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Research article

Long time dynamics for functional three-dimensional Navier-Stokes-Voigt equations

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Abstract: In this paper we consider a non-autonomous Navier-Stokes-Voigt model including a variety of delay terms in a unified formulation. Firstly, we prove the existence and uniqueness of solutions by using a Galerkin scheme. Next, we prove the existence and eventual uniqueness of stationary solutions, as well as their exponential stability by using three methods: first, a Lyapunov function which requires differentiability for the delays; next we exploit the Razumikhin technique to weaken the differentiability assumption to just continuity; finally, we use a Gronwall-like type of argument to provide sufficient conditions for the exponential stability in a general case which, in particular, for a situation of variable delay, it only requires measurability of the variable delay function.

Keywords: Navier-Stokes-Voigt model; delay; unified formulation; stationary solutions; exponential stability; Razumikhin
Mathematics Subject Classification: 35B40, 35B41, 35Q30, 35Q35, 76F20

1. Introduction

The Navier-Stokes-Voigt (NSV) system models the dynamics of a Kelvin-Voigt viscoelastic incompresible fluid and its analysis was motivated by the studies carried out by Oskolkov in [32], where it is described a model of the motion of linear viscoelastic fluids (see, for instance, [22, 33] and the references therein).

The autonomous version of (NSV) system is given by:

$$\begin{cases} \frac{\partial}{\partial t}(u - \alpha^2 \Delta u) - \nu \Delta(u) + (u \cdot \nabla)(u) + \nabla p = f, \text{ in } (0, +\infty) \times \Omega, \\ \nabla \cdot u = 0, \quad \text{in } (0, +\infty) \times \Omega, \\ u = 0, \quad \text{on } (0, +\infty) \times \partial\Omega, \\ u(0, x) = u^0(x), \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in two or three-dimensional Euclidean space, $u = (u_1, u_2, u_3)$ is the unknown velocity field of the fluid and p is the unknown pressure, ν is the kinematic viscosity coefficient, $\alpha > 0$ is a length scale parameter which characterizes the elasticity of the fluid (in the sense that the ratio α^2/ν describes the reaction time that is required for the fluid to respond to the applied force), u^0 is the initial velocity field, and f is an external force term. Obviously when $\alpha = 0$, this system becomes the classical Navier-Stokes one.

Recently, Navier-Stokes-Voigt systems have been proposed (see [4]) as regularizations of the 3D-Navier-Stokes equation for the purpose of direct numerical simulation (see also [1–3, 14, 21, 23, 31] for more details about NSV systems). In [4], some analytical studies of three-dimensional viscous and inviscid simplified Bardina turbulence models with periodic boundary conditions are carried out. In this paper, the authors prove the global well-posedness of this model for weaker initial conditions, establish an upper bound to the dimension of its global attractor and identify this dimension with the number of degrees of freedom for this model, and they establish the global existence and uniqueness of weak solutions to the inviscid model.

In this autonomous framework, the long time behavior of (1.1) has been widely studied. For example, the existence of a compact global attractor is proved in Kalantarov [18], and Kalantarov and Titi [20] investigate the long-term dynamics of the three-dimensional Navier-Stokes-Voigt model of a viscoelastic incompressible fluid. Specifically, upper bounds for the number of determining modes are derived for the 3D Navier-Stokes-Voigt equations, and subsequently used to provide information about the dimension of its global attractor. From a numerical analysis point of view, the authors consider the Navier-Stokes-Voigt model as a non-viscous (inviscid) regularization of the three-dimensional Navier-Stokes equations. Furthermore, it is also shown that the weak solutions of the Navier-Stokes-Voight equations converge, in an appropriate norm, to the weak solutions of the inviscid simplified Bardina model, as the viscosity coefficient $\nu \to 0$.

Other related results, which are worth being mentioned, are concerned with the Gévrey regularity of the global attractor when the force term is analytic of Gévrey type, and the establishment of similar statistical properties (and invariant measures) as for the 3D-Navier-Stokes equations (see [19, 24, 35] for more details).

No doubt at all, the autonomous system (1.1) can be regarded as a simplification of a more realistic model of reality. It is well understood that a realistic model should take into account non-autonomous, stochastic or random effects. Also, it is very important to notice that delay or memory terms are determining the evolution of physical models. Indeed, it is sensible to think that in the future evolution of a system not only the current state has influence but also the past history of the phenomena. Moreover, when one is interested in controlling the problem by using some feedback control, the use of delay terms is fully required and justified. Due to some of these reasons, we will consider in this paper the following non-autonomous and delay version of system (1.1):

$$\begin{cases} \frac{\partial}{\partial t}(u-\alpha^2\Delta u)-\nu\Delta(u)+(u\cdot\nabla)(u)+\nabla p=f(t)+g(t,u_t), & \text{in } (\tau,+\infty)\times\Omega,\\ \nabla\cdot u=0, & \text{in } (\tau,+\infty)\times\Omega,\\ u=0, & \text{on } (\tau,+\infty)\times\partial\Omega,\\ u(\tau,x)=u^\tau(x), & \text{in } \Omega,\\ u(\tau+t,x)=\phi(t,x), & \text{in } (-h,0)\times\Omega, \end{cases} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth enough boundary (e.g., C^2) $\partial\Omega$, $\tau\in\mathbb{R}$ is the initial time, u^τ is an initial velocity field at the initial time $\tau\in\mathbb{R}$, ϕ is a given function defined in the interval $(-h,0)$, and f is an external force term which may depend on time. Finally, the time-dependent delay term $g(t,u_t)$ represents, for instance, the influence of an external force with some kind of delay, memory or hereditary characteristics, although can also model some kind of feedback controls. Here, u_t denotes a segment of the solution, in other words, given $h>0$ and a function $u:[\tau-h,+\infty)\times\Omega\to\mathbb{R}$, for each $t\geq\tau$ we define the mapping $u_t:[-h,0]\times\Omega\to\mathbb{R}$ by

$$u_t(\theta,x)=u(t+\theta,x),\ \text{for } \theta\in[-h,0], x\in\Omega.$$

In this way, this abstract formulation includes several types of delay terms in a unified way (as it is described in [9] and in the next sections).

The non-autonomous case without delay (i.e., $g=0$) has been studied, for instance, in [40], where it is proved the asymptotic regularity of solutions as well as the existence of the uniform attractor, describing its structure and regularity. Luengo et al. [26] proved asymptotic compactness by using the energy method, and they further obtained the existence of pullback attractor for the three-dimensional non-autonomous NSV equations.

Concerning the model with delays, the analysis was initiated by Caraballo and Real [9–11] in the case $\alpha=0$ (i.e., the Navier-Stokes model) by establishing existence and eventual uniqueness of weak solutions, asymptotic behavior of the steady-state solutions, as well as the existence of pullback and uniform attractors (see also [5, 16, 27, 30, 34]). Later on, this analysis has been extended to other variants of Navier-Stokes systems such as the α-Navier-Stokes model ([7]), and the globally modified Navier-Stokes ([28, 29]).

The Navier-Stokes-Voigt equations with finite delays or with memory have been studied recently in [15,17,38] in some particular cases for the delay. In the first paper, the authors prove the existence of a uniform global attractor for a version containing some memory. In the second, the authors prove the well-posedness of the problem and the existence of pullback attractor when the term g contains variable delay, i.e., $g(u(t-\rho(t)))$, with $\rho\in C^1([0,+\infty))$, $\rho'(t)\leq\rho_0<1$, and in [38], the model is two-dimensional and contains variable and distributed delay, $g(t,u_t)=g_0(t-\rho(t),u(t-\rho(t))+\int_{-h}^0 G(s,u(t+s))ds$, with $\rho\in C^1([0,+\infty))$, $\rho'(t)\leq M<1$, and it is proved the existence of pullback attractor.

It is remarkable that, to the best of our knowledge, none of the published papers in the literature considered the existence, eventual uniqueness, and asymptotic stability of the steady state solutions for system (1.2). This is a very important feature since, in order to obtain a detailed analysis of the geometric structure of global (and/or pullback) attractors, it is necessary to know about the existence of equilibria (steady-state solutions) and their asymptotic properties, since their associate unstable manifolds play a key role in determining the geometrical structure and complexity of the attractors.

Motivated by these reasons, our main aim in this paper is to provide significant information on the asymptotic behavior of solutions for this model. We are interested in covering a wide variety of delay terms within a unified formulation. For this reason, we will analyze problem (1.2) which contains an abstract expression for the delay terms under the form of functional equations (say $g(t, u_t)$). The presence of delays in the models implies the necessity of working in a quite different phase space, for instance, the initial values must be now initial functions which must belong to some appropriate spaces, and this fact implies that we have to work in more complicated functional spaces, for instance, in a Banach space of continuous functions, rather than in a Hilbert space.

Consequently, our aim in this paper is to prove the well-posedness of system (1.2), prove the existence of stationary solutions and analyze their asymptotic behavior describing several methods which allow us to obtain different sufficient conditions. Since we will proceed with the abstract functional formulation for the delay term, our results generalize, in particular, some recent works obtained in the aforementioned literature for some particular cases of delay. This is, indeed, the main novelty of this paper, i.e., to set up a general enough framework which can include in a unified formulation most types of delay terms and carry out an analysis of the model in the sense of well-posedness of solutions and their asymptotic behavior. A more complete description of our contributions are included in the structure of our paper which is described in the next paragraph.

