are caused by pathogens such as viruses, bacteria, epiphytes or parasites like protozoans and worms. Epidemiology is the study of the distribution and determinants of disease prevalence in populations like natural environment (migration, vegetation, water ways, ...), human environment (infrastructure, intensive agriculture,...), socio-demographic drivers social inequality, prevention,...) and public health systems (healthcare system, food and water quality. One objective of epidemiology is to describe the distribution of the disease, i.e., to find out who has how much of what, where and when. Another objective is to identify the causes or risk factors for diseases in order to find out why everyone does not have the same thing uniformly [8]. A third objective of mathematical modeling in

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\* Corresponding author.

epidemiology is to develop models that will assist the decision making process [2]. In order to limit the spread of certain severe diseases, infected people are isolated as a preventive measure, this operation is called quarantine. As a helping hand to build strategies for governments to control the public health, several compartmental epidemic models have been established, to understand the mechanism and the behavior of the isolation tool. Hethcote et al. [9] introduced a deterministic SIQS model where they studied the asymptotic stability of the equilibrium points. The local and global stability of the model were also discussed by Chen [4]. The stability analysis with saturated incidence rate was investigated by Adebimpe et al. [1]. In [29], the authors studied the deterministic and stochastic dynamics of an SIQS epidemic model with non linear incidence. Yang et al. [26, 27] studied a deterministic SIQS model with saturated incidence  $\psi(S, I) = \frac{\beta SI}{1+rI}$ , where  $\beta$  is the disease transmission coefficient, and  $r$  is the saturation constant. The term  $\beta I$  represents the infection force of the disease and  $\frac{1}{1+rI}$  represents the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals. The resulting system is

$$\begin{aligned}\dot{S} &= A - \psi(S, I) - \mu S + \gamma I + \varepsilon Q, \\ \dot{I} &= \psi(S, I) - (\mu + \alpha_2 + \delta + \gamma)I, \\ \dot{Q} &= \delta I - (\mu + \alpha_3 + \varepsilon)Q,\end{aligned}\tag{1}$$

where  $S(t)$ ,  $I(t)$  and  $Q(t)$  are the population fractions of susceptible, infective and quarantined. The parameters  $A$ ,  $\beta$ ,  $\gamma$ ,  $\varepsilon$ ,  $\alpha_2$ ,  $\alpha_3$ ,  $\delta$ , are positive constants. The constant  $A$  is the recruitment rate of susceptible corresponding to births and immigration,  $\mu$  is the natural death rate of population,  $\delta$  is the rate constant for individuals leaving the compartment  $I$  for the quarantine compartment  $Q$ ,  $\alpha_2$  and  $\alpha_3$  are the disease-related death rate constant in compartments  $I$  and  $Q$  respectively,  $\gamma$  is the rate at which individuals recover and return to Susceptible  $S$  from compartment  $I$  and  $\varepsilon$  is the rate at which individuals recover and return to Susceptible  $S$  from compartment  $Q$ . The corresponding basic reproduction number [9] related to (1) is given by

$$\mathcal{R}_0 = \frac{A\beta}{\mu(\mu + \alpha_2 + \delta + \gamma)}.$$

In deterministic models, the output of the model is fully determined by the parameter values and the initial conditions. In the real world, epidemic dynamics is inevitably perturbed by the environmental noise, see references for white noise [3, 7, 17, 24, 20] and [6, 31] for Lévy noise. Stochastic models possess some inherent randomness. The same set of parameter values and initial conditions will lead to an ensemble of different outputs. In this work, we are interested in a stochastic model perturbed by white noise. In the following,  $\mathcal{B}(\mathbb{R}_+^3)$  denotes the Borel  $\sigma$ -algebra on  $\mathbb{R}_+^3$ , and  $(\mathbb{R}_+^3, \mathcal{B}(\mathbb{R}_+^3), \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$  is a complete probability space with a filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  satisfying the usual conditions (i.e., it is rightly continuous and increasing while  $\mathcal{F}_0$  contains all  $\mathcal{P}$ -null sets). Then, to characterize the effects of stochastic noises on the deterministic system (1) and to make it reasonable and realistic, a stochastic model driven by Brownian motion is proposed as follows

$$\begin{aligned}dS &= [A - \psi(S, I) - \mu S + \gamma I + \varepsilon Q] dt + \sigma_1 S dB_1(t), \\ dI &= [\psi(S, I) - (\mu + \alpha_2 + \delta + \gamma)I] dt + \sigma_2 I dB_2(t),\end{aligned}\tag{2}$$

$$dQ = [\delta I - (\mu + \alpha_3 + \varepsilon)Q] dt + \sigma_3 Q dB_3(t),$$

where  $B_1(t), B_2(t), B_3(t)$  are independent Brownian motions, and  $\sigma_1, \sigma_2, \sigma_3$  are the intensities of the white noises. See also a recent work [22], which is concerned with the dynamical behavior of a stochastic SIQR epidemic model with both white and telegraph noises. Define

$$R_S = \mathcal{R}_0 - \frac{\sigma_2^2}{2(\mu + \alpha_2 + \delta + \gamma)}.$$

In [25], Wei and Chen considered an SIQS epidemic model with a saturated incidence rate, and the authors discussed the permanence and ultimately boundedness with random perturbations related to death rates.

The stochastic model with bilinear incidence, when  $r = 0$ , has been discussed in Zhang et al. [30], where the following behaviors of model solutions, according to the value of the threshold  $R_S$ , were shown:

- If  $R_S < 1$ , then the disease will die out.
- If  $R_S > 1$ , then the disease will prevail provided that

$$\mu > \frac{\sigma^2}{2}, \tag{H1}$$

where  $\sigma^2 = \max(\sigma_1^2, \sigma_2^2, \sigma_3^2)$ .

In [31], the authors presented a stochastic model driven by Lévy noise with a mass action incidence and they discussed the existence of a stochastic threshold under the assumption of existence of  $\rho > 2$  such that

$$\mu - \frac{1}{2}(\rho - 1)\sigma^2 - \frac{\zeta}{\rho} > 0. \tag{H2}$$

where  $\zeta$  is a dependent parameter of the intensities of Lvy jumps see [31]

Our motivation, is to extend the results introduced in [30, 31]. We improve these works by deriving a stochastic threshold which does not impose any condition on the stochastic volatility as (H1) in [30] for white noise, and our analysis can be extended to the Lévy noise case without (H2) hypothesis as in [31].

The rest of this paper is organized as follows. The next section is devoted to the existence and uniqueness of the global positive solution. In the third section, we explore the stochastic boundedness and permanence of the solutions of the stochastic system (2). The fourth section is dedicated to the investigation of the stochastic threshold between the extinction and the persistence in mean of the epidemic. In the last section the presented results are demonstrated by numerical simulations. Finally, some conclusions are stated too.

We define the differential operator  $L$ , associated with the following general  $d$ -dimensional stochastic system

$$dX(t) = F(t, X(t))dt + G(t, X(t))dB(t), \tag{3}$$

where  $F(t, X(t))$  is a function in  $\mathbb{R}^d$  defined in  $[t_0, +\infty[\times\mathbb{R}^d$ ,  $G(t, X(t))$  is a  $d \times m$  matrix,  $F$  and  $G$  are locally Lipschitz functions in  $x$  and  $B(t)$  is a  $d$ -dimensional Wiener process. Let  $S_h = \{x \in \mathbb{R}^d : |x| < h\}$ . The differential operator  $L$ , acts on a function  $V \in C^{1,2}(\mathbb{R}_+ \times S_h; \mathbb{R}_+)$ . as follows

$$LV(t, x) = V_t(t, x) + V_x(t, x)F(t, x) + \frac{1}{2}\text{trace} [G^T(t, x)V_{xx}(x, t)G(x, t)]. \tag{4}$$

By Itô's formula, if  $x(t) \in S_h$ ,

$$dV(t, x(t)) = LV(t, x(t))dt + V_x(t, x(t))G(t, x(t))dB(t), \tag{5}$$

where

$$V_t = \frac{\partial V}{\partial t}, \quad V_x = \left( \frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \dots, \frac{\partial V}{\partial x_d} \right) \quad V_{xx} = \left( \frac{\partial^2 V}{\partial x_i \partial x_j} \right)_{d \times d}. \tag{6}$$

For any  $X \in \mathbb{R}^3$ , the norm  $|X|$ , as usual, is given by  $|X| = \sqrt{X_1^2 + X_2^2 + X_3^2}$ .

