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Improved delay-dependent stability criterion for uncertain networked control systems with induced time-varying delays ⋆

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Abstract: In this paper, a new stability criterion for uncertain systems controlled over a communication network is derived. Bounds on the network induced time-varying delays are assumed to be known, and no restriction on the derivative of the time-varying delay is imposed allowing for fast time-varying delay profiles. Both, L_2-bounded disturbances and norm-bounded, possibly time-varying, uncertainties are considered. Stability conditions are obtained based on the Lyapunov-Krasovskii approach, considering a new treatment of time-varying delays resorting to a polytopic covering of the delay interval. Examples are provided to demonstrate the reduced conservatism of the proposed conditions with respect to available works in the literature.

Keywords: Networked control systems, Time-varying delay, Robust control, Lyapunov-Krasovskii, L_2 disturbance.

1. INTRODUCTION

With the development of network technologies, increasing interest is being devoted to the analysis and design of networked control systems. In these control applications, serial communication networks are used to exchange information between the components, generally geographically distributed.

Although the notion of networked control systems (NCSs) is relatively new, it has captured the attention of many researchers due to the huge range of potential applications of these technologies and the challenging control problems it arises. NCSs have opened up a whole new area of real-world applications, namely, unmanned vehicles, telerobotics, e.g., long-distance telesurgery applications, Conway et al. (1990), intelligent traffic control, Varuya (1998), etc.

Nonetheless, closing a control loop over a communication network has to deal with the problem of unreliable, bandwidth limited communication links, see Zampieri (2008). Data transmissions in communication networks unavoidably introduce time delays and packet losses in the control loops, see Canudas de Wit (2006). These problems have motivated the study of stability and stabilization of systems with network induced time-varying delay and packet dropouts.

Most of recent works employ delay-dependent conditions to ensure the stability of time-delay systems. These delay-dependent conditions introduce the information of the bounds of the delay in their formulation and get better results that the delay-independent approaches. The first researches on this field supposed that the delay was constant but unknown, Park (1999), Moon et al. (2001).

However, there are many applications, including NCSs, in which the time delays are in general time-varying. In such cases, some authors have proposed delay-dependent conditions using the upper bound of the delay, Fridman and Shaked (2002), Jiang and Han (2006), He et al. (2007), Shao (2009). Recently, a growing number of works have proposed the use of the information of the lower bound of the delay, Jiang and Han (2006), Shao (2009). Also, the information of the derivative of the time-varying delay has been introduced, Fridman and Shaked (2002), He et al. (2007), Shao (2009). They show that it is possible to improve the results if this information is added to the functional.

In Fridman et al. (2004) is presented a novel input delay approach which allows to consider the effect of sampling as a time delay. With this approach, the ideas and techniques of time-delay system can be easily applied to NCS. In the aforementioned work, the stability for uncertain systems is also investigated via polytopic description. Yue et al. (2005) developed a new stability criteria and controller design method for systems with parametric uncertainties, using the information of the lower bound on the time delay to reduce the conservatism. Moreover, in Naghshtabrizi et al. (2007), resorting to concepts from infinite-dimensional impulsive systems, a new Lyapunov-Krasovskii functional is derived obtaining similar results. More recently, in Jiang et al. (2008), the upper bound of the delay for which the system remains stable and the H_∞ disturbance rejection has been improved avoiding the use of *slack matrices*. As shown in Peaucelle and Gouaisbaut (2005) the use of free weighting matrices does reduce the conservatism only when polytopic parameter-dependent Lyapunov functions are sought.

In this paper, we are concerned with the analysis and design of robust H_∞ controllers for uncertain networked control systems. We present an improved stability criterion for NCS that makes use of a Lyapunov-Krasovskii functional, providing stability conditions in terms of a set of linear matrix inequalities (LMIs).

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We introduce a continuous-time domain model in which the time-varying delays, sampling effects and packet dropouts are considered via the input delay approach. We take into account both the upper and lower bounds on the time delay in our delay-dependent condition.

The paper proposes a different treatment of the time-varying delays, resorting to a polytopic covering of the time delay interval, instead of overbounding it with the worst case. In other words, in some terms that naturally arise in the derivation of the Lyapunov-Krasovskii functionals as $\tau(t)x^T(t)Z_1x(t)$ or $(\tau(t)-\tau_m)x^T(t)Z_2x(t)$, $(Z_1, Z_2 > 0)$, we retain $\tau(t)$ instead of substitute it by its upper and lower bounds accordingly. Thus, the obtained LMIs conditions depend explicitly on $\tau(t)$. The use of slack matrices is here justified. To prove asymptotic stability, the LMIs must be feasible simultaneously for the two vertices of the polytopic covering of the time delay. To solve them, the LMI toolbox presented in Gahinet et al. (1994) can be used.