The content of this paper is as follows. In Section 2 we set up the framework for our problem and include the necessary preliminaries. The existence, uniqueness and continuous dependence of solution on the initial data are proved in Section 3 by using a Galerkin approximation scheme and the energy equality. Finally, in Section 4 we first prove the existence of stationary (steady-state) solutions of our problem, and we next analyze the asymptotic behaviour of such stationary solutions, by establishing some sufficient conditions ensuring their exponential stability. We carry out our analysis by using three different techniques. We first use a Lyapunov function which, in the particular case of variable delay, requires differentiability of the delay function; next, we weaken this assumption to only continuity by using the Razumikhin technique; finally we exploit a technique based on a Gronwall-like inequality which works for general delay terms requiring only measurability assumptions. The analysis carried out in this paper is a first step for a more complete study of the asymptotic behavior of the problem including the existence and structure of attracting sets, which will be the topic of a future paper.

2. Preliminaries

Denote by $(\cdot,\cdot)$ and $|\cdot|$, respectively, the scalar product and associate norm in $(L^2(\Omega))^3$, and by $(\nabla u, \nabla v)$ the scalar product in $(L^2(\Omega))^{3\times 3}$ for the gradients of u and v.

Let H be the closure in $(L^2(\Omega))^3$ of the following set

$$\mathcal{V} = \{v \in (C_0^\infty(\Omega))^3 : \nabla \cdot v = 0 \text{ in } \Omega\},$$

and let V be the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$. Then, H is a Hilbert space for the inner product of $(L^2(\Omega))^3$, and V is a Hilbert subspace of $(H_0^1(\Omega))^3$ with norm $\|\cdot\|$ and inner product $((\cdot,\cdot))$.

We will use $\langle\cdot,\cdot\rangle$ to denote the duality product between V and V', and $\|\cdot\|_*$ for the norm in V'. It is well known that theses spaces satisfy $V \subset H \subset V'$, where the injections are dense and compact.

Denote by A the Stokes operator defined by

$$Aw = -\mathcal{P}(\Delta w), \quad \forall\, w \in D(A), \tag{2.1}$$

where $\mathcal{P}$ is the Leray operator, i.e., is the projector operator from $((L^2(\Omega))^3$ onto H. Operator A is a linear continuous operator from V into V', satisfying

$$\langle Au, v\rangle = ((u, v)), \quad \text{for all } \ u, v \in V.$$

Taking into account that $\partial\Omega$ is regular enough, then $D(A) = (H^2(\Omega))^3 \cap V$ and $|Aw|$ defines a norm in $D(A)$ which is equivalent to the one in $(H^2(\Omega))^3$, in other words, there exists a constant $c_1(\Omega) > 0$ depending only on Ω such that

$$\|w\|_{(H^2(\Omega))^3} \leq c_1(\Omega)|Aw|, \quad \forall\, w \in D(A). \tag{2.2}$$

Consider now the trilinear form defined as

$$b(u, v, w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx,$$

for every function $u, v, w : \Omega \to \mathbb{R}^3$ for which the right-hand side is well defined.

In particular, b can be extended continuously to make sense for all $u, v, w \in V$, and is a continuous trilinear form on $V \times V \times V$, and satisfies

$$|b(u, v, w)| \leq C_1 \, \|u\| \, \|v\| \, \|w\| \,, \quad \forall u, v, w \in V, \tag{2.3}$$

$$b(u, v, w) = -b(u, w, v), \quad \forall u, v, w \in V, \tag{2.4}$$

$$b(u, v, v) = 0, \quad \forall u, v \in V, \tag{2.5}$$

and, using Agmon's inequality (e.g., cf [13]), we can assure that there exists a constant $C_2 > 0$ such that

$$|b(u, v, w)| \leq C_2 \, |Au|^{1/2} \|u\|^{1/2} \|v\| |w|, \tag{2.6}$$

for all $(u, v, w) \in D(A) \times V \times H$.

On the other hand, for any $u \in V$, we will use $B(u)$ to denote the element of V' given by

$$\langle B(u), w\rangle = b(u, u, w), \qquad \forall w \in V.$$

It follows from (2.3) that

$$\|B(u)\|_* \leq C_1 \|u\|^2, \qquad \forall u \in V, \tag{2.7}$$

and, in particular, by (2.6) and the identification of H' with H, if $u \in D(A)$, then $B(u) \in H$, with

$$|B(u)| \leq C_2 |Au|^{1/2} \|u\|^{3/2}, \qquad \forall u \in D(A). \tag{2.8}$$

We now describe the assumptions on f, g and the initial values u^τ and ϕ, for our model (1.2), and we recall the concept of variational solution.

Let Y be a Banach space, and denote $C_Y = C([-h, 0]; Y)$ and $L^2_Y = L^2(-h, 0; Y)$.

Assume $g : \mathbb{R} \times C_V \to (H^{-1}(\Omega))^3$, satisfying:

(H1) For all fixed $\xi \in C_V$, $g(\cdot, \xi)$ is measurable,
(H2) $g(t, 0) = 0, \forall t \in \mathbb{R}$,

(H3) There exists $L_g > 0$ such that for all $t \geq \tau$ and $\xi, \mu \in C_V$,

$$\|g(t,\xi) - g(t,\mu)\|_{(H^{-1}(\Omega))^3} \leq L_g \|\xi - \mu\|_{C_V},$$

(H4) There exist $m_1 \geq 0$ and $C_g > 0$ such that, for all $m \in [0, m_1], \tau \leq t < T$ and $u, v \in C^0([\tau - h, T]; V)$,

$$\int_\tau^t e^{ms}\|g(s,u_s) - g(s,v_s)\|^2_{(H^{-1}(\Omega))^3} ds \leq C_g^2 \int_{\tau-h}^t e^{ms}\|u(s) - v(s)\|^2 ds.$$

Remark 2.1. *In addition, notice that (H1)–(H4) imply that, given $u \in C([\tau - h, T]; V)$, the functional*

$$g_u : t \in [\tau, T] \to (H^{-1}(\Omega))^3$$

defined by $g_u(t) = g(t, u_t)$ for all $t \in [\tau, T]$, is measurable and the mapping

$$\mathcal{G} : u \in C([\tau - h, T]; V) \to g_u \in L^2(\tau, T; (H^{-1}(\Omega))^3)$$

possesses a unique extension to a mapping $\widetilde{\mathcal{G}}$ which is uniformly continuous from $L^2(\tau - h, T; V)$ into $L^2(\tau, T; (H^{-1}(\Omega))^3)$. From now on, we will denote $g(t, u_t) = \widetilde{\mathcal{G}}(u)(t)$ for any $u \in L^2(\tau - h, T; V)$, and thus $\forall \tau \leq t < T$, $\forall\, u, v \in L^2(\tau - h, T; V)$, condition (H4) also holds.

Assume $f \in L^2_{loc}(\mathbb{R}; (H^{-1}(\Omega))^3)$, $u^\tau \in V$, $\phi \in L^2_V$.

Definition 2.2. *It is said that u is a weak solution to (1.2) if $u \in L^2(\tau - h, T; V) \cap L^\infty(\tau, T; V)$ for all $T > \tau$, and satisfies*

$$\frac{d}{dt}(u(t) + \alpha^2 Au(t)) + \nu Au(t) + B(u(t)) = f(t) + g(t, u_t), \qquad in \quad \mathcal{D}'(\tau, \infty; V'), \tag{2.9}$$

and $u(\tau + t) = \phi(t)$ for $t \in (-h, 0)$, $u(\tau) = u^\tau$.

3. Existence and uniqueness of solution

In this section we prove existence and uniqueness of solution for (1.2). But, before studying (1.2), we will analyse the autonomous equation $u + \alpha^2 Au = \tilde{f}$. From the Lax-Milgram lemma, we deduce that for each $\tilde{f} \in V'$ there exists a unique $u_{\tilde{f}} \in V$ such that

$$u_{\tilde{f}} + \alpha^2 Au_{\tilde{f}} = \tilde{f}. \tag{3.1}$$

The mapping $\mathcal{F} : u \in V \to u + \alpha^2 Au \in V'$ is linear and bijective, with $\mathcal{F}^{-1}\tilde{f} = u_{\tilde{f}}$. From (3.1), one has $|u_{\tilde{f}}|^2 + \alpha^2\|u_{\tilde{f}}\|^2 \leq \|\tilde{f}\|_*\|u_{\tilde{f}}\|$, and in particular, $\|u_{\tilde{f}}\| \leq \alpha^{-2}\|\tilde{f}\|_*$, i.e.,

$$\|\mathcal{F}^{-1}\tilde{f}\| \leq \alpha^{-2}\|\tilde{f}\|_*, \qquad \forall \tilde{f} \in V'. \tag{3.2}$$

Observe that, by the definition of $D(A)$, we also have that $\mathcal{F}^{-1}(H) = D(A)$, and reasoning as for the obtention of (3.2), we deduce that

$$|Au_{\tilde{f}}| = \alpha^{-2}|\tilde{f} - u_{\tilde{f}}| \leq 2\alpha^{-2}|\tilde{f}|, \qquad \forall \tilde{f} \in H. \tag{3.3}$$

Remark 3.1. *If $u \in L^2(\tau, T; V)$ for all $T > \tau$ and satisfies (2.9), then the function v defined by*

$$v(t) = u(t) + \alpha^2 Au(t), \qquad t > \tau, \tag{3.4}$$

belongs to $L^2(\tau, T; V')$ for all $T > \tau$.

