15th Triennial World Congress, Barcelona, Spain

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NONLINEAR H_∞ CONTROL WITH PID STRUCTURE FOR ROBOT MANIPULATORS

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Abstract: In this paper the nonlinear H_∞ control for robot manipulators introduced in (Feng, W., Postlethwaite, I., 1994) is extended. An additional integral term is included to cope with persistent disturbances, such as constant weights at the end-efecctor. The extended controller is interpreted like a computed torque control with and external PID, whose gain matrices vary with the position and velocity of the robot joints. A particular case of the cost variable weighting matrix is studied in which the resulting external nonlinear PID does not depend on the attenuation level γ of the H_∞ formulation. Finally, experimental results are presented for the RM-10 industrial robot. *Copyright © 2002 IFAC*

Keywords: Robot manipulators, nonlinear H_∞ control, disturbance rejection.

1. INTRODUCTION

In automatic control, despite considerable effort to minimize system modeling errors, uncertainties are usually present and sometimes are significant. Research on the motion control of rigid robot manipulators has known significant progress over the last few years. An interesting question is whether control laws possess desirable rejection properties even if perfect models are assumed to be available. In this paper, a vector of disturbance signals acting on the input channels (torques) of the robot is used to represent the combined effect of modeling errors and external disturbances. The control system ability to reject these disturbances and maintain small tracking error (without excessive control effort) is measured in a L_2 gain sense. A control design -formulated into a nonlinear H_∞ optimization problem -is proposed to achieve optimal disturbance rejection. Based on this formulation, a nonlinear H_∞ suboptimal control law is derived which consists of a feedforward/feedback structure.

The remainder of the paper is organized as follows: An approach upon the concepts of L_2 gain and H_∞ optimization in the context of nonlinear systems are introduced in Section 2. In Section 3 a suboptimal nonlinear controller is derived to maximize the robot manipulator ability to reject external disturbances acting on the input channel, assuming a perfect system model. In Section 4, the nonlinear H_∞ controller is expressed in the form of a computed torque control with an external nonlinear PID controller. A nonlinear H_∞ controller for the RM-10 robot manipulator is designed and experimental results are shown in Section 5. Finally, the major conclusions to be drawn are given in Section 6.

2. NONLINEAR H_∞ CONTROL APPROACH

The dynamical equation of a nth order smooth nonlinear system which is affected by an unknown disturbance can be expressed as follows

$$\dot{x} = f(x,t) + G(x,t)\,u + K(x,t)\,d \qquad (1)$$

where $u \in \Re^p$ is the vector of control inputs, $d \in \Re^q$ is the vector of external disturbances and $x \in \Re^n$ is the vector of states. Performance can be defined using the cost variable $z \in \Re^{(m+p)}$ given by the expression

$$z = W \begin{bmatrix} h(x) \\ u \end{bmatrix} \tag{2}$$

where $h(x) \in \Re^m$ is the error vector to be controlled and $W \in \Re^{(m+p)\times(m+p)}$ is a weighting matrix. If we assume that the states x are available for measurement then the optimal H_∞ problem can be posed as follows (van der Schaft, A., 1992):

Find the smallest value $\gamma^* \geq 0$ such that for any $\gamma \geq \gamma^*$ there exists a state feedback $u = u(x,t)$ such that the L_2 gain from d to z is less than or equal to γ, that is

$$\int_0^T \|z\|_2^2 \, dt \leq \gamma^2 \int_0^T \|d\|_2^2 \, dt \tag{3}$$

The integral expression on the left-hand side of expression (3) can be written as

$$\|z\|_2^2 = z^T z = \begin{bmatrix} h^T(x) & u^T \end{bmatrix} W^T W \begin{bmatrix} h(x) \\ u \end{bmatrix}$$

and matrix $W^T W$ can be partitioned as follows

$$W^T W = \begin{bmatrix} Q & \bar{C} \\ \bar{C}^T & R \end{bmatrix}$$

where

$$Q = \begin{bmatrix} Q_1 & Q_{12} & Q_{13} \\ Q_{12} & Q_2 & Q_{23} \\ Q_{13} & Q_{23} & Q_3 \end{bmatrix} \qquad \bar{C} = \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \bar{C}_3 \end{bmatrix}$$

The matrices Q and R are symmetric positive definite and such that $Q - \bar{C}R^{-1}\bar{C}^T > 0$.

