Strong Factorizations between Couples of Operators on Banach Function Spaces

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Let $T : X_1 \to Y_1$ and $S : X_2 \to Y_2$ be two continuous linear operators between Banach function spaces related to a finite measure space. Under some lattice requirements on the spaces involved, we give characterizations by means of inequalities of when T can be strongly factorized through S, that is, $T = M_g \circ S \circ M_f$ with $M_f : X_1 \to X_2$ and $M_g : Y_2 \to Y_1$ being multiplication operators defined by some measurable functions f and g. In particular, we study the cases when S is a composition operator or a kernel operator.

Keywords: Banach function spaces, factorization of operators, multiplication operators, product spaces, vector measures

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1. Introduction

Let (Ω, Σ, μ) be a finite measure space and consider two continuous linear operators $T : X_1 \to Y_1$ and $S : X_2 \to Y_2$ between Banach function spaces related to μ. The aim of this paper is to study when it is possible to factorize T through S as $T = M_g \circ S \circ M_f$ where $M_f : X_1 \to X_2$ and $M_g : Y_2 \to Y_1$ are multiplication operators defined by measurable functions f and g. This type of factorization is called *strong factorization* for T , see for instance [16, Section III.H.§9]. There are many classical results relating inequalities for operators and factorizations,
as the well known ones coming from the Maurey-Rosenthal cycle of ideas which allow to associate some convexity/concavity inequalities for operators with strong factorizations through L^p-spaces, see for instance [16, Proposition III.H.10] and

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[5, 6, 7]. In these results, the operator S appearing in the factorization of T is constructed from the inequality satisfied by T. In the present paper we are interested in analyzing the factorization of T through a specifically given operator S. Our main results establish that, under some lattice type requirements on the spaces involved, mainly order continuity, a factorization for T as $M_g \circ S \circ M_f$ is equivalent to a domination of T by S (Theorem 4.1) or by its transpose operator S' (Theorem 4.2). Section 5 is devoted to one of these requirements, namely we give conditions under which a product space is order continuous (Proposition 5.2). The proof of Theorem 4.2 uses a Radon-Nikodým Theorem for vector measures which characterizes when, for two vector measures $n, m : \Sigma \to E$ with values in a Banach space E, there exists a bounded function h such that $n(A) = \int_A h\, dm$ for all $A \in \Sigma$. This Radon-Nikodým Theorem was proved by Musiał in [13, Theorem 1] by using a Ryll-Nardzewski lemma. In Section 3 we provide a simple proof of this characterization which is based in a completely different fundamental tool, a separation argument based on Ky Fan's Lemma.

We end the paper by studying in Section 6 the particular cases when S is a composition operator or a kernel operator. In the first case, we added a new equivalent condition for T to be strongly factorized through S (Theorem 6.1), which involves a generalization of the concept of *local operator* (see for instance [1, Definition 2.2]).

2. Basic concepts and notation

We give in this section some basic definitions and results on Banach function spaces and vector measures which will be used along this paper. Let (Ω, Σ, μ) be a *finite* measure space and $L^0(\mu)$ denote the space of all measurable real functions on Ω, where functions which are equal μ-a.e. are identified. As usual, $L^\infty(\mu)$ will denote the space of functions in $L^0(\mu)$ which are bounded μ-a.e. By *Banach function space* (briefly B.f.s.) we will mean a Banach space $X \subset L^0(\mu)$ with norm $\|\cdot\|_X$ such that if $f \in L^0(\mu)$, $g \in X$ and $|f| \le |g|$ μ-a.e. then $f \in X$ and $\|f\|_X \le \|g\|_X$. Note that every B.f.s. is a Banach lattice with the pointwise μ-a.e. order. A B.f.s. X has the *Fatou property* if for every sequence $(f_n) \subset X$ such that $0 \le f_n \uparrow f$ μ-a.e. and $\sup_n \|f_n\|_X < \infty$, it follows that $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$. We will say that X is *order continuous* if for every $f, f_n \in X$ such that $0 \le f_n \uparrow f$ μ-a.e., we have that $f_n \to f$ in norm. For issues related to B.f.s.', see [17, Ch. 15] considering the function norm ρ defined as $\rho(f) = \|f\|_X$ if $f \in X$ and $\rho(f) = \infty$ in other case.

Given two B.f.s.' X and Y, the space of *multipliers* from X to Y is defined as

$$X^Y = \left\{ h \in L^0(\mu) : hf \in Y \text{ for all } f \in X \right\}.$$

The map $\|\cdot\|_{X^Y}$ given by $\|h\|_{X^Y} = \sup_{f \in B_X} \|hf\|_Y$ for all $h \in X^Y$, where B_X is the open unit ball of X, defines a natural seminorm on X^Y which becomes a norm only in the case when X is *saturated*, i.e. there is no $A \in \Sigma$ with $\mu(A) > 0$ such that $f\chi_A = 0$ μ-a.e. for all $f \in X$. Note that X is saturated if and only if it has a weak unit (i.e. $g \in X$ such that $g > 0$ μ-a.e.). In this case, X^Y is a B.f.s. The particular multiplier space X^{L^1} is just the classical Köthe dual of X which will be denoted by X'. For each $h \in X^Y$, we denote by $M_h\colon X \to Y$ the multiplication operator defined as $M_h(f) = hf$ for all $f \in X$. For more information on these spaces see [12] and also [2].

The *product space* $X\pi Y$ of two B.f.s.' X and Y is defined as the space of functions $f \in L^0(\mu)$ such that $|f| \le \sum_{i\ge1} |x_i y_i|$ μ-a.e. for some sequences $(x_i) \subset X$ and $(y_i) \subset Y$ satisfying $\sum_{i\ge1} \|x_i\|_X \|y_i\|_Y < \infty$. For $f \in X\pi Y$, denote

$$\|f\|_{X\pi Y} = \inf \left\{ \sum_{i\ge1} \|x_i\|_X \|y_i\|_Y \right\},$$

where the infimum is taken over all sequences $(x_i) \subset X$ and $(y_i) \subset Y$ such that $|f| \le \sum_{i\ge1} |x_i y_i|$ μ-a.e. and $\sum_{i\ge1} \|x_i\|_X \|y_i\|_Y < \infty$. If X, Y and $X^{Y'}$ are saturated then $X\pi Y$ is a saturated B.f.s. with norm $\|\cdot\|_{X\pi Y}$. An study of these product spaces can be found in [8].