**2. Existence and uniqueness of the global positive solution.** In this section, using the Lyapunov analysis method (mentioned in [23]), we shall show that the model (2) has a local positive solution, then we show that this solution is global positive. Denote

$$\mathbb{R}_+^3 = \{(S, I, Q) \in \mathbb{R}^3 : S > 0, I > 0, Q > 0\}.$$

**Theorem 2.1.** *For any given initial value  $(S(0), I(0), Q(0)) \in \mathbb{R}_+^3$ , there is a unique positive solution  $(S(t), I(t), Q(t))$  of model (2) on  $t \geq 0$  and the solution will remain in  $\mathbb{R}_+^3$  with probability 1.*

*Proof.* Since the coefficients of the system (2) are locally Lipschitz continuous, for any given initial value  $(S(0), I(0), Q(0)) \in \mathbb{R}_+^3$  there is a unique local solution positive  $(S(t), I(t), Q(t))$  on  $t \in [0, \tau_\epsilon)$ , where  $\tau_\epsilon$  is the explosion time. Now, we show that the solution is global, we have only to prove that  $\tau_\epsilon = \infty$  a.s. Consider  $\epsilon_0 > 0$  such that  $S(0) > \epsilon_0, I(0) > \epsilon_0, Q(0) > \epsilon_0$ , then we define the stopping time, for all  $\epsilon > 0$  such that  $\epsilon \leq \epsilon_0$ , as follows

$$\tau_\epsilon = \inf \left\{ t \in [0, \tau_\epsilon) : S(t) \notin \left[ \epsilon, \frac{1}{\epsilon} \right] \text{ or } I(t) \notin \left[ \epsilon, \frac{1}{\epsilon} \right], \text{ or } Q(t) \notin \left[ \epsilon, \frac{1}{\epsilon} \right] \right\}. \tag{7}$$

Throughout this paper we set  $\inf \emptyset = \infty$  ( $\emptyset$  denotes the empty set). It is clear that,  $\tau_\epsilon$  is increasing as  $\epsilon \rightarrow 0$ . Set  $\tau_0 = \lim_{\epsilon \rightarrow 0} \tau_\epsilon$ , therefore  $\tau_0 < \tau_\epsilon$  a.s. If  $\tau_0 = \infty$  a.s. is true, then  $\tau_\epsilon = \infty$  a.s. and  $(S(t), I(t), R(t)) \in \mathbb{R}_+^3$  a.s. for  $t \geq 0$ . In other words, to complete the proof it is required to show that  $\tau_0 = \infty$  a.s. If this statement is false, then there exist a pair of constants  $T > 0$  and  $\delta \in (0, 1)$  such that  $\mathcal{P}\{\tau_0 \leq T\} > \delta$ . Thus there is  $\epsilon_1 > 0$  such that

$$\mathcal{P}\{\tau_\epsilon \leq T\} \geq \delta \quad \forall \epsilon \leq \epsilon_1. \tag{8}$$

Consider a  $C^2$ -function  $V_1 : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$  as follows

$$V_1(X(t)) = S(t) + \frac{1}{S(t)} + I(t) + \frac{1}{I(t)} + Q(t) + \frac{1}{Q(t)}, \tag{9}$$

where  $X(t)$  denote  $(S(t), I(t), Q(t))$  Applying Itô's formula and using the expectation (see [14] and [23]) we obtain

$$\mathbb{E}V_1(X(t \wedge \tau_\epsilon)) = V_1(X(0)) + \mathbb{E} \int_0^{t \wedge \tau_\epsilon} LV_1(X(u))du, \tag{10}$$

where  $LV_1$  is given by

$$\begin{aligned} LV_1(X(u)) = & A - \mu S - \frac{1}{S^2} \left[ A - \mu S - \frac{\beta SI}{1+rI} + \gamma I + \varepsilon Q \right] \\ & - (\mu + \alpha_2 + \delta + \gamma)I - \frac{1}{I^2} \left[ \frac{\beta S(u)I(u)}{1+rI(u)} - (\mu + \alpha_2 + \delta + \gamma)I(u) \right] \end{aligned}$$

$$\begin{aligned}
 & -(\mu + \alpha_3 + \varepsilon)Q - \frac{1}{Q^2}(\delta I - (\mu + \alpha_3 + \varepsilon)Q) + \frac{\sigma_1^2}{S} + \frac{\sigma_2^2}{I} + \frac{\sigma_3^2}{Q} \\
 \leq & A + \frac{\mu}{S} + \frac{\beta}{rS} + \frac{\mu + \alpha_2 + \delta + \gamma}{I} + \frac{\mu + \alpha_3 + \varepsilon}{Q} + \frac{\sigma_1^2}{S} + \frac{\sigma_2^2}{I} + \frac{\sigma_3^2}{Q} \\
 \leq & c_1 V(X(u)),
 \end{aligned} \tag{11}$$

where  $c_1 = \max(\mu + \beta/r + \sigma_1^2, \mu + \alpha_2 + \delta + \gamma + \sigma_2^2, \mu + \alpha_3 + \varepsilon + \sigma_3^2)$ . Substituting (11) in (10) we obtain

$$\begin{aligned}
 \mathbb{E}V_1(X(t \wedge \tau_\varepsilon)) & \leq V_1(X(0)) + \mathbb{E} \int_0^{t \wedge \tau_\varepsilon} c_1 \left( \frac{A}{c_1} + V_1(X(u)) \right) du \\
 \mathbb{E} \left( \frac{A}{c_1} + V_1(X(t \wedge \tau_\varepsilon)) \right) & \leq \frac{A}{c_1} + V_1(X(0)) + c_1 \int_0^t \mathbb{E} \left( \frac{A}{c_1} + V_1(X(u)) \right) du.
 \end{aligned}$$

Using the Gronwall inequality we obtain for  $t = T$

$$\frac{A}{c_1} + \mathbb{E}[V_1(X(T \wedge \tau_\varepsilon))] \leq \left( \frac{A}{c_1} + V_1(X(0)) \right) e^{c_1 T}. \tag{12}$$

Let  $\Omega_\varepsilon = \{\tau_\varepsilon \leq T\}$ . Then, from (8) we have,  $\mathcal{P}(\Omega_\varepsilon) \geq \delta$ . On the other hand, from (9), we have  $V_1(X(T \wedge \tau_\varepsilon)) > 0$ , thus

$$\begin{aligned}
 \mathbb{E}[V_1(X(T \wedge \tau_\varepsilon))] & = \mathbb{E}[1_{\Omega_\varepsilon} V_1(X(T \wedge \tau_\varepsilon))] + \mathbb{E}[1_{\Omega_\varepsilon^c} V_1(T \wedge \tau_\varepsilon)] \\
 & \geq \mathbb{E}[1_{\Omega_\varepsilon} V_1(X(\tau_\varepsilon))],
 \end{aligned} \tag{13}$$

where  $1_{\Omega_\varepsilon}$  is the characteristic function of  $\Omega_\varepsilon$ . For every  $\omega \in \Omega_\varepsilon$  some component of  $(X(\tau_\varepsilon, \omega))$  is equal to  $\varepsilon$ , or  $1/\varepsilon$ , then

$$V_1(X(\tau_\varepsilon)) \geq \frac{1}{\varepsilon}, \tag{14}$$

Combining (12)-(14) yields

$$\left( \frac{A}{c_1} + V_1(X(0)) \right) e^{c_1 T} \geq \mathbb{E}[1_{\Omega_\varepsilon} V_1(X(\tau_\varepsilon))] \geq \frac{1}{\varepsilon} \mathcal{P}(\Omega_\varepsilon) \geq \frac{\delta}{\varepsilon}$$

Letting  $\varepsilon$  goes to 0 we obtain

$$\infty > \left( \frac{A}{c_1} + V_1(X(0)) \right) e^{c_1 T} = \infty,$$

which is a contradiction, and therefore we must have  $\tau_\varepsilon = \infty$  a.s. Consequently  $S(t), I(t)$  and  $Q(t)$  are positive and global. □

**3. Stochastic permanence.** In this section, we will investigate the stochastic permanence of (2). First of all, we present the following definitions introduced in [13].