An optimal control signal u^* may be computed for system (1) if there exists a smooth solution $V(x,t)$, with $V(x_0,t) \equiv 0$ for $t \geq 0$, to the following Hamilton-Jacobi equation:

$$\frac{\partial V}{\partial t} + \frac{\partial^T V}{\partial x} f(x,t) + \frac{1}{2}\frac{\partial^T V}{\partial x}\left[\frac{1}{\gamma^2}K(x,t)K^T(x,t)\right.$$
$$\left. - G(x,t)R^{-1}G^T(x,t)\right]\frac{\partial V}{\partial x} - \frac{\partial^T V}{\partial x}G(x,t)R^{-1}C^T h(x)$$
$$+\frac{1}{2}h^T(x)\left(Q - \bar{C}R^{-1}\bar{C}^T\right)h(x) = 0 \tag{4}$$

for each $\gamma > \sqrt{\sigma_{\max}(R)} \geq 0$. In such case, the optimal state feedback control law –see (Feng, W., Postlethwaite, I., 1994)– is derived as

$$u^* = -R^{-1}\left(\bar{C}^T h(x) + G^T(x,t)\frac{\partial V}{\partial x}\right) \tag{5}$$

3. NONLINEAR H_∞ OPTIMIZATION IN MANIPULATOR MOTION CONTROL

The following Euler-Lagrange equations of motion are used to describe the behavior of a n degree-of-freedom (DOF) robot manipulator

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau + d_\tau \tag{6}$$

where q is the vector of joint variables (joint positions) and $\dot{q}$ is its temporal derivative (joint speeds). It is supposed that these two vectors are available for measurements. The vector τ (torques applied on the axis of the joints) is the signal input of the system and d_τ represents the total effect of system modelling errors and the external disturbances. The inertia matrix $M(q)$ is symmetric and positive definite, $V(q,\dot{q})$ is the vector of centripetal and Coriolis terms and $G(q)$ is a vector which consists of the gravitational terms.

Denoting by q_r, $\dot{q}_r$ and $\ddot{q}_r$ the desired position, speed and acceleration of the joints, respectively, the tracking error vector x and its derivative $\dot{x}$ are defined as follows:

$$x = \begin{bmatrix} \dot{e} \\ e \\ \int e dt \end{bmatrix} \quad \text{and} \quad \dot{x} = \begin{bmatrix} \ddot{e} \\ \dot{e} \\ e \end{bmatrix} \tag{7}$$

where

$$\begin{aligned} \ddot{e} &= \ddot{q} - \ddot{q}_r, \\ \dot{e} &= \dot{q} - \dot{q}_r, \\ e &= q - q_r, \\ \int e dt &= \int_o^t (q - q_r)\, dt. \end{aligned}$$

For system (6) a control law of the following structure is considered

$$\tau = M(q)\ddot{q} + V(q,\dot{q}) + G(q) - \tag{8}$$
$$-\frac{1}{\rho}\left(M(q)T\dot{x} + C(q,\dot{q})Tx\right) + \frac{1}{\rho}u$$

where

$$C(q,\dot{q}) = \frac{1}{2}\dot{M}(q) + N(q,\dot{q})$$

and the matrices $\dot{M}(q)$ and $N(q,\dot{q})$ [1] must be computed through the following expressions:

$$\dot{M}_{ij} = \frac{d}{dt}M_{ij} = \frac{\partial M_{ij}}{\partial q}\dot{q} = \sum_{k=1}^{n}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_k \tag{9}$$

[1] Note that the N matrix is skew-symmetric. We will use this propriety in the Proofs.

$$N_{ij} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i}\right)\dot{q}_k \quad (10)$$

It can be shown that

$$V(q,\dot{q}) = C(q,\dot{q})\dot{q}$$

It should be noted how vector u in control law (8) represents the additional control effort necessary for attenuating the disturbances.