The paper is organized as follows. Section 2 is devoted to the description of the NCS model to be considered. Section 3 is concerned with the main results. Section 3.1 contains a detailed exposition of the stability result for the nominal system case. Systems uncertainties and controller design are developed in section 3.2 and 3.3 respectively. Section 4 presents the obtained results. Conclusions are summarized in Section 5.

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of $n\times m$ real matrices, I is the identity matrix of appropriate dimensions, $\|\cdot\|$ stands for the Euclidean vector norm or the induced matrix 2-norm as appropriate. The notation $X>0$, for $X\in\mathbb{R}^{n\times n}$ means that the matrix X is a real symmetric matrix positive definite. For an arbitrarily real matrix B and two real symmetric matrices A and C, $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ denotes a real symmetric matrix, where $*$ denotes the entries implied by symmetry.

2. NCS MODEL AND PRELIMINARIES

Consider the following system with parametric uncertainties given by:

$$\dot{x}(t) = [A+\Delta A(t)]x(t) + [B+\Delta B(t)]u(t) + B_\omega \omega(t), \quad (1)$$

$$z(t) = Cx(t) + Du(t), \quad (2)$$

$$x(t_0) = x_0, \quad (3)$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$ and $z(t)\in\mathbb{R}^q$ are the state vector, control input vector and controlled output, respectively; $\omega(t)\in L_2[t_0,\infty)$ denotes the external perturbation; A, B, B_ω, C and D are some constant matrices of appropriate dimensions; x_0 denotes the initial conditions; $\Delta A(t)$ and $\Delta B(t)$ denote the parameter uncertainties, which satisfy the following conditions:

$$\Delta A(t) = G_1F_1(t)E_1, \quad (4)$$

$$\Delta B(t) = G_2F_2(t)E_2, \quad (5)$$

where G_i, E_i $(i=1,2)$ are known constant matrices of appropriate dimensions and $F_i(t)$ $(i=1,2)$ are unknown time-varying matrices, which are Lebesque measurable in t and satisfy $F_i^T(t)F_i(t)\le I$.

Consider system (1)-(3) being controlled through a network. The inclusion of such a network, under a linear control law, allows the system to be expressed as a hybrid one, as in Yue et al. (2005). In this approach, the system is modeled in continuous time with a piecewise constant control input, which is updated whenever a new control input reaches the plant. Networked induced delays and packet dropouts are both considered.

An alternative way, which is the one employed in this work, is modeling the system as a continuous system with delayed control input, as was firstly introduced in Mikheev et al. (1988).

Assume that the sensor nodes sample data from the plant in a time-driven manner, at time instants $t=j_kh$, with h being the sampling time, and j_k $(k=1,2,3,..)$ are some integers such that $\{j_1,j_2,j_3,...\}\subset\{1,2,3,...\}$ and $j_k<j_{k+1}$.

Let us define $t\in[t_k,t_{k+1})$ as the time intervals where the control input applied to the system is constant, where t_k is the time instant when the control signal, corresponding to the plant state at $t=j_kh$, reaches the plant.

Therefore, the control input can be written as:

$$u(t) = Kx(t_k - \tau_{sc}(k) - \tau_{ca}(k)), \quad t\in[t_k,t_{k+1}), \quad (6)$$

where $\tau_{sc}(k)$ and $\tau_{ca}(k)$ are the network induced delays of the data corresponding to the measured plant state at $t=j_kh$, from sensor to controller and from controller to actuator, respectively. The round-trip delay $\tau_{sa}(k)$ can also be defined as $\tau_{sa}(k)=\tau_{sc}(k)+\tau_{ca}(k)$.

Thus the controlled system (1)-(3) can be rewritten as:

$$\dot{x}(t) = [A+\Delta A(t)]x(t) + [B+\Delta B(t)]Kx(t-\tau(t)) + B_\omega\omega(t), (7)$$

$$z(t) = Cx(t) + DKx(t-\tau(t)), \quad \forall t\in[t_k,t_{k+1}), \quad (8)$$

$$x(t) = \phi(t), \quad t\in[t_0-\tau_M, t_0], \quad (9)$$

where $\tau(t)=t-t_k+\tau_{sc}(k)+\tau_{ca}(k)$ and τ_M is the maximum allowable delay (see Definition 4). This is the actual model that will be considered throughout the paper. It is easy to check that $\tau(t)$ is piecewise linear in t, as it represents the time difference between the $k-th$ sampling time, j_kh, and current time t. Figure 1 illustrates a possible evolution of $\tau(t)$.

