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A Methodology for Modelling Travel Distances by Bias Estimation

Francisco A. Ortega
Departamento de Matemática Aplicada I. Universidad de Sevilla.
e-mail: riejos@cica.es
Juan A. Mesa
Departamento de Matemática Aplicada II
Escuela Superior de Ingenieros. Universidad de Sevilla.
e-mail: jmesa@cica.es

Abstract

Round norms τl_p, $p \in (1,2]$ and block norms have been utilised for modelling actual distances in transportation networks. A geometric setting will permit the establishment of a relationship between bias of the road network distance and trajectory deviations, which will be used to separate the set of origin-destination pairs into two samples and also to analyse each sample using regression, thus obtaining several types of estimators. What will be demonstrated in this paper is that these functions can be combined through either a weighted sum, or by means of the introduction of the expected distance concept applied to the bias, to obtain distance predicting functions for the region considered.

Key Words: Distance functions, l_p and block norms, expected distances.

AMS subject classification: 90C99, 93A99, 90D99.

1 Introduction

A continuous formulation may be an appropriate approximation to discrete formulation of location problems when the transportation network is both well-developed in the region considered and free from barriers. Continuous modelling demands a notion of planar distance. Which of the theoretical distances should be the most suitable for approximating shortest distances in the network?

Given a set of pairs Origin-Destination of data belonging to the area considered and given a family of metrics (in a broad sense) $d(p_1, p_2, \ldots, p_m)$,

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Correspondence to: Dr. Francisco A. Ortega. Dpto. Matemática Aplicada I. Escuela Técnica Superior de Arquitectura, Avda. Reina Mercedes s/n. 41012 Sevilla, Spain.
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depending on the parameters $p_1, p_2, \ldots, p_m$, the problem, called *estimation* (Love and Morris [1988]), consists of calibrating the parameters so that a global function of the errors, which measures deviations with respect to actual distances, is minimised.

In order to calibrate the parameters involved, the dimensionless ratio

$$\eta(\vec{\mathbf{xy}}) = \frac{d_G(\mathbf{x}, \mathbf{y})}{d_2(\mathbf{x}, \mathbf{y})}$$

is used, in which $d_G(\mathbf{x}, \mathbf{y})$ is the road distance between points $\mathbf{x}$ and $\mathbf{y}$ through the network G and $d_2(\mathbf{x}, \mathbf{y})$ the corresponding Euclidean distance on the plane. This ratio, called *directional bias* function (Brimberg and Love [1993 a]), is just a function of the angle formed by the vector connecting points $\mathbf{x}$ and $\mathbf{y}$ and the positive X-axis if we suppose that $d_G(\mathbf{x}, \mathbf{y})$ is a metric on the plane.

Thus, being $d_G(\mathbf{q}_i, \mathbf{r}_i)$, $i = 1, \ldots, M$, the actual distances between pairs of points $\mathbf{q}_i$ and $\mathbf{r}_i$ through the network, let $s(p_1, \ldots, p_m; \varphi_i)$ be the bias of the considered metrics and let

$$\mathcal{N} \equiv \{(\varphi_i, s_i) : \; i = 1, \ldots, M\}$$

be one sample set, randomly obtained, where φ_i is the angle between the vector connecting the i-th pair of points and the X-axis and $s_i = \dfrac{d_G(\mathbf{q}_i, \mathbf{r}_i)}{d_2(\mathbf{q}_i, \mathbf{r}_i)}$ is the bias of the road network distance with respect to the Euclidean distance; the problem will be to determine the parameters in order to obtain one of the following objectives

$$\min_{p_1, \ldots, p_m} AD_k = \sum_{i=1}^{M} \mid s(p_1, \ldots, p_m; \varphi_i) - s_i \mid^k; \quad k = 1, 2.$$

Let us note that for $k = 2$ and the family of metrics τl_p, the objective AD_2 coincides with SND_l, introduced by Brimberg and Love [1993 b].

Two hypotheses have been employed to estimate:

– supposing that the network has a *predominant Euclidean pattern* (then the actual travel distances can be estimated by using $d_2(\mathbf{x}, \mathbf{y}; \tau) = \tau \; l_2(\vec{\mathbf{xy}})$), and

– supposing that the network has a *predominant rectangular pattern* after a rotation with angle θ.

Definition 1 (Brimberg and Love [1993 b]). Let $\eta(\varphi;\theta)$ be the directional bias relative to the angle φ measured after rotating the original axes with orientation θ. The network has a *predominant rectangular pattern* when the following relation holds: $\eta(\varphi;\theta) = \tau_2 \cdot 1 + \tau_1 R(\varphi;\theta) + \epsilon(\varphi;\theta)$; where ϵ is an independent error term with mean zero, τ_1 and τ_2 are parameters with $\tau_i \geq 0$ and $R(\varphi;\theta)$ denotes a function of φ which holds:

(a) periodicity of $\pi/2$,

(b) unimodal cycle with a maximum at the halfway point of each periodic interval, and

(c) symmetry property with respect to the maximum of each cycle.

When this pattern is assumed a previous rotation of angle θ is considered convenient (Love and Walker [1994]) in order to improve the fitting and so the following functions (denoted as appear in Love, Morris and Wesolowsky [1988]) can be used for estimating:

- $d_1(\mathbf{x},\mathbf{y};\tau) = \tau\ l_1(\vec{\mathbf{xy}})$, when $\tau > 0$.
- $d_2(\mathbf{x},\mathbf{y};\tau) = \tau\ l_2(\vec{\mathbf{xy}})$, when $\tau > 0$.
- $d_3(\mathbf{x},\mathbf{y};\tau,p) = \tau\ l_p(\vec{\mathbf{xy}})$, when $\tau > 0$ and $p \in [1,\infty)$.
- $d_4(\mathbf{x},\mathbf{y};\tau,p,s) = \tau\ \left(l_p(\vec{\mathbf{xy}})\right)^{p/s}$, when $\tau > 0$ and $p \in [1,\infty)$, $p \geq s$.
- $d_6(\mathbf{x},\mathbf{y};\tau_1,\tau_\infty) = \tau_1\ l_1(\vec{\mathbf{xy}}) + \sqrt{2}\ \tau_\infty\ l_\infty(\vec{\mathbf{xy}})$, when $\tau_1, \tau_\infty > 0$.

Another function used in the literature is $d(\mathbf{x},\mathbf{y};\tau_1,\tau_2) = \tau_1\ l_1(\vec{\mathbf{xy}}) + \tau_2\ l_2(\vec{\mathbf{xy}})$, with $\tau_1,\ \tau_2 > 0$ (Brimberg and Love [1992]).

In all these cases, the parameters involved were determined from a single sample of origin–destination pairs and the usual solution technique does not employ regression due to the distance functions not being polynomial; hence, a specialised computer programme is required to fit the model. In order to eliminate this need, Brimberg, Dowling and Love [1994] have used a linear regression analysis to determine parameters τ_1 and τ_2 in $d(\mathbf{x},\mathbf{y};\tau_1,\tau_2)$ and then to substitute a weighted l_p norm in place of the earlier weighted one-two norm as distance predictor (Brimberg, Dowling and Love [1996]).

The aim of this study is to establish a methodology for the estimation process consisting of separating the initial set of origin-destination pairs into two samples before determining the parameters using regression, and to show that the estimators obtained can be combined through either a weighted sum, or by using a density function according to the subsample weights, which permits the global expected bias for each direction to be obtained.

