Optimality of Standard Flight Procedures of Commercial Aircraft

Antonio Franco, Alfonso Valenzuela and Damián Rivas
Department of Aerospace Engineering
Escuela Técnica Superior de Ingeniería, Universidad de Sevilla, Seville, 41092, Spain
{antfranco,avalenzuela,drivas}@us.es

Abstract

Standard flight procedures for climb, cruise and descent, commonly flown by commercial aircraft, are analyzed, and the optimality of their performance is assessed by comparison with the performance of optimal trajectories, which are computed using the theory of singular optimal control. Three flight procedures are studied: CAS/Mach climb, constant-Mach cruise with fixed arrival time, and constant-CAS unpowered descent. The optimality criteria selected are minimum fuel consumption for climb and cruise, and maximum horizontal distance travelled for descent. Results are presented for a model of a Boeing 767-300ER.

1 Introduction

Trajectory optimization is, from the operational point of view, a subject of great importance in air traffic management (ATM), that aims at defining optimal flight procedures that lead to energy-efficient flights. However, in practice airlines fly standard procedures which in principle are not optimal, such as CAS/Mach climb, constant-Mach cruise or Mach/CAS descent, although they are optimized, that is, the constant values of CAS and/or Mach number are chosen so as to obtain the best possible performance.

The use of standard flight procedures is common in the generation of global trajectories. Thus, they are used, among others, by Gill and Maddock [1] in the Experimental Flight Management System of the PHARE programme, by Wu and Zhao [2] who optimize trajectories from liftoff to touchdown and quantify the deviation from actual trajectories due to modeling errors, and by Rivas et al. [3] who present a trajectory computation tool designed for a general aircraft performance model (general drag polar and general engine model), taking into account wind effects and temperature corrections for a non-standard atmosphere.

The main objective of this work is to assess the optimality of the following standard flight procedures: 1) CAS/Mach climb between two given points (given speed and altitude), 2) constant-Mach cruise at constant altitude with fixed arrival time between two given points (given speed and distance), and 3) constant-CAS unpowered descent between two given points (given speed and altitude). These procedures are optimized using parametric optimization theory (see Fletcher [4]). The optimality assessment is twofold: first, the procedures themselves are compared with the optimal procedures (which in general are not at constant values of CAS and/or Mach number), and, second, their best performance is compared with the optimal performance. The optimal trajectories are computed using the theory of singular optimal control (see Bell and Jacobson [5]); examples of application of this theory to cruise and descent problems can be found in Refs. [6,7]. The optimality criteria selected for these problems are: minimum fuel consumption for climb and cruise, and maximum horizontal distance travelled for descent. Results are presented for a model of a Boeing 767-300ER (a typical twin-engine, wide-body, long-range transport aircraft). Wind effects are not considered in this work.

2 Problem formulation

The equations of motion for a flight in a vertical plane are

$$\begin{aligned}\dot{V} &= \frac{T-D}{m} - g\gamma \\ \dot{m} &= -cT \\ \dot{h} &= V\gamma \\ \dot{x} &= V\end{aligned} \tag{1}$$

where the assumptions $\gamma \ll 1$ and $\dfrac{V\dot{\gamma}}{g}$ negligible have been taken into account (see Jackson et al. [8]). In these equations, the drag is a general known function $D(V, m, h)$, which takes into account the remaining equation of motion $L = mg$. The thrust $T(V, h)$ is given by $T(V, h) = \pi T_M(V, h)$ where π is a thrust control parameter, $0 < \pi_{min} \leq \pi \leq \pi_{max} = 1$, and $T_M(V, h)$ is a general known function. The specific fuel consumption is also a general known function $c(V, h)$. Wind effects are not considered.

In this paper three problems are analyzed:

1) Minimum-fuel climb in a vertical plane between two given points (given speed and altitude), in the case of fixed engine rating. The initial values of speed, mass, altitude and distance (V_i, m_i, h_i, x_i), and the final values of speed and altitude (V_f, h_f) are given. The final value of mass (m_f), distance (x_f), and flight time (t_f) are unspecified.

2) Minimum-fuel cruise at constant altitude with fixed arrival time between two given points (given speed and distance). The initial values of speed, aircraft mass and distance (V_i, m_i, x_i), and the final values of speed and distance (V_f, x_f) are given. The final value of aircraft mass (m_f) is unspecified, and flight time (t_f) is fixed.

3) Maximum-range unpowered descent in a vertical plane between two given points (given speed and altitude). The initial values of speed, altitude and distance (V_i, h_i, x_i), and the final values of speed and altitude (V_f, h_f) are given. The final value of distance (x_f) and the flight time (t_f) are unspecified. The aircraft mass is constant.

3 Optimal problem

In the climb problem, the objective is to minimize the following performance index

$$J_{cl} = -\int_0^{t_f} cT\mathrm{d}t, \tag{2}$$

subject to the state equations (1), with given π ($\pi = \pi_{cl}$). The flight-path angle γ is the control variable.

In the cruise problem, the objective is to minimize the following performance index

$$J_{cr} = -\int_0^{t_f} cT\,\mathrm{d}t, \tag{3}$$

subject to the state equations (1), with $\gamma = 0$ (constant altitude). The thrust control parameter π is the control variable.

In the descent problem, the objective is to minimize the following performance index

$$J_d = -\int_0^{t_f} V\,\mathrm{d}t. \tag{4}$$

subject to the state equations (1), with $\pi = 0$. The flight-path angle γ is the control variable.

In all the problems to be solved, there is one control variable which appears linearly in the equations of motion, as well as on the performance indices to be optimized (γ at climb and descent, and π at cruise). As a consequence, the Hamiltonian of the problem is also linear on the control variable, which leads to a *singular optimal control problem*.