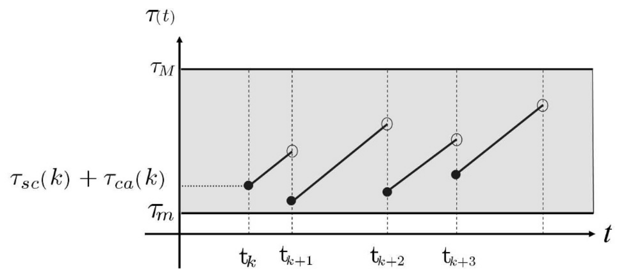

Fig. 1. Qualitative evolution of $\tau(t)$

The following assumptions and definitions will be needed throughout the paper.

Assumption 1. The sensor is clock-driven. The controller and actuator are event-driven.

Assumption 2. Two constants $\underline{\tau}_{sa}, \overline{\tau}_{sa}\ge 0$, exist such that the following inequality holds:

$$\underline{\tau}_{sa} \le \tau_{sa}(k) \le \overline{\tau}_{sa}, \forall k\in\mathbb{N}. \quad (10)$$

Assumption 3. The maximum number of consecutive data dropouts from sensor to actuator is bounded by $n_p\in\mathbb{N}$.

Furthermore, the following definitions are used throughout the paper.

Definition 4. Regarding Assumptions 2 and 3, it is possible to define two constants $\tau_m \geq 0$ and $\tau_M > \tau_m$ such that:

$$\tau(t) \geq \underline{\tau}_{sa} = \tau_m, \tag{11}$$

$$\tau(t) \leq (1+n_p)h + \overline{\tau}_{sa} = \tau_M. \tag{12}$$

Definition 5. System (7)-(9) is said to be robustly asymptotically stable with an H_∞ norm bound γ if the following hold:

(1) System (7)-(9) with $\omega(t) \equiv 0$ is robustly asymptotically stable for all admissible uncertainties $\Delta A(t)$ and $\Delta B(t)$.

(2) Under the assumption of zero initial condition, the controlled output $z(t)$ satisfies $\|z(t)\|_2 \leq \gamma \|\omega(t)\|_2$ for any nonzero $\omega(t) \in L_2[0,\infty)$.

3. MAIN RESULTS.

This section applies stability analysis techniques from time-delay systems to the NCS framework. However, some differences are introduced. In recent works, in terms as $\tau(t)x^T(t)Z_1x(t)$ or $(\tau(t)-\tau_m)x^T(t)Z_2x(t)$, $(Z_1, Z_2 > 0)$, the time delay $\tau(t)$ was substituted by its upper and lower bounds, depending on the worst case. These bounding techniques bring conservatism to the developed criteria. Next, a different way to deal with the time-varying delay is proposed, based on a polytopic description.

First, an LMI-based stability criterion for the nominal system is derived. Additionally, the H_∞ performance of the closed loop system is considered. Finally, the analysis method is extended to uncertain systems and an H_∞ controller synthesis method is developed.

3.1 Stability and H_∞ performance analysis. Nominal case

Consider the problem of analyzing the stability and H_∞ performance of an LTI system subject to time-varying network induced delays and package dropouts. Focusing our attention on the following perturbed system:

$$\dot{x}(t) = Ax(t) + BKx(t-\tau(t)) + B_\omega \omega(t),\ t \in [t_k, t_{k+1}), \tag{13}$$

$$z(t) = Cx(t) + Du(t), \quad t \in [t_k, t_{k+1}), \tag{14}$$

$$x(t_0) = x_0, \tag{15}$$

we define a Lyapunov-Krasovskii functional inspired on the functional in He et al. (2007) and conclude the following result:

Lemma 6. Given scalars $\tau_m, \tau_M, \gamma, \varepsilon > 0$, if there exist matrices $P, Q_1, Q_2, Z_1, Z_2 > 0$ and any matrices N_i, M_i, S_i, $i = 1,2$, of appropriate dimensions such that the following LMI is satisfied for the two vertices of $\tau(t)$, given by (11)-(12),

$$\begin{bmatrix} \Gamma & (\tau(t)+\varepsilon)\bar{N} & (\tau(t)+\varepsilon-\tau_m)\bar{M} & (\tau_M+\varepsilon-\tau(t))\bar{S} & \bar{A}U & \bar{C} \\ * & -(\tau(t)+\varepsilon)Z_1 & 0 & 0 & 0 & 0 \\ * & * & -(\tau(t)+\varepsilon-\tau_m)Z_2 & 0 & 0 & 0 \\ * & * & * & -(\tau_M+\varepsilon-\tau(t))(Z_1+Z_2) & 0 & 0 \\ * & * & * & * & -U & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0, \tag{16}$$

where

$$\Gamma = \begin{bmatrix} \theta_{11} & \theta_{12} & M_1 & -S_1 & PB_\omega \\ * & \theta_{22} & M_2 & -S_2 & 0 \\ * & * & -Q_1 & 0 & 0 \\ * & * & * & -Q_2 & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix},$$