In section 2, a relationship between bias and trajectory deviations is established giving rise to a new hypothesis for estimation. Sections 3 and 4 are devoted to the determination of the parameters to fit the respective subsamples. In section 5, the different weighted sums which globally permit distances to be estimated are summarised. Numerical results belonging to three networks are provided in section 6. Finally, in section 7, a density function for distributing the bias in fixed direction φ for the network is calculated and some conclusions are stated in section 8.

2 A new hypothesis

For all values $\eta \geq 1$ of the bias, it is possible to find an angle, which will be called the *least deflection of trajectory*, characterized by the following lemma.

Lemma 2. *Let* **F** *and* **F'** *be two points in the plane and* $d_G(\mathbf{F},\mathbf{F'})$ *the actual distance between them. The least trajectory deflection* ϕ *between any pair of edges* **PF** *and* **PF'** *connecting* **F** *and* **F'** *and such that* $d_2(\mathbf{F},\mathbf{P}) + d_2(\mathbf{P},\mathbf{F'}) = d_G(\mathbf{F},\mathbf{F'})$ *holds, is given by:*

$$\phi = \arccos\ (2\left(\frac{d_2(\mathbf{F},\mathbf{F'})}{d_G(\mathbf{F},\mathbf{F'})}\right)^2 - 1) = \arccos\ (\frac{2}{\eta^2} - 1).$$

In order to give a brief summary of the proof, which can be found in Ortega [1997], let us consider the ellipse with focus **F** and **F'** and sum of distances $d_G(\mathbf{F},\mathbf{F'})$, i.e. the set of points **P** holding:

$$d_2(\mathbf{F},\mathbf{P}) + d_2(\mathbf{P},\mathbf{F'}) = d_G(\mathbf{F},\mathbf{F'}).$$

A non-difficult calculation leads us to the analytical expression of trajectory deflection ϕ as a function of the bias η associated with vector $\vec{\mathbf{FF'}}$. Points in the ellipse providing least trajectory deflection are obtained considering the intersections of mediatrix line of segment **FF'** with this ellipse (see Figure 1), following that $\phi = \arccos\ (\frac{2}{\eta^2} - 1)$.

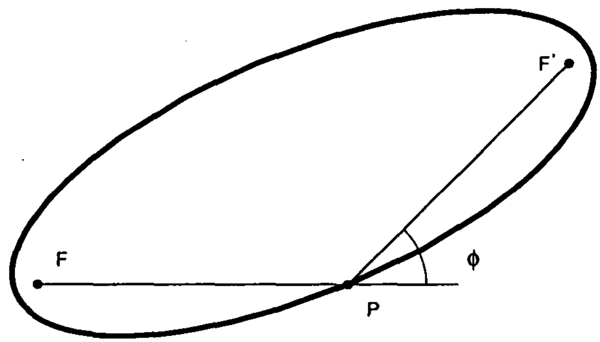

Figure 1: Least deflection of trajectory

Definition 3 (Mesa and Ortega [1996]). A ϕ-inclined block norm is defined as that whose unit ball B is the polytope with four vertices:

$$\{\pm(1,0), \pm(\cos\phi, \sin\phi)\}, \text{ where } \phi \in (0, \pi/2].$$

The ϕ-inclined block norm, denoted by $\|\cdot\|_B$, can be used in order to measure the length of the total trip on the dense ϕ-inclined network (Widmayer, Yu and Wong [1987]), and the explicit expression of the corresponding bias can be found in Mesa and Ortega [1996]:

$$r_B(\varphi) = \frac{\cos(\varphi - \phi/2)}{\cos(\phi/2)}; \ \forall\varphi \in [0, \phi].$$

¿From now on, the bias functions corresponding to the distances induced by the ϕ- and θ-inclined block norms will be called ϕ-bias and θ-bias respectively, being denoted by $r_B(\cdot)$ and $r_{B(\theta)}(\cdot)$ in each case. The following result is obtained taking into account that the inclination of line **FF'** in the previous figure was $\phi/2$.

Theorem 4. *The distance in the planar network between two points* **F** *and* **F'** *can be expressed in terms of ϕ-bias by: $d_G(\mathbf{F}, \mathbf{F'}) = \| \vec{\mathbf{FF'}} \|_2 \ r_B(\phi/2)$, where $\phi = \arccos(2\left(\frac{d_2(\mathbf{F}, \mathbf{F'})}{d_G(\mathbf{F}, \mathbf{F'})}\right)^2 - 1)$. Moreover, by using θ-bias, with $\theta \in [\phi, \pi)$ and an inflating factor $\tau_{B(\theta)} \geq 1$ it is possible to obtain:*

$$d_G(\mathbf{F}, \mathbf{F'}) = \| \vec{\mathbf{FF'}} \|_2 \ \tau_{B(\theta)} \ r_{B(\theta)}(\theta/2).$$

A one-to-one correspondence between values of $p \in [1,2)$ and angles $\phi \in (0,\pi/2]$ and a further correspondence between values of $p \in (0,1)$ and angles $\phi \in (\pi/2,\pi)$ were established (in Mesa and Ortega [1996]) by using interpolation in order to fit the points where the respective bias reach the maximum. Namely, $r_B(\phi/2) = r_p(\pi/4)$. Therefore, the relationship between *actual bias*, *least deflection angle* and *ϕ-bias* or *p-bias* has been completed.

We must note that $\varphi = 0$ and $\varphi = \phi$ represent the first unbiased directions for the ϕ-inclined block norm $(r_B(0) = r_B(\phi) = 1)$. An actual bias $\eta(\varphi)$ estimated by the bias function corresponding to the ϕ-inclined block norm, where $\phi \in (\pi/2,\pi)$, cannot hold the periodicity of $\pi/2$.

On the other hand, periodicity of π is a necessary property for the bias with central symmetry, derived from distance functions as block norms in general or as l_p functions $p \in (0,\infty]$. In this geometric setting based on the use of ϕ-inclined block norms as estimators, the following hypothesis can be formulated.

Definition 5. The network has a *predominant rectangular–wide-angled pattern* when the following relationship holds:

$$\eta(\varphi;\theta) = \tau_0 A(\varphi;\theta) + \tau_1 R(\varphi;\theta) + \epsilon(\varphi;\theta)$$

where $R(\varphi;\theta)$, $\epsilon(\varphi;\theta)$ remain as in Definition 1 and $A(\varphi;\theta)$ represents the family of functions which holds:

(a) periodicity of π, and, as previously,

(b) unimodal cycle with a maximum at the halfway point of each periodic interval and

(c) symmetry property with respect to the maximum of each cycle.

The properties of the ϕ-bias were studied in Mesa and Ortega [1996], showing that properties (a), (b) and (c) are satisfied by the bias function of $\|\cdot\|_B$.

3 Fitting subsample $\mathcal{P}$

The correspondences between values of $p \in [1,2)$ and angles $\phi \in (0,\pi/2]$ and that between values of $p \in (0,1)$ and angles $\phi \in (\pi/2,\pi)$, suggest decomposing the sample to be analysed into two subsamples.