**Definition 3.1.** The solutions of SDE (2) are called stochastically ultimately bounded, if for any  $\tilde{\varepsilon} \in (0, 1)$ , there is a positive constant  $\chi (= \chi(\tilde{\varepsilon}))$ , such that the solution of SDE (2) with any positive initial value has the property that

$$\limsup_{t \rightarrow \infty} \mathcal{P}(|X(t)| > \chi) \leq \tilde{\varepsilon}.$$

**Definition 3.2.** SDE (2) is said to be stochastically permanent if for any  $\tilde{\varepsilon} \in (0, 1)$ , there exist positive constants  $\chi (= \chi(\tilde{\varepsilon}))$ ,  $\delta (= \delta(\tilde{\varepsilon}))$ , such that

$$\liminf_{t \rightarrow \infty} \mathcal{P}(|X(t)| \leq \chi) \geq 1 - \tilde{\varepsilon}, \quad \liminf_{t \rightarrow \infty} \mathcal{P}(|X(t)| \geq \delta) \geq 1 - \tilde{\varepsilon}.$$

**Theorem 3.3.** *The solutions of model SDE (2) are stochastically ultimately bounded for any initial value  $X(0) \in \mathbb{R}_+^3$ .*

*Proof.* The total population is denoted by  $N(t) = S(t) + I(t) + Q(t)$ . Consider a  $C^2$  function  $V_2 = \vartheta_2^p$ , where  $\vartheta_2(X) = 1 + N$  and  $p$  a positive real to be chosen below. By Itô's formula we have

$$\begin{aligned} dV_2(t) = & p\vartheta_2^{p-1}(t)[A - \mu N(t) - \alpha_2 I(t) - \alpha_3 Q(t)]dt \\ & + \frac{p(p-1)}{2}\vartheta_2^{p-2}(t)[\sigma_1^2 S^2(t) + \sigma_2^2 I^2(t) + \sigma_3^2 Q^2(t)]dt \\ & + p\vartheta_2^{p-1}(t)[\sigma_1 S(t)dB_1(t) + \sigma_2 I(t)dB_2(t) + \sigma_3 Q(t)dB_3(t)]. \end{aligned} \tag{15}$$

Integrating (15) from 0 to  $t$  and taking expectations of both sides and using Fubini's Theorem, we obtain that

$$\begin{aligned} \mathbb{E}(V_2(t)) - V_2(0) = & p \int_0^t \mathbb{E}(\vartheta_2^{p-1}(u)[A + \mu - \mu\vartheta_2(u) - \alpha_2 I(u) - \alpha_3 Q(u)])du \\ & + p \int_0^t \mathbb{E} \left( \frac{p-1}{2} \vartheta_2^{p-2}(u)[\sigma_1^2 S^2(u) + \sigma_2^2 I^2(u) + \sigma_3^2 Q^2(u)] \right) du. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{d\mathbb{E}(V_2(t))}{dt} = & p\mathbb{E}(\vartheta_2^{p-1}(t)[A + \mu - \mu\vartheta_2(t) - \alpha_2 I(t) - \alpha_3 Q(t)]) \\ & + \frac{p(p-1)}{2}\mathbb{E}[\vartheta_2^{p-2}(t)(\sigma_1^2 S^2(t) + \sigma_2^2 I^2(t) + \sigma_3^2 Q^2(t))]. \end{aligned}$$

Let  $\sigma^2 = \max(\sigma_2^2, \sigma_2^2, \sigma_3^2)$  and  $p > 1$ . Since  $I, Q > 0$ , we obtain

$$\frac{d\mathbb{E}(V_2(t))}{dt} \leq p \left[ \left( \frac{p-1}{2} \sigma^2 - \mu \right) \mathbb{E}(\vartheta_2^p(t)) + (A + \mu)\mathbb{E}(\vartheta_2^{p-1}(t)) \right].$$

We choose  $p$  such that  $1 < p < 1 + \frac{2\mu}{\sigma^2}$ , and using the fact that  $x \mapsto \mathbb{E}(|\cdot|^x)^{\frac{1}{x}}$  is increasing for all  $x > 0$  (see [23, page 23]). Consequently we have

$$\begin{aligned} \frac{d\mathbb{E}(V_2(t))}{dt} \leq & p \left[ - \left( \mu - \frac{p-1}{2} \sigma^2 \right) [\mathbb{E}(\vartheta_2^{p-1}(t))]^{\frac{p}{p-1}} + (A + \mu)\mathbb{E}(\vartheta_2^{p-1}(t)) \right] \\ \leq & p\mathbb{E}(\vartheta_2^{p-1}(t)) \left[ (A + \mu) - \left( \mu - \frac{p-1}{2} \sigma^2 \right) [\mathbb{E}(\vartheta_2^{p-1}(t))]^{\frac{1}{p-1}} \right] \end{aligned}$$

Let,  $\vartheta(t) = \mathbb{E}(\vartheta_2^{p-1}(t))$ , then we have

$$\frac{d(V_2(t))}{dt} \leq p\vartheta(t) \left[ (A + \mu) - \left( \mu - \frac{p-1}{2} \sigma^2 \right) \vartheta^{\frac{1}{p-1}}(t) \right].$$

We recall that the solution of ODE defined as follow

$$\begin{cases} d\vartheta(t) = p\vartheta(t) \left[ (A + \mu) - \left( \mu - \frac{p-1}{2} \sigma^2 \right) \vartheta^{\frac{1}{p-1}}(t) \right], \\ \vartheta(0) = V_2(0) \end{cases}$$

is

$$\vartheta(t) = \left[ \frac{2\mu - (p-1)\sigma^2}{2(A + \mu)} + \left( \frac{1}{V_2(0)} - \frac{2\mu - (p-1)\sigma^2}{2(A + \mu)} \right) e^{-(A+\mu)t} \right]^{-(p-1)},$$

which leads to

$$\vartheta(t) \rightarrow \left[ \frac{2(A + \mu)}{2\mu - (p-1)\sigma^2} \right]^{p-1}, \quad \text{as } t \rightarrow \infty.$$

Then by a comparison theorem we have

$$\limsup_{t \rightarrow \infty} \mathbb{E}(V_2(t)) \leq \left[ \frac{2(A + \mu)}{2\mu - (p - 1)\sigma^2} \right]^{p-1} := L(p),$$

Which implies that there exists a  $T > 0$ , such that

$$\mathbb{E}(V_2(t)) \leq 2L(p), \quad t > T.$$

Since the function  $\mathbb{E}(V_2(t))$  is continuous, we deduce that there exists a constant  $\ell > 0$  such that for all  $t > 0$ ,

$$\mathbb{E}(1 + N(t))^p = \mathbb{E}(V_2(t)) \leq \ell. \tag{16}$$

We choose a constant  $\chi$  sufficiently large such that  $\frac{\ell}{\chi^p} < 1$ . Using Chebyshev's inequality, we have

$$\mathcal{P}(|X(t)| > \chi) \leq \mathcal{P}((1 + N(t))^p > \chi^p) \leq \frac{1}{\chi^p} \mathbb{E}((1 + N(t))^p),$$

and this implies that

$$\limsup_{t \rightarrow \infty} \mathcal{P}(|X(t)| > \chi) \leq \frac{\ell}{\chi^p} := \tilde{\varepsilon}. \tag{17}$$

□

**Theorem 3.4.** *The SDE (2) is stochastically permanent for any initial value  $X(0) \in \mathbb{R}_+^3$ .*

*Proof.* Its easy to see that inequality (17) leads to

$$\liminf_{t \rightarrow \infty} \mathcal{P}(|X(t)| \leq \chi) \leq 1 - \tilde{\varepsilon}.$$

Then to complete the proof we need to show  $\liminf_{t \rightarrow \infty} \mathcal{P}(|X(t)| \geq \delta) \leq 1 - \tilde{\varepsilon}$ .

