Optimal Domains for L^0-valued Operators Via Stochastic Measures

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Abstract. We study extension of operators $T\colon E \to L^0([0,1])$, where E is an F–function space and $L^0([0,1])$ the space of measurable functions with the topology of convergence in measure, to domains larger than E, and we study the properties of such domains. The main tool is the integration of scalar functions with respect to stochastic measures and the corresponding spaces of integrable functions.

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1. Introduction

Let $T\colon E \to L^0([0,1])$ be a linear operator, where E is a function space and $L^0([0,1])$ the space of measurable functions with the topology of convergence in measure. The aim of this paper is to study conditions on T and E that allow us to extend the operator T to domains larger than E and, in that case, to study the properties of such domains.

In the case when $T\colon E \to X$ with X (and also E) a Banach space, this problem has been considered in [7], [9], [12]. For the particular case of T the operator associated with Sobolev inequality and X a rearrangement invariant space, this study has been done in [8], [10]; and for T a convolution operator and $X = L^p(\mathbb{T})$, in [23]. In all of these cases, the main tool has been an X–valued measure ν canonically associated with the operator T and the corresponding space $L^1(\nu)$ of scalar functions which are integrable with respect to ν. The integration theory for Banach space valued measures was developed by Bartle, Dunford and Schwartz, [3], and Lewis, [16], [17]. However, $L^0([0,1])$ with the topology of convergence in

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measure is a complete metric linear space, but not locally convex. This case is essentially different from the Banach space case already studied, since the lack of duality precludes the use of the main ideas and techniques of the Banach space case. The adequate framework is to consider $L^0([0,1])$ as an F–space (and, hence, also E). This allows us to use the theory for integrating scalar functions with respect to an F–space valued measure for the study of the extension of operators $T\colon E \to L^0([0,1])$.

The paper is organized as follows. In § 2, we collect general facts on F–spaces and F–function spaces (F.f.s. in short); and recall the theory for integrating scalar functions with respect to an F–space valued measure developed by Turpin, [34], [35], Rolewicz, [25], [26], and Thomas, [32]. While the space $L^1(\nu)$, for ν a Banach space valued measure, has been thoroughly studied (see, for example, [4], [5], [6], [11], [22], [24]), this is not the case when ν is an F–space valued measure. In § 3, we study properties of the space $L^1(\nu)$ for a measure ν with values in a general F–space. Special emphasis is placed on the case of stochastic (i.e. countably additive $L^0([0,1])$–valued) measures, which are better behaved due to the properties of $L^0([0,1])$. In particular, we show that, for ν a stochastic measure, the space $L^1(\nu)$ is a C–space whenever ν is positive (Theorem 3.5). It is a remarkable fact that, for ν a stochastic measure, the space $L^1(\nu)$ always satisfies the bounded multiplier test (Theorem 3.7). We identify the class of all spaces arising as L^1 of an F–space valued measure, namely, it coincides with the class of all order continuous F.f.s. (Theorem 3.8). In § 4, by applying the results of § 3, we show that, under certain conditions, an operator $T\colon E \to X$ can be extended to $L^1(\nu)$, where ν is a measure canonically associated with T. Moreover, $L^1(\nu)$ is the largest F.f.s. with order continuous F–norm to which T can be extended (Theorem 4.3). Particular attention is given to the case when T is a kernel operator. We end in § 5 by exhibiting several examples arising from classical analysis.

2. Preliminaries

2.1. A *metric linear space* is a vector space X (which we will consider over $\mathbb{R}$) endowed with a metric d which renders continuous the operations of addition and multiplication by scalars. We can assume that the metric d is invariant under translation, that is, $d(x+z, y+z) = d(x,y)$ for $x, y, z \in X$, since there is always an equivalent metric with this property; see [26, Theorem 1.1.1]. An *F–norm* on a vector space X is a map $\|\cdot\|\colon X \to [0,\infty)$ satisfying

(i) $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|x+y\| \le \|x\| + \|y\|$, for $x, y \in X$.
(iii) $\|\alpha x\| \le \|x\|$, for $x \in X$ and $\alpha \in \mathbb{R}$ with $|\alpha| \le 1$.
(iv) $\|\alpha x\| \to 0$ when $\alpha \to 0$, for $x \in X$.

In this case, $d(x,y) := \|x-y\|$, $x, y \in X$, defines an invariant metric generating the topology of X. Conversely, if X is a metric linear space with an invariant metric

d, then $\|x\| := d(x,0)$, $x \in X$, is an F–norm in X. If the topology generated by the F–norm is complete, X is said to be an *F–space*.