Let H be the Hamiltonian of the problem and u the control variable. The derivative H_u is called the switching function. The optimal control problem reduces to finding the optimal control u^* that minimizes H. In general, u^* is determined by the necessary condition for optimality $H_u = 0$ (provided that $u_{min} <$

$u^* < u_{max}$), but in the problems to be solved the function H_u does not depend on u (hence, $H_{uu} = 0$, reason why the problem is called singular). The singular control u_{sing} now follows from the condition $\ddot{H}_u = 0$ (the function $\ddot{H}_u$ happens to be linear in u).

In the three problems considered, the corresponding optimal path lies on a singular manifold in the state space called *singular arc*, which is of order $q = 1$ (as defined in Ref. [5]) and can be obtained from the necessary conditions H =const ($H = 0$ if t_f is unspecified), $H_u = 0$ and $\dot{H}_u = 0$ (see Bryson and Ho [9]). The singular arc is in fact the locus of possible points in the state space on which optimal paths can lie, as well as a switching boundary for the optimal control (see Ben-Asher [10]). Integration along the optimal path leads to optimum performance index, provided that the generalized Legendre-Clebsch condition (see Kelley et al. [11]) given by $-\partial\ddot{H}_u/\partial u \geq 0$ (as $q = 1$) is satisfied along the singular arc.

In this work it is assumed that the initial and final points of the path are given. In that case, the optimal path is formed, in general, by three arcs: one to go from the initial point to the singular arc, the singular arc, and a final arc to go from the singular arc to the final point. The initial and final arcs are defined by the control being at its maximum or minimum value; this type of optimal control is called bang-singular-bang. Bang-singular-bang solutions are subject to additional necessary conditions for the junctions between singular and nonsingular arcs to be optimal: McDanell-Powers condition and Weierstrass-Erdman corner conditions (see Refs. [9, 12]).

In order to obtain the solution for each problem, a numerical procedure must be defined, which includes the integration of the state equations, first, with either $u = u_{min}$ or $u = u_{max}$ from the initial point (with known initial values) until the singular arc is reached (which defines the first junction point), second, with $u = u_{sing}$ from the first junction point until the second one is reached, and, third, with either $u = u_{min}$ or $u = u_{max}$ from the second junction point until the final point (with known final values) is reached. Part of the resolution procedure is also to check whether both the assumed structure for the control is correct and the generalized Legendre-Clebsch condition is satisfied. A description of this type of procedure can be found in Refs. [6, 7]. Although called optimal trajectories, the solutions are in fact extremals, that is, trajectories that satisfy the necessary conditions for optimality.

4 Standard procedures

The standard flight procedures analyzed in this work are described in this section. They are: 1) CAS-Mach climb, 2) constant-Mach cruise, and 3) constant-CAS descent.

4.1 Optimized CAS-Mach climb

First, the optimized CAS-Mach climb is analyzed. The CAS-Mach procedure considered in this work is formed by four segments, all of them with constant engine rating: 1) acceleration at constant altitude h_i from the initial speed V_i to the climb CAS (CAS_c), 2) climb with constant CAS (CAS_c) from h_i to the transition altitude h_2 at which climb Mach M_c is reached, 3) climb with constant Mach (M_c) from h_2 to the final altitude h_f, and 4) acceleration/deceleration at constant altitude h_f from M_c to the final speed V_f.

To solve the equations of motion (1) for each flight segment, a flight constraint must be given so that the control parameter γ can be determined. For the initial and final flight segments the flight constraint is $h = const$, and, therefore, $\gamma = 0$. For the constant-CAS segment, it is $CAS = const = CAS_c$, which is in fact a speed law $V = V_C(h)$ given by (see Asselin [13])

$$V_C = \sqrt{\frac{2}{k} R_a \Theta(h) \left[\left(1 + \frac{p_{SL}}{p(h)} \left[\left(1 + \frac{k}{2} \frac{\rho_{SL}}{p_{SL}} CAS_c^2 \right)^{1/k} - 1 \right] \right)^k - 1 \right]}, \tag{5}$$

where $k = (\kappa - 1)/\kappa$, κ=1.4 is the ratio of specific heats, R_a =287.053 J/(kgK) the gas constant of the air, and p, Θ, ρ the pressure, temperature and density, with p_{SL}, ρ_{SL} the reference sea-level values. For the constant-Mach segment, the flight constraint is $M = const = M_c$, which is in fact a speed law $V = V_M(h)$ given by

$$V_M = M_c \sqrt{\kappa R_a \Theta(h)}. \tag{6}$$

At the end of the integration one has the final aircraft mass m_f. The fuel consumption is therefore $m_F = m_i - m_f$. This procedure to obtain the fuel consumption for given values of CAS and Mach can be written in symbolic form as

$$m_F = m_F(CAS_c, M_c) \tag{7}$$

The CAS-Mach procedure is now optimized to give minimum fuel consumption, taking CAS_c and M_c as the optimization parameters. The optimum values of CAS_c and M_c, say CAS_c^* and M_c^*, are obtained solving the following parametric optimization problem

$$\begin{aligned} \text{minimize} \quad & m_F(CAS_c, M_c) \\ \text{subject to} \quad & CAS_f \leq CAS_c \leq CAS_i \\ & M_f \leq M_c \leq M_i \end{aligned} \tag{8}$$

where CAS_i, M_i, CAS_f and M_f are the values of CAS and Mach that correspond to V_i, h_i and V_f, h_f respectively. In this work, the optimization solver used is MATLAB's *fmincon*, a sequential quadratic programming (SQP) method (see Fletcher [4], for example).