¿From initial sample $\mathcal{N} \equiv \{(\varphi_i, s_i) : i = 1, \ldots, M\}$, the partition $\mathcal{N} = \mathcal{N}_p \cup \mathcal{N}_q$ is generated by comparing each least deflection angle corresponding to s_i, $\phi(s_i) = \arccos\left(\frac{2}{s_i^2} - 1\right)$, with $\pi/2$. Therefore, two subsamples $\mathcal{N}_p$ and $\mathcal{N}_q$, whose lengths are denoted by $\mathcal{I}_p$ and $\mathcal{I}_q$ respectively, are obtained in this way:

$$\mathcal{N}_p = \{(\varphi_i, s_i) \in \mathcal{N} : \phi(s_i) \leq \pi/2\}; \ \mathcal{N}_q = \{(\varphi_i, s_i) \in \mathcal{N} : \phi(s_i) > \pi/2\}.$$

Since the correspondence between values of $p \in [1, 2)$ and angles $\phi \in (0, \pi/2]$ was obtained by imposing $r_B(\phi/2) = r_p(\pi/4)$, the first components of pairs belonging to set $\mathcal{N}_p$ are transferred so that they remain in the interval $[0, \pi/2]$ centred at $\pi/4$. With the new set obtained

$$\mathcal{P} = \{(t_i, s_i) : \ t_i = \ \mathrm{mod}(\varphi_i, \pi/2) - \pi/4, \text{ and } (\varphi_i, s_i) \in \mathcal{N}_p\}$$

the following regression problem is solved:

$$\min_{a,b} \ E_H^{\mathcal{P}}(a, b; t) := \sum_{i=1}^{\mathcal{I}_p} \left(s_i - a(1 + bt_i^2)\right)^2.$$

The behaviour of functions $\tau r_p(\cdot)$ and $\tau r_B(\cdot)$ in a neighbourhood of $\pi/2$ is the reason for which a quadratic regression as the one described above is considered. The proof of the following lemmas is detailed in Ortega [1997].

Lemma 6. *The function $H(a, b; t) = a(1 + bt^2)$ which approximates the data set $\mathcal{P}$ in the sense of least squares is obtained by:*

$$\begin{cases} a = \dfrac{\left(\sum_{i=1}^{\mathcal{I}_p} s_i\right)\left(\sum_{i=1}^{\mathcal{I}_p} t_i^4\right) - \left(\sum_{i=1}^{\mathcal{I}_p} t_i^2\right)\left(\sum_{i=1}^{\mathcal{I}_p} s_i t_i^2\right)}{\mathcal{I}_p\left(\sum_{i=1}^{\mathcal{I}_p} t_i^4\right) - \left(\sum_{i=1}^{\mathcal{I}_p} t_i^2\right)^2} \\[2em] b = \dfrac{\mathcal{I}_p\left(\sum_{i=1}^{\mathcal{I}_p} s_i t_i^2\right) - \left(\sum_{i=1}^{\mathcal{I}_p} s_i\right)\left(\sum_{i=1}^{\mathcal{I}_p} t_i^2\right)}{\left(\sum_{i=1}^{\mathcal{I}_p} s_i\right)\left(\sum_{i=1}^{\mathcal{I}_p} t_i^4\right) - \left(\sum_{i=1}^{\mathcal{I}_p} t_i^2\right)\left(\sum_{i=1}^{\mathcal{I}_p} s_i t_i^2\right)} \end{cases}$$

whenever: *(1)* $\mathcal{I}_p \geq 2$; *(2)* $s_i \in [1, \sqrt{2}]$, $\forall i$; *(3) there exist at least two different values of* t_i.

Condition (2) is not restrictive since it is a logical consequence of the definition of subsample $\mathcal{P}$: $s_i \geq 1$, $\forall (t_i, s_i) \in \mathcal{P}$, and since $\arccos(z)$ is a decreasing function for $z \in [0, \pi]$, we have that

$$\phi(s_i) = \arccos(\frac{2}{s_i^2} - 1) \leq \pi/2 = \arccos(0)$$

implies $\frac{2}{s_i^2} - 1 \geq 0$, and so $s_i \leq \sqrt{2}$.

Lemma 7. *Whenever the previous conditions (1), (2) and (3) hold, the coefficient* a *belongs to the interval* $[1, \sqrt{2}]$.

We must note that $\forall i = 1, \ldots, \mathcal{I}_p$, $t_i \in [-\pi/4, \pi/4] \subset (-1, 1)$, and so $t_i^2 \in [0, 1)$. Property $b \in (-1, 0)$ is desirable in order to obtain, by theorem 9, a value of $p \in (0, 2)$. The conditions (4) and (5) of the following lemma establish a sufficient condition to place b in the interval $(-1, 0)$, by using three weighted sums of t_i^2 with coefficients 1, $s_i \in [1, \sqrt{2}]$ and $(1 - t_i^2) \in (0, 1]$. When conditions (4) and (5) are not satisfied in subsample $\mathcal{P}$, the polynomial approximating function $a(1 + bt^2)$ cannot determine, in the sense of least squares, other estimators in terms of $\tau r_p(\cdot)$ or $\tau r_B(\cdot)$. In all the examples included in this paper and in all the networks considered so far by the authors, conditions (4) and (5) have always been accomplished.

Lemma 8. *In addition to the previous conditions (1), (2) and (3), if the following inequalities also hold*

$$(4) \quad \frac{\sum_{i=1}^{\mathcal{I}_p} s_i t_i^2}{\sum_{i=1}^{\mathcal{I}_p} s_i} < \frac{\sum_{i=1}^{\mathcal{I}_p} t_i^2}{\mathcal{I}_p}; \qquad (5) \quad \frac{\sum_{i=1}^{\mathcal{I}_p} (1 - t_i^2) t_i^2}{\sum_{i=1}^{\mathcal{I}_p} (1 - t_i^2)} < \frac{\sum_{i=1}^{\mathcal{I}_p} s_i t_i^2}{\sum_{i=1}^{\mathcal{I}_p} s_i}$$

then the coefficient b *is located in the interval* $(-1, 0)$.

In order to reach a well-defined fitting of subsample $\mathcal{P}$ for one future application, the conditions (4) and (5), included in the previous lemma, must be added.

By using the Taylor series of the function $\tau r_p(\cdot)$, where

$$r_p(\pi/4+t) = \frac{2^{1/p}}{\sqrt{2}} \left(1 + (\frac{p}{2} - 1)\, t^2\right) + o(t^2);\ \forall t \in (-\pi/4, \pi/4),$$

and identifying the early homogeneous coefficients, we obtain the parameter values which approximate the data set $\mathcal{P}$ in the sense of least squares, as the following theorem indicates (the proof can be found in Ortega [1997]).

Theorem 9. *The function $\tau r_p(\cdot)$ which approximates the data set $\mathcal{P}$ in the sense of least squares is obtained by: $p = 2(1+b)$; $\tau = a\frac{\sqrt{2}}{2^{1/p}}$, where $\mathcal{P}$ must satisfy the conditions of lemma 8 and where a and b are the values deduced in lemma 6.*

Similarly, the function $\tau r_B(\cdot)$ which approximates the data set $\mathcal{P}$ in the sense of least squares is obtained by the following result (again in Ortega [1997]) which uses the Taylor series for the function $r_B(\varphi) = \dfrac{\cos(\varphi - \phi/2)}{\cos(\phi/2)}$ when $\varphi = \phi/2 + 2\,\dfrac{\phi t}{\pi}$; $\forall t \in [-\pi/4, \pi/4]$.