A *Riesz space* is a vector space X endowed with an order compatible with the linear structure, for which the supremum of any pair of elements exists. An *F–lattice* is an F–space X which also is a Riesz space, and where the F–norm and the order are compatible, that is, $\|x\| \le \|y\|$ whenever $x, y \in X$ with $|x| \le |y|$. An operator T between F–lattices is *positive* if $Tx \ge 0$ whenever $x \ge 0$. Linear positive operators between F–lattices are always continuous. An *F–function space* over a finite measure space $(\Omega, \Sigma, \lambda)$ is an F–space E of (classes of) real measurable, finite λ–a.e. functions, satisfying:

(i) f measurable, $g \in E$ and $|f| \le |g|$ λ–a.e., imply $f \in E$ and $\|f\| \le \|g\|$.
(ii) $\chi_A \in E$ for every $A \in \Sigma$.

This definition extends that of Banach function space given in [18, Definition 1.b.17]. Note that an F–function space is an F–lattice for the λ–a.e. order, and $L^\infty(\Omega,\Sigma,\lambda) \subset E \subset L^0(\Omega,\Sigma,\lambda)$ where the inclusions are continuous. In particular, convergence in E of a sequence implies pointwise λ–a.e. convergence for some subsequence.

For topics on metric linear spaces, see [15], [25], [26]; on Riesz spaces, see [1], [19].

2.2. We briefly recall the theory of integration of scalar measurable functions with respect to a vector measure with values in an F–space. This presentation is based on the common features of the work of Rolewicz, [25, §III.6] and [26, §3.6]; Turpin, [34] and [35, Chp.7]; and Thomas, [32].

Let (Ω, Σ) be a measurable space, X an F–space and $\nu\colon \Sigma \to X$ a *countably additive vector measure*, that is, ν satisfies that $\sum \nu(A_n)$ converges to $\nu(\cup A_n)$ in X, for every sequence (A_n) of disjoint sets in Σ. A measurable set A is *ν-null* if $\nu(B) = 0$ for every measurable set $B \subset A$. Given a simple function $\varphi = \sum_1^n a_i \chi_{A_i}$, the integral of φ with respect to ν is $\int \varphi d\nu := \sum_1^n a_i \nu(A_i) \in X$. In order to extend the integral to bounded measurable functions it is sufficient (and necessary) that the integration operator $\varphi \in S(\Sigma) \mapsto \int \varphi d\nu \in X$ is continuous when we consider the topology of uniform convergence on the space $S(\Sigma)$ of the simple functions. In this case, the integration operator is continuous from the space $L^\infty(\nu)$, of ν–essentially bounded measurable functions, to X. This condition is equivalent to the convex hull of $\nu(\Sigma)$, the range of ν, being bounded. Measures satisfying this property are called *L^∞–bounded* measures. Although this condition may fail (see examples in [27], [33] and [35, Theorem 7.4(c)]), the measures for which it holds abound. Namely, this is the case for bounded measures (i.e. with bounded range) with values in an F–space satisfying the *bounded multiplier test*, that is, $\sum b_n x_n$ converges for all $(b_n) \in \ell^\infty$ whenever the series $\sum x_n$ is unconditionally convergent. In particular, this is the case for measures with values in a Frèchet or a Banach space. All measures considered in this paper will be assumed to be L^∞–bounded.

A measurable function f is *integrable* with respect to ν if there exists a sequence of simple functions (φ_n) such that

(i) φ_n converges to f ν–a.e.
(ii) $\int \varphi_n h\, d\nu$ converges in X, for every $h \in L^\infty(\nu)$.

In this case, the integral of f is defined by $\int f d\nu := \lim \int \varphi_n d\nu$. If needed, we will also use the following notation $\nu(f) = \int f d\nu$. We denote by $L^1(\nu)$ the space of integrable functions with respect to ν, where functions which are equal ν–a.e. are identified. In order to endow the space $L^1(\nu)$ with a topology, we consider the *semivariation* of ν, that is, the map $\dot{\nu}$ defined for measurable functions f by

$$\dot{\nu}(f) = \sup\left\{ \Big\| \int \varphi\, d\nu \Big\|_X \, : \, \varphi \text{ simple}, \, |\varphi| \leq |f| \right\} ,$$

where $\|\cdot\|_X$ is the F–norm on X. Note that $\| \int \varphi d\nu \|_X \leq \dot{\nu}(\varphi)$ for every simple function φ. For $A \in \Sigma$, we denote $\dot{\nu}(A) = \dot{\nu}(\chi_A)$. Then, $A \in \Sigma$ is ν-null if and only if $\dot{\nu}(A) = 0$. Let f, g, f_n be measurable functions, the semivariation satisfies the following properties:

(a) $\dot{\nu}(f) = 0$ if and only if $f = 0$ ν–a.e.
(b) If $|f| \leq |g|$ ν–a.e., then $\dot{\nu}(f) \leq \dot{\nu}(g)$.
(c) $\dot{\nu}(f+g) \leq \dot{\nu}(f) + \dot{\nu}(g)$.
(d) $\dot{\nu}(\sum f_n) \leq \sum \dot{\nu}(f_n)$, whenever $\sum f_n$ converges ν–a.e.

Let $L^1_w(\nu)$ denote the space of all measurable functions f with $\dot{\nu}(f) < \infty$, where functions which are equal ν–a.e. are identified. The space $L^1_w(\nu)$ endowed with the ν–a.e. order is a Riesz space and an *ideal of measurable functions*, that is, $f \in L^1_w(\nu)$ whenever $|f| \leq |g|$ ν–a.e. for some $g \in L^1_w(\nu)$. The semivariation $\dot{\nu}$ would be an F–norm on $L^1_w(\nu)$ if every measurable function $f \in L^1_w(\nu)$ satisfies $\dot{\nu}(\alpha f) \to 0$ when $\alpha \to 0$. An equivalent condition is that the set $B(f) := \{\int \varphi d\nu :$ φ simple, $|\varphi| \leq |f|\}$ is bounded in X. We denote by $L^1_b(\nu)$ the subspace of $L^1_w(\nu)$ consisting of those functions f such that $B(f)$ is bounded. Then, $\dot{\nu}$ is a complete F–norm on $L^1_b(\nu)$ which is compatible with the ν–a.e. order. It follows that $L^1_b(\nu)$ is an F–lattice which is an ideal of measurable functions containing $L^\infty(\nu)$, and $L^1(\nu)$ is the closure of the simple functions in $L^1_b(\nu)$. Thus, $L^1(\nu)$ is an F–lattice for the F–norm $\dot{\nu}$ and the ν–a.e. order, and an ideal of measurable functions. A very important property of $L^1(\nu)$ follows from the dominated convergence theorem: order bounded increasing sequences are convergent, that is, $L^1(\nu)$ is *order continuous*. The inequality $\| \int f d\nu \|_X \leq \dot{\nu}(f)$ holds for all $f \in L^1(\nu)$, so the integration operator $f \in L^1(\nu) \mapsto \int f d\nu \in X$ is continuous. Note that when X is a Frèchet or a Banach space, we have $L^1_b(\nu) = L^1_w(\nu)$, and this space coincides with the space of scalarly integrable functions; see [17, §5].

3. Stochastic measures

We will focus our attention on the case when X is the particular space $L^0([0,1],$ $\mathcal{M}, m)$ of real, Lebesgue measurable, finite m–a.e. functions over $[0,1]$ ($\mathcal{M}$ is the

Lebesgue σ–algebra in [0,1] and m the Lebesgue measure on $\mathcal{M}$), where functions which are equal m–a.e. are identified. It is an F.f.s. when endowed with the m–a.e. order and the F–norm

$$\|f\|_0 = \int_0^1 \frac{|f(t)|}{1+|f(t)|}\,dt \ .$$

The topology generated is that of convergence in measure.

Let $\nu\colon \mathcal{M} \to L^0([0,1])$ be a countably additive measure, we will say in this case that ν is an *stochastic measure*. Talagrand, [31], and Kalton, Peck and Roberts, [14], have proved that ν is bounded, that is, $\nu(\mathcal{M})$ is a bounded set in $L^0([0,1])$. Maurey and Pisier have proved that $L^0([0,1])$ satisfies the bounded multiplier test, [21, Corollaire]. These two facts together imply that ν is L^∞–bounded. Thus, the space $L^1(\nu)$ always exists and it is non-trivial. Observe that, since the F–norm $\|\cdot\|_0$ is bounded, we always have $L^1_w(\nu) = L^0([0,1])$.