By the Itô formula, we have

$$\begin{aligned} d\left(\frac{1}{N(u)}\right) &= -N^{-2}(u)[A - \mu N(u) - \alpha_2 I - \alpha_3 Q(u)]du \\ &\quad + N^{-3}(u)[\sigma_1^2 S(u)^2 + \sigma_2^2 I(u)^2 + \sigma_3^2 Q(u)^2]du \\ &\quad - N^{-2}(u)[\sigma_1 S(u)dB_1(u) + \sigma_2 I(u)dB_2(u) + \sigma_3 Q(u)dB_3(u)]. \end{aligned} \tag{18}$$

Integrating (18) from 0 to  $t$ , taking expectations of both sides and using Fubini's Theorem, we obtain

$$\begin{aligned} \mathbb{E}\left(\frac{1}{N(u)}\right) - \left(\frac{1}{N(0)}\right) &= \int_0^t \mathbb{E}\left(N^{-2}(u)[-A + \mu N(u) + \alpha_2 I(u) + \alpha_3 Q(u)]\right) du \\ &\quad + \int_0^t \mathbb{E}\left(N^{-3}(u)[\sigma_1^2 S^2(u) + \sigma_2^2 I^2(u) + \sigma_3^2 Q^2(u)]\right) du. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{d\mathbb{E}(N^{-1}(t))}{dt} &= \mathbb{E}\left(N^{-2}(t)[-A + \mu N(t) + \alpha_2 I(t) + \alpha_3 Q(t)]\right) \\ &\quad + \mathbb{E}\left(N^{-3}(t)[\sigma_1^2 S^2(t) + \sigma_2^2 I^2(t) + \sigma_3^2 Q^2(t)]\right) \\ &\leq -A\mathbb{E}(N^{-2}(t)) + (\mu + \alpha_2 + \varepsilon + \sigma^2)\mathbb{E}(N^{-1}(t)) \\ &\leq -A[\mathbb{E}(N^{-1}(t))]^2 + (\mu + \alpha_2 + \varepsilon + \sigma^2)\mathbb{E}(N^{-1}(t)) \\ &= \mathbb{E}(N^{-1}(t))((\mu + \alpha_2 + \varepsilon + \sigma^2) - A[\mathbb{E}(N^{-1}(t))]). \end{aligned}$$

Therefore, letting  $z(t) = E(N^{-1}(t))$ ,

$$\frac{dz(t)}{dt} \leq z(t)[(\mu + \alpha_2 + \varepsilon + \sigma^2) - Az(t)].$$

Recall that the solution of the following ODE

$$\begin{cases} \frac{dz(t)}{dt} = z(t)[(\mu + \alpha_2 + \varepsilon + \sigma^2) - Az(t)], \\ z(0) = \frac{1}{N(0)}, \end{cases}$$

is given by

$$z(t) = \frac{1}{\frac{A}{\mu + \alpha_2 + \varepsilon + \sigma^2} + \left(N(0) - \frac{A}{\mu + \alpha_2 + \varepsilon + \sigma^2}\right) e^{-\left(\frac{A}{\mu + \alpha_2 + \varepsilon + \sigma^2}\right)t}}.$$

Therefore

$$\lim_{t \rightarrow \infty} z(t) = \frac{\mu + \alpha_2 + \varepsilon + \sigma^2}{A} := l.$$

Thus by the comparison theorem we obtain

$$\limsup_{t \rightarrow \infty} \mathbb{E} \left( \frac{1}{N(t)} \right) \leq l.$$

By the same approach as before we deduce that there is a positive real number  $\ell'$  such that for all  $t > 0$

$$\mathbb{E} \left( \frac{1}{N(t)} \right) \leq \ell'.$$

Besides, we choose  $\delta$  sufficiently small such that  $\sqrt{3}\delta\ell' < 1$ . By Chebyshev's inequality and using the inequality  $N^2 \leq 3|X|^2$  we deduce that

$$\mathcal{P}(|X| \leq \delta) = \mathcal{P} \left( \frac{1}{|X|} \geq \frac{1}{\delta} \right) \leq \delta \mathbb{E} \left( \frac{1}{|X(t)|} \right) \leq \delta \mathbb{E} \left( \frac{\sqrt{3}}{N(t)} \right) \leq \sqrt{3}\delta\ell' := \tilde{\varepsilon}.$$

Hence

$$\limsup_{t \rightarrow \infty} \mathcal{P}(|X| < \delta) \leq \tilde{\varepsilon},$$

which implies that

$$\liminf_{t \rightarrow \infty} \mathcal{P}(|X| \geq \delta) \geq 1 - \tilde{\varepsilon}.$$

□

**4. Investigation of a stochastic threshold.** In this section, we investigate a stochastic threshold between the extinction and the persistence in mean of the stochastic system (2).

**Lemma 4.1.** *Let  $(S(t), I(t), Q(t))$  be a solution of system (2) with any initial value  $(S(0), I(0), Q(0)) \in \mathbb{R}_+^3$ , then*

$$\lim_{t \rightarrow \infty} \frac{S(t)}{t} = \lim_{t \rightarrow \infty} \frac{I(t)}{t} = \lim_{t \rightarrow \infty} \frac{Q(t)}{t} = 0 \quad a.s. \tag{19}$$

*Proof.* From (15) and for all  $p$  such that  $1 < p < 1 + \frac{2\mu}{\sigma^2}$ , we can derive that

$$\begin{aligned} d(V_2(s)) &\leq p \left[ A\vartheta_2^{p-1}(s) + \frac{(p-1)\sigma^2}{2} \vartheta_2^p(s) \right] ds \\ &\quad + p\vartheta_2^{p-1}(s) [\sigma_1 S(s)dB_1(s) + \sigma_2 I(s)dB_2(s) + \sigma_3 Q(s)dB_3(s)], \end{aligned}$$



which implies that

$$\begin{aligned} \mathbb{E} \left( \sup_{k \leq u \leq k+1} V_2(u) \right) &\leq \mathbb{E}V_2(k) + p \int_k^{k+1} \mathbb{E} \left[ A\vartheta_2^{p-1}(s) + \frac{(p-1)\sigma^2}{2}\vartheta_2^p(s) \right] ds \\ &+ p\mathbb{E} \left( \sup_{k \leq u \leq k+1} \left[ \int_k^u \vartheta_2^{p-1}(s) [\sigma_1 S(s)dB_1(s) + \sigma_2 I(s)dB_2(s) + \sigma_3 Q(s)dB_3(s)] \right] \right). \end{aligned} \tag{20}$$

Let  $a$  be a positive real number such that  $1 - 4ap\sigma^2 > \frac{1}{2}$ . By Burkholder-Davis-Gundy’s inequality (see [23]) and Hölder’s inequality, we derive that

$$\begin{aligned} &\mathbb{E} \left( \sup_{k \leq u \leq k+1} \left[ \int_k^u \vartheta_2^{p-1}(s) (\sigma_1 S(s)dB_1(s) + \sigma_3 I(s)dB_2(s) + \sigma_3 Q(s)dB_3(s)) \right] \right) \\ &\leq 4\sqrt{2} \left[ \int_k^{k+1} \left( \vartheta_2^{2(p-1)}(s) [\sigma_1^2 S^2(s) + \sigma_2^2 I^2(s) + \sigma_3^2 Q^2(s)] \right) ds \right]^{\frac{1}{2}} \\ &\leq 4\sqrt{2}\sigma^2 \mathbb{E} \left[ \int_k^{k+1} \vartheta_2^{2p}(s) ds \right]^{\frac{1}{2}} \\ &\leq 4\sqrt{2}\sigma^2 \mathbb{E} \left[ \sup_{k \leq s \leq k+1} \vartheta_2^p(s) \int_k^{k+1} \vartheta_2^p(s) ds \right]^{\frac{1}{2}} \\ &\leq 4\sigma^2 \mathbb{E} \left[ \left( a \sup_{k \leq s \leq k+1} \vartheta_2^p(s) \right)^2 + \left( \frac{1}{a} \int_k^{k+1} \vartheta_2^p(s) ds \right)^2 \right]^{\frac{1}{2}} \\ &\leq 4a\sigma^2 \mathbb{E} \left( \sup_{k \leq s \leq k+1} \vartheta_2^p(s) \right) + \frac{4}{a}\sigma^2 \int_k^{k+1} \mathbb{E}\vartheta_2^p(s) ds \\ &= 4a\sigma^2 \mathbb{E} \left( \sup_{k \leq s \leq k+1} V_2(s) \right) + \frac{4}{a}\sigma^2 \int_k^{k+1} \mathbb{E}V_2(s) ds. \end{aligned}$$

This together with (20) and (16) means that there is a constant  $H > 0$  such that

$$\mathbb{E} \left( \sup_{k \leq u \leq k+1} V_2(u) \right) \leq 2\ell \left[ 1 + p \left( A + \frac{p-1}{2}\sigma^2 + \frac{4}{a}\sigma^2 \right) \right] := H, \quad k = 1, 2, \dots$$

Let  $\epsilon > 0$  be arbitrary. Then, by Chebyshev’s inequality, we have

$$\mathcal{P} \left( \sup_{k \leq u \leq k+1} V_2(u) > k^{1+\epsilon} \right) \leq \frac{H}{k^{1+\epsilon}}, \quad k = 1, 2, \dots$$

