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Nonlinear $\mathscr{H}_\infty$ Controller for the Quad-Rotor Helicopter with Input Coupling

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Abstract: This paper presents a nonlinear $\mathscr{H}_\infty$ control law for underactuated mechanical systems with input coupling. Apart from the controlled degrees of freedom (DOF), the proposed controller considers the dynamics of the remaining DOF in the cost variable, which allows to stabilize them. The underactuated mechanical system is normalized to obtain a diagonal inertia matrix allowing to weigh with various criteria different DOF. This controller is applied to the quadrotor helicopter to perform path tracking, whose mechanical structure has been modified in order to obtain coupling between longitudinal and lateral movements with roll and pitch motions. Simulation results in presence of aerodynamic disturbances, structural and parametric uncertainties are presented to corroborate the effectiveness and the robustness of the proposed controller.

Keywords: Nonlinear $\mathscr{H}_\infty$ control, Robust control, Flying robots, Guidance navigation and control

1. INTRODUCTION

Control design for underactuated mechanical systems is a great challenge to the automatic area. This problem is considerably increased due to uncertainties, that are usually present and sometimes significant. The sources of uncertainties can be unmodeled dynamics, exogenous disturbances, parameter estimation errors and noise. Thus, apart from the difficulty of controlling underactuated mechanical systems because they have fewer control inputs than degrees of freedom, an additional question is whether the proposed control law possesses desirable rejection properties even if perfect models are not assumed available.

To deal with system imperfections on the control design stage, an usual approach is the $\mathscr{H}_\infty$ control theory [van der Schaft 2000], which its aim is to achieve a bounded ratio between the energy of the cost variable and the energy of the external disturbance signals. The nonlinear $\mathscr{H}_\infty$ approach uses the L_2-gain as an extension of the $\mathscr{H}_\infty$-norm for linear systems. In a general case, the problem leads to a Hamilton-Jacobi partial differential equation (HJ PDE). However, the main problem in this approach is the absence of a general method to solve this HJ PDE. Hence, solutions have to be found for a particular case. By applying game theory to formulate the nonlinear $\mathscr{H}_\infty$ control, a constant gain similar to the results obtained with feedback linearization procedures is provided by an analytical solution. An explicit global parameterized solution to this problem, formulated as a minimax game, was developed in Chen et al. [1994] for the particular case of mechanical systems formulated via Euler-Lagrange equations, which was modified in Ortega et al. [2005].

In Siqueira and Terra [2004] a nonlinear $\mathscr{H}_\infty$ control for underactuated manipulators, as extension of the one proposed by Chen et al. [1994], was presented. Nonetheless, these results present some main restrictions, like, for example, the assumption of null-average disturbances and an exact robot model. In Raffo et al. [2007] this controller was modified taking into account the error vector with the integral term. But these nonlinear $\mathscr{H}_\infty$ controllers for underactuated mechanical systems continue to have restrictions, since the control law design considers only the same quantity of controlled DOF as control inputs they have, and the remaining ones are assumed to be a stable zero dynamic, or they are controlled in an outer-loop.

Furthermore, as stated in Chen et al. [1994], the standard formulation of the nonlinear $\mathscr{H}_\infty$ control for Euler-Lagrange mechanical systems used, for example, in Feng and Postlethwaite [1994], Ortega et al. [2005], Siqueira and Terra [2004], presents a limitation in the way to weigh the cost variable. For its appropriate formulation, some weighting matrices must be considered as some positive real scalars multiplied by the identity matrix.

In this paper a nonlinear $\mathscr{H}_\infty$ control for underactuated mechanical systems with input coupling is developed, which considers the dynamics of the remaining DOF in the cost variable allowing to maintain these coordinates stabilized. More precisely, to reach this behavior the time-derivative of their positions is considered in the error vector, which assures that the speed of the remaining DOF tends to zero when the positions are given by their coupling with the controlled DOF. Moreover, the proposed controller design allows to weigh different dynamics through various values. In the case of this paper, only two dynamics are considered: the controlled and the remaining. The procedure to weigh more dynamics can be obtained through a natural manner.

The proposed controller has been applied to a quadrotor helicopter, which is often used to develop control laws for VTOL (Vertical Take-Off and Landing). The quadrotor helicopter tries to reach a stable hovering and flight using the equilibrium

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forces produced by the four rotors [Castillo et al. 2005]. Nevertheless, these kind of vehicles are underactuated mechanical systems with six degrees of freedom and only four control actions, the four angular velocities of the rotors. Generally, two types of strategies are used to perform path tracking of the quadrotor helicopter. On one hand, the most common structures are cascade control strategies, which normally use an inner control loop for the rotational subsystem combined with an outer loop to control the translational movements [Raffo et al. 2010]. On the other hand, some control structures use an augmented state-space [Mistler et al. 2001], where a double integrator is considered on the altitude control input, which generates coupling between translational and rotational motion.

In this paper, a change on the mechanical model structure of the quadrotor helicopter is proposed to overcome cascade control strategies and an augmented state space. This consists in tilting the rotors toward the origin of the body-fixed frame of a certain angle α. This tilt creates the coupling between longitudinal and lateral motions with the roll and pitch ones. Thus, the proposed nonlinear $\mathscr{H}_\infty$ controller for underactuated mechanical systems with input coupling is synthesized to solve the path tracking problem of this modified quadrotor helicopter.

The remainder of the paper is organized as follows: in Section 2 a description of the quadrotor helicopter model with the tilt angle of the rotors is given. The proposed nonlinear $\mathscr{H}_\infty$ controller for underactuated mechanical systems with input coupling is developed in Section 3. Simulations results are presented in Section 4. Finally, the major conclusions of the work are drawn in Section 5.

2. QUADROTOR HELICOPTER MODELING

This section deals with a simplified modeling of the quadrotor helicopter for controller synthesis purposes. A more complete model is used only to emulate the vehicle during the simulation tests, which will be commented on Section 4.

The helicopter as a rigid body is characterized by a frame linked to it ($\mathscr{B} = \{\vec{x}_B, \vec{y}_B, \vec{z}_B\}$), and with the origin at its center of rotation. The inertial frame $\mathscr{I} = \{\vec{x}, \vec{y}, \vec{z}\}$ is considered fixed with respect to the earth (see Fig. 1).

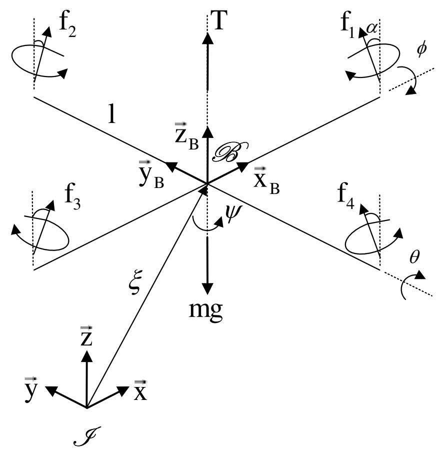

Fig. 1. QuadRotor helicopter scheme.