Theorem 10. *Let B be the polytope with vertices $\{\pm(1,0), \pm(\cos\phi, \sin\phi)\}$, where $\phi \in (0, \pi/2]$. The function $\tau r_B(\cdot)$ which approximates the data set $\mathcal{P}$ in the sense of least squares is obtained by: $\phi = \pi\sqrt{\frac{-b}{2}}$; $\tau = a\cos(\frac{\phi}{2})$, where $\mathcal{P}$ must satisfy the conditions of lemma 8 and a and b are the values deduced in lemma 6.*

4 Fitting subsample $\mathcal{Q}$

A similar procedure is employed with the complementary subsample $\mathcal{N}_q$. The first components are transferred to the interval $[0, \pi]$ centred in $\pi/2$, obtaining in this way subsample $\mathcal{Q}$ to be fitted:

$$\mathcal{Q} = \{(t_i, s_i) :\ t_i = \ \mathrm{mod}(\varphi_i, \pi) - \pi/2,\ (\varphi_i, s_i) \in \mathcal{N}_q.\}$$

If we assume conditions: (1) $\mathcal{I}_q \geq 2$; (2) $s_i \geq \sqrt{2}$, $\forall i = 1, \ldots, \mathcal{I}_q$ (condition $s_i \geq \sqrt{2}$ being a direct consequence of $(t_i, s_i) \in \mathcal{Q}$); (3) there exist at least two different values of t_i; then the corresponding regression problem

$$\min_{a,b} E_H^Q(a,b;t) := \sum_{i=1}^{\mathcal{I}_q} \left(s_i - a(1+bt_i^2)\right)^2$$

is solved when coefficients a and b are expressed as indicated in lemma 6, substituting $\mathcal{I}_q$ instead of $\mathcal{I}_p$.

Here, the condition $b \in (-1/4, 0)$ will be necessary in order to allow fitting by functions $\tau r_p(\cdot)$ and $\tau r_B(\cdot)$. This property is guaranteed if the following relationships hold (see Ortega [1997]):

$$\frac{\sum_{i=1}^{\mathcal{I}_q} (4-t_i^2)t_i^2}{\sum_{i=1}^{\mathcal{I}_q} (4-t_i^2)} < \frac{\sum_{i=1}^{\mathcal{I}_q} s_i t_i^2}{\sum_{i=1}^{\mathcal{I}_q} s_i} < \frac{\sum_{i=1}^{\mathcal{I}_q} t_i^2}{\mathcal{I}_q}$$

These inequalities have always been satisfied in the examples included in this paper and in all the networks considered to date by the authors.

Now, $\forall i = 1, \ldots, \mathcal{I}_q$, $t_i \in [-\pi/2, \pi/2] \subset (-2, 2)$, and so $t_i^2 \in [0, 4)$. Hence, the coefficients used in the weighted sums are 1, $s_i \in [\sqrt{2}, \infty]$ and $(4-t_i^2) \in (0, 4]$.

By repeating the identification process for the coefficients in the Taylor series corresponding to $\tau r_p(\pi/2 + t)$ and $\tau r_B(\phi/2 + \frac{\phi t}{\pi})$, the following theorems are obtained (more details can be found in Ortega [1997]).

Theorem 11. *The function $\tau r_p(\cdot)$ which approximates the data set Q in the sense of least squares is obtained by:* $p = 2(1+4b)$; $\tau = a\dfrac{\sqrt{2}}{2^{1/p}}$.

Theorem 12. *Let B be the polytope with vertices $\{\pm(1,0), \pm(\cos\phi, \sin\phi)\}$, where $\phi \in (0, \pi/2]$. The function $\tau r_B(\cdot)$ which approximates the data set Q in the sense of least squares is obtained by:*

$$\phi = \pi\sqrt{-2b} \ ; \quad \tau = a\cos(\frac{\phi}{2})$$

5 Combining the partial fitting functions

The polynomial functions $H_p(a_p, b_p; t) = a_p(1 + b_p t^2)$ and $H_q(a_q, b_q; t) = a_q(1 + b_q t^2)$, which partially fit subsamples $\mathcal{P}$ and $\mathcal{Q}$ respectively, can be combined according to the size of each subsample in order to obtain a global estimator. Therefore, the network distance corresponding to the trip represented by the vector $\mathbf{x}$ (argument φ) can be estimated by using

$$(I)\ \ d_G(\mathbf{x}) \approx \| \mathbf{x} \|_2 \ \Big(\frac{\mathcal{I}_p}{\mathcal{I}_p + \mathcal{I}_q}\ H_p(a_p, b_p;\ \mathrm{mod}(\varphi, \pi/2) - \pi/4) + \frac{\mathcal{I}_q}{\mathcal{I}_p + \mathcal{I}_q}\ H_q(a_q, b_q;\ \mathrm{mod}(\varphi, \pi) - \pi/2) \Big)$$

Theorems 9 and 11 permit the expression of the previous estimation in terms of l_p functions. If we denote

$$\begin{cases} p = 2(1 + b_p);\ q = 2(1 + 4b_q) \\ \tau_p = a_p \dfrac{\sqrt{2}}{2^{1/p}};\ \tau_q = a_q \dfrac{\sqrt{2}}{2^{1/q}} \end{cases}$$

then

$$(II)\ \ d_G(\mathbf{x}) \approx \| \mathbf{x} \|_2 \ \Big(\frac{\mathcal{I}_p}{\mathcal{I}_p + \mathcal{I}_q}\ \tau_p r_p(\ \mathrm{mod}(\varphi, \pi/2)) + + \frac{\mathcal{I}_q}{\mathcal{I}_p + \mathcal{I}_q}\ \tau_q r_q(\ \mathrm{mod}(\varphi/2, \pi/2)) \Big)$$

And, similarly, the expression in terms of ϕ-inclined block norms is possible by way of Theorems 9 and 11. If we denote

$$\begin{cases} \phi_p = \pi \sqrt{\dfrac{-b_p}{2}};\ \phi_q = \pi \sqrt{-2b_q} \\ \tau_p = a_p \cos(\dfrac{\phi_p}{2});\ \tau_q = a_q \cos(\dfrac{\phi_q}{2}) \end{cases}$$

then

$$(III)\ \ d_G(\mathbf{x}) \approx \| \mathbf{x} \|_2 \ \Big(\frac{\mathcal{I}_p}{\mathcal{I}_p + \mathcal{I}_q}\ \tau_p\ \frac{\cos(\frac{2\phi_p}{\pi}\ \mathrm{mod}(\varphi, \pi/2) - \phi_p/2)}{\cos(\phi_p/2)} +$$

$$\frac{\mathcal{I}_q}{\mathcal{I}_p+\mathcal{I}_q}\,\tau_q\,\frac{\cos(\frac{\phi_q}{\pi}\ \mathrm{mod}(\varphi,\pi)-\phi_q/2)}{\cos(\phi_q/2)}\Big)$$

6 Examples

(1) We have tested this methodology in a typical transportation network as in Figure 4 of the appendix ($n = 9$ nodes and $M = 36$ pairs origin-destination), where the reference axes were orientated in order to match the directional bias of the distance function (weighted l_p) with that of the network. By using a searching procedure based on the algorithm proposed by Brimberg and Love [1991] for $\Delta p = 0.05$, we obtain the minimum of the sum:

$$AD(\tau,p)=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\mid \tau \parallel \vec{\mathbf{v}_i\mathbf{v}_j} \parallel_p - d_G(\mathbf{v}_i,\mathbf{v}_j)\mid$$

The minimum of $AD(\tau,p)$ occurs when $\tau = 1.25792$ and $p = 1.45$. In relation with the directional bias of $\omega_{ij} = \mathrm{Arg}(\vec{\mathbf{v}_i\mathbf{v}_j})$, the deviation sum was

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\mid \tau r_p(\omega_{ij}) - \frac{d_G(\mathbf{v}_i,\mathbf{v}_j)}{\parallel \vec{\mathbf{v}_i\mathbf{v}_j} \parallel_2}\mid = 4.04713,$$

which represents a mean $4.04713/M$ equal to 0.11242. If the criterion AD_k (for $k = 1$), described in section 1, had been used then the parameters minimizing the earlier deviation sum would have been $\tau = 1.31324$ and $p = 1.65$, and the total error would be 3.99409.