Let $m : \Sigma \to E$ be a *vector measure*, that is, a countably additive set function, where E is a real Banach space. A set $A \in \Sigma$ is *m-null* if $m(B) = 0$ for every $B \in \Sigma$ with $B \subset A$. For each x^* in the topological dual E^* of E, we denote by $|x^*m|$ the variation of the real measure x^*m given by the composition of m with x^*. There exists $x_0^* \in E^*$ such that $|x_0^*m|$ has the same null sets as m, see [9, Ch. IX. 2]. We will call $|x_0^*m|$ a *Rybakov control measure* for m.

A measurable function $f\colon \Omega \to \mathbb{R}$ is *integrable with respect to m* if:

(i) $\int |f|\, d|x^*m| < \infty$ for all $x^* \in E^*$.
(ii) For each $A \in \Sigma$, there exists $x_A \in E$ such that

$$x^*(x_A) = \int_A f\, dx^*m, \quad \text{for all } x^* \in E.$$

The element x_A will be written as $\int_A f\, dm$. Denote by $L^1(m)$ the space of integrable functions with respect to m, where functions which are equal m-a.e. are identified. The space $L^1(m)$ is a Banach space endowed with the norm

$$\|f\|_m = \sup_{x^* \in B_{E^*}} \int |f|\, d|x^*m|.$$

Note that $L^\infty(|x_0^*m|) \subset L^1(m)$. In particular every measure of the type $|x^*m|$ is finite as $|x^*m|(\Omega) \le \|x^*\| \cdot \|\chi_\Omega\|_m$.

Given $f \in L^1(m)$, the set function $m_f\colon \Sigma \to E$ given by $m_f(A) = \int_A f\, dm$ for all $A \in \Sigma$ is a vector measure. Moreover, $g \in L^1(m_f)$ if and only if $gf \in L^1(m)$ and in this case $\int g\, dm_f = \int gf\, dm$.

For a complete overview about integration with respect to vector measures we refer to [4], [14, Ch. 3] and the references therein.

3. The Radon-Nikodým theorem for vector measures

In the next section we will use the following Radon-Nikodým theorem for vector measures proved by Musiał in [13, Theorem 1]. We include here a totally different proof for the real Banach space case based on Ky Fan's lemma (see for instance [15, E. 4]).

Theorem 3.1. *Let $m, n\colon \Sigma \to E$ be two vector measures and take $|x_0^* m|$ a Rybakov control measure for m. The following statements are equivalent.*

(*1*) *There exists a positive constant K such that*

$$x^*n(A) \leq K|x^*m|(A) \quad \text{for all } A \in \Sigma \text{ and } x^* \in E^*.$$

(*2*) *There exists a function $g \in L^\infty(|x_0^* m|)$ such that*

$$n(A) = \int_A g\, dm \quad \text{for all } A \in \Sigma.$$

Proof. Let us show that (*1*) implies (*2*). Denote $\xi = |x_0^* m|$. For each $x^* \in E^*$, since the measure x^*m is absolutely continuous with respect to ξ, there exists $h_{x^*} \in L^1(\xi)$ such that

$$x^*m(A) = \int_A h_{x^*}\, d\xi$$

for all $A \in \Sigma$, i.e. h_{x^*} is the Radon-Nikodým derivative of x^*m with respect to ξ. For every finite measurable partition $\{A_1, ..., A_n\}$ of Ω and every finite set of vectors $x_1^*, ..., x_n^* \in E^*$, we define the function $\phi\colon B_{L^\infty(\xi)} \to \mathbb{R}$ by

$$\phi(f) = \sum_{i=1}^n x_i^* n(A_i) - K \sum_{i=1}^n \int_{A_i} f h_{x_i^*}\, d\xi$$

for all $f \in B_{L^\infty(\xi)}$. Note that ϕ is convex. Considering the weak* topology on $B_{L^\infty(\xi)}$, we have that ϕ is a continuous function on a compact convex set. Let $\mathcal{F}$ denote the family of the functions ϕ defined in this way. If ϕ_1 and ϕ_2 are two functions in $\mathcal{F}$ and $0 < \alpha < 1$, since

$$\begin{aligned}
\alpha\phi_1(f) + (1-\alpha)\phi_2(f) = {} & \sum_{i=1}^n \alpha x_{1,i}^* n(A_i^1) - K \sum_{i=1}^n \int_{A_i^1} f h_{\alpha x_{1,i}^*}\, d\xi \\
& + \sum_{j=1}^k (1-\alpha) x_{2,j}^* n(A_j^2) - K \sum_{j=1}^k \int_{A_j^2} f h_{(1-\alpha)x_{2,j}^*}\, d\xi \\
= {} & \sum_{i=1}^n \sum_{j=1}^k (\alpha x_{1,i}^* + (1-\alpha) x_{2,j}^*)\, n(A_i^1 \cap A_j^2) \\
& - K \sum_{i=1}^n \sum_{j=1}^k \int_{A_i^1 \cap A_j^2} f\, h_{\alpha x_{1,i}^* + (1-\alpha)x_{2,j}^*}\, d\xi,
\end{aligned}$$

we have that $\alpha\phi_1 + (1-\alpha)\phi_2 \in \mathcal{F}$ as $\{A_i^1 \cap A_j^2 : i = 1, ..., n,\ j = 1, ..., k\}$ is a partition of Ω. Then, it follows that $\mathcal{F}$ is a concave family of real functions. If (*1*) holds, for each $\phi \in \mathcal{F}$, taking $f_\phi = \sum_{i=1}^n \operatorname{sign}(h_{x_i^*})\chi_{A_i} \in B_{L^\infty(\xi)}$, we have that

$$\sum_{i=1}^n x_i^* n(A_i) \leq K \sum_{i=1}^n |x_i^* m|(A_i) = K \sum_{i=1}^n \int_{A_i} |h_{x_i^*}|\, d\xi = K \sum_{i=1}^n \int_{A_i} f_\phi h_{x_i^*}\, d\xi,$$

that is $\phi(f_\phi) \leq 0$. Therefore, by applying Ky Fan's lemma, there exists a function $f_0 \in B_{L^\infty(\xi)}$ such that $\phi(f_0) \leq 0$ for all $\phi \in \mathcal{F}$. Consequently, for every $A \in \Sigma$ and $x^* \in E^*$, if we consider the partition $\{A, \Omega \setminus A\}$ and the vectors x^* and 0, we obtain

$$x^* n(A) \leq K \int_A f_0 h_{x^*}\, d\xi$$

and

$$-x^* n(A) \leq K \int_A f_0 h_{-x^*}\, d\xi = -K \int_A f_0 h_{x^*}\, d\xi,$$

that is $x^* n(A) = K \int_A f_0 h_{x^*}\, d\xi$. Since this happens for every $x^* \in E^*$, taking into account that $\int_A f_0 h_{x^*}\, d\xi = \int_A f_0\, dx^* m = x^* \left(\int_A f_0\, dm\right)$, we have that $n(A) = K \int_A f_0\, dm$ for every $A \in \Sigma$ and so (*2*) holds for $g = K f_0$.