Matrix T in equation (8) can be partitioned as follows:

$$T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix} \quad (11)$$

with $T_1 = \rho I$, where ρ is a positive scalar and I is the nth-order identity matrix.

On substituting the expression of the control law from (8) into the model equation of the robot and defining $d = \rho d_\tau$ yields

$$MT\dot{x} + CTx = u + d \quad (12)$$

which is a $3n$th order equation of the nonlinear dynamics of the error. Thereby, the control problem is to minimize the tracking error x in the presence of d without excessive control effort u.

Equation (12) can be rewritten into the standard form of equation (1) as follows:

$$f(x,t) = T_o^{-1}\ P\ T_o x \quad (13)$$

where

$$P = \begin{bmatrix} -M^{-1}C & O & O \\ \frac{1}{\rho}I & I - \frac{1}{\rho}T_2 & -I - \frac{1}{\rho}(T_3 - T_2) \\ O & I & -I \end{bmatrix}$$

and

$$G(x,t) = K(x,t) = T_o^{-1}\begin{bmatrix} M^{-1} \\ O \\ O \end{bmatrix}$$

where I is the identity matrix, O the zero matrix, both of n-th order and

$$T_o = \begin{bmatrix} T_1 & T_2 & T_3 \\ O & I & I \\ O & O & I \end{bmatrix}$$

If function $h(x)$ in expression (2) is chosen equal to the error vector x, then the following expression constitutes a solution of the Hamilton-Jacobi equation (4):

$$V(x,t) = \frac{1}{2}x^T T_o^T \begin{bmatrix} M & O & O \\ O & Y & S - Y \\ O & S - Y & Z + Y \end{bmatrix} T_o x \quad (14)$$

where the matrices Y, S, Z and $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$ can be obtained solving the equation

$$\begin{bmatrix} O & Y & S \\ Y & 2S & Z + 2S \\ S & Z + 2S & O \end{bmatrix} + Q + \frac{1}{\gamma^2}T^T T - \quad (15)$$

$$-\left(\bar{C}^T + T\right)^T R^{-1}\left(\bar{C}^T + T\right) = 0$$

The proof of this result can be found in appendix A at the end of the paper.

The algorithm to obtain matrix T is the following:

(1) Compute T_1 and T_3 solving the following Riccati algebraic equations:

$$T_1^T\left(\frac{1}{\gamma^2}I - R^{-1}\right)T_1 - \bar{C}_1 R^{-1} T_1 - $$
$$-T_1^T R^{-1}\bar{C}_1^T - \bar{C}_1 R^{-1}\bar{C}_1^T + Q_1 = O,$$

$$T_3^T\left(\frac{1}{\gamma^2}I - R^{-1}\right)T_3 - \bar{C}_3 R^{-1} T_3 - $$
$$-T_3^T R^{-1}\bar{C}_3^T - \bar{C}_3 R^{-1}\bar{C}_3^T + Q_3 = O$$

(2) Compute matrix S given by

$$S = -\Big(T_1^T\left(\frac{1}{\gamma^2}I - R^{-1}\right)T_3 - \bar{C}_1 R^{-1} T_3 - $$
$$-T_1^T R^{-1}\bar{C}_3^T - \bar{C}_1 R^{-1}\bar{C}_3^T + Q_{13}\Big)$$

(3) Compute T_2 solving the Riccati algebraic equation:

$$T_2^T\left(\frac{1}{\gamma^2}I - R^{-1}\right)T_2 - \bar{C}_2 R^{-1} T_2 - $$
$$-T_2^T R^{-1}\bar{C}_2^T - \bar{C}_2 R^{-1}\bar{C}_2^T + Q_2 + 2S = O$$