On the other hand, by (H2) and (H4) we have

$$\begin{aligned} \int_\tau^t \|g(s, u_s)\|_{V'}^2 ds &\leq C_g^2 \int_{\tau-h}^t \|u(s)\|^2 ds \\ &= C_g^2 \int_{\tau-h}^\tau \|\phi(s-\tau)\|^2 ds + C_g^2 \int_\tau^t \|u(s)\|^2 ds, \end{aligned} \tag{3.5}$$

and thanks to (2.7) and (3.6), we deduce that $v' \in L^1(\tau, T; V')$ for all $T > \tau$.
Consequently, $v \in C([\tau, +\infty); V')$, and therefore, by (3.2), $u \in C([\tau, +\infty); V)$. Moreover, again by (2.7), (2.9) and (3.6), $v' \in L^2(\tau, T; V')$ for all $T > \tau$, and therefore, as $u' = \mathcal{F}^{-1} v'$, we deduce that $u' \in L^2(\tau, T; V)$ for all $T > \tau$.
From previous considerations, it is clear that u is a weak solution to (1.2) if and only if $u \in C([\tau, +\infty); V), u' \in L^2(\tau, T; V)$ for all $T > \tau$, and

$$\begin{aligned} &u(t) + \alpha^2 Au(t) + \int_\tau^t (\nu Au(s) + B(u(s)))ds \\ &\qquad = u^\tau + \alpha^2 Au^\tau + \int_\tau^t f(s)ds + \int_\tau^t g(s, u_s)ds \ (\text{equality in } V'), \end{aligned} \tag{3.6}$$

for all $t \geq \tau$.

Remark 3.2. *Let u be a weak solution of (1.2). Then, u satisfies the energy equality:*

$$\begin{aligned} &|u(t)|^2 + \alpha^2 \|u(t)\|^2 + 2\nu \int_s^t \|u(r)\|^2 dr \\ &= |u(s)|^2 + \alpha^2 \|u(s)\|^2 + 2 \int_s^t \langle f(r), u(r) \rangle dr \\ &+2 \int_s^t \langle g(r, u_r), u(r) \rangle dr \quad \forall s, t \in [\tau, \infty). \end{aligned}$$

Our main result in this section is the following.

Theorem 3.3. *Assume that g satisfies assumptions (H1)–(H4) and the following convergence one:*

(H5) For all $T > 0$ and for any sequence $\{v^n(\cdot)\}_{n \geq 1} \subset L^2(-h, T; V)$ such that $v^n \rightharpoonup v$ weakly in $L^2(-h, T; V)$ and $v^n \to v$ strongly in $L^2(-h, T; H)$, it follows that $g(s, v_s^n) \rightharpoonup g(s, v_s)$ weakly in $L^2(0, T; (H^{-1}(\Omega))^3)$.

Then, for each $\tau \in \mathbb{R}$, $u^\tau \in V$ and $\phi \in L_V^2$, there exists a unique weak solution $u = u(\cdot; \tau, u^\tau, \phi)$ of (1.2). Moreover, if $u^\tau \in D(A)$ and $f \in L_{loc}^2(\mathbb{R}; H)$, then $u \in C([\tau, \infty); D(A))$ and $u' \in L^2(\tau, T; D(A))$, for all $T > \tau$, and

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|u(t)\|^2 + \alpha^2 |Au(t)|^2) + \nu |Au(t)|^2 + (B(u(t)), Au(t)) \\ &\qquad = (f(t) + g(t, u_t), Au(t)), \text{ a.e. } t > \tau. \end{aligned} \tag{3.7}$$

Proof. For simplicity, we will argue in the case $\tau = 0$. The general case is similar. We split the proof of existence into three steps.

A Galerkin scheme. First a priori estimates. Let us consider $\{v_j\} \subset V$, the orthonormal basis of H of all the eigenfunctions of the operator A ($Av_j = \lambda_j v_j$)). Denote $V_m = \text{span}[v_1, \ldots, v_m]$ and consider the projector $P_m u = \sum_{j=1}^{m} (u, v_j) v_j$.

Define also

$$u^m(t) = \sum_{j=1}^{m} \gamma_{m,j}(t) v_j,$$

where the upper script m will be used instead of (m) since no confusion is possible with powers of u, and where the coefficients $\gamma_{m,j}$ are required to satisfy the following system of ordinary differential equations:

$$\begin{aligned} &\frac{d}{dt}(u^m(t) + \alpha^2 A u^m(t), v_j) + \langle \nu A u^m(t) + B(u^m(t)), v_j \rangle \\ &\qquad = \langle f(t), v_j \rangle + \langle g(t, u_t^m), v_j \rangle, \quad \text{a.e. } t > 0, \quad 1 \le j \le m, \end{aligned} \tag{3.8}$$

and the initial condition $u^m(s) = P_m \phi(s)$ for $s \in [-h, 0]$.

Thanks to an appropriate modification of the Picard Theorem (see [9, Appendix]), the above system of ordinary functional differential Eq (3.8) possesses a unique local solution defined in $[0, t_m)$, with $0 < t_m \le \infty$.

We prove now that the solutions do exist for all time $t \in [0, +\infty)$.

Multiplying (3.8) by $\gamma_{m,j}(t)$ and summing in j, and taking into account the properties of the operator b, we obtain for a.e. $t \ge 0$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|u^m(t)|^2 + \alpha^2 \|u^m(t)\|^2) + \nu \|u^m(t)\|^2 \\ &\qquad = \langle f(t) + g(t, u_t^m), u^m(t) \rangle, \end{aligned} \tag{3.9}$$

and therefore, using Young's inequality, and taking into account (3.6),

$$\begin{aligned} |u^m(t)|^2 + \alpha^2 \|u^m(t)\|^2 &\le |u^0|^2 + \alpha^2 \|u^0\|^2 + C_g^2 \int_{-h}^{0} \|\phi(s)\|^2 ds \\ &+ \int_0^t \|f(s)\|^2_{(H^{-1}(\Omega))^3} ds + (2 + C_g^2) \int_0^t \|u^m(s)\|^2 ds, \quad \text{a.e. } t \in (0, T). \end{aligned} \tag{3.10}$$

From the above inequality and the Gronwall Lemma, the sequence $\{u^m\}$ is bounded in $L^2(0, T; V)$ and in $L^\infty(0, T; V)$, for any $T > 0$.

Now observe that by (3.8), if we denote $v^m = \mathcal{F} u^m$, then v^m satisfies

$$\frac{d}{dt}(v^m(t)) = \tilde{P}_m(-\nu A u^m(t) - B(u^m(t)) + f(t) + g(t, u_t)), \quad \text{a.e. } t > 0, \tag{3.11}$$

where

$$\langle \tilde{P}_m g, w \rangle = \langle g, P_m w \rangle \quad \forall g \in V', \ w \in V.$$

Consequently, as $\|\tilde{P}_m\|_{\mathcal{L}(V')} \leq 1$ for all $m \geq 1$, we deduce that the sequence $\{dv^m/dt\}_{m\geq 1}$ is bounded in $L^2(0,T;V')$ for all $T > 0$, and therefore, taking into account that $du^m/dt = \mathcal{F}^{-1}(dv^m/dt)$, we have that the sequence $\{du^m/dt\}_{m\geq 1}$ is bounded in $L^2(0,T;V)$ for all $T > 0$.

Then, since the injection of V into H is compact, the Ascoli-Arzelà theorem implies that there exist a subsequence $\{u^m\}_{m\geq 1}$ (we relabel the same) and a function $u \in W^{1,2}(0,T;V)$ for all $T > 0$ with $u = \phi$ in $(-h, 0)$, such that

$$\begin{cases} u^m \overset{*}{\rightharpoonup} u \text{ weakly-star in } L^\infty(0,T;V), \\ u^m \rightharpoonup u \text{ weakly in } L^2(0,T;V), \\ u^m \to u \text{ strongly in } C([0,T];H), \\ u^m \to u \text{ a.e. in } \Omega \times (0,T), \\ \dfrac{du^m}{dt} \rightharpoonup \dfrac{du}{dt} \text{ weakly in } L^2(0,T;V), \\ \dfrac{dv^m}{dt} = \mathcal{F}\left(\dfrac{du^m}{dt}\right) \rightharpoonup \mathcal{F}\left(\dfrac{du}{dt}\right) \text{ weakly in } L^2(0,T;V'), \end{cases} \tag{3.12}$$

for all $T > 0$.

Thanks to the properties of operator A and (3.12), we obtain that $Au^m \rightharpoonup Au$ weakly in $L^2(0,T;V')$ for all $T > 0$. Reasoning as in [25], Chapter 1, Lemma 1.3, we deduce that $Bu^m \rightharpoonup Bu$ weakly in $L^2(0,T;V')$, for all $T > 0$.

On the other hand, thanks to assumption (H5), the convergences in (3.12) and the definition of ϕ^m, we deduce that

$$g(t, u_t^m) \to g(t, u_t) \quad \text{in } \; L^2(0,T;V').$$

From all the convergences above, and (3.11), we can take limits and we prove that u is a global solution of (1.2) in the sense of Definition 2.2.

Regularity. Assume now that $u^0 \in D(A)$ and $f \in L^2_{loc}(\mathbb{R};H)$.

Multiplying in (3.8) by $\lambda_j \gamma_{m,j}(t)$, and summing from $j = 1$ to $j = m$, we obtain that a.e. $t > 0$,

$$\begin{aligned} \frac{d}{dt}(\|u^m(t)\|^2 + \alpha^2 |Au^m(t)|^2) + 2\nu |Au^m(t)|^2 + 2\left(B(u^m(t)), Au^m(t)\right) \\ = 2\left(f(t) + g(t, u_t^m), Au^m(t)\right). \end{aligned} \tag{3.13}$$

But by (2.8) and the Young inequality,

$$2|\left(B(u^m(t)), Au^m(t)\right)| \leq C_\nu \|u^m(t)\|^6 + \nu |Au^m(t)|^2,$$

where $C_\nu = 27C_2^4(16\nu^3)^{-1}$.