The vector $\boldsymbol{\xi} = [x \quad y \quad z]' \in \Re^3$ represents the position of the body-fixed frame origin of the helicopter expressed at the origin of the inertial frame $\mathscr{I}$.[1] The vehicle orientation is given by a rotation matrix $\boldsymbol{R}_\mathscr{I} : \mathscr{B} \to \mathscr{I}$, where $\boldsymbol{R}_\mathscr{I} \in SO(3)$ is an orthonormal rotation matrix [Fantoni and Lozano 2002]. In this paper the XYZ Euler angles, $\boldsymbol{\eta} = [\phi \quad \theta \quad \psi]' \in \Re^3$, have been used to describe the helicopter rotation matrix, $\boldsymbol{R}_\mathscr{I}$, with respect to the ground. These angles are bounded as follows: the roll angle, ϕ, by $(-\pi/2 < \phi < \pi/2)$, the pitch angle, θ, by $(-\pi/2 < \theta < \pi/2)$ and the yaw angle, ψ, by $(-\pi < \psi < \pi)$.

The movement of the unmanned aerial vehicle (UAV) results from changes on the lift force caused by adjusting the velocities of the rotors. Longitudinal motions are achieved by means of front and rear rotors velocity, while lateral displacements are performed through the speed of the right and left propellers. Yaw movements are obtained from the difference in the counter-torque between each pair of propellers.

It can be observed in Fig. 1 that the four propellers are tilted toward the origin of the body-fixed frame of the same angle α. This tilt, proposed in this work, provides a certain coupling between lateral and longitudinal movements with the roll and pitch motions, which makes it possible to choose the x and y positions like the controlled variables, without the necessity to employ an augmented state vector nor cascade control strategies. Therefore, the components of the propeller forces projected on $\vec{x}_B$ and $\vec{y}_B$, and the applied thrust, T, (i.e. the propeller forces projected on $\vec{z}_B$) are given by:

$$\boldsymbol{f_a} = \begin{bmatrix} f_{x_a} \\ f_{y_a} \\ f_{z_a} \end{bmatrix} = \begin{bmatrix} \sin(\alpha)(f_3 - f_1) \\ \sin(\alpha)(f_4 - f_2) \\ \left(\sum_{i=1}^{4} \cos(\alpha) f_i\right) \end{bmatrix} = \begin{bmatrix} \sin(\alpha) b \left(\Omega_3^2 - \Omega_1^2\right) \\ \sin(\alpha) b \left(\Omega_4^2 - \Omega_2^2\right) \\ \left(\sum_{i=1}^{4} \cos(\alpha) b \Omega_i^2\right) \end{bmatrix} \tag{1}$$

where f_i is the force generated by the ith rotor, Ω_i is the angular velocity of the ith rotor around its axis and b is the thrust coefficient of the rotors.

The applied torque vector on the three body-fixed axes is given by:

$$\boldsymbol{\tau_a} = \begin{bmatrix} \tau_{\phi_a} \\ \tau_{\theta_a} \\ \tau_{\psi_a} \end{bmatrix} = \begin{bmatrix} (f_2 - f_4) l \cos(\alpha) \\ (f_3 - f_1) l \cos(\alpha) \\ \sum_{i=1}^{4} \tau_{M_i} \cos(\alpha) \end{bmatrix} = \begin{bmatrix} lb\cos(\alpha)\left(\Omega_2^2 - \Omega_4^2\right) \\ lb\cos(\alpha)\left(\Omega_3^2 - \Omega_1^2\right) \\ k_\tau \cos(\alpha)\left(\Omega_1^2 + \Omega_3^2 - \Omega_2^2 - \Omega_4^2\right) \end{bmatrix} \tag{2}$$

where l is the distance between the rotors and the center of rotation, $k_\tau > 0$ is a constant, and τ_{M_i} is the torsion effort generated by each electrical motor considering the dynamic of each disc of the motor as a uncoupled system in the generalized variable Ω_i.

Besides, some assumptions are made to compute the model for control purposes. The ground effect is neglected and the helicopter airframe is assumed to be symmetric, which results in a moment of inertia tensor of the center of the body-fixed frame with just diagonal inertia terms. Moreover, for control purposes, the center of mass and the body-fixed frame origin are assumed congruent.

Under these assumptions, the helicopter motion equations can be obtained by the Euler-Lagrange formalism based on the kinetic and potential energy concept:

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial \mathscr{L}}{\partial \boldsymbol{q}} = \begin{bmatrix} \boldsymbol{\tau_\eta} \\ \boldsymbol{f_\xi} \end{bmatrix} \tag{3}$$

where $\mathscr{L}$ is the Lagrangian of the system. $\boldsymbol{\tau_\eta} = \boldsymbol{\tau_{\eta_a}} + \boldsymbol{\delta_\eta} \in \Re^3$ represents the total roll, pitch and yaw moments expressed

[1] The notation prime $'$ denotes transpose.

in the inertial reference frame, which join the applied torque vector, $\boldsymbol{\tau}_{\boldsymbol{\eta}_a} = \boldsymbol{W}'_{\boldsymbol{\eta}}\boldsymbol{\tau}_{\boldsymbol{a}}$, and the torques generated by the total effect of the error system modeling and external disturbances, $\boldsymbol{\tau}_{\boldsymbol{\eta}_d}$. $\boldsymbol{W}_{\boldsymbol{\eta}}$ is the Euler matrix, which relates the time-derivative of the Euler angles with the angular rates in the body-fixed frame (see Raffo et al. [2010]).

The translational force vector $\boldsymbol{f}_{\boldsymbol{\xi}} = \boldsymbol{R}_{\mathscr{I}}\boldsymbol{f}_{\boldsymbol{a}} + \boldsymbol{\delta}_{\boldsymbol{\xi}} \in \Re^3$ is also divided into two parts: first the term $\boldsymbol{R}_{\mathscr{I}}\boldsymbol{f}_{\boldsymbol{a}}$ constitutes the applied force vector to the helicopter. The second part, $\boldsymbol{\delta}_{\boldsymbol{\xi}}$, combines the parametric uncertainties with the external disturbances. The generalized coordinates of a rigid body evolutioning in a three-dimensional space can be written by $\boldsymbol{q} = [\boldsymbol{\eta}' \quad \boldsymbol{\xi}']' \in \Re^n$, with $n = 6$.