Substituting for the value of $V(x,t)$ into equation (5), the control law u^* which corresponds to the H_∞ optimal index γ is

$$u^* = -R^{-1}\left(\bar{C}^T + T\right)x$$

4. THE CONTROL LAW LIKE A NONLINEAR PID

There exists in the literature several control methods in which robot controllers have been posed as PID control -see, for example, (Ramirez J.A., Cervantes I., Kelly R., 2000). In this section it is shown how the obtained control law may be interpreted like a computed torque control with an external nonlinear PID controller. Substituting for the expressions of T, $\dot{x}$ and u^* in (8), and after some computations the optimal control law can be written as:

$$\tau^* = M\ddot{q}_r + V + G - \frac{1}{\rho}\left(M\begin{bmatrix} T_2 & T_3 \end{bmatrix}\begin{bmatrix} \dot{e} \\ e \end{bmatrix} + \right.$$

$$+CTx\Big) - \frac{1}{\rho}R^{-1}\left(\bar{C}^T + T\right)x$$

Keeping in mind the definition of x, the control law can be rewritten as follows

$$\tau^* = M\ddot{q}_r + V + G - \frac{1}{\rho}\left(M\left[\,T_2\ T_3\ 0\,\right] + \right.$$

$$\left. + CT + R^{-1}\left(\bar{C}^T + T\right)\right)x$$

or in a more compact expression:

$$\tau^* = M(q)\,\ddot{q}_r + V(q,\dot{q}) + G(q) -$$

$$-\left(K_D\dot{e} + K_P e + K_I \textstyle\int edt\right)$$

where

$$K_D = \frac{1}{\rho}\left(MT_2 + \left(\frac{1}{2}\dot{M} + N\right)T_1 + R^{-1}\left(\bar{C}_1^T + T_1\right)\right)$$

$$K_P = \frac{1}{\rho}\left(MT_3 + \left(\frac{1}{2}\dot{M} + N\right)T_2 + R^{-1}\left(\bar{C}_2^T + T_2\right)\right)$$

$$K_I = \frac{1}{\rho}\left(\left(\frac{1}{2}\dot{M} + N\right)T_3 + R^{-1}\left(\bar{C}_3^T + T_3\right)\right)$$

A particular case is obtained if components of the weighting matrix W^TW satisfy $Q_1 = w_1^2 I$, $Q_2 = w_2^2 I$, $Q_3 = w_3^2 I$, $R = w_u^2 I$, $Q_{12} = Q_{13} = Q_{23} = O$, and $\bar{C}_1 = \bar{C}_2 = \bar{C}_3 = O$. In this case the gain matrices take the form

$$K_D = \frac{\sqrt{w_2^2 + 2w_1w_3}}{w_1}M + \frac{1}{2}\dot{M} + N + \frac{1}{w_u^2}I$$

$$K_P = \frac{w_3}{w_1}M + \frac{\sqrt{w_2^2 + 2w_1w_3}}{w_1}\left(\frac{1}{2}\dot{M} + N + \frac{1}{w_u^2}I\right)$$

$$K_I = \frac{w_3}{w_1}\left(\frac{1}{2}\dot{M} + N + \frac{1}{w_u^2}I\right)$$

These expressions have an important property: *they do not depend on the parameter* γ. Thereby we have algebraic expressions to compute the general optimal solution for this particular case.

5. EXPERIMENTAL RESULTS FOR THE RM-10 INDUSTRIAL ROBOT

The RM-10, shown in Figure 1, is a six-degree-of-freedom revolute joint manipulator arm.

All the six joints are driven by DC-brushless low-inertia electric motors which provide a uniform torque for all joint positions, and enables high control torque peaks. Torque is delivered to the joint axis through gear reductions, thus RM-10 is an indirect-drive manipulator. The joints also provide an electric brake to block the manipulator arm in any position -see (System Robot, 1991).

Coupled to each motor axis there is a two-pole resolver device which provides an accurate measurement of the correspondent joint position.