Applying our methodology, the data set $\mathcal{P}$ ($\mathcal{I}_p = 22$) is fitted by the function $H_p(t) = 1.30229(1 - 0.0928454\ t^2)$, $t \in [-\pi/4, \pi/4]$. The coefficients a and b generate:

$$\begin{cases} p = 1.81431 \\ \tau_p = 1.25691 \end{cases} ; \quad \begin{cases} \phi(p) = 0.67688 \ (38.78 \text{ degrees}) \\ \tau_{\phi(p)} = 1.22842 \end{cases}$$

and the partial error in the subsample $\mathcal{P}$ is:

$$\sum_{(t_i,s_i)\in\mathcal{P}} \mid \tau_p r_p(t_i+\pi/4) - s_i \mid = 1.47505;\ \text{mean } 1.47505/\mathcal{I}_p = 0.06704$$

$$\sum_{(t_i,s_i)\in\mathcal{P}} \mid \tau_\phi r_B(\frac{2\phi t_i}{\pi}) - s_i \mid = 1.75294; \text{ mean } 1.75294/\mathcal{I}_p = 0.079679.$$

On the other hand, the complementary data set $\mathcal{Q}$ ($\mathcal{I}_q = 14$) is fitted by $H_q(t) = 1.53076(1 - 0.01616\ t^2)$, $t \in [-\pi/2, \pi/2]$, and the respective coefficients a and b give rise to:

$$\begin{cases} q = 1.87072 \\ \tau_q = 1.49453 \end{cases} ; \quad \begin{cases} \phi(q) = 0.56478 \text{ (32.36 degrees)} \\ \tau_{\phi(q)} = 1.47013 \end{cases}$$

The partial error in the subsample $\mathcal{Q}$ is now:

$$\sum_{(t_i,s_i)\in\mathcal{Q}} \mid \tau_q r_q(t_i/2 + \pi/4) - s_i \mid = 0.844882; \text{ mean } 0.844882/\mathcal{I}_q = 0.060348$$

$$\sum_{(t_i,s_i)\in\mathcal{Q}} \mid \tau_\phi r_B(\frac{\phi t_i}{\pi}) - s_i \mid = 0.919363; \text{ mean } 0.919363/\mathcal{I}_q = 0.065667.$$

Summarising, the partial errors in subsamples $\mathcal{P}$ and $\mathcal{Q}$ were smaller than those obtained following the previous methodology (when partial means and the global mean are compared), but the total error (being 4.13848 when type $\tau_p l_p$ estimations are combined by using approximation (II) and 4.08999 when type $\tau_\phi r_B$ estimations are used by means of (III)) was similar to the smallest error.

(2) We have also tested this methodology in a network circumnavigating a barrier, as in Figure 5 of the appendix ($n = 9$ nodes and $M = 36$ pairs origin-destination again).

Here, the minimum of $AD(\tau, p)$ occurs when $\tau = 1.6736$ and $p = 1.7$. The global error, in relation with the directional bias, is:

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mid \tau r_p(\omega_{ij}) - \frac{d_G(\mathbf{v}_i, \mathbf{v}_j)}{\| \vec{\mathbf{v}_i\mathbf{v}_j} \|_2} \mid = 13.1504,$$

whose mean $13.1504/M$ is 0.365289. If the criterion AD_k (for $k = 1$) had been applied then the total error would be 12.9419 by using the parameters $\tau = 1.71814$ and $p = 3$.

Repeating the process for this network, we obtain $\mathcal{I}_p = 14$ and $\mathcal{I}_q = 22$ and the polynomial fitting functions:

$$H_p(t) = 1.36299(1 - 0.185413\, t^2),\ t \in [-\pi/4, \pi/4];$$

$$H_q(t) = 2.30711(1 - 0.133364\, t^2),\ t \in [-\pi/2, \pi/2]$$

The respective estimation coefficients for each type have been calculated

$$\text{subsample } \mathcal{P} \begin{cases} p = 1.62917 \\ \tau_p = 1.2596 \end{cases} ; \begin{cases} \phi(p) = 0.95654 \text{ (54.80 degrees)} \\ \tau_{\phi(p)} = 1.21005 \end{cases}$$

$$\text{subsample } \mathcal{Q} \begin{cases} q = 0.933085 \\ \tau_q = 1.55227 \end{cases} ; \begin{cases} \phi(q) = 1.62249 \text{ (92.96 degrees)} \\ \tau_{\phi(q)} = 1.58866 \end{cases}$$

and the partial errors are

$$\sum_{(t_i,s_i)\in\mathcal{P}} \mid \tau_p r_p(t_i + \pi/4) - s_i \mid = 0.707134; \text{ mean } 0.707134/\mathcal{I}_p = 0.0505096,$$

$$\sum_{(t_i,s_i)\in\mathcal{P}} \mid \tau_\phi r_B(\frac{2\phi t_i}{\pi}) - s_i \mid = 1.56913; \text{ mean } 1.56913/\mathcal{I}_p = 0.112081,$$

$$\sum_{(t_i,s_i)\in\mathcal{Q}} \mid \tau_q r_q(t_i/2 + \pi/4) - s_i \mid = 6.54733; \text{ mean } 6.54733/\mathcal{I}_q = 0.297606,$$

$$\sum_{(t_i,s_i)\in\mathcal{Q}} \mid \tau_\phi r_B(\frac{\phi t_i}{\pi}) - s_i \mid = 11.4851; \text{ mean } 11.4851/\mathcal{I}_q = 0.52205.$$

The final combination has involved an improvement upon the best estimation previously obtained in both cases, because the total error was 12.1989, using type $\tau_p l_p$ functions by means of (II), and 12.2293, using type $\tau_\phi r_B$ functions by employing (III).

(3) In order to test the new methodology in an actual network, as can be found in Figure 6 of the appendix, we have considered a geographical region in Southwestern Spain in which the road network consists of $n = 44$ nodes and $M = 946$ pairs origin-destination, containing the main cities (as Sevilla, Málaga, Córdoba, Cádiz and Huelva), and circumnavigates the protected area called the National Park of Doñana.

The reference axes have had to be rotated through an angle of 23 degrees with respect to North-South orientation so that new axes and directional bias of the distance function derived from l_p norm are in phase (this best orientation has been selected by using a grid search for $\Delta\theta = 1$ degree and $\Delta p = 0.05$ between $p = 1$ and $p = 1.95$).