Now, the Borel-Cantelli Lemma implies that, with probability 1, we have

$$\sup_{k \leq u \leq k+1} V_2(u) \leq k^{1+\epsilon},$$

for all but finitely many  $k$ . Thus for almost all  $\omega \in \Omega$ , there exists a positive integer  $k_0(\omega)$  such that

$$\sup_{k \leq u \leq k+1} V_2(u) \leq k^{1+\epsilon}, \quad \text{whenever } k \geq k_0.$$

Hence, for almost all  $\omega \in \Omega$ , if  $k \geq k_0$  and  $k \leq t \leq k + 1$ , we obtain

$$\frac{\log(N(t))^p}{\log(t)} \leq \frac{\log(1 + N(t))^p}{\log(t)} \leq \frac{(1 + \epsilon)\log(k)}{\log(k)} = 1 + \epsilon.$$

Letting  $\epsilon \rightarrow 0$ , we obtain

$$\limsup_{t \rightarrow \infty} \frac{\log(N(t))}{\log(t)} \leq \frac{1}{p} \quad \text{a.s.}$$

Then, for any arbitrarily small positive constant  $\epsilon_1 (\epsilon_1 < 1 - \frac{1}{p})$ , there exist a constant  $T_1(\omega)$  and a set  $\Omega_1$ , such that  $\mathcal{P}(\Omega_1) \geq 1 - \epsilon_1$  and for  $t \geq T_1(\omega)$ ,  $\omega \in \Omega_1$

$$\frac{\log(N(t))}{\log(t)} \leq \frac{1}{p} + \epsilon_1.$$

Hence

$$\limsup_{t \rightarrow \infty} \frac{N(t)}{t} \leq \limsup_{t \rightarrow \infty} \frac{t^{\frac{1}{p} + \epsilon_1}}{t} = 0 \quad \text{a.s.}$$

By Theorem 2.1, the solution  $(S(t), I(t), Q(t))$  with positive initial value will remain in  $\mathbb{R}_+^3$ , and

$$\lim_{t \rightarrow \infty} \frac{S(t)}{t} = \lim_{t \rightarrow \infty} \frac{I(t)}{t} = \lim_{t \rightarrow \infty} \frac{Q(t)}{t} = 0 \quad \text{a.s.}$$

□

**Lemma 4.2.** *Let  $(S(t), I(t), Q(t))$  be a solution of system (2) with an initial value  $(S(0), I(0), Q(0)) \in \mathbb{R}_+^3$ . Then*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t S(u) dB_1(u) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t I(u) dB_2(u) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t Q(u) dB_3(u) = 0 \quad \text{a.s.} \tag{21}$$

*Proof.* The proof is analogous to [30]. □

**4.1. Extinction.** In this section, we shall establish sufficient conditions for extinction of the disease in the stochastic model (2). In the sequel, we set  $\langle f(t) \rangle = \frac{1}{t} \int_0^t f(u) du$ .

**Theorem 4.3.** *Let  $(S(t), I(t), Q(t))$  be a solution of system (2) with initial value  $(S(0), I(0), Q(0)) \in \mathbb{R}_+^3$ . Then, if  $R_S < 1$ , the disease of system (2) will go to extinction almost surely, i.e.,*

$$\lim_{t \rightarrow \infty} I(t) = 0 \quad \text{a.s.}$$

Moreover

$$\lim_{t \rightarrow \infty} \langle Q(t) \rangle = 0 \quad \text{a.s.} \quad \text{and} \quad \lim_{t \rightarrow \infty} \langle S(t) \rangle = \frac{A}{\mu} \quad \text{a.s.}$$

*Proof.* By Itô's formula, we have

$$d(\log(I(t))) = \left( \frac{\beta S(t)}{1 + rI(t)} - (\mu + \alpha_2 + \delta + \gamma) - \frac{\sigma_2^2}{2} \right) dt + \sigma_2 dB_2(t). \tag{22}$$

Integrating both sides of (22) from 0 to  $t$  and dividing by  $t$ ,

$$\frac{\log(I(t))}{t} = \frac{\log(I(0))}{t} + \left\langle \frac{\beta S(t)}{1 + rI(t)} \right\rangle - \left( \mu + \alpha_2 + \delta + \gamma + \frac{\sigma_2^2}{2} \right) + \sigma_2 \frac{B(t)}{t}. \tag{23}$$

By Theorem 2.1, the solution  $(S(t), I(t), Q(t))$  with positive initial value will remain in  $\mathbb{R}_+^3$ , leads that

$$\frac{\log(I(t))}{t} \leq \frac{\log(I(0))}{t} + \beta \langle S(t) \rangle - \left( (\mu + \alpha_2 + \delta + \gamma) + \frac{\sigma_2^2}{2} \right) + \sigma_2 \frac{B(t)}{t}. \tag{24}$$

Define the function

$$\phi_1(t) = S(t) + I(t) + \frac{\varepsilon}{\mu + \alpha_3 + \varepsilon}Q(t).$$

From the system (2), we obtain

$$\begin{aligned} d\phi_1(t) = & \left[ A - \mu S(t) + \left( \frac{\varepsilon\delta}{\mu + \alpha_3 + \varepsilon} - (\mu + \alpha_2 + \delta) \right) I(t) \right] dt \\ & + \sigma_1 S(t)dB_1(t) + \sigma_2 I(t)dB_2(t) + \frac{\sigma_3\varepsilon\delta}{\mu + \alpha_3 + \varepsilon}Q(t)dB_3(t). \end{aligned}$$

By integration we deduce that

$$\langle S(t) \rangle = \frac{A}{\mu} + \left( \frac{\varepsilon\delta}{\mu(\mu + \alpha_3 + \varepsilon)} - \frac{\mu + \alpha_2 + \delta}{\mu} \right) \langle I(t) \rangle + \frac{\Psi_1(t)}{t}. \tag{25}$$

where

$$\begin{aligned} \Psi_1(t) = & \frac{1}{\mu} \left( \sigma_1 \int_0^t S(u)dB_1(u)du + \sigma_2 \int_0^t I(u)dB_2(u)du \right. \\ & \left. + \frac{\sigma_3\varepsilon\delta}{\mu + \alpha_3 + \varepsilon} \int_0^t Q(u)dB_3(u)du + \phi_1(0) - \phi_1(t) \right). \end{aligned}$$

It is easy to see that

$$\langle S(t) \rangle \leq \frac{A}{\mu} + \frac{\Psi_1(t)}{t}. \tag{26}$$

Using (24) into (26), we obtain

$$\begin{aligned} \frac{\log(I(t))}{t} \leq & \frac{\log(I(0))}{t} + \frac{\beta A}{\mu} - \left( \mu + \alpha_2 + \delta + \gamma + \frac{\sigma_2^2}{2} \right) \frac{B(t)}{t} + \beta \frac{\Psi_1(t)}{t} \\ = & \frac{\log(I(0))}{t} + (\mu + \alpha_2 + \delta + \gamma)(R_S - 1) + \sigma_2 \frac{B(t)}{t} + \beta \frac{\Psi_1(t)}{t}. \end{aligned} \tag{27}$$

Taking into account now the law of large numbers and Lemma 4.1 and Lemma 4.2, we have  $\lim_{t \rightarrow \infty} \frac{B(t)}{t} = 0$  and

$$\lim_{t \rightarrow \infty} \frac{\Psi_1(t)}{t} = 0. \tag{28}$$

This together with (27) implies that

$$\limsup_{t \rightarrow \infty} \frac{\log(I(t))}{t} \leq (\mu + \alpha_2 + \delta + \gamma)(R_S - 1),$$

which leads to

$$\lim_{t \rightarrow \infty} I(t) = 0 \quad \text{a.s.} \tag{29}$$

From the third equation of system (2) we deduce that

$$\frac{Q(t) - Q(0)}{t} = \delta \langle I(s) \rangle - (\mu + \alpha_3 + \varepsilon) \langle Q(t) \rangle + \frac{\sigma_3}{t} \int_0^t Q(s)dB(s).$$

From lemma (4.2) and (29). Hence, we conclude that

$$\lim_{t \rightarrow \infty} \langle Q(t) \rangle = 0 \quad \text{a.s.} \tag{30}$$

Using (25), (28), (29) and Lemmas 4.1 and 4.2 we obtain

$$\lim_{t \rightarrow \infty} \langle S(t) \rangle = \frac{A}{\mu} \quad \text{a.s.}$$