The converse implication is obtained easily as follows. If (*2*) holds, for every $A \in \Sigma$ and $x^* \in E^*$ we have that

$$\begin{aligned} x^* n(A) \leq |x^* n(A)| &= \left| x^* \left(\int_A g\, dm \right) \right| = \left| \int_A g\, dx^* m \right| \\ &\leq \int_A |g|\, d|x^* m| \leq \|g\|_\infty |x^* m|(A). \end{aligned}$$

□

4. Strongly factorized operators between B.f.s.'

Let (Ω, Σ, μ) be a fixed finite measure space and X_1, X_2, Y_1, Y_2 B.f.s.' related to μ such that $L^\infty(\mu) \subset X_1 \subset X_2$ and $L^\infty(\mu) \subset Y_2 \subset Y_1$. This guarantees that $X_1^{X_2}$ and $Y_2^{Y_1}$ are B.f.s.' containing $L^\infty(\mu)$.

Consider two continuous linear operators $T : X_1 \to Y_1$ and $S : X_2 \to Y_2$. This section is devoted to characterize when T *factorizes strongly* through S, that is, the following diagram commutes

$$\begin{tikzcd} X_1 \arrow[r, "T"] \arrow[d, dotted, "M_f"'] & Y_1 \\ X_2 \arrow[r, dotted, "S"] & Y_2 \arrow[u, dotted, "M_g"'] \end{tikzcd} \tag{1}$$

for some $f \in X_1^{X_2}$ and $g \in Y_2^{Y_1}$. We will give two different characterizations. In the first one we will use the Ky Fan's lemma ([15, E. 4]) and the product space $Y_2 \pi Y_1'$ which under our conditions is a saturated B.f.s. (see [8, Proposition 2.2]).

Theorem 4.1. *Suppose that $Y_1, Y_2 \pi Y_1'$ are order continuous and moreover Y_1 has the Fatou property. The following statements are equivalent:*

(i) *There exists a function* $h \in X_1^{X_2}$ *such that*

$$\sum_{i=1}^{n} \int T(x_i) y_i' \, d\mu \leq \left\| \sum_{i=1}^{n} S(hx_i) y_i' \right\|_{Y_2 \pi Y_1'}$$

for every $x_1, ..., x_n \in X_1$ *and* $y_1', ..., y_n' \in Y_1'$.

(ii) *There exist functions* $f \in X_1^{X_2}$ *and* $g \in Y_2^{Y_1}$ *such that* $T(x) = gS(fx)$ *for all* $x \in X_1$, *i.e.* T *factorizes strongly through* S *as in* (1).

Proof. First note that since $Y_2 \pi Y_1'$ is order continuous, the Köthe dual $(Y_2 \pi Y_1')'$ can be identified with the topological dual $(Y_2 \pi Y_1')^*$ (see [10, p. 29]). On the other hand, $Y_1 = Y_1''$ as Y_1 has the Fatou property (see [10, p. 30]), and so by [8, Proposition 2.2] we have that $(Y_2 \pi Y_1')' = Y_2^{Y_1''} = Y_2^{Y_1}$. Then, applying the Hahn-Banach theorem, for each $\xi \in Y_2 \pi Y_1'$ there exists $\xi' \in B_{Y_2^{Y_1}}$ such that $\|\xi\|_{Y_2 \pi Y_1'} = \langle \xi', \xi \rangle = \int \xi \, \xi' \, d\mu$.

Suppose (i) holds for some $h \in X_1^{X_2}$. For every $x_1, ...x_n \in X_1$ and $y_1', ..., y_n' \in Y_1'$, we define the convex function $\phi \colon B_{Y_2^{Y_1}} \to \mathbb{R}$ by

$$\phi(g) = \sum_{i=1}^{n} \int T(x_i) y_i' \, d\mu - \sum_{i=1}^{n} \int g S(hx_i) y_i' \, d\mu,$$

for all $g \in B_{Y_2^{Y_1}}$. Since $(Y_2 \pi Y_1')^* = Y_2^{Y_1}$, considering the weak* topology we have that $B_{Y_2^{Y^1}}$ is a compact convex set and ϕ is continuous. Denote by $\mathcal{F}$ the family of all functions ϕ defined in this way, which is obviously concave. Let $\phi \in \mathcal{F}$ defined through $x_1, ...x_n \in X_1$ and $y_1', ..., y_n' \in Y_1'$. As we have noted before, for $\sum_{i=1}^{n} S(hx_i) y_i' \in Y_2 \pi Y_1'$ there exists $g_\phi \in B_{Y_2^{Y_1}}$ such that

$$\left\| \sum_{i=1}^{n} S(hx_i) y_i' \right\|_{Y_2 \pi Y_1'} = \int g_\phi \sum_{i=1}^{n} S(hx_i) y_i' \, d\mu.$$

Then, from (i) we have that $\phi(g_\phi) \leq 0$. By Ky Fan's lemma, there exists a function $g \in B_{Y_2^{Y_1}}$ such that $\phi(g) \leq 0$ for all $\phi \in \mathcal{F}$. In particular,

$$\int T(x) y' \, d\mu \leq \int g S(hx) y' \, d\mu \tag{2}$$

for every $x \in X_1$ and $y' \in Y_1'$. Since this holds for every $y' \in Y_1'$, by taking $-y'$ we obtain also

$$-\int T(x) y' \, d\mu \leq -\int g S(hx) y' \, d\mu.$$

Then

$$\int T(x) y' \, d\mu = \int g S(hx) y' \, d\mu$$

for every $x \in X_1$ and $y' \in Y_1'$. Since Y_1 is order continuous it follows that $T(x) = gS(hx)$ for all $x \in X_1$, that is, (ii) holds.

Suppose now that (ii) holds for $f \in X_1^{X_2}$ and $g \in Y_2^{Y_1}$. For every $x_1, ..., x_n \in X_1$ and $y_1', ..., y_n' \in Y_1'$ we have that

$$\sum_{i=1}^{n} \int T(x_i) y_i' \, d\mu = \sum_{i=1}^{n} \int gS(fx_i) y_i' \, d\mu \le \|g\|_{Y_2^{Y_1}} \cdot \left\| \sum_{i=1}^{n} S(fx_i) y_i' \right\|_{Y_2 \pi Y_1'},$$

that is, (i) holds for $h = \|g\|_{Y_2^{Y_1}} f$. □

Note that the previous theorem requires conditions on the B.f.s.' Y_1, Y_2 and the operator S. The second characterization of when T factorizes strongly through S is a dual result of Theorem 4.1 in the sense that the conditions are required on X_1, X_2 and the transpose operator S' of S. For the proof we will use also the Radon-Nikodým theorem proved in Section 3.