By solving the derivatives required by the Euler-Lagrange equations (3), which have been omitted here in order to avoid an unnecessary explanation, the equations of motion of the quadrotor helicopter result in:

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{B}(\boldsymbol{q})\boldsymbol{\Gamma} + \boldsymbol{\delta}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}},\boldsymbol{\Gamma}_{\boldsymbol{d}}) \qquad (4)$$

$$\begin{bmatrix} \mathscr{J}(\boldsymbol{\eta}) & \mathbb{O} \\ \mathbb{O} & m\mathbb{1} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{\eta}} \\ \ddot{\boldsymbol{\xi}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_{\boldsymbol{\eta\eta}} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \dot{\boldsymbol{\xi}} \end{bmatrix} + \begin{bmatrix} \mathbb{O} \\ mg\boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_{\boldsymbol{\eta}} \\ \boldsymbol{B}_{\boldsymbol{\xi}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau}_a \\ \boldsymbol{f}_a \end{bmatrix} + \begin{bmatrix} \boldsymbol{\delta}_{\boldsymbol{\eta}} \\ \boldsymbol{\delta}_{\boldsymbol{\xi}} \end{bmatrix}$$

where m is the helicopter mass, g is the gravitational acceleration and $\boldsymbol{e}_3$ represents the vector $\boldsymbol{e}_3 = [0\ 0\ 1]'$. $\boldsymbol{M}(\boldsymbol{q})$ is the symmetric positive definite inertia matrix, $\boldsymbol{G}(\boldsymbol{q})$ represents the gravitational forces vector and $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ is the so-called Coriolis and centrifugal forces vector. In this model, the inertia matrix of the rotational subsystem is given by $\mathscr{J}(\boldsymbol{\eta}) = \boldsymbol{W}'_{\boldsymbol{\eta}}\boldsymbol{I}\boldsymbol{W}_{\boldsymbol{\eta}}$, with $\boldsymbol{I}$ being the moment of inertia tensor. $\mathbb{1}$ represents the identity matrix and $\mathbb{O}$ the zero matrix, both with proper dimensions. The Coriolis and centrifugal forces matrix $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ is obtained by using Christoffel symbols and is given by the following expression [Spong et al. 2006]:

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{M}}(\boldsymbol{q}) + \mathscr{N}(\boldsymbol{q},\dot{\boldsymbol{q}})$$

being $\mathscr{N}(\boldsymbol{q},\dot{\boldsymbol{q}})$ skew-symmetric. The force matrix is given by:

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{W}'_{\boldsymbol{\eta}} & \mathbb{O} \\ \mathbb{O} & \boldsymbol{R}_{\mathscr{I}} \end{bmatrix} \qquad (5)$$

However, the vector of the applied forces/torques can be written in terms of the angular velocity of each rotor as follows:

$$\boldsymbol{B}(\boldsymbol{q})\boldsymbol{\Gamma} = \boldsymbol{B}(\boldsymbol{q})\boldsymbol{B}_{\boldsymbol{M}}\boldsymbol{u}_{\boldsymbol{M}} = \boldsymbol{B}_{\mathscr{I}}(\boldsymbol{q})\boldsymbol{u}_{\boldsymbol{M}} = \qquad (6)$$

$$= \begin{bmatrix} \boldsymbol{W}'_{\boldsymbol{\eta}} & \mathbb{O} \\ \mathbb{O} & \boldsymbol{R}_{\mathscr{I}} \end{bmatrix} \begin{bmatrix} 0 & lb\cos(\alpha) & 0 & -lb\cos(\alpha) \\ -lb\cos(\alpha) & 0 & lb\cos(\alpha) & 0 \\ d\cos(\alpha) & -d\cos(\alpha) & d\cos(\alpha) & -d\cos(\alpha) \\ -b\sin(\alpha) & 0 & b\sin(\alpha) & 0 \\ 0 & -b\sin(\alpha) & 0 & b\sin(\alpha) \\ b\cos(\alpha) & b\cos(\alpha) & b\cos(\alpha) & b\cos(\alpha) \end{bmatrix} \begin{bmatrix} \Omega_1^2 \\ \Omega_2^2 \\ \Omega_3^2 \\ \Omega_4^2 \end{bmatrix}$$

where $\boldsymbol{B}_{\mathscr{I}}(\boldsymbol{q})$ is the input coupling matrix that transforms the input signal represented in the body-fixed frame to the inertial frame, and $rank(\boldsymbol{B}_{\mathscr{I}}(\boldsymbol{q})) = n_a < n$, being n_a the number of actuators available in the system.

The angular velocities of the rotors are the control signal generated by the controller designed in this paper.

3. CONTROL LAW DESIGN

As stated at the introduction, the objective of the paper is to synthesize a controller for the quadrotor helicopter without the necessity of dealing with an augmented state-space, nor any cascade control strategy.

Therefore, in this section a control law design based on the nonlinear $\mathscr{H}_\infty$ theory for a class of underactuated mechanical system is proposed. The main objective is to perform path tracking of the controlled degrees of freedom (DOF), while the remaining ones are maintained stabilized. Thus, the controller proposed considers the whole dynamic of the system into its structure. Furthermore, this control law allows to achieve robustness in presence of sustained disturbances, unmodeled dynamics, and parametric and structural uncertainties. An improvement of this controller with respect to ones proposed by Chen et al. [1994], Siqueira and Terra [2004], is the flexibility to weigh different dynamics of the system.

As commented before, the quadrotor helicopter represents an underactuated mechanical system since there are more degrees of freedom than actuators. It is well known that no more than n_a degrees of freedom can be controlled (i.e regulated at an operation point) at each moment by external generalized forces/torques. Thus, the dynamic equations of these kind of systems with n DOF can be partitioned into two components, one corresponding to the controlled generalized coordinates, $\boldsymbol{q}_{\boldsymbol{c}} \in \Re^{n_c}$, and the other to the remaining ones, $\boldsymbol{q}_{\boldsymbol{r}} \in \Re^{n_r}$, where $n = n_c + n_r$. Accordingly, for the case of the quadrotor helicopter, $\boldsymbol{q}_{\boldsymbol{r}} = [\phi \quad \theta]'$ and $\boldsymbol{q}_{\boldsymbol{c}} = [\psi \quad x \quad y \quad z]'$.