□

**4.2. Persistence in mean.** To study the persistence of the disease, we need the following lemmas presented in [12].

**Lemma 4.4.** *Let  $f \in C[[0, \infty) \times \Omega, (0, \infty)]$ . If there exist positive constants  $\lambda_0, \lambda$  such that*

$$\log(f(t)) \geq \lambda t - \lambda_0 \int_0^t f(t)dt + F(t), \quad a.s.$$

for all  $t \geq 0$ , where  $F(t) \in C[[0, \infty) \times \Omega, \mathbb{R}]$  and  $\lim_{t \rightarrow \infty} \frac{F(t)}{t} = 0$  a.s. Then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(t)dt \geq \frac{\lambda}{\lambda_0} \quad a.s.$$

**Lemma 4.5.** *Let  $f \in C[[0, \infty) \times \Omega, (0, \infty)]$ . If there exist positive constants  $\lambda_0, \lambda$  such that*

$$\log(f(t)) \leq \lambda t - \lambda_0 \int_0^t f(t)dt + F(t), \quad a.s.$$

for all  $t \geq 0$ , where  $F(t) \in C[[0, \infty) \times \Omega, \mathbb{R}]$  and  $\lim_{t \rightarrow \infty} \frac{F(t)}{t} = 0$  a.s. Then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(t)dt \leq \frac{\lambda}{\lambda_0} \quad a.s.$$

**Theorem 4.6.** *Let  $(S(t), I(t), Q(t))$  be a solution of system (2) starting from a positive data  $(S(0), I(0), Q(0))$ . If  $R_S > 1$ , then the disease will be persistent in mean. Moreover we have*

$$\lim_{t \rightarrow \infty} \langle I(t) \rangle = \frac{(\mu + \alpha_2 + \delta + \gamma)(R_S - 1)}{\left(\gamma + \frac{\delta\varepsilon}{\mu + \alpha_3 + \varepsilon}\right)r + (\mu r + \beta) \left(\frac{\mu + \alpha_2}{\mu} + \frac{(\mu + \alpha_3)\delta}{\mu(\mu + \alpha_3 + \varepsilon)}\right)}, \quad (31)$$

$$\lim_{t \rightarrow \infty} \langle S(t) \rangle = \frac{A}{\mu} + \frac{\left(\frac{\varepsilon\delta}{\mu(\mu + \alpha_3 + \varepsilon)} - \frac{\mu + \alpha_2 + \delta}{\mu}\right)(\mu + \alpha_2 + \delta + \gamma)(R_S - 1)}{\left(\gamma + \frac{\delta\varepsilon}{\mu + \alpha_3 + \varepsilon}\right)r + (\mu r + \beta) \left(\frac{\mu + \alpha_2}{\mu} + \frac{(\mu + \alpha_3)\delta}{\mu(\mu + \alpha_3 + \varepsilon)}\right)}, \quad (32)$$

$$\lim_{t \rightarrow \infty} \langle Q(t) \rangle = \frac{\delta(\mu + \alpha_2 + \delta + \gamma)(R_S - 1)}{(\gamma + \delta\varepsilon)(\mu + \alpha_3 + \varepsilon)r + (\mu r + \beta) \left(\frac{(\mu + \alpha_2)(\mu + \alpha_3 + \varepsilon)}{\mu} + \frac{(\mu + \alpha_3)\delta}{\mu}\right)}. \quad (33)$$

*Proof.* Define a function

$$\phi_2(t) = S(t) + \frac{\varepsilon}{\mu + \alpha_3 + \varepsilon}Q(t).$$

Let

$$M_1(t) = \sigma_1 \int_0^t S(u)dB(u), \quad M_2(t) = \sigma_2 \int_0^t I(u)dB(u),$$

and

$$M_3(t) = \sigma_3 \int_0^t R(u)dB(u).$$

Integrating the first and the third equation of system (2), we obtain

$$\begin{aligned} \frac{\phi_2(t) - \phi_2(0)}{t} &= A - \left(\mu + \frac{\beta}{r}\right) \langle S(t) \rangle + \frac{\beta}{r} \left\langle \frac{S(t)}{1 + rI(t)} \right\rangle + \frac{M_1(t)}{t} \\ &\quad + \left(\gamma + \frac{\delta\varepsilon}{\mu + \alpha_3 + \varepsilon}\right) \langle I(t) \rangle + \frac{\varepsilon}{\mu + \alpha_3 + \varepsilon} \frac{M_3(t)}{t}. \end{aligned} \quad (34)$$

Combining (23), (25) and (34) we compute that

$$\begin{aligned} \frac{\log(I(t))}{t} &= \frac{A\beta}{\mu} - \left( \mu + \alpha_2 + \delta + \gamma + \frac{\sigma_2^2}{2} \right) + \frac{F(t)}{t} \\ &\quad - r \left[ \gamma + \frac{\delta\varepsilon}{\mu + \alpha_3 + \varepsilon} + \left( \mu + \frac{\beta}{r} \right) \left( \frac{\mu + \alpha_2}{\mu} + \frac{(\mu + \alpha_3)\delta}{\mu(\mu + \alpha_3 + \varepsilon)} \right) \right] < I(t) > \\ &= - \left[ \left( \gamma + \frac{\delta\varepsilon}{\mu + \alpha_3 + \varepsilon} \right) r + (\mu r + \beta) \left( \frac{\mu + \alpha_2}{\mu} + \frac{(\mu + \alpha_3)\delta}{\mu(\mu + \alpha_3 + \varepsilon)} \right) \right] < I(t) > \\ &\quad + (\mu + \alpha_2 + \delta + \gamma)(R_S - 1) + \frac{F(t)}{t}, \end{aligned} \tag{35}$$

where

$$F(t) = \log(I(0)) + \left( \frac{\mu r}{\beta} + 1 \right) \phi_2(t) + \frac{r}{\beta} \left( M_1(t) + \frac{\varepsilon}{\mu + \alpha_3 + \varepsilon} M_3(t) \right).$$

Using Lemma 4.4 and Lemma 4.5 we deduce assertion (31). Combining (25),(28) and (31) we obtain assertion (32). Integrating now the third equation of system (2),

$$\frac{Q(t) - Q(0)}{t} = \delta < I(t) > - (\mu + \alpha_3 + \varepsilon) < Q(t) > ,$$

making  $t$  go to  $\infty$  and using (31) and Lemma 4.1 we obtain the desired assertion (33). □

**5. Numerical simulations.** To illustrate numerically [10, 15] our theoretical results, we consider a realistic infective disease as pneumococcus which is a bacterium infecting the area of the upper throat named the nasopharynx which can be transmitted by direct contact as coughs and sneezes of an infected person. Usually antibiotics were necessarily used to treat pneumococcus infections. Therefore a resistance has increased across the time against this treatment. For this matter vaccines have been developed to cure the infection. We consider a population of children under 2 years of age in Scotland (see [19]). Due to direct spread of the disease, we consider the quarantine measurement as a tool to control the transmission of the Pneumococcus infection as their immune systems are not sufficiently well developed. In the following, we choose initial values  $S(0) = 156015$ ,  $I(0) = 1990$  and  $Q(0) = 995$  and the time step  $\Delta t = 10^{-3}$ . For the parameters we take the same values as Zhang et al. in [30] as follows.

$A$	$\beta$	$\mu$	$\gamma$	$\varepsilon$
206.04	$2.865 \times 10^{-7}$	$1.3736 \times 10^{-3}$	0.02011	0.1

TABLE 1. Table of parameter used in the numerical simulation.

Numerical simulations are performed using different values of  $\alpha_2$ ,  $\alpha_3$ ,  $\sigma_1$ ,  $\sigma_2$ ,  $\sigma_3$ , and  $\delta$ .

### Extinction when $R_S < 1$ .

**Example 1** ( $\mu > \sigma^2/2$ ). We choose the parameters  $r = 0.0001$ ,  $\alpha_2 = 0.001$ ,  $\alpha_3 = 0.011$ ,  $\sigma_1 = 0.001$ ,  $\sigma_2 = 0.052$ ,  $\sigma_3 = 0.02$ , to get,  $\mathcal{R}_0 = 1.0116$  and  $\mathcal{R}_S = 0.9797$ . Hence, according to Theorem 4.3, if  $R_s < 1$ , for the positive solution  $(S(t), I(t), Q(t))$  of the system (2) the disease will extinct a.s. Fig 1 clearly support the theoretical result. Therefore, we illustrated the same parameters simulated in [30] for the bilinear incidence rate.