Theorem 4.2. *Suppose that X_1, Y_1, X_2 and $X_1 \pi X_2'$ are order continuous and X_2 has the Fatou property. The following statements are equivalent.*

(i) *There exists a function $h \in Y_2^{Y_1}$ such that*

$$\sum_{i=1}^{n} \int T(x_i) y_i' \, d\mu \le \left\| \sum_{i=1}^{n} |S'(hy_i') x_i| \right\|_{X_1 \pi X_2'}$$

for every $x_1, ..., x_n \in X_1$ and $y_1', ..., y_n' \in Y_1'$.

(ii) *There exist functions $f \in X_1^{X_2}$ and $g \in Y_2^{Y_1}$ such that $T(x) = gS(fx)$ for all $x \in X_1$.*

Proof. Since $X_1 \pi X_2'$ is order continuous and X_2 has the Fatou property it follows that $(X_1 \pi X_2')^* = X_1^{X_2}$.

Suppose (i) holds for some $h \in Y_2^{Y_1}$. It is direct to check that $Y_2^{Y_1} \subset Y_1'^{Y_2'}$, then since X_2 is order continuous, we have that $S'(hy') \in X_2'$ for all $y' \in Y_1'$. For every $x_1, ...x_n \in X_1$ and $y_1', ..., y_n' \in Y_1'$, we define the convex function $\phi \colon B_{X_1^{X_2}} \to \mathbb{R}$ by

$$\phi(f) = \sum_{i=1}^{n} \int T(x_i) y_i' \, d\mu - \sum_{i=1}^{n} \int f |S'(hy_i') x_i| \, d\mu,$$

for all $f \in B_{X_1^{X_2}}$. Considering the weak* topology, ϕ is continuous on a compact convex set. Let $\mathcal{F}$ be the concave family of all functions ϕ defined in this way. If $\phi \in \mathcal{F}$ is defined through $x_1, ...x_n \in X_1$ and $y_1', ..., y_n' \in Y_1'$, taking $f_\phi \in B_{X_1^{X_2}}$ such that

$$\left\| \sum_{i=1}^{n} |S'(hy_i') x_i| \right\|_{X_1 \pi X_2'} = \int f_\phi \sum_{i=1}^{n} |S'(hy_i') x_i| \, d\mu,$$

from (i) we have that $\phi(f_\phi) \leq 0$. By Ky Fan's lemma, there exists a function $f \in B_{X_1{}^{X_2}}$ such that $\phi(f) \leq 0$ for all $\phi \in \mathcal{F}$. In particular,

$$\int T(x)y'\,d\mu \leq \int f|S'(hy')x|\,d\mu \tag{3}$$

for every $x \in X_1$ and $y' \in Y_1'$. We can assume that $f \geq 0$.

On the other hand, consider the continuous linear operator $R\colon X_1 \to Y_1$ given by $R = M_h \circ S \circ M_f$ and the set function $m_R : \Sigma \to Y_1$ defined as $m_R(A) = R(\chi_A)$ for all $A \in \Sigma$. Since X_1 is order continuous we have that m_R is a vector measure and for every $x \in X_1$ it follows that $x \in L^1(m_R)$ with $\int x\,dm_R = R(x)$, see for instance [3, Section 3]. For each $y' \in Y_1'$ and $A \in \Sigma$, we have that

$$\begin{aligned} y'm_R(A) = \langle y', R(\chi_A)\rangle &= \int y'hS(f\chi_A)\,d\mu \\ &= \langle y'h, S(f\chi_A)\rangle = \langle S'(y'h), f\chi_A\rangle = \int_A S'(y'h)f\,d\mu, \end{aligned}$$

where in the last inequality we have used that X_2 is order continuous and so $S'\colon Y_2^* \to X_2^* = X_2'$. Then,

$$|y'm_R|(A) = \int_A |S'(y'h)|f\,d\mu. \tag{4}$$

Now, considering the vector measure $m_T : \Sigma \to Y_1$ given by $m_T(A) = T(\chi_A)$ for all $A \in \Sigma$ and applying (3) for $x = \chi_A$ and (4), we have that

$$y'm_T(A) = \int T(\chi_A)y'\,d\mu \leq \int_A f|S'(hy')|\,d\mu = |y'm_R|(A).$$

Hence, since Y_1 is order continuous, Theorem 3.1 provides a function $f_1 \in L^\infty(\xi)$, where ξ is a Rybakov control measure for m_R, such that

$$m_T(A) = \int_A f_1\,dm_R$$

for all $A \in \Sigma$. Since $X_1 \subset L^1(m_T)$ and the integration operator with respect to m_T restricted to X_1 coincides with T, considering a m_R-null set Z such that $f_2 = f_1\chi_{\Omega\setminus Z}$ is bounded in all Ω, for each $x \in X_1$ it follows that

$$T(x) = \int x\,dm_T = \int xf_1\,dm_R = \int xf_2\,dm_R.$$

Noting that $xf_2 \in X_1$, we have that

$$T(x) = R(xf_2) = hS(fxf_2)$$

where $ff_2 \in X_1^{X_2}$. So (ii) holds.

Suppose now that (ii) holds for $f \in X_1^{X_2}$ and $g \in Y_2^{Y_1}$. For every $x_1, ..., x_n \in X_1$ and $y'_1, ..., y'_n \in Y'_1$ we have that

$$\begin{aligned}
\sum_{i=1}^{n} \int T(x_i) y'_i \, d\mu &= \sum_{i=1}^{n} \int g S(f x_i) y'_i \, d\mu \\
&= \sum_{i=1}^{n} \int f x_i S'(g y'_i) \, d\mu \\
&\leq \int |f| \sum_{i=1}^{n} |x_i S'(g y'_i)| \, d\mu \\
&\leq \|f\|_{X_1^{X_2}} \left\| \sum_{i=1}^{n} |x_i S'(g y'_i)| \right\|_{X_1 \pi X'_2},
\end{aligned}$$

that is, (i) holds for $h = g \|f\|_{X_1^{X_2}}$. □

An easy example is given by the case when X_1, X_2, Y_1, Y_2 all coincide with an order continuous B.f.s. X having the Fatou property and containing $L^\infty(\mu)$. By a classical Lozanovskii's result ([11, Theorem 6]) we have that $X \pi X' = L^1(\mu)$ (with equal norms) is order continuous. Also note that $X^X = L^\infty(\mu)$ (with equal norms), see [12, Theorem 1]. Then, Theorems 4.1 and 4.2 produce the following result.