Since the helicopter is underactuated, that is, there is only four control actions, $\boldsymbol{u}_{\boldsymbol{M}} = [\Omega_1 \quad \Omega_1 \quad \Omega_3 \quad \Omega_4]'$, but six generalized coordinates, $\boldsymbol{\eta} = [\phi \quad \theta \quad \psi]'$ and $\boldsymbol{\xi} = [x \quad y \quad z]'$, for the suitable partition between controlled and remaining DOF, the system (4) can be written as follows:

$$\begin{bmatrix} \boldsymbol{M}_{rr} & \boldsymbol{M}_{rc} \\ \boldsymbol{M}_{cr} & \boldsymbol{M}_{cc} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}}_r \\ \ddot{\boldsymbol{q}}_c \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_{rr} & \boldsymbol{C}_{rc} \\ \boldsymbol{C}_{cr} & \boldsymbol{C}_{cc} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}_r \\ \dot{\boldsymbol{q}}_c \end{bmatrix} + \begin{bmatrix} \boldsymbol{G}_r \\ \boldsymbol{G}_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_{\mathscr{I}_r} \\ \boldsymbol{B}_{\mathscr{I}_c} \end{bmatrix} \boldsymbol{u}_{\boldsymbol{M}} + \begin{bmatrix} \boldsymbol{\delta}_r \\ \boldsymbol{\delta}_c \end{bmatrix} \qquad (7)$$

where $\boldsymbol{\delta}_{\boldsymbol{r}}$ and $\boldsymbol{\delta}_{\boldsymbol{c}}$ represent the total effect of the system modeling errors and external disturbances on the remaining and controlled DOF, respectively. Without loss of generality, $\boldsymbol{B}_{\mathscr{I}}(\boldsymbol{q})$ can be written as:

$$\boldsymbol{B}_{\mathscr{I}}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{B}_{\mathscr{I}_r}(\boldsymbol{q}) \\ \boldsymbol{B}_{\mathscr{I}_c}(\boldsymbol{q}) \end{bmatrix} \qquad (8)$$

such that $\boldsymbol{B}_{\mathscr{I}_c}(\boldsymbol{q})$ is an invertible $n_a \times n_a$ matrix. From the input coupling follows that $\boldsymbol{B}_{\mathscr{I}_r}(\boldsymbol{q}) \neq 0$ for all $\boldsymbol{q}$ [Olfati-Saber 2001].

Taking into account this partition, where the inertia matrix presents cross terms between the DOFs, this system can be normalized to obtain an inertia matrix with such terms equal to zero through the following form:

$$\overline{\boldsymbol{M}}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \overline{\boldsymbol{C}}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \overline{\boldsymbol{G}}(\boldsymbol{q}) = \overline{\boldsymbol{\Gamma}}(\boldsymbol{q}) + \overline{\boldsymbol{\delta}}\ , \qquad (9)$$

$$\begin{bmatrix} \boldsymbol{M}_{or} & \mathbb{O} \\ \mathbb{O} & \boldsymbol{M}_{ic} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}}_r \\ \ddot{\boldsymbol{q}}_c \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_{or} & \boldsymbol{C}_{oc} \\ \boldsymbol{C}_{ir} & \boldsymbol{C}_{ic} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}_r \\ \dot{\boldsymbol{q}}_c \end{bmatrix} + \begin{bmatrix} \boldsymbol{G}_{or} \\ \boldsymbol{G}_{ic} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Gamma}_{or} \\ \boldsymbol{\Gamma}_{ic} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\delta}_{or} \\ \boldsymbol{\delta}_{is} \end{bmatrix}$$

where:

$$\overline{\boldsymbol{\Gamma}}(\boldsymbol{q}) = \boldsymbol{T}_{\boldsymbol{M}}(\boldsymbol{q})\boldsymbol{B}_{\mathscr{I}}(\boldsymbol{q})\boldsymbol{u}_{\boldsymbol{M}}, \quad \overline{\boldsymbol{\delta}}(\boldsymbol{q}) = \boldsymbol{T}_{\boldsymbol{M}}(\boldsymbol{q})\boldsymbol{\delta},$$
$$\overline{\boldsymbol{M}}(\boldsymbol{q}) = \boldsymbol{T}_{\boldsymbol{M}}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q}), \quad \overline{\boldsymbol{C}}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{T}_{\boldsymbol{M}}(\boldsymbol{q})\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}),$$
$$\overline{\boldsymbol{G}}(\boldsymbol{q}) = \boldsymbol{T}_{\boldsymbol{M}}(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q})$$

with:

$$\boldsymbol{T}_{\boldsymbol{M}}(\boldsymbol{q}) = \begin{bmatrix} \mathbb{1} & -\boldsymbol{M}_{rc}(\boldsymbol{q})\boldsymbol{M}_{cc}^{-1}(\boldsymbol{q}) \\ -\boldsymbol{M}_{cr}(\boldsymbol{q})\boldsymbol{M}_{rr}^{-1}(\boldsymbol{q}) & \mathbb{1} \end{bmatrix}$$

This normalization of the system will allow to weigh different dynamics through different weighting parameters, and to consider only the time-derivative of the remaining DOF into the tracking error vector.

As the control objective for the quadrotor helicopter is to perform a path tracking of the controlled DOF, $\boldsymbol{q_c} = [\psi\ x\ y\ z]'$, while the remaining ones, $\boldsymbol{q_r} = [\phi\ \theta]'$, are maintained stabilized, the tracking error vector is defined as follows:

$$\boldsymbol{x} = \begin{bmatrix} \dot{\tilde{\boldsymbol{q}}}_r \\ \dot{\tilde{\boldsymbol{q}}}_c \\ \tilde{\boldsymbol{q}}_c \\ \int \tilde{\boldsymbol{q}}_c dt \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}}_r^d \\ \dot{\boldsymbol{q}}_c - \dot{\boldsymbol{q}}_c^d \\ \boldsymbol{q}_c - \boldsymbol{q}_c^d \\ \int \boldsymbol{q}_c - \boldsymbol{q}_c^d dt \end{bmatrix} . \quad (10)$$

being $\boldsymbol{q}_c^d$, $\dot{\boldsymbol{q}}_c^d$, $\dot{\boldsymbol{q}}_r^d$ and $\ddot{\boldsymbol{q}}_c^d$ the desired trajectory and the corresponding velocities and acceleration, respectively. For the quadrotor helicopter application, $\dot{\boldsymbol{q}}_r^d$ is equal to zero. Note that an integral term has been included in the error vector. This term will allow the achievement of a null steady-state error when persistent disturbances are acting on the system.

3.1 Nonlinear $\mathscr{H}_\infty$ Controller

To design the nonlinear $\mathscr{H}_\infty$ controller, in a previous step, the following state transformation is defined:

$$\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \boldsymbol{T_o x} = \begin{bmatrix} \boldsymbol{T}_{11} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \boldsymbol{T}_{22} & \boldsymbol{T}_{23} & \boldsymbol{T}_{24} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} & \mathbb{1} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \dot{\tilde{\boldsymbol{q}}}_r \\ \dot{\tilde{\boldsymbol{q}}}_c \\ \tilde{\boldsymbol{q}}_c \\ \int \tilde{\boldsymbol{q}}_c dt \end{bmatrix} \quad (11)$$

with $\boldsymbol{T}_{11} = \rho\mathbb{1}$ and $\boldsymbol{T}_{22} = \nu\mathbb{1}$, where ρ and ν are positive scalars.