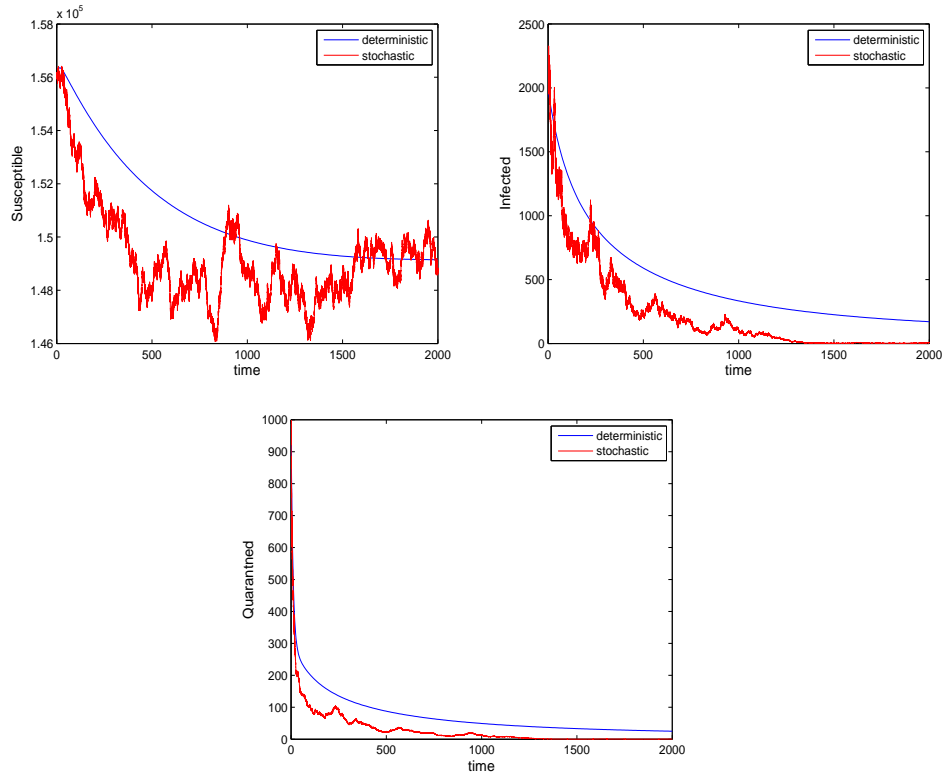


FIGURE 1. Paths of stochastic and deterministic systems as given in Example 1

**Example 2** ( $\mu \leq \sigma^2/2$ ). We choose parameters  $r = 0.0001$ ,  $\alpha_2 = 0.006$ ,  $\alpha_3 = 0.001$ ,  $\delta = 0.015$ ,  $\sigma_1 = 0.001$ ,  $\sigma_2 = 0.15$ ,  $\sigma_3 = 0.2$ . In this case,  $\mathcal{R}_0 = 1.0116$  and  $\mathcal{R}_S = 0.7468 < 1$ . Hence, according to Theorem 4.3, if the condition on the stochastic threshold is fulfilled for the positive solution  $(S(t), I(t), Q(t))$ . The pneumococcus disease goes to extinction a.s. as illustrated in Fig 2. Thus, the simulation clearly support the theoretical result which alleviate the condition mentioned in the previous example on the volatility for the stochastic epidemic model (2) with the saturated incidence rate.

### Persistence when $R_S > 1$ .

**Example 3** ( $\mu > \sigma^2/2$ ). We choose parameters  $r = 0.001$ ,  $\alpha_2 = 0.0001$ ,  $\alpha_3 = 0.001$ ,  $\delta = 0.005$ ,  $\sigma_1 = 0.001$ ,  $\sigma_2 = 0.006$ ,  $\sigma_3 = 0.02$ , to get,  $\mathcal{R}_0 = 1.6166$  and  $\mathcal{R}_S =$

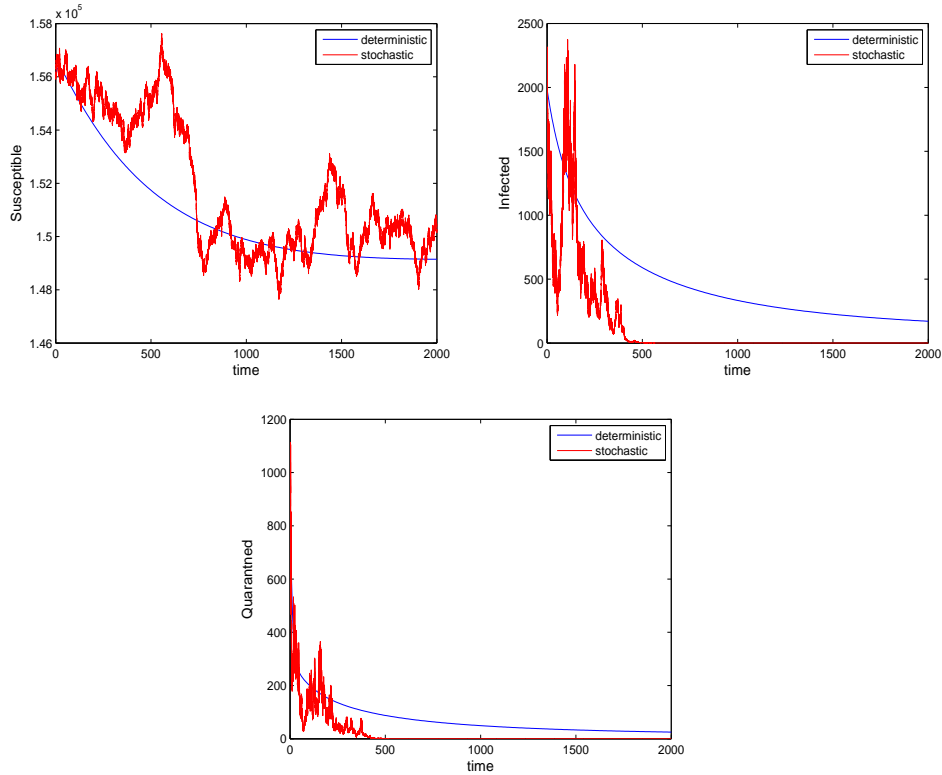


FIGURE 2. Paths of stochastic and deterministic systems as given in Example 2

$1.5696 > 1$ . Hence, according to Theorem 4.6, the positive solution  $(S(t), I(t), Q(t))$  of the stochastic system (2) is persistent in mean. Obviously, the numerical simulation results, see Fig 3, are consistent with the conclusions of our Theorem 4.6. The numerical simulation of the Pneumococcus disease with quarantine measurement were performed for the incidence rate  $r$  under a condition on the volatility and the stochastic threshold to generalize the bilinear case in [30].

**Example 4** ( $\mu \leq \sigma^2/2$ ).  $r = 0.001, \alpha_2 = 0.001, \alpha_3 = 0.001, \delta = 0.007, \sigma_1 = 0.001, \sigma_2 = 0.06, \sigma_3 = 0.05$ , to get,  $\mathcal{R}_0 = 1.4576$  and  $\mathcal{R}_S = 1.3965 > 1$ . Hence, according to Theorem 4.6, the disease will persist a.s. as illustrated in Fig 4 . The performed simulation for the pneumococcus disease support our theoretical result where we assumed that the persistence will hold in a population of children under two years if the condition on the stochastic threshold is ensured regardless of the intensity of the white noise and the considered quarantine tool to control the transmission of the disease for a saturated incidence rate.