The best-fit values of τ and p for AD criterion were $\tau = 1.82078$ and $p = 1.95$. In terms of directional bias, the deviation sum was:

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mid \tau r_p(\omega_{ij}) - \frac{d_G(\mathbf{v}_i, \mathbf{v}_j)}{\| \mathbf{v}_i\vec{\mathbf{v}}_j \|_2} \mid = 297.951; \text{ mean } 297.951/M = 0.314959.$$

On the other hand, the best-fit values of τ and p for AD_1 criterion were $\tau = 1.41898$ and $p = 2.2$, and the deviation sum was 289.762.

Data set $\mathcal{P}$ ($\mathcal{I}_p = 25$) was fitted by the function:

$$H_p(t) = 1.29714(1 - 0.135138\, t^2),\ t \in [-\pi/4, \pi/4],$$

following our methodology, and coefficients $a = 1.29714$ and $b = -0.135138$ allow us to express the parameters

$$\begin{cases} p = 1.72972 \\ \tau_p = 1.22876 \end{cases} ; \quad \begin{cases} \phi(p) = 46.7892 \text{ degrees} \\ \tau_{\phi(p)} = 1.1905 \end{cases}$$

necessary to build new estimations. Hence, the partial errors for these estimations were:

$$\sum_{(t_i,s_i)\in\mathcal{P}} \mid \tau_p r_p(t_i + \pi/4) - s_i \mid = 2.3989; \text{ mean } 2.3989/\mathcal{I}_p = 0.095956,$$

$$\sum_{(t_i,s_i)\in\mathcal{P}} \mid \tau_\phi r_B(\frac{2\phi t_i}{\pi}) - s_i \mid = 2.50528; \text{ mean } 2.50528/\mathcal{I}_p = 0.100211.$$

By repeating the process for data set $\mathcal{Q}$ ($\mathcal{I}_q = 921$), we obtain the polynomial fitting function

$$H_q(t) = 2.20503(1 - 0.120548\, t^2),\ t \in [-\pi/2, \pi/2]$$

The respective estimation coefficients were calculated, being

$$\begin{cases} q = 1.03561 \\ \tau_q = 1.5968 \end{cases} ; \quad \begin{cases} \phi(q) = 88.3829 \text{ degrees} \\ \tau_{\phi(q)} = 1.58104 \end{cases}$$

and the partial errors obtained were

$$\sum_{(t_i,s_i)\in Q} \mid \tau_q r_q(t_i/2+\pi/4) - s_i \mid = 277.948; \text{ mean } 277.948/\mathcal{I}_q = 0.301789,$$

$$\sum_{(t_i,s_i)\in Q} \mid \tau_\phi r_B(\frac{\phi t_i}{\pi}) - s_i \mid = 299.899; \text{ mean } 299.899/\mathcal{I}_q = 0.325623.$$

The weighted combination, in accordance with the subsample sizes, involved an improvement with respect to value 297.951, because the total error was 289.426, using type $\tau_p l_p$ functions by means of (II), and 289.287, using type $\tau_\phi r_B$ functions by applying (III).

7 Directional bias distribution

The previous methodology generated two approximating formulas $e_p \equiv e_p(\varphi)$ and $e_q \equiv e_q(\varphi)$ of average bias for each subsample depending on the fixed angle φ considered. In relation to the estimation in terms of l_p functions, $e_p(\varphi)$ and $e_q(\varphi)$ are given by:

$$e_p(\varphi) = \tau_p r_p(\text{ mod}(\varphi, \pi/2)); \; e_q(\varphi) = \tau_q r_q(\text{ mod}(\varphi/2, \pi/2)).$$

Hence, the total directional bias in the network can be estimated by

$$\eta(\varphi) \approx \lambda_p e_p(\varphi) + \lambda_q e_q(\varphi); \; \lambda_p = \frac{\mathcal{I}_p}{M}, \; \lambda_q = \frac{\mathcal{I}_q}{M}, \; \lambda_p + \lambda_q = 1$$

The expression $\lambda_p e_p(\varphi) + \lambda_q e_q(\varphi)$ can be assumed to be the mean of a density function $\lambda_p f_p(\eta(\varphi)) + \lambda_q f_q(\eta(\varphi))$, where $e_p(\varphi)$ and $e_q(\varphi)$ could be the respective means of density functions $f_p(\eta(\varphi))$ and $f_q(\eta(\varphi))$. Therefore, we have assumed the existence of two density functions, $f_p(\eta(\varphi))$ and $f_q(\eta(\varphi))$, whilst conserving the property that both e_p and e_q ($e_p \leq e_q$) are centralised measures for the bias distributions in the respective ranges.

The total range of bias (for each angle φ considered) can be partitioned into three intervals $\eta \equiv \eta(\varphi) \in [1,\infty) = [1,e_p) \cup [e_p,e_q) \cup [e_q,\infty)$. Although there exist other possibilities, we have considered a triangular function $f_p(\eta)$ in the range $[1,c]$, for some $c \geq 1$, and a negative exponential function $f_q(\eta)$ in the range $[c,\infty)$, for the same c, in order to represent the behaviour of the second component of pairs (t,s) belonging to $\mathcal{P}$ and belonging to $\mathcal{Q}$, respectively. The reason for this choice is the extreme simplicity for computing their corresponding indefinite integrals.

Combining the shapes (triangular in the beginning and negative exponential at the end) of functions $f_p(\eta)$ and $f_q(\eta)$, we have built one piecewise function $f(\eta)$ as follows:

$$f(\eta) = \begin{cases} \dfrac{h_p}{e_p - 1}(\eta - 1) & \text{if } 1 \leq \eta < e_p \\ h_p + \dfrac{h_q - h_p}{e_q - e_p}(\eta - e_p) & \text{if } e_p \leq \eta < e_q \\ h_q \ \mathrm{Exp}[e_q - \eta] & \text{if } \eta \geq e_q \end{cases}$$

where the parameters h_p and h_q indicate the heights for the respective points whose abscises are e_p and e_q (see Figure 2). These parameters are determined by demanding that the function $f(\eta)$ is effectively a density function.

The calculation of partial areas in the partition $[1,e_p) \cup [e_p,e_q) \cup [e_q,\infty)$ (denoted by A_1, A_2 and A_3, respectively) is easy due to the linear and exponential shapes used in building $f(\eta)$.

The respective values are

$$A_1 = \frac{(e_p - 1)}{2}\, h_p; \;\; A_2 = \frac{(e_q - e_p)}{2}\, h_p + \frac{(e_q - e_p)}{2}\, h_q; \;\; A_3 = h_q.$$

Since the mean of a typical density function $g(x) = \mu e^{-\mu x}$, where $x \in [0,\infty)$ and $\mu > 0$, is $\overline{x} = 1/\mu$ and the area proportion, with respect to the total $\int_0^\infty g(x)dx = 1$, outside the range $[0,\overline{x}]$ is $\int_{\overline{x}}^\infty g(x)dx = \mathrm{Exp}[-1]$, we have weighted $\mathrm{Exp}[-1]$ by $\lambda_q = \frac{I_q}{M}$ (the percentage of trips included in subsample $\mathcal{Q}$) and by a new parameter $k \in (0,1]$ whose meaning is the percentage in $\mathcal{Q}$ of bias values which escape from $[1,e_q)$. The result is