Despite of this state-space transformation, a change of variables over the control action and disturbances must be considered (see Johansson [1990]). Thus, to minimize the necessary forces/torques for the worst case of all possible disturbances acting on the system, the following change of variables is defined:

$$\begin{bmatrix} \boldsymbol{\omega}_{or} + \boldsymbol{u}_{or} \\ \boldsymbol{\omega}_{1c} + \boldsymbol{u}_{1c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}_{or} & \mathbb{0} & \boldsymbol{C}_{or} & \boldsymbol{C}_{oc} \\ \mathbb{0} & \boldsymbol{M}_{ic} & \boldsymbol{C}_{ir} & \boldsymbol{C}_{ic} \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ z_1 \\ z_2 \end{bmatrix} \quad (12)$$

$$\overline{\boldsymbol{M}}(\boldsymbol{q})\boldsymbol{T}\dot{\boldsymbol{x}} + \overline{\boldsymbol{C}}(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{T}\boldsymbol{x} = \overline{\boldsymbol{u}} + \overline{\boldsymbol{d}} \quad (13)$$

where matrix $\boldsymbol{T}$ can be partitioned as follows:

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{T}_{11} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \boldsymbol{T}_{22} & \boldsymbol{T}_{23} & \boldsymbol{T}_{24} \end{bmatrix} ,$$

and

$$\overline{\boldsymbol{u}} + \overline{\boldsymbol{d}} = \boldsymbol{T_M}(\boldsymbol{q})\,(\boldsymbol{u} + \boldsymbol{d}) . \quad (14)$$

By expanding this transformation, which includes reference trajectories, forces/torques affecting kinetic energy and the state-space transformation (11), the *dynamic equation of the system error* can be written as follows:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x},\boldsymbol{q_r},t) + g(\boldsymbol{x},\boldsymbol{q_r},t)\overline{\boldsymbol{u}} + k(\boldsymbol{x},\boldsymbol{q_r},t)\overline{\boldsymbol{d}} , \quad (15)$$

$$f(\boldsymbol{x},\boldsymbol{q_r},t) = \boldsymbol{T_o}^{-1} \begin{bmatrix} -\boldsymbol{M}_{or}^{-1}\boldsymbol{C}_{or} & -\boldsymbol{M}_{or}^{-1}\boldsymbol{C}_{oc} & \mathbb{0} & \mathbb{0} \\ -\boldsymbol{M}_{ic}^{-1}\boldsymbol{C}_{ir} & -\boldsymbol{M}_{ic}^{-1}\boldsymbol{C}_{ic} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \boldsymbol{T}_{22}^{-1} & \mathbb{1} - \boldsymbol{T}_{22}^{-1}\boldsymbol{T}_{23} & -\mathbb{1} - \boldsymbol{T}_{22}^{-1}(\boldsymbol{T}_{23} - \boldsymbol{T}_{24}) \\ \mathbb{0} & \mathbb{0} & \mathbb{1} & -\mathbb{1} \end{bmatrix} \boldsymbol{T_o x} ,$$

$$g(\boldsymbol{x},\boldsymbol{q_r},t) = k(\boldsymbol{x},\boldsymbol{q_r},t) = \boldsymbol{T_o}^{-1} \begin{bmatrix} \boldsymbol{M}_{or}^{-1} & \mathbb{0} \\ \mathbb{0} & \boldsymbol{M}_{ic}^{-1} \\ \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{bmatrix} ,$$

Without loss of generality, through equations (9) and (15), the transformed external disturbance vector $\overline{\boldsymbol{d}}$ and control input $\overline{\boldsymbol{u}}$ are obtained as follows:

$$\overline{\boldsymbol{d}} = \overline{\boldsymbol{M}}(\boldsymbol{q})\boldsymbol{T_c}\overline{\boldsymbol{M}}^{-1}(\boldsymbol{q})\overline{\boldsymbol{\delta}} \quad (16)$$

$$\overline{\boldsymbol{u}} = \boldsymbol{T_c}\left(-\boldsymbol{F}(\boldsymbol{x_e}) + \overline{\boldsymbol{\Gamma}}\right) \quad (17)$$

where:

$$\boldsymbol{T_c} = \begin{bmatrix} \boldsymbol{T}_{11} & \mathbb{0} \\ \mathbb{0} & \boldsymbol{T}_{22} \end{bmatrix} .$$

The relationship between the applied forces/torques, $\overline{\boldsymbol{\Gamma}}$, and the control input, $\overline{\boldsymbol{u}}$, is given by equation (17). So that, by isolating $\overline{\boldsymbol{\Gamma}}$, the following control law is obtained:

$$\overline{\boldsymbol{\Gamma}} = \overline{\boldsymbol{M}}\ddot{\boldsymbol{q}} + \overline{\boldsymbol{C}}\dot{\boldsymbol{q}} + \overline{\boldsymbol{G}} - \boldsymbol{T_c}^{-1}\left(\overline{\boldsymbol{M}}\boldsymbol{T}\dot{\boldsymbol{x}} + \overline{\boldsymbol{C}}\boldsymbol{T}\boldsymbol{x}\right) + \boldsymbol{T_c}^{-1}\overline{\boldsymbol{u}} \quad (18)$$

which is arranged in terms of the error vector and its time derivative.

Taking into account equation (15) and the following cost variable, $\boldsymbol{\zeta} = \boldsymbol{W}\left[\, h(\boldsymbol{x})'\ \boldsymbol{u}' \,\right]'$, where $h(\boldsymbol{x}) \in \Re^{n_r+3n_c}$ represents a function of the vector of the states to be controlled and stabilized, $\boldsymbol{W} \in \Re^{(3n_c+n_r+n)\times(3n_c+n_r+n)}$ is a weighting matrix, and $\boldsymbol{u}$ is the control signal without the transformation $\boldsymbol{T_M}$, the nonlinear $\mathscr{H}_\infty$ control problem can be posed as follows [van der Schaft 2000]:

"Find the smallest value $\gamma^* \geq 0$ such that for any $\gamma \geq \gamma^*$ exists a control law $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},\boldsymbol{q_r},t)$, such that the L_2 gain from the disturbance signals $\boldsymbol{d}$ to the cost variable $\boldsymbol{\zeta} = \boldsymbol{W}\ [h'(\boldsymbol{x})\ \boldsymbol{u}']'$ is less than or equal to a given attenuation level γ, that is:".

$$\int_0^T \|\boldsymbol{\zeta}\|_2^2 dt \leq \gamma^2 \int_0^T \|\boldsymbol{d}\|_2^2 dt . \quad (19)$$

The internal term of the integral expression on the left-hand side of inequality (19) can be written as:

$$\|\boldsymbol{\zeta}\|_2^2 = \boldsymbol{\zeta}'\boldsymbol{\zeta} = \left[\, h'(\boldsymbol{x})\ \boldsymbol{u}' \,\right] \boldsymbol{W}'\boldsymbol{W} \begin{bmatrix} h(\boldsymbol{x}) \\ \boldsymbol{u} \end{bmatrix}$$

and the symmetric positive definite matrix $\boldsymbol{W}'\boldsymbol{W}$ can be partitioned as follows:

$$\boldsymbol{W}'\boldsymbol{W} = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}' & \boldsymbol{R} \end{bmatrix} \quad (20)$$

Matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ are symmetric positive definite and the fact that $\boldsymbol{W}'\boldsymbol{W} > \mathbb{0}$ guarantees that $\boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}' > \mathbb{0}$.