5.1. The quarantine effect.

**Example 5.** In this example, choose steep  $\Delta = 0.1$  with initial values  $S(0) = 100015, I(0) = 57990, Q(0) = 995$ , and the parameters  $r = 0, \alpha_2 = 0.005, \alpha_3 = 0, \delta = [0.0009 \ 0.005 \ 0.015], \sigma_1 = 0.001, \sigma_2 = 0.18, \sigma_3 = 0.01$ , to get,  $\mathcal{R}_0 = [1.5694 \ 1.3650 \ 1.0360]$  and  $\mathcal{R}_S = [0.9778 \ 0.8504 \ 0.6454]$ . Hence, according to

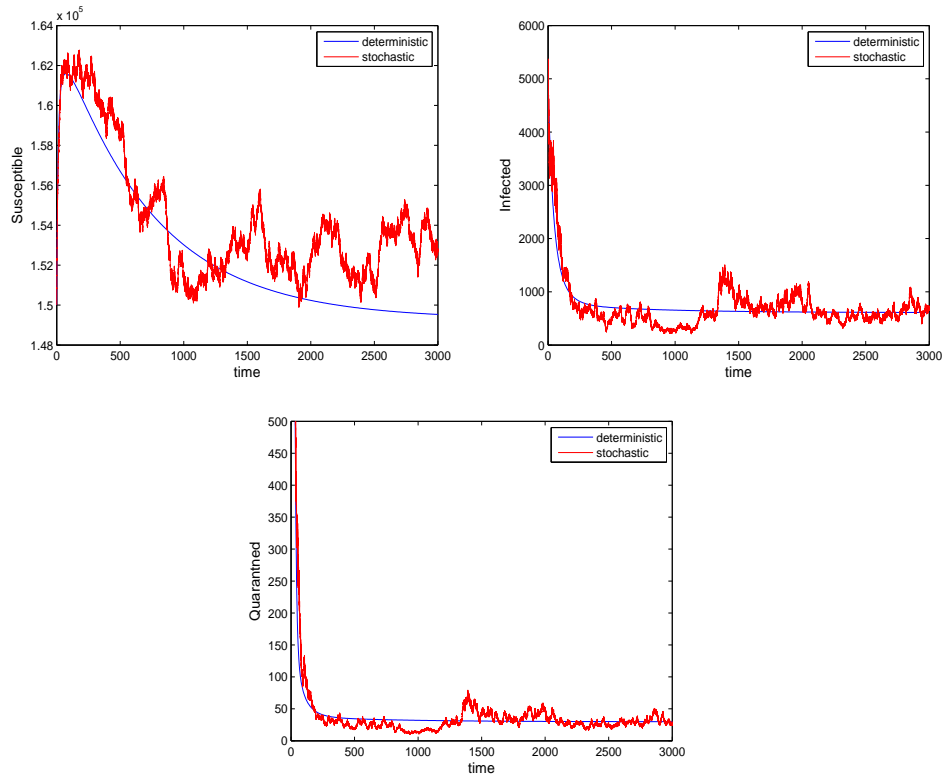


FIGURE 3. Paths of stochastic and deterministic systems as given in Example 3

Theorem 4.3, all positive solutions of the system (2) if  $R_s < 1$  the disease extinct a.s. Fig 5a clearly support the theoretical result. The presented illustration for the extinction of the Pneumococcus disease shows the impact of the quarantine measurement upon the time of the stochastic epidemic model. Thus the adopted strategy of increasing the isolation measure will lead to a quick extinction of the disease.

**Example 6.** In this example, we choose step  $\Delta = 0.01$  with initial values  $S(0) = 156015$ ,  $I(0) = 1990$ ,  $Q(0) = 995$  and the parameters  $r = 0$ ,  $\alpha_2 = 0.001$ ,  $\alpha_3 = 0$ ,  $\delta = [0 \ 0.005 \ 0.01]$ ,  $\sigma_1 = 0.01$ ,  $\sigma_2 = 0.02$ ,  $\sigma_3 = 0.01$ . By computation, we get that,  $\mathcal{R}_0 = [2,004 \ 1,6227 \ 1.3650]$  and  $\mathcal{R}_S = [1.9911 \ 1.6152 \ 1.3586]$ . Hence, according to Theorem 4.6, all positive solutions of the system (2) are persistent in mean in both cases of no quarantine measure taken and the use of isolation with different rate of individuals leaving the infected compartment I to the quarantine Q. Fig 5b clearly support the theoretical result. The performed simulation shows the role of the isolation measurement to maintain the persistence of the disease under a reasonable number of infected for a population of children under two years in Scotland. In other words, as the adoption of the quarantine method has been proven efficient for the limitation of the disease spread.



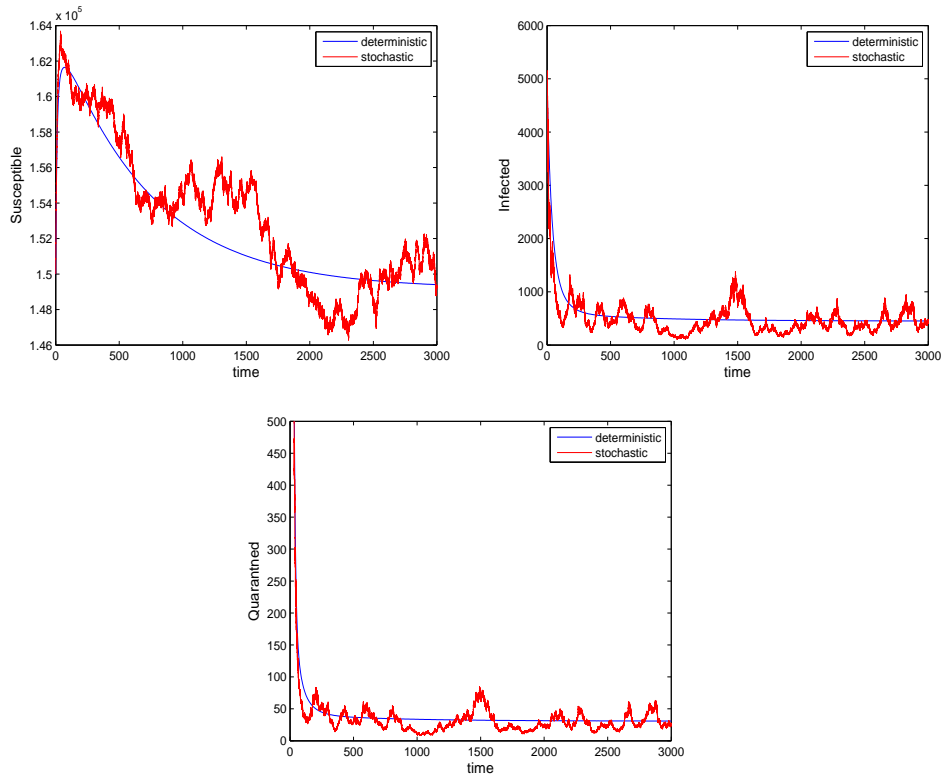
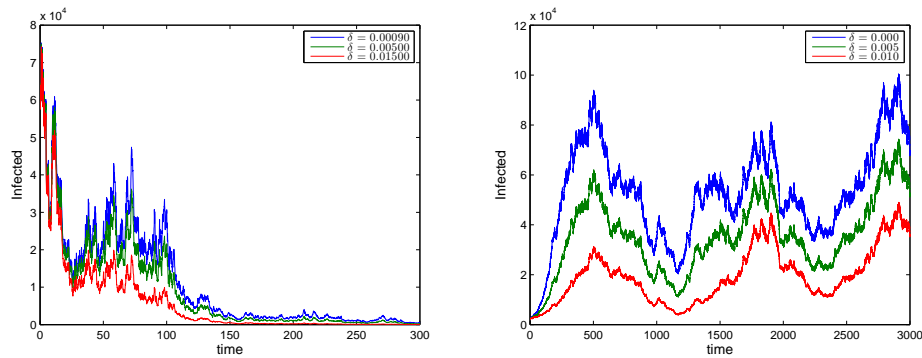


FIGURE 4. Paths of stochastic and deterministic systems as given in Example 4



(A) Paths of stochastic and deterministic systems as given in Example 5

(B) Paths of stochastic and deterministic systems as given in Example 6

FIGURE 5. Effect of quarantine

**Concluding remarks and discussions.** In this paper we have established the existence and uniqueness of the global positive solution for a stochastic solution *SIQS* epidemic model with saturated incidence rate. We showed the solutions are

ultimately bounded and stochastic permanence for the stochastic *SIQS* model. We investigated the extinction and persistence of the disease according to the stochastic threshold  $R_s$ . Our main motivation was to alleviate the assumptions on the threshold in the extinction and the persistence in mean cases. Throughout this paper, this work provides theoretical and numerical guidance for prevention and control of infectious diseases such as pneumococcus. More precisely the results show that the stochastic threshold  $R_s$  completely determines the global behavior of the stochastic *SIQS* model; if  $R_s < 1$  the disease goes to extinction; if  $R_s > 1$  the disease will persist in mean. Thus, if the stochastic threshold condition is fulfilled, the adopted strategy of increasing the isolation measure will lead to a quick extinction of the disease. However if there is a lack of mobilization from the government health policy to use this tool, the disease will take time to be extinct.

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*E-mail address:* [melfatini@gmail.com](mailto:melfatini@gmail.com)

*E-mail address:* [idrisssekkak@gmail.com](mailto:idrisssekkak@gmail.com)

*E-mail address:* [taki.reg@gmail.com](mailto:taki.reg@gmail.com)

*E-mail address:* [laaribiazize@gmail.com](mailto:laaribiazize@gmail.com)