$$\begin{aligned} \theta_{11} &= PA + A^TP + Q_1 + Q_2 + N_1 + N_1^T, \\ \theta_{12} &= PBK - N_1 + S_1 - M_1 + N_2^T, \\ \theta_{22} &= -N_2 - N_2^T + S_2 + S_2^T - M_2 - M_2^T, \\ \Delta\tau &= \tau_M - \tau_m, \\ U &= \tau_M Z_1 + \Delta\tau Z_2, \end{aligned}$$

$$\bar{C}^T = [\,C \ DK \ 0 \ 0 \ 0\,], \qquad \bar{N}^T = \left[\,N_1^T \ N_2^T \ 0 \ 0 \ 0\,\right],$$
$$\bar{A}^T = [\,A \ BK \ 0 \ 0 \ B_\omega\,], \qquad \bar{M}^T = \left[\,M_1^T \ M_2^T \ 0 \ 0 \ 0\,\right],$$
$$\bar{S}^T = \left[\,S_1^T \ S_2^T \ 0 \ 0 \ 0\,\right],$$

then system (13)-(15) with a control network satisfying Assumptions 1-3 is asymptotically stable with an H_∞ norm bound γ.

Proof. Construct the following Lyapunov-Krasovskii functional:

$$V(t) = x^T(t)Px(t) + \int_{t-\tau_m}^{t} x^T(s)Q_1x(s)ds + \int_{t-\tau_M}^{t} x^T(s)Q_2x(s)ds + \int_{-\tau_M}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z_1\dot{x}(s)dsd\theta + \int_{-\tau_M}^{-\tau_m}\int_{t+\theta}^{t} \dot{x}^T(s)Z_2\dot{x}(s)dsd\theta. \tag{17}$$

Taking the time derivative of $V(t)$ along the trajectory of (13) yields that, for $t \in [t_k, t_{k+1})$:

$$\begin{aligned} \dot{V}(t) &= 2x^T(t)P\dot{x}(t) + x^T(t)(Q_1+Q_2)x(t) - x^T(t-\tau_m)Q_1x(t-\tau_m) \\ &- x^T(t-\tau_M)Q_2x(t-\tau_M) + \dot{x}^T(t)(\tau_M Z_1 + \Delta\tau Z_2)\dot{x}(t) \\ &- \int_{t-\tau_M}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds - \int_{t-\tau_M}^{t-\tau_m} \dot{x}^T(s)Z_2\dot{x}(s)ds. \end{aligned} \tag{18}$$

Integral terms in (18) can be rewritten as follows:

$$\begin{aligned} \int_{t-\tau_M}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds &= \int_{t-\tau_M}^{t-\tau(t)} \dot{x}^T(s)Z_1\dot{x}(s)ds + \int_{t-\tau(t)}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds, \\ \int_{t-\tau_M}^{t-\tau_m} \dot{x}^T(s)Z_2\dot{x}(s)ds &= \int_{t-\tau_M}^{t-\tau(t)} \dot{x}^T(s)Z_2\dot{x}(s)ds + \int_{t-\tau(t)}^{t-\tau_m} \dot{x}^T(s)Z_2\dot{x}(s)ds. \end{aligned} \tag{19}$$

The following null terms are added to the right hand side of (18):

$$0 = 2[x^T(t)N_1 + x^T(t-\tau(t))N_2]\left[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{x}(s)ds\right], \tag{20}$$

$$0 = 2[x^T(t)S_1 + x^T(t-\tau(t))S_2]\left[x(t-\tau(t)) - x(t-\tau_M) - \int_{t-\tau_M}^{t-\tau(t)} \dot{x}(s)ds\right],$$

$$0 = 2[x^T(t)M_1 + x^T(t-\tau(t))M_2]\left[x(t-\tau_m) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t-\tau_m} \dot{x}(s)ds\right],$$

$$0 = \gamma^2\omega^T(t)\omega(t) - \gamma^2\omega^T(t)\omega(t),$$

$$0 = [Cx(t) + DKx(t-\tau(t))]^T[Cx(t) + DKx(t-\tau(t))] - z^T(t)z(t).$$

Defining the augmented state as:

$$\xi^T(t) = \left[\,x^T(t) \ x^T(t-\tau(t)) \ x^T(t-\tau_m) \ x^T(t-\tau_M) \ \omega(t)\,\right],$$

equation (18) can be rewritten as:

$$\begin{aligned} \dot{V}(t) &= \xi^T(t)(\Gamma + \bar{C}\bar{C}^T)\xi(t) + \dot{x}^T(t)(\tau_M Z_1 + \Delta\tau Z_2)\dot{x}(t) + \\ &- \int_{t-\tau(t)}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds - 2\xi^T(t)\bar{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds \\ &- \int_{t-\tau(t)}^{t-\tau_m} \dot{x}^T(s)Z_2\dot{x}(s)ds - 2\xi^T(t)\bar{M}\int_{t-\tau(t)}^{t-\tau_m} \dot{x}(s)ds \\ &- \int_{t-\tau_M}^{t-\tau(t)} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)ds - 2\xi^T(t)\bar{S}\int_{t-\tau_M}^{t-\tau(t)} \dot{x}(s)ds \\ &+ \gamma^2\omega^T(t)\omega(t) - z^T(t)z(t). \end{aligned} \tag{21}$$