4.2 Constant-Mach cruise

Now the constant-Mach cruise is analyzed. The constant-Mach procedure considered in this work is formed by three segments, all of them with constant altitude: 1) acceleration/deceleration from the given initial speed V_i to the cruise speed V_{cr}, with maximum cruise/idle engine rating, 2) cruise with constant speed V_{cr}, and 3) acceleration/deceleration from V_{cr} to the given final speed V_f, with maximum cruise/idle engine rating.

To solve the equations of motion (1) for each flight segment, a flight constraint must be given so that the control parameter π can be determined. For the initial and final flight segments the flight constraint is $\pi = \pi_{max}$ or $\pi = \pi_{min}$. For the constant-Mach segment, the flight constraint is $M = const = M_{cr}$, hence the cruise speed is $V_{cr} = M_{cr} a(h)$ where $a(h)$ is the speed of sound at the given altitude h.

To perform the integration, the variables V_{cr} and x_2 (distance flown during the second segment) are guessed. At the end of the integration one has values of the flight distance and the flight time which in general are different from x_f and t_f. One must then iterate on the two free variables V_{cr} and x_2 until distance and time coincide with x_f and t_f to within some prescribed tolerance. The iteration is started with the initial guess $V_{cr} = \dfrac{x_f}{t_f}$ and $x_2 = x_f$. Once the iteration ends, one has the final aircraft mass m_f; the fuel consumption is $m_F = m_i - m_f$.

4.3 Optimized constant-CAS descent

Finally, the optimized constant-CAS descent is analyzed. The constant-CAS procedure considered in this work is formed by three segments, all of them with zero thrust: 1) deceleration at constant altitude h_i from the initial speed V_i to the descent CAS (CAS_d), 2) descent with constant CAS (CAS_d) from h_i to the final altitude h_f, and 3) acceleration/deceleration at constant altitude h_f from CAS_d to the final speed V_f.

As in the CAS-Mach climb, to solve the equations of motion (1) for each flight segment, a flight constraint must be given so that the control parameter γ can be determined. For the initial and final flight segments the flight constraint is $h = const$, and, therefore, $\gamma = 0$. For the constant-CAS segment, it is $CAS = const = CAS_d$, which leads to the same speed law $V = V_C(h)$ given by Eq. (5), except for considering CAS_d instead of CAS_c.

At the end of the integration one has the distance travelled x_f. This procedure to obtain the range for a given value of CAS can be written in symbolic form as

$$x_f = x_f(CAS_d) \tag{9}$$

The constant-CAS procedure is now optimized to give maximum range, taking CAS_d as the optimization parameter. The optimum value of CAS_d, say CAS_d^*, is obtained solving the following parametric optimization problem

$$\begin{aligned} \text{minimize} \quad & -x_f(CAS_d) \\ \text{subject to} \quad & CAS_f \leq CAS_d \leq CAS_i \end{aligned} \tag{10}$$

where CAS_i and CAS_f are the values of CAS that correspond to V_i, h_i and V_f, h_f respectively. The optimization solver used is again MATLAB's *fmincon*.

5 Results

The aerodynamic and propulsion models considered in this paper for the numerical applications (corresponding to a Boeing 767-300ER) are described in the Appendix, and the atmosphere model is the International Standard Atmosphere (ISA).

Climb results are presented for the case of initial and final γ_{min}-arcs, which require that the initial and final speeds be sufficiently low and high respectively. In particular, $\gamma_{min} = 0$ has been considered so that the initial and final arcs are horizontal segments, as in the optimized CAS-Mach procedure, with which the optimum results are to be compared. The initial conditions (corresponding to a hypothetical SID final fix) are $CAS_i = 250$ kt, $h_i = 10000$ ft, and the final conditions are $M_f = 0.8$, $h_f = 33000$ ft. The initial aircraft weight ranges from $W_i = 1650$ kN to $W_i = 1750$ kN.

Cruise results are presented for a flight defined by a range $x_f = 8000$ km, and by initial and final speeds $V_i = 240$ m/s and $V_f = 180$ m/s. The fixed arrival time ranges from $t_f = 9.17$ h to $t_f = 10$ h. The initial aircraft weight ranges from $W_i = 1650$ kN to $W_i = 1750$ kN. The cruise altitude is taken to be $h = 10000$ m.

Descent results are presented for the case of two initial and final γ_{max}-arcs, which require that the initial and final speeds be sufficiently high and low respectively. In particular, $\gamma_{max} = 0$ has been considered so that the initial and final arcs are horizontal segments, as in the optimized constant-CAS procedure, with which the optimum results are to be compared. The initial conditions are $M_i = 0.8$, $h_i = 33000$ ft, and the final conditions (corresponding to a hypothetical approach fix within the TMA) are $CAS_f = 210$ kt, $h_f = 9000$ ft. The aircraft weight during the descent ranges from $W_i = 1100$ kN to $W_i = 1300$ kN.

In the following, optimal trajectories and control laws as well as global results (fuel consumption, distance travelled, final time) are presented along with the optimized, standard ones. A comparison between both sets of results is made. The influence of the aircraft weight on the results is analyzed.

5.1 Climb problem

In this section the optimized standard CAS-Mach climb is compared with the optimal climb problem. The optimal and the optimized CAS-Mach climb trajectories $V(h)$ are represented in Fig. 1, for different values of the initial aircraft weight. The climb trajectories start and finish with horizontal accelerations. In the optimal trajectories, these horizontal segments correspond to the γ_{min}-arcs, from the given initial point to the singular arc, and from the singular arc to the given final point. In the optimized CAS-Mach climbs, these horizontal segments correspond to the initial and final horizontal accelerations from the given initial CAS to CAS_c^*, and from M_c^* to the final Mach number. The speed continuously increases, reaches a maximum and then slowly decreases. As W_i increases, the speed along the singular arc and the constant-CAS/Mach segments slightly increases. Note that the first part of the optimal profile is not at constant CAS; in that sense, the CAS/Mach procedure itself is not close to optimal.