Corollary 4.3. *The following statements are equivalent:*

(i) *T factorizes strongly through S, i.e.*

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, dotted, "M_f"'] & X \\
X \arrow[r, dotted, "S"] & X \arrow[u, dotted, "M_g"']
\end{tikzcd}$$

for some $f, g \in L^\infty(\mu)$.

(ii) *There exists a function $h \in L^\infty(\mu)$ such that*

$$\sum_{i=1}^{n} \int T(x_i) x'_i \, d\mu \leq \int \left| \sum_{i=1}^{n} S(h x_i) x'_i \right| d\mu$$

for every $x_1, ..., x_n \in X$ and $x'_1, ..., x'_n \in X'$.

(iii) *There exists a function $h \in L^\infty(\mu)$ such that*

$$\int T(x) x' \, d\mu \leq \int |S'(h x') x| \, d\mu$$

for every $x \in X$ and $x' \in X'$.

Note that condition (*iii*) in Corollary 4.3 holds if there exists a constant $K > 0$ such that

$$\int T(x)x'\,d\mu \leq K \int |S'(x')x|\,d\mu$$

for every $x \in X$ and $x' \in X'$.

5. Order continuity for product spaces

For convenient use of Theorem 4.1 and Theorem 4.2, it is desirable to know when a product space is order continuous. For particular cases, we can check this property through direct computations.

Example 5.1. (See for instance [8, p. 202]).

(a) If $1 \leq p, q < \infty$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1$, then $L^p(\mu)\pi L^q(\mu) = L^r(\mu)$ is order continuous.

(b) For any B.f.s. X it follows that $L^\infty(\mu)\pi X = X$ and so the order continuity depends on X.

(c) For any B.f.s. X, Lozanovskii's theorem ensures that $X\pi X' = L^1(\mu)$ is order continuous.

The next result provides conditions under which a product space is order continuous.

Proposition 5.2. *Let X and Y be two B.f.s.' containing $L^\infty(\mu)$ such that X is order continuous, the simple functions are dense in Y and $X^{Y'}$ is saturated. Then $X\pi Y$ is order continuous.*

Proof. Note that from [8, Proposition 2.2], the product space $X\pi Y$ is a saturated B.f.s. Let us prove that $(X\pi Y)^* = (X\pi Y)'$ and so we will have that $X\pi Y$ is order continuous, see for instance [10, p. 29].

Let $0 \leq \phi \in (X\pi Y)^*$ and consider the finitely additive set function $\mu_\phi \colon \Sigma \to [0, \infty)$ defined by $\mu_\phi(A) = \phi(\chi_A)$ for all $A \in \Sigma$. Note that μ_ϕ is well defined as for every $A \in \Sigma$ we have that $\chi_A = \chi_A\chi_\Omega \in X\pi Y$. Moreover, since

$$\mu_\phi(A) \leq \|\phi\|_{(X\pi Y)^*}\|\chi_A\|_{X\pi Y} \leq \|\phi\|_{(X\pi Y)^*}\|\chi_A\|_X\|\chi_\Omega\|_Y,$$

and X is order continuous, it follows that μ_ϕ is countably additive. If $\mu(A) = 0$ then $\|\chi_A\|_X = 0$ and so $\mu_\phi(A) = 0$, that is, μ_ϕ is absolutely continuous with respect to μ. The scalar Radon-Nikodým theorem gives a function $0 \leq g \in L^1(\mu)$ such that $\mu_\phi(A) = \int_A g\,d\mu$ for all $A \in \Sigma$. Note that every simple function h satisfies

$$\phi(h) = \int h\,d\mu_\phi = \int hg\,d\mu.$$

Let $f \in X\pi Y$ and take a sequence of simple functions (h_n) such that $0 \leq h_n \uparrow |f|$ μ-a.e. Then, since $0 \leq h_n g \uparrow |f|g$ μ-a.e., applying the monotone convergence theorem, it follows that

$$\int |f| g \, d\mu = \lim_n \int h_n g \, d\mu = \lim_n \phi(h_n)$$
$$\leq \|\phi\|_{(X\pi Y)^*} \lim_n \|h_n\|_{X\pi Y} \leq \|\phi\|_{(X\pi Y)^*} \|f\|_{X\pi Y}.$$

Hence, $g \in (X\pi Y)'$. Denote by ϕ_g the functional on $X\pi Y$ defined via g as $\phi_g(f) = \int fg \, d\mu$. We will finish the proof if we show that $\phi = \phi_g$. We have already seen that $\phi(h) = \phi_g(h)$ for all simple function h.

Note that $X \subset X\pi Y$ as $f = f\chi_\Omega \in X\pi Y$ for all $f \in X$. Let us see that $\phi = \phi_g$ on X. Take $0 \leq f \in X$ and a sequence of simple functions (h_n) such that $0 \leq h_n \uparrow f$. Since X is order continuous, (h_n) also converges to f in X and so in $X\pi Y$, as the inclusion between Banach lattices is always continuous (see [10, p. 2]). Then,

$$\phi(f) = \lim_n \phi(h_n) = \lim_n \phi_g(h_n) = \lim_n \int h_n g \, d\mu = \int fg \, d\mu = \phi_g(f).$$

For a general $f \in X$, we only have to consider the positive and negative parts of f.

Let now $f \in X\pi Y$. Given $\varepsilon > 0$, there are two sequences $(x_i) \subset X$ and $(y_i) \subset Y$ such that $f = \sum_{i=1}^\infty x_i y_i$ and $\sum_{i=1}^\infty \|x_i\|_X \|y_i\|_Y \leq \|f\|_{X\pi Y} + \varepsilon$, see [8, Remark 2.3]. Take n_ε such that $\sum_{i>n_\varepsilon} \|x_i\|_X \|y_i\|_Y \leq \frac{\varepsilon}{2}$ and for each i take a simple function $y_{i,\varepsilon} \in Y$ such that $\|y_i - y_{i,\varepsilon}\|_Y \leq \frac{\varepsilon}{2^{i+1}\|x_i\|_X}$. Noting that $\sum_{i=1}^{n_\varepsilon} x_i y_{i,\varepsilon} \in X$, we have that

$$\begin{aligned} |\phi(f) - \phi_g(f)| &\leq \left| \phi(f) - \phi\left(\sum_{i=1}^{n_\varepsilon} x_i y_{i,\varepsilon} \right) \right| + \left| \phi_g\left(\sum_{i=1}^{n_\varepsilon} x_i y_{i,\varepsilon} \right) - \phi_g(f) \right| \\ &\leq K \left\| \sum_{i=1}^{n_\varepsilon} x_i (y_i - y_{i,\varepsilon}) + \sum_{i>n_\varepsilon} x_i y_i \right\|_{X\pi Y} \\ &\leq K \left(\sum_{i=1}^{n_\varepsilon} \|x_i\|_X \|y_i - y_{i,\varepsilon}\|_Y + \sum_{i>n_\varepsilon} \|x_i\|_X \|y_i\|_Y \right) \leq K\varepsilon \end{aligned}$$

where $K = \|\phi\|_{(X\pi Y)^*} + \|\phi_g\|_{(X\pi Y)^*}$. Since ε is arbitrary, it follows that $\phi(f) = \phi_g(f)$. $\square$

Taking into account that the simple functions are dense in every order continuous B.f.s. and that $L^\infty(\mu) \subset X^Y$ whenever $X \subset Y$, we obtain the following corollary.