Now, using the well-known upper bound for the inner product of two vectors:

$$-2b^T a - a^T X a \le b^T X^{-1} b, \quad X > 0, \tag{22}$$

and introducing a positive scalar $\varepsilon > 0$, the following upper bounds for the integral terms in (21) can be found:

$$-\int_{t-\tau(t)}^{t} \dot{x}^T(s) Z_1 \dot{x}(s) ds - 2z^T(t)\bar{N}\int_{t-\tau(t)}^{t} \dot{x}(s) ds \tag{23}$$
$$\le (\tau(t)+\varepsilon) z^T(t)\bar{N}Z_1^{-1}\bar{N}^T z(t),$$
$$-\int_{t-\tau(t)}^{t-\tau_m} \dot{x}^T(s) Z_2 \dot{x}(s) ds - 2z^T(t)\bar{M}\int_{t-\tau(t)}^{t-\tau_m} \dot{x}(s) ds \tag{24}$$
$$\le (\tau(t)+\varepsilon-\tau_m) z^T(t)\bar{M}Z_2^{-1}\bar{M}^T z(t),$$
$$-\int_{t-\tau_M}^{t-\tau(t)} \dot{x}^T(s)(Z_1+Z_2)\dot{x}(s) ds - 2z^T(t)\bar{S}\int_{t-\tau_M}^{t-\tau(t)} \dot{x}(s) ds \tag{25}$$
$$\le (\tau_M+\varepsilon-\tau(t)) z^T(t)\bar{S}(Z_1+Z_2)^{-1}\bar{S}^T z(t).$$

Next, instead of substituting $\tau(t)$ by different bounds, we retain it. Then, combining (21) with (23)-(25), we can show that, for $t \in [t_k, t_{k+1})$,

$$\begin{aligned}\dot{V}(t) \le\ & \xi^T(t)(\Gamma + (\tau(t)+\varepsilon)\bar{N}Z_1^{-1}\bar{N}^T + (\tau(t)+\varepsilon-\tau_m)\bar{M}Z_2^{-1}\bar{M}^T \\ & + (\tau_M+\varepsilon-\tau(t))\bar{S}(Z_1+Z_2)^{-1}\bar{S}^T + \bar{C}\bar{C}^T + \bar{A}U\bar{A}^T)\xi(t) \\ & + \gamma^2\omega^T(t)\omega(t) - z^T(t)z(t),\end{aligned} \tag{26}$$

By Schur complement it is easy to see from (26) that if (16) holds then

$$\dot{V}(t) \le -z^T(t)z(t) + \gamma^2\omega^T(t)\omega(t). \tag{27}$$

To deduce asymptotic stability the external perturbation $\omega(t)$ is assumed to be zero. From (27) one can obtain that $V(t)$ decreases for $t \in [t_k, t_{k+1})$. Since $V(t)$ is continuous in $[t_0, \infty)$, due to the continuity of $x(t)$ in t, then $\dot{V}(x_t) \le -\rho\|x(t)\|^2$ for a sufficient small $\rho > 0$, which ensure asymptotic stability of system (13)-(15), see e.g. Hale and Verduyn Lunel (1993).

Next, the H_∞ disturbance rejection is proved. In this case the external perturbations are not assumed to be zero. Integrating both sides of (27) from t_k to $t \in [t_k, t_{k+1})$, yields

$$V(t) - V(t_k) \le -\int_{t_k}^{t} z^T(s)z(s)ds + \int_{t_k}^{t} \gamma^2\omega^T(s)\omega(s)ds. \tag{28}$$

Since $\bigcup_{k=1}^{\infty}[t_k, t_{k+1}) = [t_0, \infty)$ and $V(t)$ is continuous in t since $x(t)$ is continuous in t, one can see that

$$V(t) - V(t_0) \le -\int_{t_0}^{t} z^T(s)z(s)ds + \int_{t_0}^{t} \gamma^2\omega^T(s)\omega(s)ds. \tag{29}$$

Under zero initial condition, the following yields $V(t_0) \equiv 0$. Moreover, $V(t)$ is always, by definition, greater or equal than zero. Then, letting $t \to \infty$ it can be shown that

$$\int_{t_0}^{\infty} z^T(s)z(s)ds \le \int_{t_0}^{\infty} \gamma^2\omega^T(s)\omega(s)ds, \tag{30}$$

thus $\|z(t)\|_2 \le \gamma\|\omega(t)\|_2$.