Under these assumptions, an optimal state feedback control law $\boldsymbol{u}^*(\boldsymbol{x},\boldsymbol{q_r},t)$ can be computed if a smooth solution $V(\boldsymbol{x},\boldsymbol{q_r},t)$, with $\boldsymbol{x_0} = 0$ and $V(\boldsymbol{x_0},\boldsymbol{q_{r_0}},t) \equiv 0$ for $t \geq 0$, is found for the following HJ equation:

$$\frac{\partial V}{\partial t} + \frac{\partial' V}{\partial \boldsymbol{x}} f(\boldsymbol{x},\boldsymbol{q_r},t) + \frac{\partial' V}{\partial \boldsymbol{q_r}}\dot{\boldsymbol{q}}_r + \frac{1}{2}\frac{\partial' V}{\partial \boldsymbol{x}}\left[\frac{1}{\gamma^2}k(\boldsymbol{x},\boldsymbol{q_r},t)k'(\boldsymbol{x},\boldsymbol{q_r},t) - g(\boldsymbol{x},\boldsymbol{q_r},t)\boldsymbol{R}^{-1}g'(\boldsymbol{x},\boldsymbol{q_r},t)\right]\frac{\partial V}{\partial \boldsymbol{x}} - \frac{\partial' V}{\partial \boldsymbol{x}} g(\boldsymbol{x},\boldsymbol{q_r},t)\boldsymbol{R}^{-1}\boldsymbol{S}'h(\boldsymbol{x}) + \frac{1}{2}h'(\boldsymbol{x})\left(\boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}'\right)h(\boldsymbol{x}) = 0 \quad (21)$$

for each $\gamma > \sqrt{\sigma_{\max}(\boldsymbol{R})} \geq 0$, where $\sigma_{\max}$ stands for the maximum singular value. In such a case, the optimal state feedback *additional control* effort is derived as (see Feng and Postlethwaite [1994]):

$$\boldsymbol{u}^* = -\boldsymbol{R}^{-1}\left(\boldsymbol{S}'h(\boldsymbol{x}) + g'(\boldsymbol{x},\boldsymbol{q_r},t)\frac{\partial V(\boldsymbol{x},\boldsymbol{q_r},t)}{\partial \boldsymbol{x}}\right) . \quad (22)$$

As stated before, the solution of the HJ equation depends on the choice of the cost variable, $\boldsymbol{\zeta}$, and particularly on the selection of function $h(\boldsymbol{x})$. In this paper, this function is taken to be equal to the error vector, that is, $h(\boldsymbol{x}) = \boldsymbol{x}$. Once this function has been selected, computing the control law, $\boldsymbol{u}$, will require finding the solution, $V(\boldsymbol{x}, \boldsymbol{q_r}, t)$, to the HJ equation (21). The following theorem will help do this.

Theorem 1. Let $V(\boldsymbol{x}, \boldsymbol{q_r}, t)$ be the parameterized scalar function:

$$V(\boldsymbol{x}, \boldsymbol{q_r}, t) = \frac{1}{2}\boldsymbol{x}'\boldsymbol{T_o}' \begin{bmatrix} \boldsymbol{M_{rr}} & \boldsymbol{M_{rc}} & \mathbb{0} & \mathbb{0} \\ \boldsymbol{M_{cr}} & \boldsymbol{M_{cc}} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \boldsymbol{Y} & \boldsymbol{X}-\boldsymbol{Y} \\ \mathbb{0} & \mathbb{0} & \boldsymbol{X}-\boldsymbol{Y} & \boldsymbol{Z}+\boldsymbol{Y} \end{bmatrix} \boldsymbol{T_o}\boldsymbol{x} \quad , \tag{23}$$

where $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z} \in \Re^{n_c \times n_c}$ are constant, symmetric, and positive definite matrices such that $\boldsymbol{Z} - \boldsymbol{X}\boldsymbol{Y}^{-1}\boldsymbol{X} + 2\boldsymbol{X} > \mathbb{0}$, and $\boldsymbol{T_o}$ is as defined in (11). Let $\boldsymbol{T}$ be the matrix appearing in (18). If these matrices verify the following equation:

$$\begin{bmatrix} \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \boldsymbol{Y} & \boldsymbol{X} \\ \mathbb{0} & \boldsymbol{Y} & 2\boldsymbol{X} & \boldsymbol{Z}+2\boldsymbol{X} \\ \mathbb{0} & \boldsymbol{X} & \boldsymbol{Z}+2\boldsymbol{X} & \mathbb{0} \end{bmatrix} + \boldsymbol{Q} + \frac{1}{\gamma^2}\boldsymbol{T}'\boldsymbol{T} - (\boldsymbol{S}'+\boldsymbol{T})'\boldsymbol{R}^{-1}(\boldsymbol{S}'+\boldsymbol{T}) = \mathbb{0} \tag{24}$$

then, function $V(\boldsymbol{x}, \boldsymbol{q_r}, t)$ constitutes a solution to the HJ, for a sufficiently high value of γ.

The proof of this theorem is derived following the steps presented in Ortega et al. [2005].