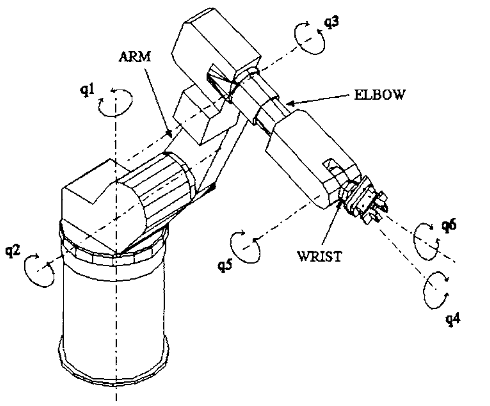

Fig. 1. The RM-10 Robot Manipulator

These measures will allow, as usual, the closed-loop control of the system. The RM-10 system employs a VME bus based architecture, providing independent control boards for every joint. Particularly, the real-time DS1103 control board, dSPACE trade, was employed (Implementation Guide, 1999), (MOO, 1998), (MOO, 1990). The control board was plugged into the expansion bus of a commercial PC, holding a 333MHz PowerPC as a main processor and an additional DSP as an input/output processor.

Before accomplishing the design of a controller it is necessary to obtain a dynamic model of the robot manipulator. According to the Euler-Lagrange formulation (Craig, J.J., 1989), the dynamic model of a general n-link rigid-body robot is a second order nonlinear equation, as shown in Equation 6.

In this case, the motion equation is complex and contains a number of hard-to-handle nonlinear terms (Perez, C., 1999). In order to simplify the controller design, friction terms in equation (6) have been neglected, and included as model uncertainty. This yields a very simplified model that only takes into account diagonal terms.

A number of additional parameters are required to characterize the dynamic model of the robot manipulator, such as link masses and inertias. These parameters have been estimated by geometric measurements and dynamical experiments of the robot arm. In Table 1, the estimated masses of the different links of the robot are shown. These values may help to make an idea of the characteristics of the robot.

link	mass (Kg)
1	38.65
2	51.80
3	84.10
4	33.89
5	7.36
6	5.00

Table 1. Estimated masses of the links

A diagonal $W^T W$ weighting matrix has been considered to design the controller. Table 2 shows the values for the diagonal weighting submatrices used for the RM-10 control synthesis.

Signal to minimize	**Weighting matrix**
Speed error $\dot{e}$	$Q_1 = \left(\frac{1}{2}\right)^2 I$
Position error e	$Q_2 = I$
Integral error $\int edt$	$Q_3 = 3^2 I$
Additional control effort u	$R = 0.4^2 I$

Table 2. Weights for the controller

In the experiments presented in this paper, the position references (which are computed by a trajectory generator) are fifth degree polynomials between the initial position $[q_1\ q_2\ q_3\ q_4\ q_5\ q_6] = [0.0\ 0.0\ 0.0\ 0.0\ 0.0\ 0.0\ ]$ rad to final position equal to $[0.6\ -0.3\ -0.3\ 0.6\ 0.6\ 0.6]$ rad with initial and final speeds and accelerations equal to zero. The transition time is 4.3 seconds.

The position errors and speed errors are shown in Figure 2 and 3 respectively, while the control signals generated by the controller are presented in Figure 4. It can be seen how the performance is aceptable for all axes, with small errors in both angular positions and velocities. In the same way, it should be noted how the position error does not tend to zero despite of the integral action. This effect is due to a dead zone because of friction in the actuators, which is not compensated in this work. This fact agrees with the evolution of some control signals, whose magnitudes increase until their respective actuators leave their dead zones.