Also,

$$2|\left(f(t), Au^m(t)\right)| \leq \frac{\nu}{2}|Au^m(t)|^2 + 2\nu^{-1}|f(t)|^2.$$

and

$$2|\left(g(t, u_t^m), Au^m(t)\right)| \leq \frac{\nu}{2}|Au^m(t)|^2 + 2\nu^{-1}|g(t, u_t^m)|^2.$$

Observing that $|AP_m u^0| \le |Au^0|$ and $\|P_m u^0\| \le \|u^0\|$, from (3.13) we deduce in particular that

$$\begin{aligned} \alpha^2|Au^m(t)|^2 \le\ & \|u^0\|^2 + \alpha^2|Au^0|^2 + 2\nu^{-1}\int_0^t |f(s)|^2\,ds \\ & +2\nu^{-1}\int_0^t |g(s,u_s^m)|^2\,ds + C_\nu t \sup_{s\in[0,t]} \|u^m(s)\|^6, \end{aligned} \tag{3.14}$$

for all $t \ge 0$, and any $m \ge 1$.

Consequently, as $\{u^m\}_{m\ge1}$ is bounded in $C([0,T];V)$, from (3.6) and (3.14), we have that $\{u^m\}_{m\ge1}$ is bounded in $C([0,T];D(A))$, for all $T > 0$, and therefore, extracting a subsequence weakly-star convergent in $L^\infty(0,T;D(A))$, we see that $u \in L^\infty(0,T;D(A))$, for all $T > 0$.

But then, $v = u + \alpha^2 Au \in L^\infty(0,T;H)$, with $v' = -\nu Au - B(u) + f(t) + g(t,u_t) \in L^2(0,T;H)$, for all $T > 0$, and therefore, $v \in C([0,\infty);H)$.

Thus, $Au = \alpha^{-2}(v-u) \in C([0,\infty);H)$, i.e., $u \in C([0,\infty);D(A))$.

Moreover, as $v' \in L^2(0,T;H)$, by (3.3), then $u' = \mathcal{F}^{-1}v' \in L^2(0,T;D(A))$, for all $T > 0$.

Identity (3.7)**.** If $u^0 \in D(A)$ and $f \in L^2_{loc}(\mathbb{R};H)$, we have seen that $u \in W^{1,2}(0,T;D(A))$ and $v = \mathcal{F}u \in W^{1,2}(0,T;H)$, for all $T > 0$. Then,

$$\frac{d}{dt}|v(t)|^2 = 2(v'(t),v(t)), \quad \text{a.e. } t > 0$$

and taking into account that $\mathcal{F}$ is self-adjoint and that $v'(t) = \mathcal{F}u'(t)$, we have

$$\frac{d}{dt}(u(t),v(t)) = 2(u(t),v'(t)), \quad \text{a.e. } t > 0.$$

Thus,

$$\begin{aligned} \frac{d}{dt}(Au(t),v(t)) &= \alpha^{-2}\frac{d}{dt}(v(t)-u(t),v(t)) \\ &= 2(v'(t),Au(t)), \quad \text{a.e. } t > 0. \end{aligned}$$

From this identity, taking into account (2.5) and (2.9), we have (3.7).

Uniqueness. Let $u^{(1)}$ and $u^{(2)}$ be two weak solutions to (1.2), corresponding to the same data u^0 and ϕ. Let us denote $\hat{u} = u^{(1)} - u^{(2)}$. It is obvious that $\hat{u}(0,x) = 0$ in Ω and $\hat{u}(t,x) = 0$ in $(-h,0)\times\Omega$. From Definition (2.2) is easy to deduce that $\hat{u}$ satisfies the following equality

$$\begin{aligned} |\hat{u}(t)|^2 + \alpha^2\|\hat{u}(t)\|^2 + 2\nu\int_0^t \|\hat{u}(s)\|^2 ds + 2\int_0^t \langle B(u^1(s)) - B(u^2(s)), \hat{u}(s)\rangle ds \\ = 2\int_0^t \langle g(s,u_s^1) - g(s,u_s^2), \hat{u}(s)\rangle ds, \end{aligned} \tag{3.15}$$

for all $t > 0$. But, on the one hand, by (2.3) we have that,

$$\begin{aligned} & \|B(u^{(1)}(s)) - B(u^{(2)}(s))\|_* \\ = & \sup_{w\in V, \|w\|=1} |b(u^{(1)}(s) - u^{(2)}(s), u^{(1)}(s), w) - b(u^{(2)}(s), u^{(2)}(s) - u^{(1)}(s), w)| \\ \le & C_1(\|u^{(1)}(s)\| + \|u^{(2)}(s)\|)\|u^{(1)}(s) - u^{(2)}(s)\|. \end{aligned}$$

Thus, if we fix an arbitrary $T > 0$, and denote $R_T = C_1 \max_{s\in[0,T]}(\|u^{(1)}(s)\| + \|u^{(2)}(s)\|)$, we have

$$\|B(u^{(1)}(s)) - B(u^{(2)}(s))\|_* \leq R_T\|u^{(1)}(s) - u^{(2)}(s)\| \quad \text{for all } s \in [0, T]. \tag{3.16}$$

On the other hand, by (H3) we deduce that

$$\begin{aligned} & 2\int_0^t \langle g(s, u_s^1) - g(s, u_s^2), \hat{u}(s)\rangle ds \\ \leq\ & 2\int_0^t \|g(s, u_s^1) - g(s, u_s^2)\|_*\|\hat{u}(s)\|ds \\ \leq\ & 2\left(\int_0^t \|g(s, u_s^1) - g(s, u_s^2)\|_*^2 ds\right)^{1/2}\left(\int_0^t \|\hat{u}(s)\|^2 ds\right)^{1/2} \\ \leq\ & 2C_g\left(\int_{-h}^t \|u^1(s) - u^2(s)\|^2 ds\right)^{1/2}\left(\int_0^t \|\hat{u}(s)\|^2 ds\right)^{1/2} \\ =\ & 2C_g\int_0^t \|\hat{u}(s)\|^2 ds \end{aligned} \tag{3.17}$$

Then, as $\|A\hat{u}(s)\|_* = \|\hat{u}(s)\|$, from (3.15), (3.16) and (3.17) we deduce that

$$\|\hat{u}(t)\| \leq \alpha^{-2}(\nu + R_T + 2C_g)\int_0^t \|\hat{u}(s)\|\, ds \quad \text{for all } t \in [0, T].$$

From this inequality and Gronwall's lemma, we deduce that $\|\hat{u}(t)\| = 0$ for all $t \in [0, T]$, and therefore, the uniqueness of weak solution to (1.2). □

Remark 3.4. *We emphasize that assumption (H5) does not need to be assumed if the delay term $g : \mathbb{R} \times C_H \to H$, and satisfies:*

$(H1)'$ *For all fixed $\xi \in C_H$, $g(\cdot, \xi)$ is measurable,*
$(H2)'$ $g(t, 0) = 0, \forall t \in \mathbb{R}$,
$(H3)'$ *There exists $L_g > 0$ such that for all $t \geq \tau$ and $\xi, \mu \in C_H$,*

$$|g(t, \xi) - g(t, \mu)| \leq L_g\, \|\xi - \mu\|_{C_H},$$

$(H4)'$ *There exist $m_1 \geq 0$ and $C_g > 0$ such that, for all $m \in [0, m_1]$, $\tau \leq t < T$ and $u, v \in C^0([\tau - h, T]; H)$,*

$$\int_\tau^t e^{ms}|g(s, u_s) - g(s, v_s)|^2 ds \leq C_g^2\int_{\tau-h}^t e^{ms}|u(s) - v(s)|^2 ds.$$

See [9] for more details.

We can also obtain a result on the continuous dependence on the initial data.

Lemma 3.5. *Let $(u^\tau, \phi), (v^\tau, \psi) \in V \times L_V^2$ be two pairs of initial values for the problem (1.2), and $\tau \in \mathbb{R}$ an initial time. Denote by $u(\cdot) = u(\cdot; \tau, (u^\tau, \phi))$ and $v(\cdot) = v(\cdot; \tau, (v^\tau, \psi))$ their corresponding associated solutions to (1.2). Then,*

$$\|u(t) - v(t)\|^2 \leq \left(\left(1 + \lambda_1^{-1}\alpha^{-2}\right)\|u^\tau - v^\tau\|^2 + \frac{\alpha^{-2}C_g^2}{\nu}\|\phi - \psi\|_{L_V^2}^2\right)\times \tag{3.18}$$

$$\times \exp\left(\int_\tau^t \left(\frac{\alpha^{-2}C_g^2}{\nu} + \frac{\alpha^{-2}C_1^2}{\nu}\|v(s)\|^2\right)ds\right),\ \forall t \geq \tau\ .$$

As a consequence, it also follows

$$\|u_t - v_t\|_{C_V}^2 \leq \left(\left(1 + \lambda_1^{-1}\alpha^{-2}\right)\|u^\tau - v^\tau\|^2 + \frac{\alpha^{-2}C_g^2}{\nu}\|\phi - \psi\|_{L_V^2}^2\right)\times \times \exp\left(\int_\tau^t \left(\frac{\alpha^{-2}C_g^2}{\nu} + \frac{\alpha^{-2}C_1^2}{\nu}\|v(s)\|^2\right)ds\right),\ \forall t \geq \tau + h. \tag{3.19}$$