◇

Once matrix $\boldsymbol{T}$ is computed by solving some Riccati algebraic equations, substituting $V(\boldsymbol{x}, \boldsymbol{q_r}, t)$ into the optimal state feedback control law (22), the *additional control* effort $\overline{\boldsymbol{u}}^*$ corresponding to the $\mathscr{H}_\infty$ optimal index γ is given by

$$\overline{\boldsymbol{u}}^* = -\boldsymbol{T_M}\boldsymbol{R}^{-1}\left(\boldsymbol{S}'+\boldsymbol{T}\right)\boldsymbol{x} \tag{25}$$

Finally, if the *additional control* effort (25) is replaced into (13) under the assumption that $\overline{\boldsymbol{d}} = 0$, and after some manipulations, the *control acceleration* can be obtained as follows:

$$\ddot{\boldsymbol{q}} = \ddot{\boldsymbol{q}}^{\boldsymbol{d}} - \boldsymbol{K_D}\dot{\tilde{\boldsymbol{q}}} - \boldsymbol{K_P}\tilde{\boldsymbol{q}} - \boldsymbol{K_I}\int \tilde{\boldsymbol{q}}dt \tag{26}$$

with

$$\boldsymbol{K_D} = \begin{bmatrix} \boldsymbol{K_{D_{rr}}} & \boldsymbol{K_{D_{rc}}} \\ \boldsymbol{K_{D_{cr}}} & \boldsymbol{K_{D_{cc}}} \end{bmatrix}, \quad \boldsymbol{K_P} = \begin{bmatrix} \boldsymbol{K_{P_{rc}}} \\ \boldsymbol{K_{P_{cc}}} \end{bmatrix}, \quad \boldsymbol{K_I} = \begin{bmatrix} \boldsymbol{K_{I_{rc}}} \\ \boldsymbol{K_{I_{cc}}} \end{bmatrix}$$

A particular case can be obtained when the elements of the weighting compound $\boldsymbol{W}'\boldsymbol{W}$ verify:

$$\boldsymbol{Q_1} = \omega_{1r}^2\mathbb{1}, \quad \boldsymbol{Q_2} = \omega_{1c}^2\mathbb{1}, \quad \boldsymbol{Q_3} = \omega_{2c}^2\mathbb{1}, \quad \boldsymbol{Q_4} = \omega_{3c}^2\mathbb{1},$$

$$\boldsymbol{R} = \begin{bmatrix} \omega_{ur}^2\mathbb{1} & \mathbb{0} \\ \mathbb{0} & \omega_{uc}^2\mathbb{1} \end{bmatrix}, \quad \boldsymbol{Q_{12}} = \boldsymbol{Q_{13}} = \boldsymbol{Q_{23}} = \mathbb{0},$$

$$\boldsymbol{S_{11}} = \boldsymbol{S_{12}} = \boldsymbol{S_{21}} = \boldsymbol{S_{22}} = \boldsymbol{S_{31}} = \boldsymbol{S_{32}} = \mathbb{0}.$$

In this case, the analytical equations for the gain matrices can be expressed as follows:

$$\begin{aligned}
\boldsymbol{K_{D_{rr}}} &= \boldsymbol{M_{or}}^{-1}\left(\boldsymbol{C_{or}} + \frac{1}{\omega_{ur}^2}\mathbb{1}_{n_r\times n_r}\right) \\
\boldsymbol{K_{D_{rc}}} &= \boldsymbol{M_{or}}^{-1}\left(\boldsymbol{C_{oc}} - \boldsymbol{M_{rc}}\boldsymbol{M_{cc}}^{-1}\frac{1}{\omega_{uc}^2}\right)\frac{\omega_{uc}\omega_{1c}}{\sqrt{\gamma^2-\omega_{uc}^2}}\frac{\sqrt{\gamma^2-\omega_{ur}^2}}{\omega_{ur}\omega_{1r}} \\
\boldsymbol{K_{P_{rc}}} &= \boldsymbol{M_{or}}^{-1}\left(\boldsymbol{C_{oc}} - \boldsymbol{M_{rc}}\boldsymbol{M_{cc}}^{-1}\frac{1}{\omega_{uc}^2}\right)\frac{\omega_{uc}\sqrt{\omega_{2c}^2+2\omega_{1c}\omega_{3c}}}{\sqrt{\gamma^2-\omega_{uc}^2}}\frac{\sqrt{\gamma^2-\omega_{ur}^2}}{\omega_{ur}\omega_{1r}} \\
\boldsymbol{K_{I_{rc}}} &= \boldsymbol{M_{or}}^{-1}\left(\boldsymbol{C_{oc}} - \boldsymbol{M_{rc}}\boldsymbol{M_{cc}}^{-1}\frac{1}{\omega_{uc}^2}\right)\frac{\omega_{uc}\omega_{3c}}{\sqrt{\gamma^2-\omega_{uc}^2}}\frac{\sqrt{\gamma^2-\omega_{ur}^2}}{\omega_{ur}\omega_{1r}} \\
\boldsymbol{K_{D_{cr}}} &= \boldsymbol{M_{ic}}^{-1}\left(\boldsymbol{C_{ir}} - \boldsymbol{M_{cr}}\boldsymbol{M_{rr}}^{-1}\frac{1}{\omega_{ur}^2}\right)\frac{\omega_{ur}\omega_{1r}}{\sqrt{\gamma^2-\omega_{ur}^2}}\frac{\sqrt{\gamma^2-\omega_{uc}^2}}{\omega_{uc}\omega_{1c}} \\
\boldsymbol{K_{D_{cc}}} &= \frac{\sqrt{\omega_{2c}^2+2\omega_{1c}\omega_{3c}}}{\omega_{1s}}\mathbb{1} + \boldsymbol{M_{ic}}^{-1}\left(\boldsymbol{C_{ic}} + \frac{1}{\omega_{uc}^2}\mathbb{1}\right) \\
\boldsymbol{K_{P_{cc}}} &= \frac{\sqrt{\omega_{2c}^2+2\omega_{1c}\omega_{3c}}}{\omega_{1s}}\boldsymbol{M_{ic}}^{-1}\left(\boldsymbol{C_{ic}} + \frac{1}{\omega_{uc}^2}\mathbb{1}\right) + \frac{\omega_{3c}}{\omega_{1s}}\mathbb{1} \\
\boldsymbol{K_{I_{cc}}} &= \boldsymbol{M_{ic}}^{-1}\left(\boldsymbol{C_{ic}} + \frac{1}{\omega_{uc}^2}\mathbb{1}\right)\frac{\omega_{3c}}{\omega_{1s}}\mathbb{1}
\end{aligned}$$

where ω_{1r} and ω_{1c} are the weighting parameters of the time-derivative of the position error of the remaining and controlled DOF, respectively; ω_{2c} and ω_{3c} are the weighting values of position error and its integral of the controlled DOF, respectively; and the weighting of the *additional control* effort for the remaining and controlled degrees of freedom are ω_{ur} and ω_{uc}.

Equation (26) gives the necessary acceleration of the DOF to track the reference trajectory. Therefore, the applied forces/torques to the quadrotor helicopter can be computed substituting this *control acceleration* in (9) and multiplying it by the inverse of $\boldsymbol{T_M}(\boldsymbol{q})$. With the forces/torques necessary to follow the desired trajectory, the velocities of the rotors can be computed, that is:

$$\boldsymbol{u_M} = \boldsymbol{B_{\mathscr{I}}}(\boldsymbol{q})^{\#}\boldsymbol{T_M}(\boldsymbol{q})^{-1}\overline{\boldsymbol{\Gamma}} \tag{27}$$

where # means the pseudoinverse operator of a matrix.