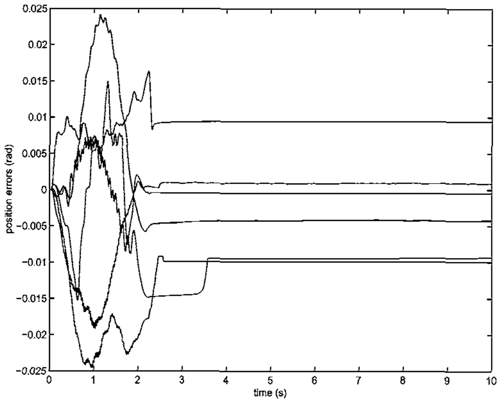

Fig. 2. Position errors in the six axes of the RM-10

6. SUMMARY

In this paper a nonlinear H_∞ control for robot manipulators has been developed which copes with persistent disturbances due to the inclusion of an integral term. This controller can be interpreted like a computed torque control with an external nonlinear PID controller. A particular case has been obtained in which the nonlinear PID gains do not depend on the value of the attenuation level γ. Finally, experimental results have been presented by using an industrial robot which proves the good performance supplied by this controller.

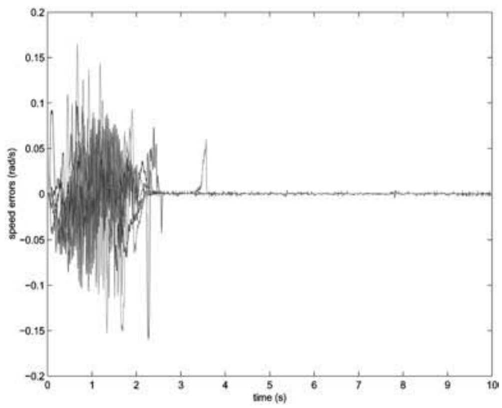

Fig. 3. Speed errors in the six axes of the RM-10

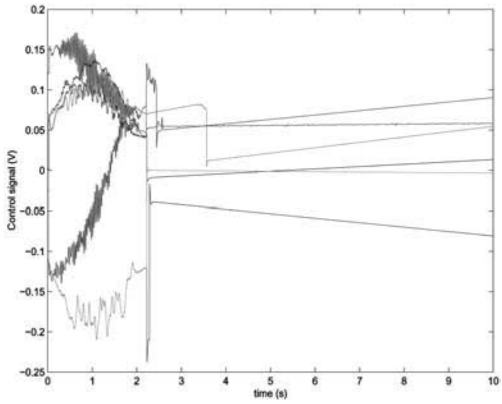

Fig. 4. Control signals generated by the controller

Acknowledgments: *The authors would like to acknowledge CICYT and FEDER for funding this work under grants TAP98-0541, DPI2000-1218-C01 and 1FD97-0783. Likewise, the authors appreciate the work by Rafael Romero Vicente in the realization of the real experiments.*

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Appendix A. PROOF OF RESULTS IN SECTION 3

We will show that the scalar function $V(x,t)$ – equation (14)– is the solution for the Hamilton-Jacobi equation (4). The gradient of $V(x,t)$ is given by

$$\frac{\partial V^T(x,t)}{\partial x} = x^T T_o^T \begin{bmatrix} M & O & O \\ O & Y & S-Y \\ O & S-Y & Z+Y \end{bmatrix} T_o + \frac{1}{2}\left[0_{1\times n}\ \omega\ 0_{1\times n}\right]$$

with $\omega \in \Re^{1\times n}$ and

$$\omega_i = x^T T_o^T \begin{bmatrix} \frac{\partial^T M}{\partial q_i} & O & O \\ O & O & O \\ O & O & O \end{bmatrix} T_o x,\ i = 1\ldots n.$$

It is easy to validate that

$$\left[0_{1\times n}\ \omega\ 0_{1\times n}\right](G(x,t)u + K(x,t)d) = 0.$$

Making some computations we can obtain this expression

$$\frac{\partial V^T(x,t)}{\partial x} f = x^T T_o^T \begin{bmatrix} M & O & O \\ O & Y & S-Y \\ O & S-Y & Z+Y \end{bmatrix} T_o f + \frac{1}{2}\left[0_{1\times n}\ \omega\ 0_{1\times n}\right]\dot{x}.$$