Proof. From (3.6) we obtain

$$\begin{aligned} &u(t) - v(t) + \alpha^2 A\,(u(t) - v(t)) + \nu\int_\tau^t A\,(u(s) - v(s))\,ds \\ &+ \int_\tau^t (B(u(s)) - B(v(s)))\ ds \\ &= u^\tau - v^\tau + \alpha^2 A\,(u^\tau - v^\tau) + \int_\tau^t (g(s,u_s) - g(s,v_s))\ ds,\ \forall\, t \in [\tau, T]. \end{aligned}$$

Denoting $w = u - v$, we have

$$\frac{1}{2}\frac{d}{dt}\left(|w(t)|^2 + \alpha^2\|w(t)\|^2\right) + \nu\|w(t)\|^2 + \langle B(u(t)) - B(v(t)), w(t)\rangle = \langle g(t,u_t) - g(t,v_t), w(t)\rangle. \tag{3.20}$$

Thanks to (2.3) and (2.5), we have

$$\begin{aligned} |\langle B(u(t)) - B(v(t)), w(t)\rangle| &= |\langle b(w(t),(v(t),w(t)\rangle| \leq C_1\|w(t)\|\|v(t)\|\|w(t)\| \\ &\leq \frac{C_1^2}{2\nu}\|w(t)\|^2\|v(t)\|^2 + \frac{\nu}{2}\|w(t)\|^2 \end{aligned}$$

and then, from (3.20), we easily deduce

$$\frac{d}{dt}\left(|w(t)|^2 + \alpha^2\|w(t)\|^2\right) \leq \frac{C_1^2}{\nu}\|v(t)\|^2\|w(t)\|^2 + \frac{1}{\nu}\|g(t,u_t) - g(t,v_t)\|_*^2.$$

Now, (H4) implies, for $t \geq \tau$,

$$\begin{aligned} \|w(t)\|^2 - \|w(\tau)\|^2 &\leq \frac{\alpha^{-2}C_1^2}{\nu}\int_\tau^t \|v(s)\|^2\|w(s)\|^2 ds \\ &+ \frac{\alpha^{-2}C_g^2}{\nu}\int_{\tau-h}^t \|u(s) - v(s)\|^2 ds + \alpha^{-2}|w(\tau)|^2 \\ &= \lambda^{-1}\alpha^{-2}\|w(\tau)\|^2 + \frac{\alpha^{-2}C_g^2}{\nu}\|\phi - \psi\|_{L_V^2}^2 \\ &+ \int_\tau^t \left(\frac{\alpha^{-2}C_g^2}{\nu} + \frac{\alpha^{-2}C_1^2}{\nu}\|v(s)\|^2\right)\|w(s)\|^2 ds. \end{aligned}$$

Thus,

$$\begin{aligned}\|w(t)\|^2 \quad \leq \quad & \left(1+\lambda^{-1}\alpha^{-2}\right)\|u^\tau - v^\tau\|^2 + \frac{\alpha^{-2}C_g^2}{\nu}\|\phi-\psi\|^2_{L^2_V} \\ & + \int_\tau^t \left(\frac{\alpha^{-2}C_g^2}{\nu} + \frac{\alpha^{-2}C_1^2}{\nu}\|v(s)\|^2\right)\|w(s)\|^2 ds, \quad \forall t \geq \tau,\end{aligned}$$

and, (3.18) holds by applying the Gronwall lemma. Finally, (3.19) is a straightforward consequence of (3.18). □

4. Stationary solutions and their stability

In this section we prove that, under additional hypotheses, there exists a unique stationary solution to problem (1.2) and it is globally asymptotically exponentially stable.

4.1. Existence and uniqueness of stationary solutions.

From now on we assume that $f(t) = f \in (H^{-1}(\Omega))^3$ for all $t \geq \tau$, a constant function, and that $g : \mathbb{R}\times C_V \to (H^{-1}(\Omega))^3$ satisfies (H1)–(H4), but is autonomous, in the sense that there exists a function $g_0 : V \to (H^{-1}(\Omega))^3$ such that

$$g(t, w) = g_0(w), \text{ for all } (t, w) \in [\tau, \infty) \times V,$$

where, with a slight abuse of notation, we identify every element $w \in V$ with the constant function in C_V which is equal to w for any time $t \in [-h, 0]$.

Two examples, which can be considered as canonical within this situation are the following:

- *(Forcing term with variable delay)* Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ be a measurable function satisfying $G(0) = 0$, and assume that there exists $M > 0$ such that

  $$|G(u) - G(v)|_{\mathbb{R}^3} \leq M|u - v|_{\mathbb{R}^3}, \forall u, v \in \mathbb{R}^3.$$

  Now, consider a function $\rho(t)$, which is going to play the role of the delay. Assume that $\rho(\cdot)$ is measurable and define $g(t, \xi)(x) = G(\xi(-\rho(t))(x))$ for each $\xi \in C_V$, $x \in \Omega$ and $t \in [0, T]$. In this case, the delayed term g in our problem becomes

  $$g(t, u_t) = G(u(t - \rho(t))).$$

- *(Forcing term with distributed delay)* Let $G : [-h, 0] \times\mathbb{R}^3 \to \mathbb{R}^3$ be a measurable function satisfying $G(s, 0) = 0$ for all $s \in [-h, 0]$, and there exists a function $\beta \in L^2(-h, 0)$ such that

  $$|G(s, u) - G(s, v)|_{\mathbb{R}^3} \leq \beta(s)|u - v|_{\mathbb{R}^3}, \forall u, v \in \mathbb{R}^3, \forall s \in [-h, 0].$$

  Then, g is given by

  $$g(t, \xi)(x) = \int_{-h}^0 G(s, \xi(s)(x))\, ds$$

  for each $\xi \in C_V$, $t \in [0, T]$, $x \in \Omega$, and the delayed term in our problem becomes

  $$g(t, u_t)(x) = \int_{-h}^0 G(s, u(t + s))\, ds$$

Observe that both situations are within our framework, and hypothesis (H1)–(H4) are fulfilled under appropriate assumptions on the variable delay (see [9] for more details). In fact, these two examples are within the particular case mentioned in Remark 3.4. Therefore, we can ensure the existence and uniqueness of solutions of our model because ($H5$) is automatically fulfilled.

We consider the following equation,

$$\frac{d}{dt}(u + \alpha^2 Au) + \nu Au + B(u) = f + g(t, u_t) \quad t > \tau. \tag{4.1}$$

A stationary (steady-state) solution to (4.1) is an element $u^* \in V$ such that

$$\nu\langle Au^*, v\rangle + \langle B(u^*), v\rangle = \langle f, v\rangle + \langle g_0(u^*), v\rangle \quad \forall v \in V. \tag{4.2}$$

Now we recall a result ensuring existence of steady-state solutions for Eq (4.1) which, obviously, is the same as for the Navier-Stokes case considered in [6].

Theorem 4.1. *Under the above conditions, if $\nu > L_g$, then:*

(a) Problem (4.1) admits at least one stationary solution u^*, which indeed belongs to $D(A)$. Moreover, any stationary solution satisfies the estimate

$$(\nu - L_g)\|u^*\| \leq \|f\|_{(H^{-1}(\Omega))^3}. \tag{4.3}$$

(b) If the following condition holds,

$$2C_1\|f\|_{(H^{-1}(\Omega))^3} < \left(\nu - L_g\right)^2, \tag{4.4}$$

then the stationary solution of (4.1) is unique.

Proof. In [6] the authors prove an analogous result to the 2D-Navier-Stokes models, but the proof of the existence is valid for any dimension, while the uniqueness can be ensured for dimension less than or equal to 4 (see [39]). Therefore, we omit the proof of this theorem. □

4.2. Exponential convergence of solutions: A direct approach for the model.

Now, we will prove the existence and uniqueness of stationary solution, u^*, and that every weak solution approaches u^* exponentially.

Theorem 4.2. *Assume that assumptions in Theorem 3.3 and Theorem 4.1 hold true. Moreover, assume that $\nu > \max\{C_g, L_g\}$ and*

$$C_1\|f\|_{(H^{-1}(\Omega))^3} < (\nu - C_g)(\nu - L_g). \tag{4.5}$$

Then, there exists $\lambda > 0$ such that for the solution $u(\cdot, 0, u^0, \phi)$ of (1.2) with $\tau = 0$ and $\phi \in L^2_V$, the following estimate holds for all $t \geq 0$:

$$|u(t, 0, u^0, \phi) - u^*|^2 \leq e^{-\lambda t}\left(|u^0 - u^*|^2 + \alpha^2\|u^0 - u^*\|^2 + C_g\|\phi - u^*\|^2_{L^2_V}\right), \tag{4.6}$$

where u^* is the unique stationary solution of (4.1) given by Theorem 4.1.

Proof. Let us denote $w(t) = u(t) - u^*$. Considering equations (4.1) for $u(t)$ and (4.2) for u^*, one has

$$\frac{d}{dt}(w(t) + \alpha^2 Aw(t), v) + \nu((w(t), v)) + \langle B(u(t), v\rangle - \langle B(u^*, v\rangle = \langle g(t, u_t) - g(u^*), v\rangle,$$

for $t > 0$, for any $v \in V$.