□

Remark 7. It is necessary to solve the LMI in each vertex of the polytopic covering of the time delay to guarantee the asymptotic stability of the system. The scalar parameter $\varepsilon > 0$ needs to be introduced in order to make the LMIs feasible. Otherwise, some null matrices appears in the diagonal of the LMIs. It is worth to mention that this modification does not introduce more conservatism, since $\varepsilon > 0$ can be chosen as small as necessary, i.e., $\varepsilon \to 0^+$.

Given network conditions and an LTI system, Lemma 6 can be used to check the asymptotic stability and H_∞ disturbance attenuation of the closed loop dynamics. However, it does not offer any guarantees on the stability of uncertain systems. Models of real systems are always affected by modeling errors. Therefore, to provide robust analysis methods applicable to real systems, uncertainties need to be considered. In the following, the stability criterion is extended in order to consider uncertain systems.

3.2 Robust stability and H_∞ analysis of uncertain systems

Aiming at increasing the practical applicability of the previous result, the method will be extended to uncertain systems. Therefore, system (7)-(9) described in Section 2 will be considered. The stability of this system can be studied by applying the following theorem.

Theorem 8. Given scalars $\tau_m, \tau_M, \gamma, \varepsilon > 0$, if there exist matrices $P, Q_1, Q_2, Z_1, Z_2 > 0$, any matrices N_i, M_i, S_i $(i = 1,2)$ of appropriate dimensions and scalars $e_1, e_2 > 0$ such that the following LMI is satisfied for the two vertices of $\tau(t)$, given by (11)-(12),

$$\begin{bmatrix} \Pi & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 \\ * & -e_1 I & 0 & 0 & 0 \\ * & * & -e_1 I & 0 & 0 \\ * & * & * & -e_2 I & 0 \\ * & * & * & * & -e_2 I \end{bmatrix} < 0, \tag{31}$$

where Π is the matrix required to be negative definite by (16) and

$$\begin{aligned}\alpha_1 &= [G_1^T P \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad G_1^T U]^T;\\ \alpha_2 &= [G_2^T P \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad G_2^T U]^T;\\ \beta_1 &= [e_1 E_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T;\\ \beta_2 &= [0 \quad e_2 E_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T;\end{aligned}$$

then, system (7)-(9) with a control network satisfying Assumptions 1-3 is robustly asymptotically stable for all admisible uncertainties $\Delta A(t), \Delta B(t)$, with an H_∞ norm bound γ.

Proof. Substituting A and B for $A + \Delta A(t)$ and $B + \Delta B(t)$ respectively in (16), taking into account equations (4) and (5), $\Pi < 0$ in (16) can be written as

$$\Pi + \alpha_1 F_1(t)\beta_1^{*T} + \beta_1^* F_1^T(t)\alpha_1^T + \alpha_2 F_2(t)\beta_2^{*T} + \beta_2^* F_2^T(t)\alpha_2^T < 0, \tag{32}$$

where

$$\beta_i^* = \frac{1}{e_i}\beta_i, \quad i = 1,2.$$

By Lemma 2.4 in Xie (1996), the conditions above hold if and only if there exists some scalars $\lambda_1, \lambda_2 > 0$ such that

$$\Pi+\lambda_1\alpha_1\alpha_1^T+\frac{1}{\lambda_1}\beta_1^*\beta_1^{*T}+\lambda_2\alpha_2\alpha_2^T+\frac{1}{\lambda_2}\beta_2^*\beta_2^{*T}<0. \quad (33)$$

Using Schur complements and naming $e_i=\frac{1}{\lambda i}$, $i=1,2$, yields (31).

□

3.3 Robust H_∞ controller design

Since controller K appears in (31) multiplying variable matrices of the LMI, Theorem 8 can not be directly used to controller design. For this purpose, Theorem 8 is modified in order to design the robust feedback controller K that makes the system robustly asymptotically stable with an H_∞ norm bound γ.

Theorem 9. Given scalars $\rho_1>0,\rho_2>0$ and scalars τ_m,τ_M,γ, $\varepsilon>0$. If there exist matrices $X,\tilde{Q}_1,\tilde{Q}_2>0$, any matrices $Y,\tilde{N}_i,\tilde{M}_i,\tilde{S}_i$, $i=1,2$, of appropriate dimensions and scalars $\mu_1,\mu_2>0$, such that the LMI (34) is satisfied for the two vertices of $\tau(t)$, given by (11)-(12), then, under the controller system $u(t)=Kx(t)$, with $K=YX^{-1}$, system (7)-(9) with a control network satisfying Assumptions 1-3 is robustly asymptotically stable for all admisible uncertainties $\Delta A(t),\Delta B(t)$ with an H_∞ norm bound γ.