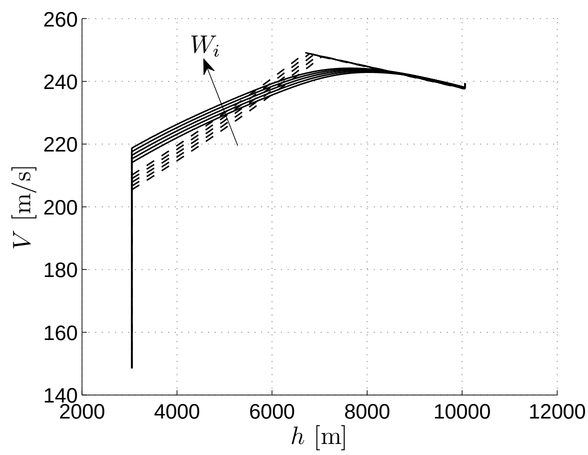

Figure 1: $V(h)$ comparison for $W_i = 1650$, 1675, 1700, 1725 and 1750 kN. Solid lines: optimal climb. Dashed lines: optimized CAS-Mach climb.

The optimal control $\gamma(h)$ and the optimal altitude law $h(x)$ are represented in Fig. 2, along with the flight path angle law and the altitude law corresponding to the optimized CAS-Mach climb, for the same values of the initial aircraft weight. The control is discontinuous: for the optimal trajectories, one has the two arcs with $\gamma_{min} = 0$ (hardly seen in the figure) and the singular arc, whereas, for the optimized CAS-Mach trajectories, one has the four constitutive segments. Note that the control γ decreases as the climb progresses. As W_i increases, the control slightly decreases, and the horizontal distance travelled increases.

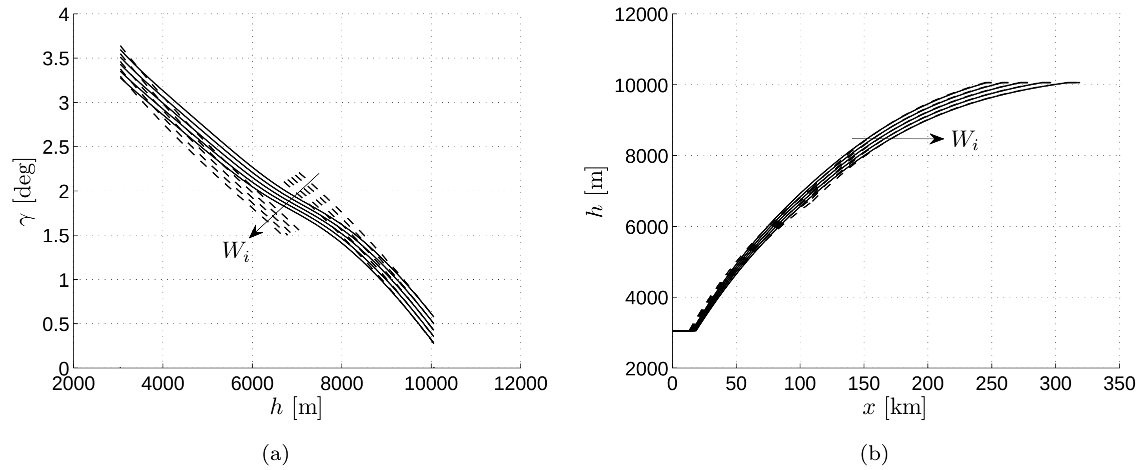

Figure 2: $\gamma(h)$ and $h(x)$ comparison for $W_i = 1650$, 1675, 1700, 1725 and 1750 kN. Solid lines: optimal climb. Dashed lines: optimized CAS-Mach climb.

In Figs. 3 and 4, the minimum fuel consumption, the flight time and the range for both problems are represented as functions of the initial aircraft weight. As expected, heavier aircraft require larger values of fuel consumption, time to climb and distance travelled, as compared to lighter aircraft. One can see that the differences between both sets of results are almost negligible in all cases (less than 3.4 kg in minimum fuel consumption, less than 3.3 s in flight time and less than 188 m in range). Hence, it can be concluded that the performance of the CAS-Mach procedure is very close to optimal, provided that the optimum values of CAS_c and M_c are used in the climb.

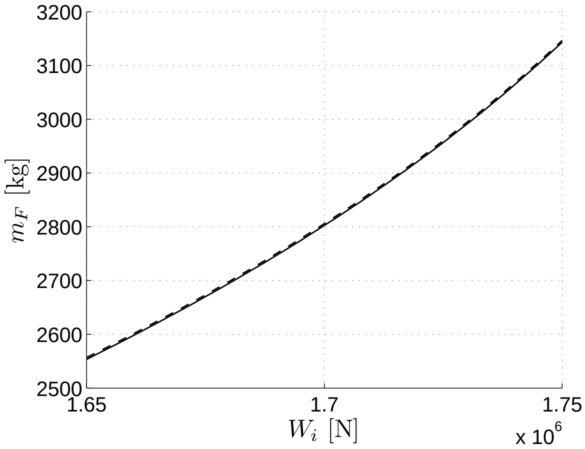

Figure 3: Minimum fuel consumption vs. initial aircraft weight. Solid lines: optimal climb. Dashed lines: optimized CAS-Mach climb.