Due to the properties of $L^0([0,1])$, integrability with respect to a stochastic measure has a rich variety of equivalent conditions. Recall that a sequence (x_n) in an F–space X is a *C–sequence* if the series $\sum a_n x_n$ converges for every $(a_n) \in c_0$. The space X is a *C–space* if, for any C–sequence (x_n), the series $\sum x_n$ is convergent. Schwartz has shown that $L^0([0,1])$ is a C–space, [29].

Proposition 3.1. *Let ν be an stochastic measure. For a measurable function f, the following conditions are equivalent:*

- (i) *f is integrable with respect to ν.*
- (ii) *There exists a sequence (φ_n) of simple functions converging to f ν–a.e., and satisfying that $(\int_A \varphi_n\, d\nu)$ converges in $L^0([0,1])$, for every $A \in \mathcal{M}$.*
- (iii) *The sequence (g_n) is a C–sequence in $L^1(\nu)$, whenever (g_n) are disjoint simple functions with $|g_n| \le |f|$ ν–a.e.*
- (iv) *The sequence $(\nu(g_n))$ is a C–sequence in $L^0([0,1])$, whenever (g_n) are disjoint simple functions with $|g_n| \le |f|$ ν–a.e.*
- (v) *$\sum \left|\nu(g_n)(t)\right|^2 < \infty$, for m–a.e. $t \in [0,1]$, whenever (g_n) are disjoint simple functions with $|g_n| \le |f|\nu$–a.e.*

Proof. (i) and (ii) are equivalent since ν takes its values in $L^0([0,1])$, a space satisfying the bounded multiplier test; see [34, 2.15].

(i) $\Rightarrow$ (iii) Let $f \in L^1(\nu)$ and (g_n) disjoint simple functions with $|g_n| \le |f|$ ν–a.e. Since $\sum |g_n| \le |f|$ and $L^1(\nu)$ is an order continuous ideal of measurable functions, we have that $\sum |g_n|$ converges in $L^1(\nu)$. Hence, $\sum a_n g_n$ is convergent in $L^1(\nu)$ for every $(a_n) \in c_0$.

(iii) $\Rightarrow$ (iv) Follows from the continuity of the integration map.

(iv) $\Rightarrow$ (v) Follows from the Kolmogorov–Kintchine inequality: $\sum |f_n|^2$ converges m–a.e. whenever (f_n) is a C–sequence in $L^0([0,1])$; see [26, Proposition 3.10.7].

(v) $\Rightarrow$ (iv) Suppose there exist disjoint simple functions (g_n) with $|g_n| \le |f|$ ν–a.e. and $(a_n) \in c_0$ such that $\sum a_n \nu(g_n)$ does not converge in $L^0([0,1])$. Then, there exist $\delta > 0$, and $m_k > n_k > m_{k-1} > n_{k-1}$ such that $\|\sum_{n_k}^{m_k} a_j \nu(g_j)\|_0 > \delta$, for $k \geq 1$. Consider the functions $\phi_k := \sum_{n_k}^{m_k} a_j g_j$. They are disjoint, simple, and satisfy $|\phi_k| \leq |f|$ ν–a.e. for large enough k such that $\sup_{j \geq n_k} |a_j| \leq 1$. Then, by assumption, $\sum \left|\nu(\phi_k)(t)\right|^2 < \infty$, for m–a.e. $t \in [0,1]$. In particular, $\nu(\phi_k) \to 0$ m–a.e. and so, $\nu(\phi_k) \to 0$ in $L^0([0,1])$. This contradicts $\|\nu(\phi_k)\|_0 > \delta$.

(iv) $\Rightarrow$ (iii) Suppose there exist disjoint simple functions (g_n) with $|g_n| \leq |f|$ ν–a.e. and $(a_n) \in c_0$ such that $\sum a_n g_n$ does not converge in $L^1(\nu)$. Then, there exist $\delta > 0$, and $m_k > n_k > m_{k-1} > n_{k-1}$ such that $\dot{\nu}(\sum_{n_k}^{m_k} a_j g_j) > \delta$, for $k \geq 1$. By definition of the semivariation of ν, there exist simple functions (φ_k) such that $|\varphi_k| \leq |\sum_{n_k}^{m_k} a_j g_j|$ and $\|\nu(\varphi_k)\|_0 > \delta$. Note that (φ_k) are disjoint and $|\varphi_k| \leq |f|$ ν–a.e. for large enough k. Then, $(\nu(\varphi_k))$ is a C–sequence in $L^0([0,1])$. Since $L^0([0,1])$ is a C–space, $\sum \nu(\varphi_k)$ converges in $L^0([0,1])$. Hence, $\nu(\varphi_k)$ tends to zero in $L^0([0,1])$. We have arrived at a contradiction.