Using equation (13) the first term of the last expression can be written

$$x^T T_o^T \begin{bmatrix} M & O & O \\ O & Y & S-Y \\ O & S-Y & Z+Y \end{bmatrix} T_o f = x^T T_o^T \times$$

$$\times \begin{bmatrix} -\left(\frac{1}{2}\dot{M}+N\right) & O & O \\ \frac{1}{\rho}Y & S-\frac{1}{\rho}YT_2 & -S-\frac{1}{\rho}Y(T_3-T_2) \\ \frac{1}{\rho}(S-Y) & \begin{matrix} S+Z- \\ \frac{1}{\rho}(S-Y)T_2 \end{matrix} & \begin{matrix} -(S+Z) \\ -\frac{1}{\rho}(S-Y)(T_3-T_2) \end{matrix} \end{bmatrix} T_o x,$$

and for the second term we have

$$\frac{1}{2}\left[0_{1\times n}\ \omega\ 0_{1\times n}\right]\dot{x} = \frac{1}{2}x^T T_o^T \times$$

$$\times \begin{bmatrix} \sum_{k=1}^{n} \frac{\partial^T M}{\partial q_k}(\dot{q}_k - \dot{q}_{rk}) & O & O \\ O & O & O \\ O & O & O \end{bmatrix} T_o x.$$

The derivative of $V(x,t)$ with respect to the time is

$$\frac{\partial V}{\partial t} = \frac{1}{2}x^T T_o^T \begin{bmatrix} \sum_{k=1}^{n} \frac{\partial^T M}{\partial q_k}\dot{q}_{rk} & O & O \\ O & O & O \\ O & O & O \end{bmatrix} T_o x.$$

Adding the previous expressions we arrive at

$$\frac{\partial V}{\partial t} + \frac{\partial V^T(x,t)}{\partial x} f = x^T T_o^T \times$$

$$\times \begin{bmatrix} -N & O & O \\ \frac{1}{\rho}Y & S-\frac{1}{\rho}YT_2 & -S-\frac{1}{\rho}Y(T_3-T_2) \\ \frac{1}{\rho}(S-Y) & \begin{matrix} S+Z- \\ \frac{1}{\rho}(S-Y)T_2 \end{matrix} & \begin{matrix} -(S+Z)- \\ \frac{1}{\rho}(S-Y)(T_3-T_2) \end{matrix} \end{bmatrix} T_o x.$$

Since matrix N is skew-symmetric and due to the particular structure of T_o we can write

$$\frac{\partial V}{\partial t} + \frac{\partial V^T(x,t)}{\partial x} f = \frac{1}{2}x^T \begin{bmatrix} O & Y & S \\ Y & 2S & Z+2S \\ S & Z+2S & O \end{bmatrix} x.$$

Also we can compute $G^T(x,t)\frac{\partial V(x,t)}{\partial x} = Tx$, and since $G(x,t) = K(x,t)$ we have

$$\frac{1}{2}\frac{\partial^T V}{\partial x}\left(\frac{1}{\gamma^2}K(x,t)K(x,t) - G(x,t)R^{-1}G^T(x,t)\right)\frac{\partial V}{\partial x} =$$

$$= \frac{1}{2}x^T T^T\left(\frac{1}{\gamma^2}I - R^{-1}\right)Tx.$$

Using the computed expressions for the terms of the Hamilton-Jacobi equation given in (4) and the value of $h(x) = x$ we have

$$\frac{1}{2}x^T \begin{bmatrix} O & Y & S \\ Y & 2S & Z+2S \\ S & Z+2S & O \end{bmatrix} x +$$

$$+\frac{1}{2}x^T T^T\left(\frac{1}{\gamma^2}I - R^{-1}\right)Tx - x^T T^T R^{-1} C^T x +$$

$$+\frac{1}{2}x^T\left(Q - CR^{-1}C^T\right)x = 0,$$

and by simplifying we obtain expression (15).