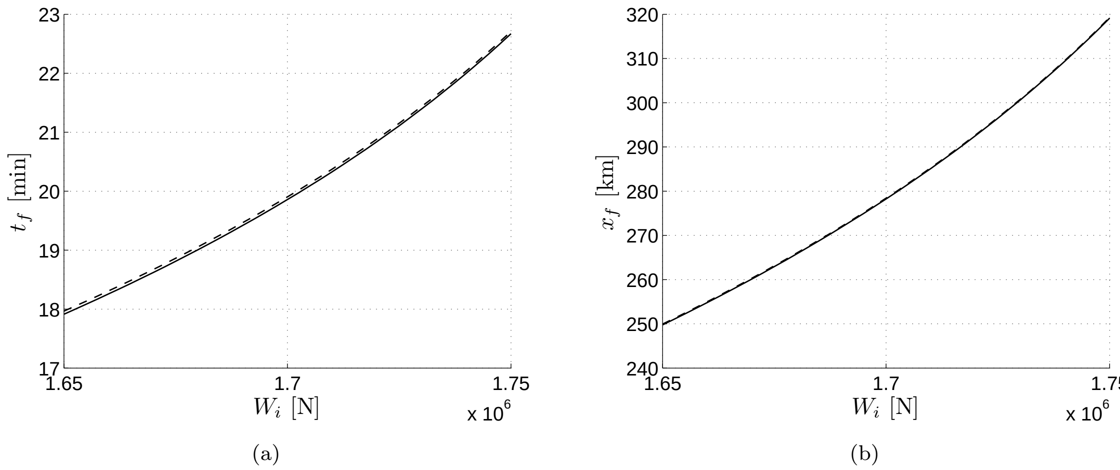

Figure 4: Flight time (a) and Range (b) vs. initial aircraft weight. Solid lines: optimal climb. Dashed lines: optimized CAS-Mach climb.

5.2 Cruise problem

Now the standard constant-Mach cruise is compared with the optimal cruise problem. The optimal trajectories and the constant-Mach trajectories for different initial aircraft weights and an arrival time $t_f = 9.5$ h are shown in Fig. 5a. The corresponding controls are shown in Fig. 5b. For the optimal trajectories, the structure is minimum-thrust arc, singular arc, minimum-thrust arc, in all cases shown. For the constant-Mach trajectories, one has the three constitutive segments. In this problem where the final distance and time are fixed, the speed is so constrained that the influence of the initial aircraft weight on the speed profiles is very small (almost negligible). However, the singular control and the constant-Mach cruise control (which are almost identical) increase as W_i increases, and they decrease along the singular arc and the constant-Mach segment.

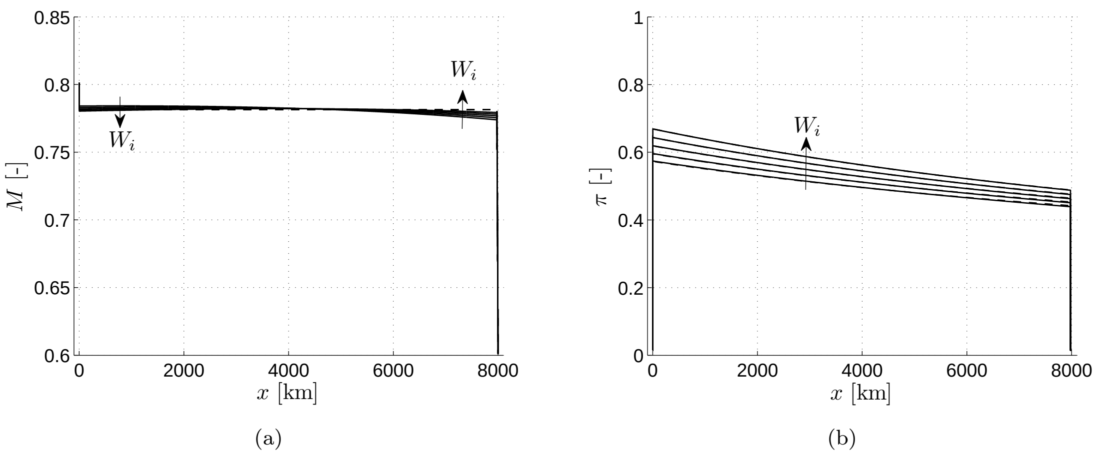

Figure 5: Trajectories (a) and control (b) for $W_i = 1500, 1550, 1600, 1650, 1700$ kN ($t_f = 9.5$ h, $h = 10000$ m). Solid lines: optimal cruise. Dashed lines: constant-Mach cruise.

To see the influence of the arrival time, the optimal trajectories and the constant-Mach trajectories for different arrival times (t_f = 9.17, 9.33, 9.5, 9.67, 9.83, 10 h) and for $W_i = 1600$ kN are shown in Fig. 6a. The corresponding controls are shown in Fig. 6b. As expected, the Mach number decreases as the arrival time increases. For small values of t_f the Mach number of the optimal trajectory is roughly constant, however for large values of t_f it decreases along the cruise. The results also show that the variation of both the singular control and the constant-Mach cruise control with the arrival time is quite small.