Now, pick $\lambda \in (0, m_1)$ to be fixed later. Then

$$\begin{aligned} \frac{d}{dt}(e^{\lambda t}(|w(t)|^2 + \alpha^2\|w(t)\|^2)) &= \lambda e^{\lambda t}(|w(t)|^2 + \alpha^2\|w(t)\|^2) \\ &\quad + e^{\lambda t}\frac{d}{dt}(|w(t)|^2 + \alpha^2\|w(t)\|^2) \\ &= \lambda e^{\lambda t}(|w(t)|^2 + \alpha^2\|w(t)\|^2) - 2\nu e^{\lambda t}\|w(t)\|^2 + \\ &\quad + 2e^{\lambda t}\langle B(u(t)) - B(u^*), w(t)\rangle \\ &\quad + 2e^{\lambda t}\langle g(t, u_t) - g(u^*), w(t)\rangle \end{aligned}$$

for $t > 0$.

Thanks to (2.3), (2.4) and (4.3),

$$\begin{aligned} |\langle B(u(t)) - B(u^*), w(t)\rangle| &= |\langle b(w(t), v(t), w(t)\rangle| \\ &\leq C_1\|w(t)\|\|u^*\|\|w(t)\| \\ &\leq \frac{C_1\|f\|_{(H^{-1}(\Omega))^3}}{\nu - L_g}\|w(t)\|^2 \end{aligned} \tag{4.7}$$

Hence, using a Young inequality we conclude that

$$\begin{aligned} &\frac{d}{dt}(e^{\lambda t}(|w(t)|^2 + \alpha\|w(t)\|^2)) \\ &\quad \leq e^{\lambda t}(\lambda\lambda_1^{-1} + \lambda\alpha^2 - 2\nu + \frac{2C_1\|f\|_{(H^{-1}(\Omega))^3}}{\nu - L_g} + C_g)\|w(t)\|^2 \\ &\quad\quad + \frac{1}{C_g}e^{\lambda t}\|g(t, u_t) - g(u^*)\|^2_{(H^{-1}(\Omega))^3}. \end{aligned}$$

Therefore, integrating from 0 to t, we have

$$\begin{aligned} e^{\lambda t}|w(t)|^2 &\leq |w(0)|^2 + \alpha^2\|w(0)\|^2 + \frac{1}{C_g}\int_0^t e^{\lambda s}\|g(s, u_s) - g(u^*)\|^2_{(H^{-1}(\Omega))^3} ds \\ &\quad + (\lambda\lambda_1^{-1} + \lambda\alpha^2 - 2\nu + \frac{2C_1\|f\|_{(H^{-1}(\Omega))^3}}{\nu - L_g} + C_g)\int_0^t e^{\lambda s}\|w(s)\|^2 ds, \end{aligned} \tag{4.8}$$

and taking into account hypothesis (H4), we obtain that

$$e^{\lambda t}|w(t)|^2 \leq |w(0)|^2 + \alpha^2\|w(0)\|^2 + C_g\int_{-h}^0 e^{\lambda s}\|w(s)\|^2 ds$$

$$+(\lambda\lambda_1^{-1} + \lambda\alpha^2 - 2\nu + \frac{2C_1\|f\|_{(H^{-1}(\Omega))^3}}{\nu - L_g} + 2C_g)\int_0^t e^{\lambda s}\|w(s)\|^2 ds$$
$$\leq |u^0 - u^*|^2 + \alpha^2\|u^0 - u^*\|^2 + C_g\|\phi - u^*\|^2_{L^2_V}$$
$$+(\lambda\lambda_1^{-1} + \lambda\alpha^2 - 2\nu + \frac{2C_1\|f\|_{(H^{-1}(\Omega))^3}}{\nu - L_g} + 2C_g)\int_0^t e^{\lambda s}\|w(s)\|^2 ds. \tag{4.9}$$

If we have (4.5), then we can conclude that there exists $\lambda > 0$ such that

$$e^{\lambda t}|w(t)|^2 \quad \leq |u^0 - u^*|^2 + \alpha^2\|u^0 - u^*\|^2 + C_g\|\phi - u^*\|^2_{L^2_V}.$$

□

Remark 4.3. *It is worth noticing that in some applications, the constants C_g and L_g are closely related to each other (see for instance Example 3.5 in [10] or Example 1 in [5]), and it happens that $C_g \geq L_g$. In this case, a sufficient condition implying* (4.5) *is*

$$C_1\|f\|_{(H^{-1}(\Omega))^3} < (\nu - C_g)^2. \tag{4.10}$$

4.3. A Razumikhin-type approach

In the previous subsection we have obtained a result on the exponential convergence of solutions of our problem to the unique stationary solution, but we need the delay term satisfies some additional hypothesis, for example, when g contains a variable delay which is continuously differentiable. However, we will prove here that, using a different approach and weakening the assumptions it is possible obtain a result for these more general terms. This technique has been developed by Razumikhin (see Razumikhin [36] [37]) in the framework of delay ordinary differential equations, and has recently been applied to some stochastic ordinary and partial differential equations (e.g., Caraballo et al. [8]). But it is worth mentioning that this approach requires some kind of continuity concerning the operators in the model and the delay term, and we also need to work with strong solutions.

First, we prove the following result.

Theorem 4.4. *Assume g satisfies conditions (H1)–(H4) for any $T > 0$, and for each $\xi \in C_V$ the map $t \in [0, +\infty) \mapsto g(t, \xi) \in (H^{-1}(\Omega))^3$ is continuous. Suppose that for $f \in (H^{-1}(\Omega))^3$ there exists a stationary solution u^* for the problem (4.1), and such that, for some $\lambda > 0$,*

$$\begin{aligned} &-\nu\langle A(\phi(0) - u^*), \phi(0) - u^*\rangle - \langle B(\phi(0)) - B(u^*), \phi(0) - u^*\rangle \\ &+ \langle g(t, \phi) - g(t, u^*), \phi(0) - u^*\rangle \\ &< -\lambda\left(|\phi(0) - u^*|^2 + \alpha^2\|\phi(0) - u^*\|^2\right), \quad t \geq 0, \end{aligned} \tag{4.11}$$

provided that $\phi \in C_V$ satisfying $\phi(0) \neq u^*$, and

$$\|\phi - u^*\|^2_{C_V} \leq e^{\lambda h}\left(|\phi(0) - u^*|^2 + \alpha^2\|\phi(0) - u^*\|^2\right). \tag{4.12}$$

Then, u^* is the unique stationary solution of (4.1) and for all $\psi \in C_V$, the solution of (1.2) corresponding to the initial data $u = \psi$ in $[-h, 0]$, which is denoted by $u(t; \psi)$, satisfies

$$|u(t;\psi) - u^*|^2 + \alpha^2\|u(t;\psi) - u^*\|^2 \leq e^{-\lambda t}\|\psi - u^*\|^2_{C_V}, \ \forall t \geq 0. \tag{4.13}$$

Proof. Arguing by contradiction, assume that there is a $\psi \in C_V$ such that (4.13) does not hold. Then, denoting

$$\sigma = \inf\{t > 0;\ |u(t;\psi) - u^*|^2 + \alpha^2\|u(t;\psi) - u^*\|^2 > e^{-\lambda t}\|\psi - u^*\|^2_{C_V}\},$$

we obtain

$$\begin{aligned} e^{\lambda t}\left(|u(t;\psi) - u^*|^2 + \alpha^2\|u(t;\psi) - u^*\|^2\right) &\le e^{\lambda\sigma}\left(|u(\sigma;\psi) - u^*|^2 + \alpha^2\|u(\sigma;\psi) - u^*\|^2\right) \\ &= \|\psi - u^*\|^2_{C_V}, \end{aligned} \tag{4.14}$$

for all $0 \le t \le \sigma$, and there exists a sequence $\{t_k\}_{k\ge1} \subset \mathbb{R}^+$ such that $t_k \downarrow \sigma$ as $k \to \infty$, and

$$e^{\lambda t_k}\left(|u(t_k;\psi) - u^*|^2 + \alpha^2\|u(t_k;\psi) - u^*\|^2\right) > e^{\lambda\sigma}\left(|u(\sigma;\psi) - u^*|^2 + \alpha^2\|u(\sigma;\psi) - u^*\|^2\right). \tag{4.15}$$

On the other hand, it follows from (4.14) that

$$|u(\sigma + \theta;\psi) - u^*|^2 \le e^{\lambda h}\left(|u(\sigma;\psi) - u^*|^2 + \alpha^2\|u(\sigma;\psi) - u^*\|^2\right),$$

for all $-h \le \theta \le 0$, which, taking into account (4.11), implies

$$\begin{aligned} &- \nu\langle A(u(\sigma;\psi) - u^*), u(\sigma;\psi) - u_\infty\rangle - \langle B(u(\sigma;\psi)) - B(u^*), u(\sigma;\psi) - u^*\rangle \\ &+ \langle g(\sigma, u_\sigma(\cdot;\psi)) - g(\sigma, u^*), u(\sigma;\psi) - u^*\rangle \\ &< -\lambda\left(|u(\sigma;\psi) - u^*|^2 + \alpha^2\|u(\sigma;\psi) - u^*\|^2\right). \end{aligned} \tag{4.16}$$