Corollary 5.3. *If X and Y are two order continuous B.f.s.' containing $L^\infty(\mu)$ such that $X \subset Y'$, then $X\pi Y$ is order continuous.*

From the previous corollary and noting that $X \subset X''$ for every B.f.s. X, in Section 4 we get that $Y_2 \pi Y_1'$ is order continuous if Y_2, Y_1' are order continuous and $L^\infty(\mu) \subset Y_1'$ (equivalently, $Y_1 \subset L^1(\mu)$). Similarly for $X_1 \pi X_2'$.

Remark 5.4. In Proposition 5.2 we have proved that $\phi = \phi_g$ on X without using the condition of the density of the simple functions in Y. We obtain the same conclusion if we replace this condition by any of the following ones:

(i) X is dense in $X\pi Y$.

(ii) $X \cap B_{X\pi Y}$ is *norming* for $(X\pi Y)^*$, i.e. $\|\phi\|_{(X\pi Y)^*} = \sup_{f\in X\cap B_{X\pi Y}} |\phi(f)|$ for every $\phi \in (X\pi Y)^*$.

6. Particular cases

As in Section 4, let X_1, X_2, Y_1, Y_2 be B.f.s.' related to a finite measure space (Ω, Σ, μ) such that $L^\infty(\mu) \subset X_1 \subset X_2$ and $L^\infty(\mu) \subset Y_2 \subset Y_1$ and let $T : X_1 \to Y_1$ be a continuous linear operator. In this section we study the strong factorization of T through two particular classes of operators, namely, the composition operators and the kernel operators.

6.1. Strong factorization through a composition operator

Consider a *measure-preserving transformation* $\sigma\colon \Omega \to \Omega$, that is, a measurable function such that $\mu\left(\sigma^{-1}(A)\right) = \mu(A)$ for all $A \in \Sigma$. Then, the composition operator $S_\sigma\colon L^0(\mu) \to L^0(\mu)$ given by $S_\sigma(f) = f \circ \sigma$ is well defined and positive. Suppose that $S_\sigma\colon X_2 \to Y_2$ is well defined and so automatically continuous as it is a positive operator between Banach lattices.

Under conditions of Theorem 4.1 (or 4.2), statement (i) holds if and only if T factorizes strongly through S_σ, that is, there exist $f \in X_1^{X_2}$ and $g \in Y_2^{Y_1}$ such that

$$T(x)(\omega) = g(\omega) f\left(\sigma(\omega)\right) x\left(\sigma(\omega)\right)$$

for all $x \in X_1$ and μ-a.e. $\omega \in \Omega$.

We will give a new equivalent condition for T to be strongly factorized through S_σ, in which the following concept is involved. Given $Q\colon X \to Y$ and $R\colon X \to Z$ two continuous linear operators between B.f.s.' related to μ, we will say that Q is *R-local* if for each $x \in X$ it follows that $Q(x)(w) = 0$ μ-a.e. on $\{w \in \Omega : R(x)(w) = 0\}$, that is, $\mathrm{Supp}(Q(x)) \subset \mathrm{Supp}(R(x)) \cup N$ for some μ-null set N. In the case when $X = Y = Z$ and R is the identity map, Q will be said to be *local*.

Theorem 6.1. *If the simple functions are dense in X_1 and $T(\chi_\Omega) \in Y_2^{Y_1}$, then the following statements are equivalent:*

(i) *T factorizes strongly through S_σ.*

(ii) *There exists $h \in X_1^{X_2}$ such that T is $S_\sigma \circ M_h$-local.*

Proof. If (i) holds for some $f \in X_1^{X_2}$ and $g \in Y_2^{Y_1}$, since $T(x) = M_g \circ S_\sigma \circ M_f(x)$ for all $x \in X_1$, then T is clearly $S_\sigma \circ M_f$-local.

Suppose that (ii) holds. Then, for every $A \in \Sigma$ we have that

$$\mathrm{Supp}(T(\chi_A)) \subset \mathrm{Supp}(S_\sigma \circ M_h(\chi_A)) \cup N \subset \sigma^{-1}(A) \cup N$$

for some μ-null set N. So,

$$T(\chi_A) = T(\chi_A) \cdot \chi_{\sigma^{-1}(A)} = T(\chi_\Omega) \cdot \chi_{\sigma^{-1}(A)},$$

where in the last equality we use again that $\mathrm{Supp}(T(\chi_{\Omega \setminus A})) \subset \sigma^{-1}(\Omega \setminus A) \cup \tilde{N}$ for some μ-null set $\tilde{N}$. Hence, for a simple function $x = \sum_{i=1}^n \alpha_i \chi_{A_i}$ it follows that

$$T(x) = \sum_{i=1}^n \alpha_i T(\chi_{A_i}) = T(\chi_\Omega) \left(\sum_{i=1}^n \alpha_i \chi_{\sigma^{-1}(A_i)} \right) = T(\chi_\Omega) S_\sigma(x).$$

Let $x \in X_1$ and take a sequence (x_n) of simple functions converging to x in X_1. Then,

$$T(x) = \lim T(x_n) = \lim T(\chi_\Omega) S_\sigma(x_n).$$

Since X_1 is continuously included in X_2, we have that $S_\sigma(x_n) \to S_\sigma(x)$ in Y_2, and since $T(\chi_\Omega) \in Y_2^{Y_1}$, it follows that $T(\chi_\Omega) S_\sigma(x_n) \to T(\chi_\Omega) S_\sigma(x)$ in Y_1. So, $T(x) = T(\chi_\Omega) S_\sigma(x)$ and (i) holds for $g = T(\chi_\Omega)$ and $f = \chi_\Omega$. □

Unfortunately, Theorem 6.1 works because of the particular composition operator S_σ. For a general operator S, if there exist $f \in X_1^{X_2}$ and $g \in Y_2^{Y_1}$ such that $T(x) = M_g \circ S \circ M_f(x)$ for all $x \in X_1$, then T is clearly $S \circ M_f$-local, but the converse may fail.