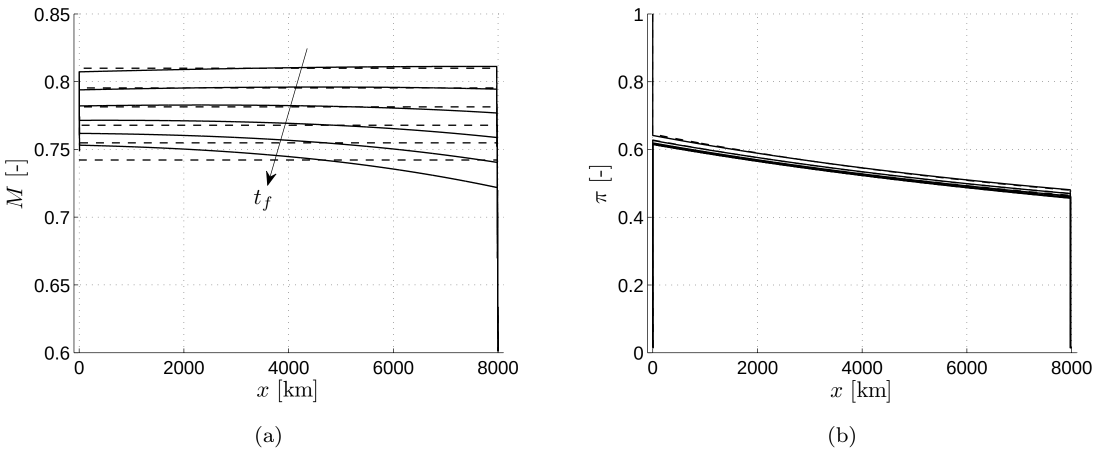

Figure 6: Trajectories (a) and control (b) for different arrival times ($t_f = 9.17, 9.33, 9.5, 9.67, 9.83, 10$ h, $h = 10000$ m, $W_i = 1600$ kN). Solid lines: optimal cruise. Dashed lines: constant-Mach cruise.

The minimum fuel consumption for both problems as a function of the flight time is shown in Fig. 7 for different initial aircraft weights. As expected, heavier aircraft require larger values of fuel consumption, as compared to lighter aircraft. One can see that the differences between the optimal and the constant-Mach results are almost negligible in all cases (less than 62 kg). Hence, it can be concluded that the performance of the constant-Mach procedure is very close to optimal.

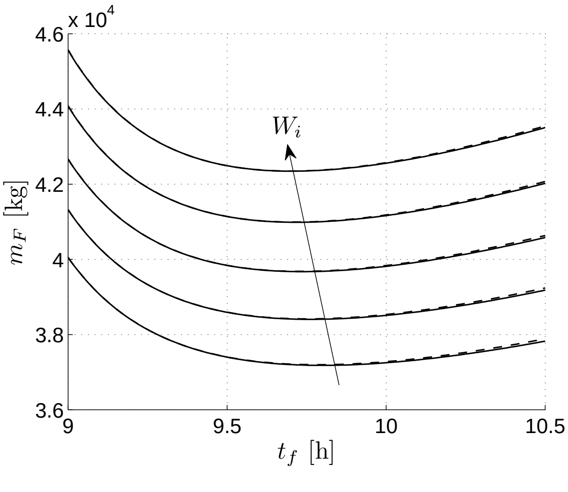

Figure 7: Minimum fuel consumption vs flight time, for $W_i = 1500, 1550, 1600, 1650, 1700$ kN ($h = 10000$ m). Solid lines: optimal cruise. Dashed lines: constant-Mach cruise.

5.3 Descent problem

Finally, the optimized standard constant-CAS unpowered descent is compared with the optimal descent problem. The speed laws $V(x)$ corresponding to the optimal problem and to the optimized constant-CAS descent are presented in Fig. 8 for different values of the aircraft weight. The descent trajectories start and finish with horizontal decelerations. In the optimal trajectories, these horizontal segments correspond to the γ_{max}-arcs, from the given initial point to the singular arc, and from the singular arc to the given final point. In the optimized constant-CAS descent, these horizontal segments correspond to the initial and final horizontal decelerations from the given initial Mach number to CAS_d^*, and from CAS_d^* to the final CAS. The speed continuously decreases along the entire trajectory. As the aircraft weight increases, the speed during the singular arc and the constant-CAS segment increases. The differences between both sets of profiles are very small, so that one can conclude that the constant-CAS procedure is very close to optimal.

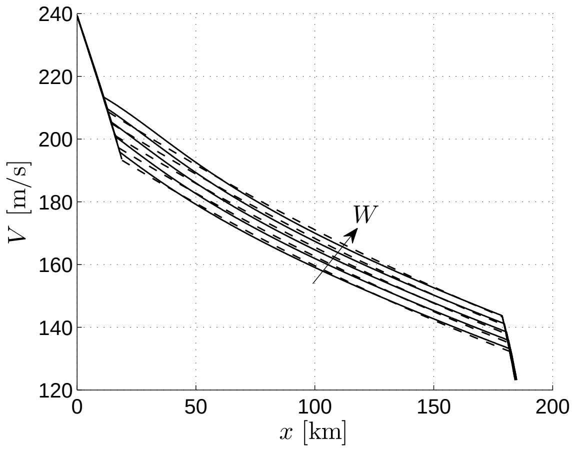

Figure 8: $V(x)$ comparison for $W = 1100, 1150, 1200, 1250$ and 1300 kN. Solid lines: optimal descent. Dashed lines: optimized constant-CAS descent.

The optimal control $\gamma(h)$ and the flight path angle corresponding to the optimized constant-CAS descent are represented in Fig. 9a for the same values of the aircraft weight. They are discontinuous (one has the two arcs with $\gamma_{max} = 0$ and the singular arc). The results show that both the optimal control and the ground path angle are roughly constant, which can be also seen in Fig. 9b, where one has a roughly linear altitude profile during the singular arc. The aircraft weight W has very little influence both on the singular optimal control and on the flight path angle during the constant-CAS segment.

The optimal altitude law and the altitude law corresponding to the optimized constant-CAS descent $h(x)$ are represented in Fig. 9b for the same values of the aircraft weight. The altitude continuously decreases, except for the initial and final decelerations at constant altitude. As the aircraft weight increases, the altitude during both the singular arc and the constant-CAS segment slightly decreases (the influence of W is very small).

The maximum range and the flight time for the optimized constant-CAS descent and for the optimal problem are represented as functions of the aircraft weight in Fig. 10. The influence of W in the maximum range is negligible, however, as the aircraft weight increases, the flight time decreases. One can see that differences between the optimized constant-CAS results and the optimal results are negligible (less than 28 m in maximum range and 0.32 s in flight time). Hence, it can be concluded that the performance of the constant-CAS procedure is very close to optimal, provided that the optimum value of CAS_d is used in the descent.