Proof. Define $P=Z_0,Z_1=\rho_1Z_0,Z_2=\rho_2Z_0,\rho_1,\rho_2>0$ in (31) and denote it (31)'. Obviously, $Z_0>0$. Pre- and post-multiplying both sides of (31)' with $diag[X,X,X,X,I,X,X,X,X,X,I,I,I,I]$ and its transpose, where $X=Z_0^{-1}>0$, and introducing new variables $\tilde{Q}_i=XQ_iX,\tilde{M}_i=XM_iX,\tilde{N}_i=XN_iX,\tilde{S}_i=XS_iX,\mu_i=1/e_1$, $i=1,2$, we can obtain (34) by Schur complement. It is easy to see that (34) implies (31)'. Therefore, from Theorem 9, we can complete the proof.

□

Solving LMI (34), a feasible feedback controller can be obtained. Using available computational software it is possible to find the controller that minimizes the disturbance rejection factor γ.

4. NUMERICAL EXAMPLES

4.1 Example 1

Consider the following system, from Zhang et al. (2001).

$$\dot{x}(t)=\begin{bmatrix}0&1\\0&-0.1\end{bmatrix}x(t)+\begin{bmatrix}0\\0.1\end{bmatrix}u(t). \quad (35)$$

To compare our result with another works, we will employ the same feedback controller as in Zhang et al. (2001), that is, $u(t)=[\,-3.75\ \ -11.5\,]$. The maximum delays τ_M that guarantee the stability of system (35) controlled over a network are given in Table 1 for different methods. It is assumed that $\tau_m=0$.

Table 1. Maximum delay for different methods

Method	τ_M
Zhang et al. (2001)	4.5×10^{-4}
Yue et al. (2005)	0.8871
Naghshtabrizi et al. (2007)	<0.8871
Jiang et al. (2008)	1.0081
Lemma 6	1.0432

In this example there are not disturbances, so the factor γ can be chosen as big as necessary. One can observe that our method is less conservative than others in the literature.

If we consider the effect of the external perturbation on the system, (35) can be described by,

$$\dot{x}(t)=\begin{bmatrix}0&1\\0&-0.1\end{bmatrix}x(t)+\begin{bmatrix}0\\0.1\end{bmatrix}u(t)+\begin{bmatrix}0.1\\0.1\end{bmatrix}\omega(t).$$
$$z(t)=[\,0\ \ 1\,]x(t)+0.1u(t). \quad (36)$$

Next, we consider the H_∞ performance of system (36) under the given controller. For the case of $\tau_m=0$ and $\tau_M=0.8695$, the value of γ_{min} for different methods are given in Table 2.

Table 2. Disturbance attenuation obtained for different methods

Method	γ_{min}
Yue et al. (2005)	6.82
Jiang et al. (2008)	1.0005
Lemma 6	0.8724

We obtain better values of γ_{min} than other methods existing in the literature.

4.2 Example 2

Consider the following uncertain system controlled over a network, Yue et al. (2005),

$$\dot{x}(t)=\left(\begin{bmatrix}-1&0&-0.5\\1&-0.5&0\\0&0&0.5\end{bmatrix}+\Delta A(t)\right)x(t)+\begin{bmatrix}0\\0\\1\end{bmatrix}u(t)+\begin{bmatrix}1\\1\\1\end{bmatrix}\omega(t).$$
$$z(t)=[\,0\ \ 1\,]x(t)+0.1u(t). \quad (37)$$

where $\|\Delta A(t)\|\leq0.01$. With the controller given by $u(t)=[\,-0.5425\ \ -0.0014\ \ -1.3858\,]x(t)$ and $\tau_M=0.5$, $\tau_m=0.1$, in Yue et al. (2005) is found a $\gamma_{min}=1.9$. For the same situation, in Jiang et al. (2008) is found a $\gamma_{min}=1.6242$. Using Theorem 8 is found a $\gamma_{min}=1.6246$.

With a controller given by $u(t)=[\,-0.6085\ \ -0.0072\ \ -1.4456\,]$ $x(t)$, in Jiang et al. (2008) is found a $\gamma_{min}=1.62$. Using (31), we find $\gamma_{min}=1.6163$. We obtain for this example a similar disturbance rejection.

4.3 Example 3

Consider the following system,

$$\dot{x}(t)=\begin{bmatrix}0&1\\-1&-2\end{bmatrix}x(t)+\begin{bmatrix}0\\1\end{bmatrix}[\,-1\ \ 1\,]x(t). \quad (38)$$

The admissible upper bound of the delay τ_M, which guarantee the asymptotic stability of the system, are listed in Table 3 with given τ_m. Again, for this example the disturbance are not considered, so the value of γ can be chosen freely.