Example 6.2. Let X be a B.f.s.' containing $L^\infty(\mu)$ and consider two functions $h_T > 0$ and $h_S > 0$ in $L^\infty(\mu)$ such that $\frac{h_T}{h_S} \notin L^\infty(\mu)$. Take an element $0 \neq x' \in X'$ and define the operators $T : X \to X$ and $S : X \to X$ as $T(x) = \langle x', x \rangle h_T$ and $S(x) = \langle x', x \rangle h_S$. Then T is S-local. However, if there exist $f, g \in L^\infty(\mu)$ such that $T(x) = gS(fx)$ for all $x \in X$, by taking $x \in X$ such that $\langle x', x \rangle \neq 0$ we have that $\frac{h_T}{h_S} = g \frac{\langle x', fx \rangle}{\langle x', x \rangle} \in L^\infty(\mu)$ which is a contradiction.

Consider now the case when X_1, X_2, Y_1, Y_2 all coincide with an order continuous B.f.s. X having the Fatou property and containing $L^\infty(\mu)$. Suppose that σ is bijective with measurable inverse σ^{-1}. Note that in this case σ^{-1} is a measure-preserving transformation and $S_{\sigma^{-1}} : X' \to X'$ is well defined. Indeed, $S_{\sigma^{-1}} = S'_\sigma$, since for every $x' \in X'$ and $x \in X$ we have that

$$\begin{aligned} \langle S'_\sigma(x'), x \rangle = \langle x', S_\sigma(x) \rangle &= \int x'(\omega)\, x(\sigma(\omega))\, d\mu(\omega) \\ &= \int x'\left(\sigma^{-1}(\omega)\right) x(\omega)\, d\mu(\omega) = \langle S_{\sigma^{-1}}(x'), x \rangle. \end{aligned}$$

Corollary 6.3. *The following statements are equivalent:*

(i) *T factorizes strongly through S_σ.*

(ii) *There exists $h \in L^\infty(\mu)$ such that*

$$\int T(x)(\omega) x'(\omega)\, d\mu(\omega) \leq \int |h(\omega) x'(\omega) x(\sigma(\omega))|\, d\mu(\omega)$$

for all $x \in X$ and $x' \in X'$.

(*iii*) *There exists* $h \in L^\infty(\mu)$ *such that*

$$|T'(x')(\omega)| \le \left|h\left(\sigma^{-1}(\omega)\right) x'\left(\sigma^{-1}(\omega)\right)\right| \quad \mu\text{-a.e.}(\omega)$$

for all $x' \in X'$.

Proof. The equivalence between (*i*) and (*ii*) is just the equivalence between (*i*) and (*iii*) in Corollary 4.3, as $S'_\sigma = S_{\sigma^{-1}}$.

Suppose (*ii*) holds. Given $x' \in X'$ we take $B^+ = \{\omega \in \Omega : T'(x')(\omega) \ge 0\}$ and $B^- = \{\omega \in \Omega : T'(x')(\omega) < 0\}$. For every $A \in \Sigma$, considering the element $x_0 = \chi_A(\chi_{B^+} - \chi_{B^-})$, we have that

$$\begin{aligned}
\int_A |T'(x')(\omega)|\, d\mu(\omega) &= \int T'(x')(\omega) x_0(\omega)\, d\mu(\omega) = \langle T'(x'), x_0\rangle \\
&= \langle x', T(x_0)\rangle = \int T(x_0)(\omega) x'(\omega)\, d\mu(\omega) \\
&\le \int |h(\omega) x'(\omega) x_0(\sigma(\omega))|\, d\mu(\omega) \\
&= \int \left|h\left(\sigma^{-1}(\omega)\right) x'\left(\sigma^{-1}(\omega)\right) x_0(\omega)\right|\, d\mu(\omega) \\
&\le \int_A \left|h\left(\sigma^{-1}(\omega)\right) x'\left(\sigma^{-1}(\omega)\right)\right|\, d\mu(\omega),
\end{aligned}$$

so (*iii*) holds. Conversely, if (*iii*) holds, for every $x \in X$ and $x' \in X'$, it follows that

$$\begin{aligned}
\int T(x)(\omega) x'(\omega)\, d\mu(\omega) = \int T'(x')(\omega) x(\omega)\, d\mu(\omega) &\le \int |T'(x')(\omega) x(\omega)|\, d\mu(\omega) \\
&\le \int \left|h\left(\sigma^{-1}(\omega)\right) x'\left(\sigma^{-1}(\omega)\right) x(\omega)\right|\, d\mu(\omega).
\end{aligned}$$

□

Moreover, if $S_{\sigma^{-1}} \colon X \to X$ is well defined, then conditions (*i*)–(*iii*) in Corollary 6.3 are equivalent to

(*iv*) *There exists* $h \in L^\infty(\mu)$ *such that* T *is* $S_\sigma \circ M_h$*-local.*

Indeed, if (*iv*) holds, in the same way as in the proof of Theorem 6.1 we obtain that $T(x) = T(\chi_\Omega) S_\sigma(x)$ for all simple function x. By taking $x \circ \sigma^{-1}$ which is also a simple function, it follows that $T(x \circ \sigma^{-1}) = T(\chi_\Omega)\, x$. Note that $T(\chi_\Omega)\, x \in X$ for every simple function x. Let us prove that this holds for every $x \in X$, so we will have that $T(\chi_\Omega) \in X^X$ and we can follow the proof of Theorem 6.1 to obtain that (*i*) holds. Let $0 \le x \in X$ and take a sequence (x_n) of simple functions such that $0 \le x_n \uparrow x$ μ-a.e. Then, $0 \le |T(\chi_\Omega)|\, x_n \uparrow |T(\chi_\Omega)|\, x$ μ-a.e. and

$$\begin{aligned}
\sup_n \||T(\chi_\Omega)|\, x_n\|_X = \sup_n \|T(\chi_\Omega)\, x_n\|_X &= \sup_n \|T(x_n \circ \sigma^{-1})\|_X \\
&\le \|T\| \sup_n \|x_n \circ \sigma^{-1}\|_X \le \|T\|\, \|x \circ \sigma^{-1}\|_X < \infty.
\end{aligned}$$

Since X has the Fatou property, we have that $|T(\chi_\Omega)|\, x \in X$ and so $T(\chi_\Omega)\, x \in X$. For a general $x \in X$, considering x^+ and x^- the positive and negative parts of x, we have that $T(\chi_\Omega)\, x = T(\chi_\Omega)\, x^+ - T(\chi_\Omega)\, x^- \in X$.