As $u(\cdot;\psi) \in C([-h, +\infty); V)$, from the continuity of the operators in our problem, we ensure the existence of $\epsilon_* > 0$ such that for each $\epsilon \in (0, \epsilon_*]$,

$$\begin{aligned} &- \nu\langle A(u(t;\psi) - u^*), u(t;\psi) - u^*\rangle - \langle B(u(t;\psi)) - B(u^*), u(t;\psi) - u^*\rangle \\ &+ \langle g(t, u_t(\cdot;\psi)) - g(t, u^*), u(t;\psi) - u^*\rangle \\ &\le -\lambda\left(|u(t;\psi) - u^*|^2 + \alpha^2\ |u(t;\psi) - u^*\|^2\right), \end{aligned} \tag{4.17}$$

for all $t \in [\sigma, \sigma + \epsilon]$. Thus, writing $w(t) = u(t;\psi) - u^*$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(|w(t)|^2 + \alpha^2\|w(t)\|^2\right) &= -\nu\langle Aw(t), w(t)\rangle - \langle B(u(t;\psi)) - B(u^*), w(t)\rangle \\ &\quad + \langle g(t, u_t(\cdot;\psi)) - g(t, u^*), w(t)\rangle \end{aligned}$$

for all $t \in [\sigma, \sigma + \epsilon]$, and integrating now we obtain

$$\begin{aligned} &e^{\lambda(\sigma+\epsilon)}\left(|w(\sigma + \epsilon;\psi)|^2 + \alpha^2\|w(\sigma + \epsilon;\psi)\|^2\right) \\ &- e^{\lambda\sigma}\left(|u(\sigma;\psi) - u^*|^2 + \alpha^2\|u(\sigma;\psi) - u^*\|^2\right) \\ &\quad = \int_\sigma^{\sigma+\epsilon} \lambda e^{\lambda t}\left(|w(t;\psi)|^2 + \alpha^2\|w(t;\psi)\|^2\right) dt \\ &\qquad + \int_\sigma^{\sigma+\epsilon} e^{\lambda t}\left(-2\nu\langle Aw(t), w(t)\rangle - 2\langle B(u(t;\psi)) - B(u^*), w(t)\rangle\right) dt \\ &\qquad + \int_\sigma^{\sigma+\epsilon} e^{\lambda t}\langle g(t, u_t(\cdot;\psi)) - g(t, u^*), w(t)\rangle dt \le 0. \end{aligned}$$

Obviously this contradicts (4.15), and the proof is complete.

For the uniqueness of the stationary solution, if $\hat{u}^*$ is another stationary solution to (4.1), then $u(t) \equiv \hat{u}^*$ is a solution of (1.2) with $u^0 = \hat{u}^*$ and $\phi = \hat{u}^*$, and therefore, by applying (4.6) with t going to $+\infty$, we have that $|\hat{u}^* - u^*|^2 \leq 0$. □

Remark 4.5. *It would be very interesting to have a sufficient condition which could be checked more easily in applications than (4.11). We establish one in the next Corollary.*

Corollary 4.6. *Assume g satisfies conditions (H1)–(H4) for any $T > 0$, and suppose that for any $\xi \in C_V$ the mapping $t \in [0, +\infty) \mapsto g(t, \xi) \in (H^{-1}(\Omega))^3$ is continuous. Suppose that $f \in (H^{-1}(\Omega))^3$ is such that there exists a stationary solution u^* for the problem (4.1). If $\nu - L_g > 0$ and*

$$-\nu + L_g\left(\lambda_1^{-1} + \alpha^2\right) + \frac{C_1\|f\|_{(H^{-1}(\Omega))^3}}{\nu - L_g} < 0, \tag{4.18}$$

then the stationary solution u^* of (4.1) is unique and there exists $\lambda > 0$ such that for each $\psi \in C_V$, the solution $u(t;\psi)$ of (1.2) corresponding to this initial datum, satisfies (4.13), i.e.,

$$|u(t;\psi) - u^*|^2 + \alpha^2\|u(t;\psi) - u^*\|^2 \leq e^{-\lambda t}\|\psi - u^*\|^2_{C_V},\ \forall t \geq 0.$$

Proof. Let $\phi \in C_V$ be such that $\phi(0) \neq u^*$ and

$$\|\phi - u^*\|^2_{C_V} \leq e^{\lambda h}\left(|\phi(0) - u^*|^2 + \alpha^2\|\phi(0) - u^*\|^2\right) \tag{4.19}$$

where $\lambda > 0$ is to be chosen later. Then,

$$\begin{aligned}
&-\nu\langle A(\phi(0) - u^*), \phi(0) - u^*\rangle - \langle B(\phi(0)) - B(u^*), \phi(0) - u^*\rangle \\
&+ \langle g(t,\phi) - g(t,u^*), \phi(0) - u^*\rangle \\
&\leq -\nu\|\phi(0) - u^*\|^2 + |\langle B(\phi(0)) - B(u^*), \phi(0) - u^*\rangle| \\
&+ L_g\|\phi - u^*\|_{C_V}\|\phi(0) - u^*\| \\
&\leq -\nu\|\phi(0) - u^*\|^2 + L_g e^{\lambda h}\left(\lambda_1^{-1} + \alpha^2\right)\|\phi(0) - u^*\|^2 \\
&+ |\langle B(\phi(0)) - B(u^*), \phi(0) - u^*\rangle|.
\end{aligned}$$

Using now (4.7), we obtain

$$\begin{aligned}
&-\nu\langle A(\phi(0) - u^*), \phi(0) - u^*\rangle - \langle B(\phi(0)) - B(u^*), \phi(0) - u^*\rangle \\
&+ \langle g(t,\phi) - g(t,u^*), \phi(0) - u^*\rangle \\
&\leq \left(-\nu + L_g e^{\lambda h}\left(\lambda_1^{-1} + \alpha^2\right) + \frac{C_1\|f\|_{(H^{-1}(\Omega))^3}}{\nu - L_g}\right)\|\phi(0) - u^*\|^2.
\end{aligned} \tag{4.20}$$

If we denote

$$\mu = \nu - L_g e^{\lambda h}\left(\lambda_1^{-1} + \alpha^2\right) - \frac{C_1\|f\|_{(H^{-1}(\Omega))^3}}{\nu - L_g},$$

is easy to deduce that

$$-\mu\|\phi(0) - u^*\|^2 \leq -\min\{\frac{\mu\lambda_1^{-1}}{2}, \frac{\mu}{2\alpha^2}\}\left(|\phi(0) - u^*|^2 + \alpha^2\|\phi(0) - u^*\|^2\right)$$

If (4.18) holds, then there exists $\lambda > 0$ such that

$$\lambda - \min\{\frac{\mu\lambda_1^{-1}}{2}, \frac{\mu}{2\alpha^2}\} < 0,$$

and for this fixed λ, we can obtain from (4.20)

$$\begin{aligned}
&-\nu\langle A(\phi(0)-u^*), \phi(0)-u^*\rangle - \langle B(\phi(0)) - B(u^*), \phi(0)-u^*\rangle \\
&+ \langle g(t,\phi) - g(t,u^*), \phi(0)-u^*\rangle \\
&\leq -\lambda\left(|\phi(0)-u^*|^2 + \alpha^2\|\phi(0)-u^*\|^2\right).
\end{aligned}$$

The proof is now complete. □

4.4. Exponential stability via a Gronwall-like lemma.

Our aim in this subsection is to prove a sufficient condition for the exponential stability of stationary solutions to the Navier-Stokes-Voigt model with delay, using a Gronwall-like lemma. In this case, we do not need to specify any particular form of the delay as in the previous cases, but in the particular case of variable delay it is not necessary to impose neither differentiability nor continuity but only measurability of the variable delay function. Let us first recall the key tool for our analysis, which is a Gronwall-like inequality proved by H. Chen in [12].

Lemma 4.7. (*[12, Lemma 3.2]) Let $y(\cdot) : [-h, +\infty) \to [0, +\infty)$ be a function. Assume that there exist positive numbers γ, α_1 and α_2 such that $\gamma > \alpha_2$, and the following inequality holds:*

$$y(t) \leq \begin{cases} \alpha_1 e^{-\gamma t} + \alpha_2 \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-h,0]} y(s+\theta)\, ds, & t \geq 0, \\ \alpha_1 e^{-\gamma t}, & t \in [-h, 0]. \end{cases} \tag{4.21}$$

Then,

$$y(t) \leq \alpha_1 e^{-\mu t}, \text{ for } t \geq -h,$$

where $\mu \in (0, \gamma)$ is given by the unique root of the equation

$$\frac{\alpha_2}{\gamma - \mu} e^{\mu h} = 1$$

in this interval.

We state our stability result in the next theorem.

Theorem 4.8. *Assume that $g(\cdot, \cdot)$ satisfies conditions (H1)–(H3) and $f \in (H^{-1}(\Omega))^3$. Assume also that $u^* \in V$ is the unique stationary solution to* (4.1)*. Then, u^* is exponentially stable if*

$$\nu > C_1\|u^*\| + L_g. \tag{4.22}$$

Proof. Let λ be a positive constant such that

$$\frac{\lambda}{2}\left(\lambda_1^{-1} + \alpha^2\right) - \left(2\nu - 2C_1\|u^*\| - 2L_g\right) < 0.$$