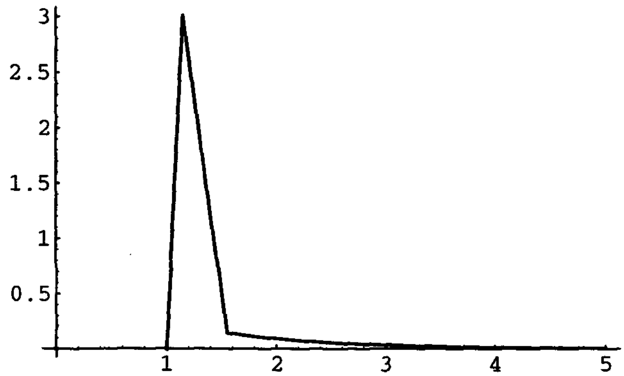

Figure 2: A density function for distributing η

assumed equal to A_3, as the second equation indicates in the system

$$\begin{cases} A_1 + A_2 + A_3 = 1 \\ A_3 = \dfrac{k}{\text{Exp}[1]}\,\lambda_q \end{cases}$$

The parameters h_p and h_q are calculated by imposing normalisation at function $f(\eta)$ (first equation in the previous system), and the results are sensitive to parameter $k \in (0, 1]$:

$$\begin{cases} h_q &= \dfrac{k}{\text{Exp}[1]}\,\lambda_q \\ h_p &= \dfrac{2 - (2 + e_q - e_p)h_q}{e_q - 1} \end{cases}$$

The expected bias $E[\eta]$ can be assumed in a similar way to expected distance (Muñoz and Puerto [1996]) from the origin to a value $\eta(\varphi)$ belonging to the interval $[1, \infty)$ in fixed direction φ, weighted by $f(\eta(\varphi))$:

$$E[\eta] = \int_1^{\infty} \eta f(\eta)\, \mathrm{d}\eta$$

The average bias in fixed direction φ for the network can be calculated by partitioning the integral expression into intervals, taking into account that

$$E_1(\eta) = \int_1^{e_p} \frac{h_p}{e_p - 1} (\eta^2 - \eta)\, \mathrm{d}\eta = \frac{h_p(2e_p^2 - e_p - 1)}{6},$$

$$E_2(\eta) = \int_{e_p}^{e_q} \Big(h_p + \frac{h_q - h_p}{e_q - e_p}(\eta - e_p)\Big)\, \eta\, \mathrm{d}\eta = \Big(\frac{e_q^2 - e_p^2}{2} - \frac{2e_q^2 - e_p e_q - e_p^2}{6}\Big) h_p +$$

$$+ \Big(\frac{2e_q^2 - e_p e_q - e_p^2}{6}\Big) h_q,$$

$$E_3(\eta) = \int_{e_q}^{\infty} h_q\, \eta\, \mathrm{Exp}[e_q - \eta]\, \mathrm{d}\eta = h_q(1 + e_q).$$

Hence, the result $E\Big(\eta(\varphi)\Big)$ is a quadratic expression (instead of the linear expression $\lambda_p e_p(\varphi) + \lambda_q e_q(\varphi)$) which combines partial estimation functions $e_p(\varphi)$ and $e_q(\varphi)$:

$$E(\eta) = E_1(\eta) + E_2(\eta) + E_3(\eta) =$$

$$= \begin{pmatrix} 1 & e_p & e_q \end{pmatrix} \cdot \begin{pmatrix} \frac{6h_q - h_p}{6} & \frac{-h_p}{12} & \frac{h_q}{2} \\ \frac{-h_p}{12} & \frac{-h_q}{6} & \frac{h_p - h_q}{12} \\ \frac{h_q}{2} & \frac{h_p - h_q}{12} & \frac{h_p + 2h_q}{6} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ e_p \\ e_q \end{pmatrix}$$

The estimation of network distances in terms of the expected bias corresponding to the trip expressed by the vector $\mathbf{x}$ (argument φ) which connects the origin and destination points, is obtained by

$$(IV) \quad d_G(\mathbf{x}) \approx \| \mathbf{x} \|_2 \; E(\eta(\varphi))$$

where

$$\begin{cases} e_p = \tau_p r_p(\bmod(\varphi, \pi/2)); \quad e_q = \tau_q r_q(\bmod(\varphi/2, \pi/2)) \\ h_q = \dfrac{k}{\mathrm{Exp}[1]}\, \lambda_q \\ h_p = \dfrac{2 - (2 + e_q - e_p)h_q}{e_q - 1} \end{cases}$$

Finally, several values of parameter k were considered in order to reduce the minimal value obtained relative to the total error (4.13848) in example 1 by means of approximation (IV), when a combination of l_p functions is considered as global estimator.

values for k	total error measured by AD
1	4.7096
0.65	4.1637
0.5	4.1075
0.45	**4.08877**
6/14($\approx$ 0.43)	4.09869
0.35	4.23853

We can note an improvement in the total error with respect to the usual convex combination when the expected bias is built by using $k = 0.45$.

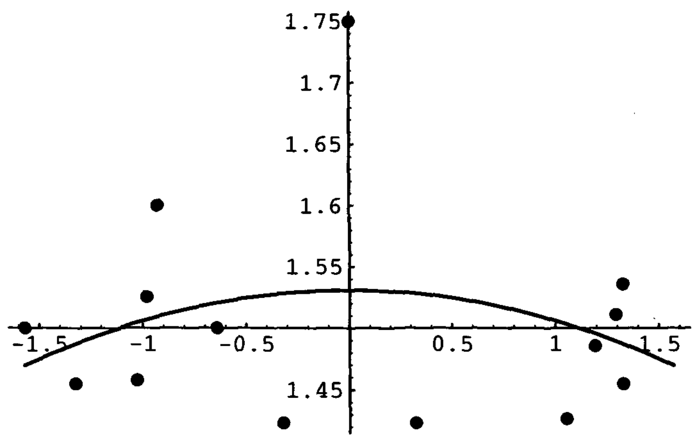

Figure 3: Fitting $\mathcal{Q}$ by polynomial function

In example 1, subsample $\mathcal{Q}$ ($\mathcal{I}_q = 14$) was fitted by polynomial function $H_q(t) = 1.53076(1 - 0.01616\, t^2)$, $t \in [-\pi/2, \pi/2]$ and six points $(t_i, s_i) \in \mathcal{Q}$ satisfied condition $H_q(t_i) < s_i$, i.e. they were *outside* $[1, e_q]$ (in Figure 3, the points above the curve). Hence, the percentage of bias which escapes from $[1, e_q]$ (according to the meaning of k) was $6/14 \approx 0.43$, which is close to the best value of k (0.45) obtained in the table.

In order to limit the search for a good value of k, inside the region $[0, 1]$, we can consider $k_0 = | \mathcal{Q}^+ | / \mathcal{I}_q$ as the initial point, where

$$\mathcal{Q}^+ = \{(t_i, s_i) \in \mathcal{Q} : \; H_q(t_i) < s_i\}.$$

Then the procedure is continued by taking points around k_0.

8 Conclusions

A methodology for the estimation process has been presented and evaluated in three cases corresponding to different types of networks. After rotating coordinate axes so that the directional bias in the transportation network is in phase with the new axes, the steps have been:

- Separating the initial set of origin–destination pairs into two subsamples according to the observed bias value.

- Using regression in order to determine the fitting parameters corresponding to polynomial functions.

- The generation, using polynomial coefficients, of other estimators (l_p functions or ϕ-inclined block norms) which maintain the approximation reached in each part.