In the particular case when σ is the identity map, the equivalence between (i) and (iv) in Corollary 6.3 is a known result saying that T is local if and only if it is a multiplication operator, see [1, Proposition 1.7] and [18, Theorem 8].

Example 6.4. Let $([0,1], \mathcal{B}[0,1], \lambda)$ be the measure space with $\mathcal{B}[0,1]$ being the Borel σ-algebra of $[0,1]$ and λ the Lebesgue measure. For the measure-preserving transformation $\sigma\colon [0,1] \to [0,1]$ given by $\sigma(s) = 1-s$ for all $s \in [0,1]$, the composition operator $S_\sigma\colon L^p[0,1] \to L^p[0,1]$ $(1 \le p < \infty)$ is well defined. The continuous linear operators $T\colon L^p[0,1] \to L^p[0,1]$ which factorize strongly through S_σ are characterized by conditions (ii)–(iv) of Corollary 6.3.

6.2. Strong factorization through a kernel operator

Consider a measurable function $K : \Omega \times \Omega \to [0,\infty)$ such that the operator $S_K\colon X_2 \to Y_2$ given by

$$S_K(f)(s) = \int f(t) K(s,t)\, d\mu(t)$$

for all $f \in X_2$ and $s \in \Omega$, is well defined and so continuous.

Under conditions of Theorem 4.1 (or 4.2), statement (i) holds if and only if T factorizes strongly through S_K, that is, there exist $f \in X_1^{X_2}$ and $g \in Y_2^{Y_1}$ such that

$$T(x)(s) = g(s) \int f(t) x(t) K(s,t)\, d\mu(t)$$

for all $x \in X_1$ and $s \in \Omega$. In this case, T is also a kernel operator with kernel $\widetilde{K}(s,t) = g(s) f(t) K(s,t)$.

In the case when X_1, X_2, Y_1, Y_2 all coincide with an order continuous B.f.s. X having the Fatou property and containing $L^\infty(\mu)$, using the equivalence between (i) and (iii) in Corollary 4.3, we obtain that T factorizes strongly through S_K if and only if there exists $h \in L^\infty(\mu)$ such that

$$\int T(x)(t) x'(t)\, d\mu(t) \le \int \left| x(t) \int h(s) x'(s) K(s,t)\, d\mu(s) \right| d\mu(t) \tag{5}$$

for all $x \in X$ and $x' \in X'$. Indeed, $S_K'\colon X' \to X'$ satisfies that

$$\begin{aligned} \langle S_K'(x'), x\rangle = \langle x', S_K(x)\rangle &= \int x'(s) \int x(t) K(s,t)\, d\mu(t)\ d\mu(s) \\ &= \int x(t) \int x'(s) K(s,t)\, d\mu(s)\ d\mu(t) \\ &= \left\langle \int x'(s) K(s,\cdot)\, d\mu(s), x \right\rangle \end{aligned}$$

for all $x' \in X'$ and $x \in X$.

Example 6.5. Consider the measure space given by the interval $[0,1]$, its Borel σ-algebra and the Lebesgue measure. Let K be the kernel given by $K(s,t) = \chi_{[0,s]}(t)$ for all $s,t \in [0,1]$. Then, S_K is just the Volterra operator. Suppose that $S_K\colon X \to X$ is well defined and continuous (e.g. $X = L^p[0,1]$ with $1 \le p < \infty$). The following statements are equivalent:

(i) There exist $g, f \in L^\infty[0,1]$ such that

$$T(x)(s) = g(s) \int_0^s f(t)x(t)\,dt \quad \text{a.e.}(s)$$

for all $x \in X$.

(ii) There exists $h \in L^\infty[0,1]$ such that

$$\int_0^1 T(x)(t)x'(t)\,dt \le \int_0^1 \left| x(t) \int_t^1 h(s)x'(s)\,ds \right| dt$$

for all $x \in X$ and $x' \in X'$.

(iii) There exists $h \in L^\infty[0,1]$ such that

$$|T'(x')(t)| \le \left| \int_t^1 h(s)x'(s)\,ds \right| \quad \text{a.e.}(t)$$

for all $x' \in X'$.

Note that the equivalence between (i) and (ii) follows from applying (5). If (ii) holds, given $x' \in X'$ we consider the sets $B^+ = \{t \in [0,1] : T'(x') \ge 0\}$ and $B^- = \{t \in [0,1] : T'(x') < 0\}$. For each measurable set A, taking $x = \chi_A(\chi_{B^+} - \chi_{B^-})$, we have that

$$\begin{aligned}
\int_A |T'(x')(t)|dt &= \int_0^1 T'(x')(t)x(t)dt = \langle T'(x'), x\rangle = \langle x', T(x)\rangle \\
&= \int_0^1 T(x)(t)x'(t)dt \le \int_0^1 \left| x(t) \int_t^1 h(s)x'(s)\,ds \right| dt \\
&\le \int_A \left| \int_t^1 h(s)x'(s)\,ds \right| dt.
\end{aligned}$$

so (iii) holds. Conversely, if (iii) holds, for every $x \in X$ and $x' \in X'$, it follows that

$$\begin{aligned}
\int_0^1 T(x)(t)x'(t)\,dt &= \int_0^1 T'(x')(t)x(t)dt \le \int_0^1 |T'(x')(t)x(t)|dt \\
&\le \int_0^1 \left| x(t) \int_t^1 h(s)x'(s)\,ds \right| dt.
\end{aligned}$$

If we consider now the Hardy operator which is given by the kernel $K(s,t) = \frac{1}{s}\chi_{[0,s]}(t)$ for all $s,t \in [0,1]$ and suppose that $S_K\colon X \to X$ is well defined and continuous (e.g. $X = L^p[0,1]$ with $1 < p < \infty$), in a similar way we obtain that the following statements are equivalent:

(i) There exist $g, f \in L^\infty[0,1]$ such that

$$T(x)(s) = \frac{g(s)}{s} \int_0^s f(t)x(t)\,dt \quad \text{a.e.}(s)$$

for all $x \in X$.

(ii) There exists $h \in L^\infty[0,1]$ such that

$$\int_0^1 T(x)(t)x'(t)\,dt \le \int_0^1 \left| x(t) \int_t^1 \frac{h(s)x'(s)}{s}\,ds \right| dt$$

for all $x \in X$ and $x' \in X'$.

(iii) There exists $h \in L^\infty[0,1]$ such that

$$|T'(x')(t)| \le \left| \int_t^1 \frac{h(s)x'(s)}{s}\,ds \right| \quad \text{a.e.}(t)$$

for all $x' \in X'$.

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