4. SIMULATION RESULTS

Simulations have been carried out in order to corroborate the proposed controller when the quadrotor helicopter tracks some trajectory. The simulations have been executed considering a more accurate model, which emulates a real quadrotor helicopter. This model considers that the axes of rotation of the body-fixed frame are parallel to the axes passing through the center of mass, and its origin is displaced by a distance $\boldsymbol{r}$ from the center of mass. This assumption results in a strongly-coupled dynamic model, with crossed terms in the inertia matrix and in the Coriolis and centrifugal matrix between $\ddot{\boldsymbol{\xi}}$ and $\ddot{\boldsymbol{\eta}}$, and in the gravitational forces vector. Moreover, the moment of inertia tensor also presents crossed terms, which are obtained through the Steiner's parallel-axis theorem. On the other hand, this model also takes into account saturated control inputs. These facts imply that structural uncertainty has been considered with respect to the simplified nominal model used for control synthesis purposed.

Furthermore, an amount of $\pm 40\%$ in the uncertainty of the mass has also been considered to test the robustness provided by the control strategy with respect to parametric uncertainty. Finally, sustained disturbances affecting all the degrees of freedom have been applied in different instants of time to check the disturbance rejection capability of the proposed control strategy. The following persistent steps have been applied as aerodynamic moments and forces disturbances in the simulations: $A_r = 0.5$Nm at $t = 5$s; $A_z = -1.0$N at $t = 10$s; $A_p = 0.5$Nm at $t = 15$s; $A_x = 1.0$N at $t = 20$s; $A_q = -0.5$Nm at $t = 25$s and $A_y = 1.0$N at $t = 30$s.

The simulations have been executed with a reference trajectory made up of a set of several kinds of stretches. The initial conditions of the helicopter are $\boldsymbol{\xi_0} = [0 \quad 0.5 \quad 0.5]'$m and $\boldsymbol{\eta_0} = [0 \quad 0 \quad 0.5]'$rad. The values of the model parameters used for simulations are the following: $m = 0.74$kg, $l = 0.21$m, $g = 9.81\text{m/s}^2$, $I_{xx} = I_{yy} = 0.004\text{kgm}^2$, $I_{zz} = 0.0084\text{kgm}^2$, $b = 2.9e-5\text{Ns}^2$ and $d = 6.0e-6\text{Nms}^2$. The tilt angle of the rotors has been designed as $\alpha = 5°$. The nonlinear $\mathscr{H}_\infty$ controller gains were tuned with the following values: $\omega_{1r} = 1.5$, $\omega_{1c} = 1.0$, $\omega_{2c} = 0.5$, $\omega_{3c} = 6.0$, $\omega_{ur} = 2.5$, $\omega_{uc} = 0.7$ and $\gamma = 3.0$.

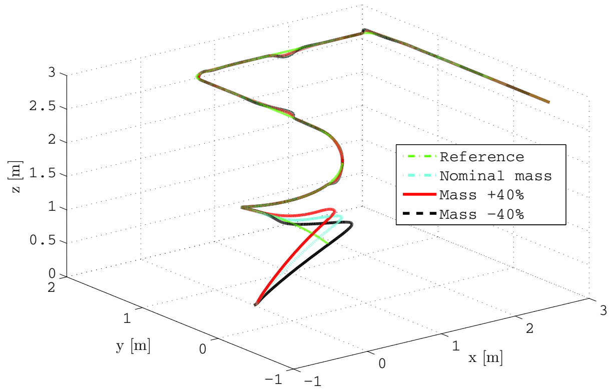

Fig. 2. Path following.

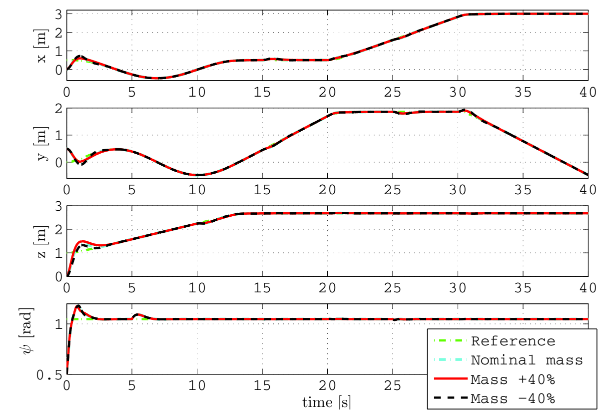

Fig. 3. Position of the controlled DOF (x, y, z, ψ).

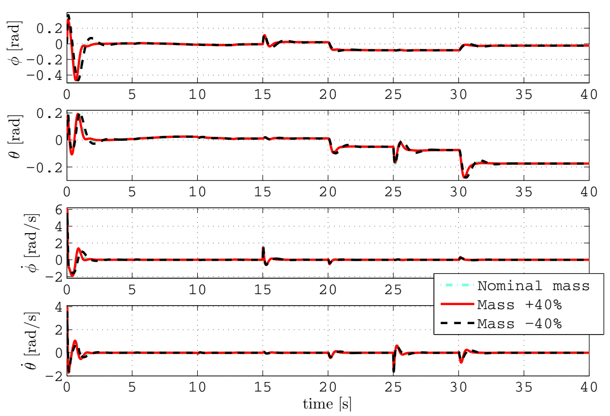

Fig. 4. Position and velocity of the remaining DOF $(\phi, \theta, \dot{\phi}, \dot{\theta})$.

Figs. 2-4 illustrate a good performance of the quadrotor helicopter to perform path tracking when sustained disturbances, and structural and parametric uncertainties are considered, which confirm the robustness provided by the proposed controller. Fig 3 shows that the controlled DOF achieve null steady state error when aerodynamics forces and moments are acting on the whole system; this is due to the inclusion of an integral term in the error vector. Moreover, it can be observed in Fig. 4 that the remaining DOF are maintained stable, which verify the use of their velocities in the objective vector.

5. CONCLUSIONS

In this paper a new approach of the nonlinear $\mathscr{H}_\infty$ control design for a class of underactuated mechanical system under input coupling has been presented. To perform the proposed controller a normalization of the equation of motions of the system is used, which allows to consider the dynamic of the remaining DOF in the $\mathscr{H}_\infty$ control design. Besides, it enables to weigh the velocity error of the DOF through different criteria. Furthermore, in this paper, a modified model of quadrotor helicopter is used, which adds coupling between the translational and rotational movements. This coupling avoids the necessity to use cascade control strategies, or to consider an augmented state vector through a double integrator. Finally, the proposed controller have been corroborated by simulation results to solve the path tracking problem of the quadrotor helicopter, when sustained disturbances were acting on the six degrees of freedom, and structural and parametric uncertainties have been considered.

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