The linear combination of partial estimators, according to the size of each subset, logically worsens the final result for the global estimation. However, a quadratic combination, based on the expected bias for network trips in each fixed direction, can improve the final estimation if a sensitivity coefficient is appropriately chosen. The consideration of the rotation angle of the axes as a further parameter in the process may give rise to better results.

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Appendix

Example I:

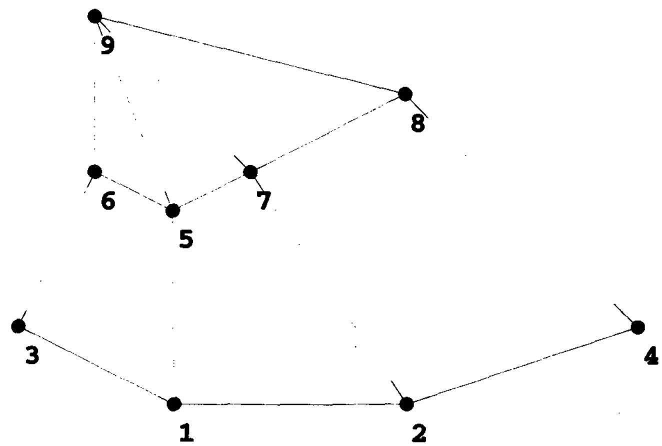

Figure 4: A typical transportation network

Nodes	Coordinates	Distance matrix								
1	(2, 0)	0	3.5	3	7.5	3	4.5	4.5	7.5	6.5
2	(5, 0)	–	0	6.5	4	5.5	7	4	7	7.5
3	(0, 1)	–	–	0	10.5	4	2.5	5.5	8.5	5
4	(8, 1)	–	–	–	0	9.5	11	8	6	11.5
5	(2, 2.25)	–	–	–	–	0	1.5	1.5	4.5	3.5
6	(1, 3)	–	–	–	–	–	0	3	6	2.5
7	(3, 3)	–	–	–	–	–	–	0	3	3.5
8	(5, 4)	–	–	–	–	–	–	–	0	6
9	(1, 5)	–	–	–	–	–	–	–	–	0

Example II:

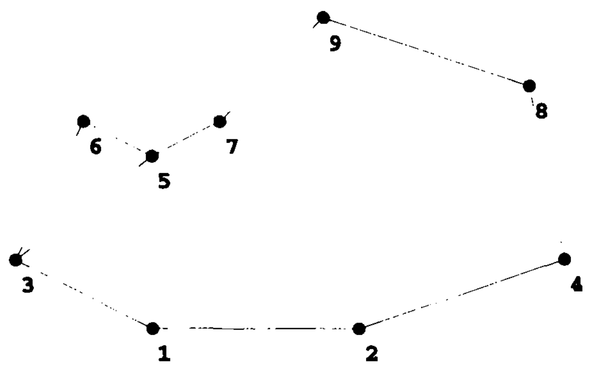

Figure 5: A network circumnavigating a barrier

Nodes	**Coordinates**	**Distance matrix**								
1	(2, 0)	0	3.5	3	7.5	6.5	5.5	8	11.5	11
2	(5, 0)	–	0	6.5	4	10	9	11.5	8	12.5
3	(0, 1)	–	–	0	10.5	3.5	2.5	5	13	8
4	(8, 1)	–	–	–	0	13	13	11.5	4	9
5	(2, 2.25)	–	–	–	–	0	1.5	1.5	9.5	4.5
6	(1, 3)	–	–	–	–	–	0	3	11	6
7	(3, 3)	–	–	–	–	–	–	0	8	3
8	(7.5, 3.5)	–	–	–	–	–	–	–	0	5
9	(4.5, 4.5)	–	–	–	–	–	–	–	–	0

Example III:

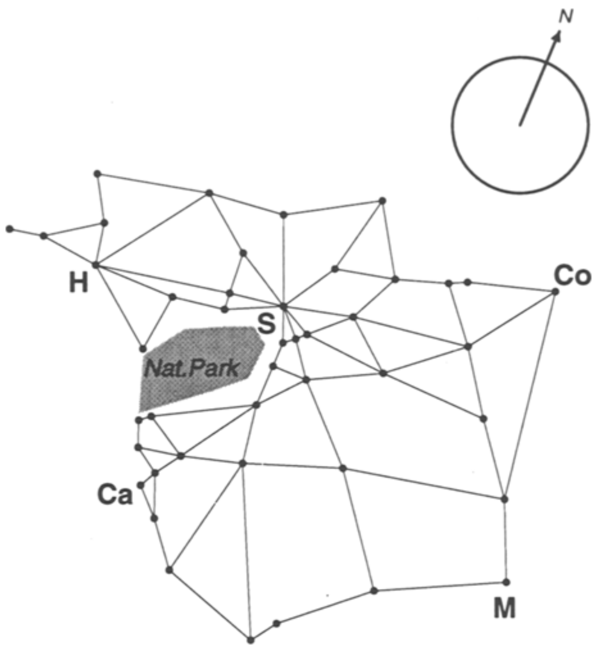

Figure 6: An actual network in Southwestern Spain

Abbreviations					
Ca	Co	H	M	S	Nat.Park
Cádiz	Córdoba	Huelva	Málaga	Sevilla	P.N. Doñana

References

Brimberg, J. and R.F. Love (1991). "Estimating Travel Distances by the Weighted l_p Norm". *Naval Research Logistics* **38**, 241–259.

Brimberg, J. and R.F. Love (1992). "A New Distance Function for Modeling Travel Distances in a Transportation Network". *Transportation Science* **26**, 129–137.

Brimberg, J. and R.F. Love (1993 a). "Directional Bias of the l_p-Norm". *European Journal of Operational Research* **67**, 287-294.

Brimberg, J. and R.F. Love (1993 b). "General Considerations on the Use of the Weighted l_p Norm as an Empirical Distance Measure". *Transportation Science* **27**, 341–349.

Brimberg, J., P.D. Dowling and R.F. Love (1994). "The Weighted One-Two Norm Distance Model: Empirical Validation and Confidence Interval Estimation". *Location Science* **2**, 91–100.

Brimberg, J., P.D. Dowling and R.F. Love (1996). "Estimating the Parameters of the Weighted l_p Norm by Linear Regression". *IIE Transactions* **28**, 363–367.

Love, R.F. and J.G. Morris (1988). "On Estimating Road Distances by Mathematical Functions". *European Journal of Operational Research* **36**, 251–253.

Love, R.F., J.G. Morris and G.O. Wesolowsky (1988). "Facilities Location: Models and Methods". *North-Holland.* New York (EEUU).

Love, R.F. and J.H. Walker (1994). "An Empirical Comparison of Block and Round Norms for Modeling Actual Distances". *Location Science* **2**, 21–43.

Mesa, J.A. and F.A. Ortega (1996). "Determining Regular Networks for Each Given l_p Function ($p \in [1,2]$)". *Studies in Locational Analysis* **10**, 91–108.

Muñoz, M. and J. Puerto (1996). "Problemas de Localización Estocástica Continua". *Lecturas en Teoría de Localización*, 167–195. Universidad de Sevilla. Sevilla (España).

Ortega, F.A. (1997). "Localización y Trazado de Redes". *Ph.D. Thesis*, Universidad de Sevilla. Sevilla, Spain.

Widmayer, P., Y. F. Yu and C.K. Wong (1987). "On Some Distance Problems in Fixed Orientations". *SIAM J. Comput.* **16**, 728–746.