From Table 3, it can be seen that our stability results are less conservative than those in Jiang and Han (2006), He et al. (2007) and Shao (2009).

$$
\begin{bmatrix}
\Upsilon & (\tau(t)+\varepsilon)\tilde{N} & (\tau(t)+\varepsilon-\tau_m)\tilde{M} & (\tau_M+\varepsilon-\tau(t))\tilde{S} & \bar{\alpha} & \tilde{C} & \bar{E}_1 & \bar{E}_2 \\
* & -(\tau(t)+\varepsilon)\rho_1 X & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -(\tau(t)+\varepsilon-\tau_m)\rho_2 X & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -(\tau_M+\varepsilon-\tau(t))(\rho_1+\rho_2)X & 0 & 0 & 0 & 0 \\
* & * & * & * & \alpha_{99} & 0 & 0 & 0 \\
* & * & * & * & * & -I & 0 & 0 \\
* & * & * & * & * & * & -\mu_1 I & 0 \\
* & * & * & * & * & * & * & -\mu_2 I
\end{bmatrix} < 0, \tag{34}
$$

$$
\text{where} \quad \Upsilon = \begin{bmatrix}
\alpha_{11} & \alpha_{12} & \tilde{M}_1 & -\tilde{S}_1 & B_\omega \\
* & \alpha_{22} & \tilde{M}_2 & -\tilde{S}_2 & 0 \\
* & * & -\tilde{Q}_1 & 0 & 0 \\
* & * & * & -\tilde{Q}_2 & 0 \\
* & * & * & * & -\gamma^2 I
\end{bmatrix},
$$

$$\tilde{N}^T = [\tilde{N}_1^T \quad \tilde{N}_2^T \quad 0 \quad 0 \quad 0];$$

$$\tilde{M}^T = [\tilde{M}_1^T \quad \tilde{M}_2^T \quad 0 \quad 0 \quad 0];$$

$$\tilde{S}^T = [\tilde{S}_1^T \quad \tilde{S}_2^T \quad 0 \quad 0 \quad 0];$$

$$\bar{\alpha}^T = [\alpha_{19}^T \quad \alpha_{29}^T \quad 0 \quad 0 \quad \alpha_{59}^T];$$

$$\tilde{C}^T = [CX^T \quad DY \quad 0 \quad 0 \quad 0];$$

$$\bar{E}_1^T = [E_1 X^T \quad 0 \quad 0 \quad 0 \quad 0];$$

$$\bar{E}_2^T = [0 \quad E_2 Y \quad 0 \quad 0 \quad 0];$$

$$\alpha_{11} = AX + XA^T + \tilde{Q}_1 + \tilde{Q}_2 + \tilde{N}_1 + \tilde{N}_1^T + \mu_1 D_1 D_1^T + \mu_2 D_2 D_2^T$$

$$\alpha_{12} = BY - \tilde{N}_1 + \tilde{S}_1 - \tilde{M}_1 + \tilde{N}_2^T$$

$$\alpha_{19} = (\tau_M \rho_1 + \Delta\tau\rho_2)(XA^T + \mu_1 D_1 D_1^T + \mu_2 D_2 D_2^T)$$

$$\alpha_{22} = -\tilde{N}_2 - \tilde{N}_2^T + \tilde{S}_2 + \tilde{S}_2^T - \tilde{M}_2 - \tilde{M}_2^T$$

$$\alpha_{29} = (\tau_M \rho_1 + \Delta\tau\rho_2) Y^T B^T$$

$$\alpha_{59} = (\tau_M \rho_1 + \Delta\tau\rho_2) B_\omega^T$$

$$\alpha_{99} = -(\tau_M \rho_1 + \Delta\tau\rho_2)X + (\tau_M \rho_1 + \Delta\tau\rho_2)^2(\mu_1 D_1 D_1^T + \mu_2 D_2 D_2^T)$$

Table 3. Maximum delay for various τ_m

Method	τ_m	0.3	0.5	0.8	1
Jiang and Han (2006)	τ_M	0.91	1.07	1.33	1.50
He et al. (2007)	τ_M	0.9431	1.0991	1.3476	1.5187
Shao (2009)	τ_M	1.0715	1.2191	1.4539	1.6169
Lemma 6	τ_M	1.1986	1.3399	1.5697	1.7298

5. CONCLUSIONS

The stability and disturbance attenuation problem have been investigated using a Lyapunov-Krasovskii functional. By covering time-varying delay with a polytope, less conservative LMI criteria can be obtained. Some examples are given to illustrate the reduced conservatism of this result.

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