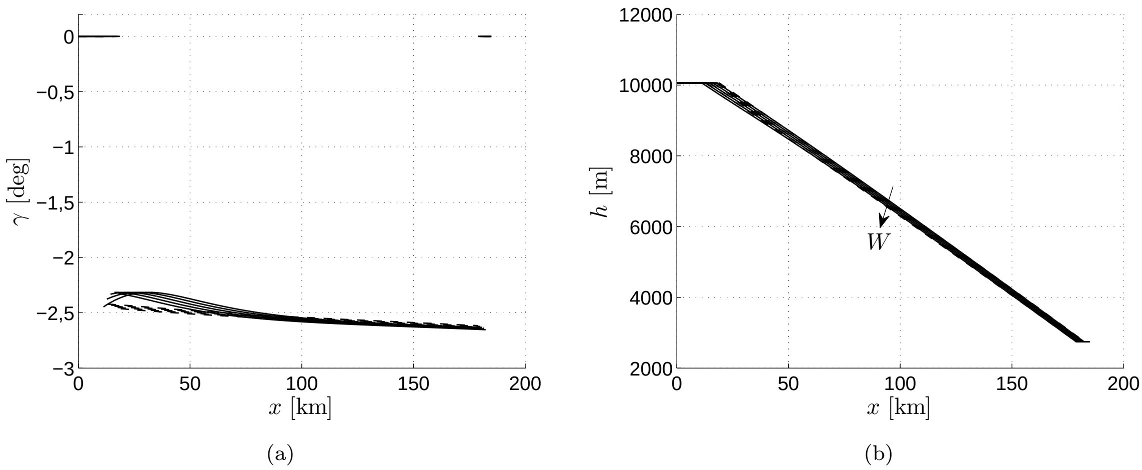

Figure 9: $\gamma(h)$ and $h(x)$ comparison for $W = 1100, 1150, 1200, 1250$ and 1300 kN. Solid lines: optimal descent. Dashed lines: optimized constant-CAS descent.

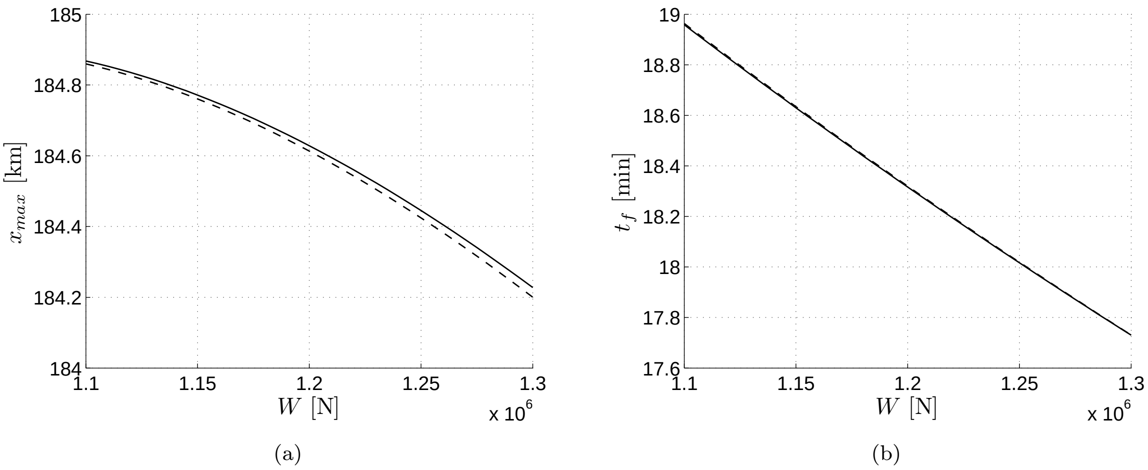

Figure 10: Maximum range (a) and flight time (b) vs. aircraft weight. Solid line: optimal descent. Dashed line: optimized constant-CAS descent.

6 Conclusions

In this paper, standard flight procedures for climb, cruise and descent, commonly flown by commercial aircraft, have been analyzed, and the optimality of their performance assessed by comparison with the performance of optimal trajectories computed using singular optimal control theory.

The main conclusion is that, in all cases studied (CAS/Mach climb, constant-Mach cruise and constant-CAS descent), the performance of the standard procedures has been shown to be very close to optimal, which justifies their use in operational practice.

However, the procedure itself may differ from the optimal one, as in the case of the CAS/Mach climb, in which the first part of the climb is not at constant CAS, not even approximately constant, and in the case of the constant-Mach cruise with large arrival times, in which the optimal speed varies along the cruise. Clearly, in these cases the standard procedure represents some sort of average which leads to performance close to optimal. On the contrary, in the case of the constant-Mach cruise with small arrival times, the Mach number is approximately constant, and in the case of constant-CAS descent, the optimal trajectory is at CAS approximately constant, so that in these cases one has that the procedure itself is close to optimal.