Then, taking into account condition (H3) and the fact that $\nu Au^* + Bu^* f - g(t, u^*) = 0$, if $u(\cdot)$ is a weak solution to the model (1.2) for the initial datum ϕ, we obtain, for $t \geq 0$,

$$\begin{aligned}
&\frac{d}{dt}\left(e^{\lambda t}\left(|u(t)-u^*|^2+\alpha^2\|u(t)-u^*\|^2\right)\right)\\
&= \lambda e^{\lambda t}\left(|u(t)-u^*|^2+\alpha^2\|u(t)-u^*\|^2\right)\\
&\quad + e^{\lambda t}\frac{d}{dt}\left(|u(t)-u^*|^2+\alpha^2\|u(t)-u^*\|^2\right)\\
&= \lambda e^{\lambda t}\left(|u(t)-u^*|^2+\alpha^2\|u(t)-u^*\|^2\right)\\
&\quad + 2e^{\lambda t}\left\langle \frac{d}{dt}(u(t)-u^*), u(t)-u^*\right\rangle\\
&= \lambda e^{\lambda t}\left(|u(t)-u^*|^2+\alpha^2\|u(t)-u^*\|^2\right)\\
&\quad + 2e^{\lambda t}\langle -\nu A(u(t)-u^*), u(t)-u^*\rangle\\
&\quad + 2e^{\lambda t}\langle -(B(u(t))-B(u^*)), u(t)-u^*\rangle\\
&\quad + 2e^{\lambda t}\langle g(t,u_t)-g(t,u^*), u(t)-u^*\rangle\\
&\leq \lambda e^{\lambda t}\left(|u(t)-u^*|^2+\alpha^2\|u(t)-u^*\|^2\right) - 2e^{\lambda t}\nu\|u(t)-u^*\|^2\\
&\quad + 2e^{\lambda t}\langle -(B(u(t))-B(u^*)), u(t)-u^*\rangle\\
&\quad + 2e^{\lambda t}L_g\|u_t-u^*\|^2_{C_V}.
\end{aligned}$$

Thanks to (2.3) and (2.5),

$$\begin{aligned}
|\langle B(u(t))-B(u^*), w(t)\rangle| &= \langle b(u(t)-u^*, u^*, u(t)-u^*)\rangle|\\
&\leq C_1\|u^*\|\|u(t)-u^*\|^2 \qquad (4.23)
\end{aligned}$$

and we arrive at

$$\begin{aligned}
&\frac{d}{dt}\left(e^{\lambda t}\left(|u(t)-u^*|^2+\alpha^2\|u(t)-u^*\|^2\right)\right)\\
&\leq \lambda e^{\lambda t}\left(|u(t)-u^*|^2+\alpha^2\|u(t)-u^*\|^2\right) - 2e^{\lambda t}\nu\|u(t)-u^*\|^2\\
&\quad + 2C_1 e^{\lambda t}\|u^*\|\|u(t)-u^*\|^2 + 2e^{\lambda t}L_g\|u_t-u^*\|^2_{C_V}\\
&\leq e^{\lambda t}\left[\frac{\lambda}{2}\left(\lambda_1^{-1}+\alpha^2\right) - 2\nu + 2C_1\|u^*\| + 2L_g\right]\|u_t-u^*\|^2_{C_V}\\
&\quad + \frac{\lambda}{2}e^{\lambda t}\left(|u(t)-u^*|^2+\alpha^2\|u(t)-u^*\|^2\right). \qquad (4.24)
\end{aligned}$$

Integrating (4.24) over the interval $[0, t]$ gives

$$\begin{aligned}
&e^{\lambda t}\left(|u(t)-u^*|^2+\alpha^2\|u(t)-u^*\|^2\right)\\
&\quad \leq |u(0)-u^*|^2+\alpha^2\|u(0)-u^*\|^2\\
&\qquad + \frac{\lambda}{2}\int_0^t e^{\lambda s}\left(|u(s)-u^*|^2+\alpha^2\|u(s)-u^*\|^2\right) ds,
\end{aligned}$$

and consequently,

$$\begin{aligned}
&|u(t) - u^*|^2 + \alpha^2\|u(t) - u^*\|^2 \\
&\quad \le e^{-\lambda t}\left(|u(0) - u^*|^2 + \alpha^2\|u(0) - u^*\|^2\right) \\
&\qquad + \frac{\lambda}{2}\int_0^t e^{-\lambda(t-s)} \sup_{\theta\in[-h,0]} \left(|u(s+\theta) - u^*|^2 + \alpha^2\|u(s+\theta) - u^*\|^2\right) ds.
\end{aligned}$$

Now, we can apply Lemma 4.7 denoting $\gamma = \lambda, \alpha_2 = \frac{\lambda}{2}$ and

$$\alpha_1 = \sup_{\theta\in[-h,0]} (|u(\theta) - u^*|^2 + \alpha^2\|u(\theta) - u^*\|^2),$$

since it is straightforward to see that, for $t \in [-h, 0]$,

$$|u(t) - u^*|^2 + \alpha^2\|u(t) - u^*\|^2 \le e^{-\lambda t} \sup_{\theta\in[-h,0]} \left(|u(\theta) - u^*|^2 + \alpha^2\|u(\theta) - u^*\|^2\right).$$

Then, the exponential stability of u^* follows from Lemma 4.7, namely, there exists $\mu \in (0, \lambda)$ such that

$$|u(t) - u^*|^2 + \alpha^2\|u(t) - u^*\|^2 \le \alpha_1 e^{-\lambda t}$$

for all $t \ge -h$. □

4.5. An illustrative example to compare the three stability methods

In this subsection, we will compare in the particular case of forcing term with variable delay, the main results obtained by the three different methods described previously. Notice that, by straightforward computations, the value of constants L_g and C_g are

$$L_g = \frac{M}{\lambda_1}, \quad C_g = \frac{Me^{m_1 h/2}}{\lambda_1\sqrt{1-\rho^*}}.$$

Indeed, to check (H3),

$$\begin{aligned}
\|g(t,\xi) - g(t,\mu)\|_{(H^{-1}(\Omega))^3} &\le \lambda_1^{-1/2}|G(\xi(-\rho(t))) - G(\mu(-\rho(t)))| \\
&\le M\lambda_1^{-1/2}|\xi(-\rho(t)) - \mu(-\rho(t))| \\
&\le M\lambda_1^{-1}\|\xi(-\rho(t)) - \mu(-\rho(t))\| \\
&\le M\lambda_1^{-1}\, \|\xi - \mu\|_{C_V},
\end{aligned}$$

and, to check (H4),

$$\begin{aligned}
&\int_\tau^t e^{ms}\|g(s,u_s) - g(s,v_s)\|^2_{(H^{-1}(\Omega))^3} ds \\
&\le \lambda_1^{-1}\int_\tau^t e^{ms}|G(u(s-\rho(s))) - G(v(s-\rho(s)))|^2 ds \\
&\le \lambda_1^{-1}M^2\int_\tau^t e^{ms}|u(s-\rho(s)) - v(s-\rho(s))|^2 ds
\end{aligned}$$

$$\le \lambda_1^{-2} M^2 \int_\tau^t e^{ms} \|u(s-\rho(s)) - v(s-\rho(s))\|^2 ds$$
$$\le \frac{M^2 e^{m_1 h}}{\lambda_1^2 (1-\rho^*)} \int_{\tau-h}^t e^{m\theta} \|u(\theta) - v(\theta)\|^2 d\theta,$$

whenever $\rho \in C^1([0,+\infty))$, $\rho \ge 0$ for all $t \ge 0$, $h = \sup_{t\ge 0} \rho(t)$ and $\rho^* = \sup_{t\ge 0} \rho'(t) < 1$.

Then, the sufficient condition provided by Theorem 4.2 reads $\nu > \frac{M e^{m_1 h/2}}{\lambda_1 \sqrt{1-\rho^*}}$ and

$$C_1 \|f\|_{(H^{-1}(\Omega))^3} < \left(\nu - \frac{M e^{m_1 h/2}}{\lambda_1 \sqrt{1-\rho^*}}\right)\left(\nu - \frac{M}{\lambda_1}\right), \tag{4.25}$$

and we need to impose that the delay function is continuously differentiable.

Next, Corollary 4.6 ensures exponential stability by simply assuming that ρ is a continuous function and the following condition holds:

$$C_1 \|f\|_{(H^{-1}(\Omega))^3} < \left(\nu - \frac{M}{\lambda_1}\left(\lambda_1^{-1} + \alpha^2\right)\right)\left(\nu - \frac{M}{\lambda_1}\right) \tag{4.26}$$

Finally the condition in Theorem 4.8 is

$$C_1 \|f\|_{(H^{-1}(\Omega))^3} < \left(\nu - \frac{M}{\lambda_1}\right)^2. \tag{4.27}$$

Observe that $\frac{e^{m_1 h/2}}{\sqrt{1-\rho^*}} > 1$, therefore, $\frac{M e^{m_1 h/2}}{\lambda_1 \sqrt{1-\rho^*}} > \frac{M}{\lambda_1}$, and then (4.25) implies (4.27). As for condition (4.26), observe that this depends on the value of $\lambda_1^{-1} + \alpha^2$, and this is related to the size of the domain Ω, since the first eigenvalue λ_1 depends on this. Therefore, this condition can provide better or worse stability regions depending on the shape of the domain, and on the value of α.

5. Conclusion

We have proved some results on the existence, uniqueness and asymptotic behaviour of the solutions for a three-dimensional Navier-Stokes-Voigt model with delay forcing term. Our assumptions are general enough to include several types of delay in the formulation (constant delay, variable delay with only measurable delay function, distributed delay, etc.). In particular, we have analyzed the exponential stability of the stationary solutions.

We have studied the local stability of stationary solutions by several different methods: the classical Lyapunov function method, the Razumikhin-Lyapunov technique and another one based in a Gronwall-like lemma. On the other hand, the study of global long-time behavior of solutions to Navier-Stokes-Voigt equations, in 2D and 3D, has been recently considered in some particular cases of special delays (see, for instance, [15,17,38]). Therefore, it will be also interesting to carry out a similar global analysis of this model by proving the existence of attractors as well as the study of their geometrical structure in the abstract functional formulation that we have considered in this paper, and it is the reason we have first studied their stationary solutions and their stability. We plan to investigate these issues in a subsequent paper.

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Conflict of interest

The authors declare that they have no competing interest.

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