Appendix: Aerodynamic and Propulsion Models

The aerodynamic model of a Boeing 767-300ER considered for the numerical applications is described next. From the definition of drag coefficient C_D and lift coefficient C_L one has $D = \frac{1}{2}\rho V^2 S C_D(C_L, M)$ and $C_L = \dfrac{mg}{\frac{1}{2}\rho V^2 S}$, where the equation of motion $L = mg$ has been taken into account (ρ is the air density and S =283.3 m^2 the reference wing surface area). The aerodynamic model defines the drag polar $C_D = C_D(C_L, M)$ which is given by (see Cavcar and Cavcar [14])

$$C_D = \left(C_{D_{0,i}} + \sum_{j=1}^{5} k_{0j} K^j (M) \right) + \left(C_{D_{1,i}} + \sum_{j=1}^{5} k_{1j} K^j (M) \right) C_L + \left(C_{D_{2,i}} + \sum_{j=1}^{5} k_{2j} K^j (M) \right) C_L^2 \tag{11}$$

where $M = V/a$ is the Mach number (a being the speed of sound) and

$$K(M) = \frac{(M - 0.4)^2}{\sqrt{1 - M^2}} \tag{12}$$

The incompressible drag polar coefficients are $C_{D_{0,i}} = 0.01322$, $C_{D_{1,i}} = -0.00610$, $C_{D_{2,i}} = 0.06000$, and the compressible coefficients are given in Table 1.

Table 1: Compressible drag-polar coefficients for the model aircraft

j	1	2	3	4	5
k_{0j}	0.0067	−0.1861	2.2420	−6.4350	6.3428
k_{1j}	0.0962	−0.7602	−1.2870	3.7925	−2.7672
k_{2j}	−0.1317	1.3427	−1.2839	5.0164	0.0000

The propulsion model defines the thrust available and the specific fuel consumption. The maximum thrust is defined by (see Torenbeek [15])

$$T_M = W_{TO} \delta C_T \tag{13}$$

where W_{TO} is the reference take-off weight, $\delta = p/p_{SL}$ is the pressure ratio (p_{SL} being the ISA pressure at sea level), and the thrust coefficient is given by (see Mattingly et al. [16] and Barman and Erzberger [17])

$$C_T = \frac{T_{SL}}{W_{TO}} \left(1 + \frac{\kappa - 1}{2} M^2 \right)^{\frac{\kappa}{\kappa - 1}} \left(1 - 0.49\sqrt{M} \right) \frac{1}{\theta} \tag{14}$$

where $\kappa = 1.4$, the maximum thrust at sea level and for $M = 0$ is $T_{SL} = 5.0 \times 10^5$ N, and $\theta = \Theta/\Theta_{SL}$ is the temperature ratio (Θ_{SL} being the ISA temperature at sea level).

The specific fuel consumption is defined by (see Torenbeek [15])

$$c = \frac{a_{SL}\sqrt{\theta}}{L_H} C_C(M) \tag{15}$$

with the specific fuel consumption coefficient given by (see Mattingly et al. [16])

$$C_C = c_{SL} \frac{L_H}{a_{SL}} (1.0 + 1.2M) \tag{16}$$

where a_{SL} is the ISA speed of sound at sea level, the specific fuel consumption at sea level and for $M = 0$ is $c_{SL} = 9.0 \times 10^{-6}$ kg/(s N), and the fuel latent heat is $L_H = 43 \times 10^6$ J/kg. In general, C_C is a function of M and the thrust coefficient C_T, however, the dependence of C_C with C_T is in practice very weak and can be neglected (see Torenbeek [15]).

References

[1] Gill, W., and R. Maddock. 1997. EFMS prediction of optimal 4D trajectories in the presence of time and altitude constraints. PHARE DOC 97-70-09, EUROCONTROL.

[2] Wu, D., and Y.J. Zhao. 2009. Performances and sensitivities of optimal trajectory generation for air traffic control automation. AIAA 2009-6167: 1-22.

[3] Rivas, D., A. Valenzuela, and J.L. de Augusto. 2013. Computation of global trajectories of commercial transport aircraft. Proc. IME, part G, Journal of Aerospace Engineering 227(1):141-157.

[4] Fletcher, R. 1987. Practical methods of optimization. John Wiley & Sons, Ltd., Chichester, England.

[5] Bell, D.J., and D.H. Jacobson. 1975. Singular optimal control problems. Academic Press, New York.

[6] Franco, A., and D. Rivas. 2011. Minimum-cost cruise at constant altitude of commercial aircraft including wind effects. Journal of Guidance, Control, and Dynamics 34(4):1253-1260.

[7] Franco, A., D. Rivas, and A. Valenzuela. 2012. Optimization of unpowered descents of commercial aircraft in altitude-dependent winds. Journal of Aircraft 49(5):1460-1470.

[8] Jackson, M.R., Y. Zhao, and R.A. Slattery. 1999. Sensitivity of trajectory prediction in air traffic management. Journal of Guidance, Control, and Dynamics 22(2):219-228.

[9] Bryson, A.E., and Yu-Chi Ho. 1975. Applied optimal control. Hemisphere Publishing Corporation, Washington DC.

[10] Ben-Asher, J.Z. 2010. Optimal control theory with aerospace applications. AIAA Education Series, Reston, VA.

[11] Kelley, H.J., R.E. Kopp, and H.G. Moyer. 1967. Singular extremals. In: Topics in optimization. Edited by G. Leitman. Academic Press New York.

[12] McDanell, J.P., and W.F. Powers. 1971. Necessary conditions for joining optimal singular and nonsingular subarcs. SIAM Journal on Control 9(2):161-173.

[13] Asselin, M. 1997. An introduction to aircraft performance. AIAA Education Series, Reston, VA.

[14] Cavcar, A., and M. Cavcar. 2004. Approximate solutions of range for constant altitude-constant high subsonic speed flight of transport aircraft. Aerospace Science and Technology 8:557-567.

[15] Torenbeek, E. 1997. Cruise performance and range prediction reconsidered. Progress in Aerospace Sciences 33:285-321.

[16] Mattingly, J.D., W.H. Heiser, and D.T. Pratt. 2002. Aircraft engine design, 2nd edition. AIAA Education Series, Reston, VA.

[17] Barman, J.F., and H. Erzberger. 1976. Fixed-range optimum trajectories for short-haul aircraft. Journal of Aircraft 10:748-754.