(iii) $\Rightarrow$ (i) Consider the sets $A_n = \{x \in [0,1] : n-1 \leq |f(x)| < n\}$, for $n \geq 1$. We have $\sum (n-1)\chi_{A_n} \leq |f| < \sum n\chi_{A_n}$ pointwise. The simple functions $g_n = (n-1)\chi_{A_n}$ are disjoint and satisfy $|g_n| \leq |f|$, so (g_n) is a C–sequence in $L^1(\nu)$. Suppose that $\sum g_n$ does not converges in $L^1(\nu)$. Then, there exist $\delta > 0$, and $m_k > n_k > m_{k-1} > n_{k-1}$ such that $\dot{\nu}(\sum_{n_k}^{m_k} g_j) > \delta$, for $k \geq 1$. Let (φ_k) be simple functions such that $|\varphi_k| \leq |\sum_{n_k}^{m_k} g_j|$ and $\|\nu(\varphi_k)\|_0 > \delta$, for $k \geq 1$. Since (φ_k) are disjoint and $|\varphi_k| \leq |f|$ ν–a.e., we have that (φ_k) is a C–sequence in $L^1(\nu)$. Then, $(\nu(\varphi_k))$ is a C–sequence in $L^0([0,1])$, which is a C–space, and so $\nu(\varphi_k)$ tends to zero in $L^0([0,1])$. We have arrived at a contradiction. Consequently, $\sum g_n$ converges in $L^1(\nu)$. Hence, $g = \sum g_n \in L^1(\nu)$ and, since $|f| < g + \chi_{[0,1]}$, we have that $f \in L^1(\nu)$. □

Remark 3.2. Condition (ii) is precisely the definition, by Bartle, Dunford and Schwartz, of integrability with respect to a Banach space–valued measure; [3].

There is a particular class of vector measures for which we obtain special properties. Namely, the positive vector measures. Let X be an F–lattice and $\nu\colon \Sigma \to X$ a vector measure, we say that ν is *positive* if $\nu(A) \geq 0$ (in the order of X), for all $A \in \Sigma$. Note that positive vector measures are always L^∞–bounded. To see this let ν be a positive measure and $\varphi = \sum_1^n a_i \chi_{A_i}$ a simple function, then

$$\Big|\int \varphi d\nu\Big| \leq \sum_1^n |a_i| \nu(A_i) \leq \|\varphi\|_\infty \nu(\cup_1^n A_i) \leq \|\varphi\|_\infty \nu(\Omega)\,.$$

Thus, $\|\int \varphi d\nu\|_X \leq \|\,\|\varphi\|_\infty \cdot \nu(\Omega)\|_X$, and so the integration operator is continuous on the space of the simple functions with the topology of the uniform convergence. Since for a positive vector measure ν we have $\int f d\nu \geq 0$, for every non-negative function $f \in L^1(\nu)$, the following equivalent expressions for the semivariation of $f \in L^1(\nu)$ hold:

$$\sup_{A \in \Sigma} \Big\|\int_A f d\nu\Big\|_X \leq \dot{\nu}(f) = \Big\|\int |f|\, d\nu\Big\|_X \leq 2 \sup_{A \in \Sigma} \Big\|\int_A f d\nu\Big\|_X. \tag{3.1}$$

An important property for a vector measure is the existence of a control measure. Given a measure $\nu\colon \Sigma \to X$, a *control measure* for ν is a real measure $\mu\colon \Sigma \to [0,\infty)$ such that $\mu(A) \to 0$ implies $\nu(A) \to 0$. This equivalent to $\mu(A) = 0$ implies $\nu(A) = 0$ since μ is finite. When μ and ν have the same null sets (that is, $\mu(A) = 0$ if and only if $\dot{\nu}(A) = 0$) they are said to be *equivalent*.

Remark 3.3. A measure $\nu\colon \Sigma \to X$, where X is a F.f.s over a finite measure space (Δ, Ξ, λ), always has an equivalent control measure. To see this for ν being positive, we only have to consider the vector measure ξ defined, for $A \in \Sigma$, by $\xi(A)(\delta) := \frac{\nu(A)}{\nu(\Omega)}(\delta)$ when δ belongs to the support of $\nu(\Omega)$, and $\xi(A) = 0$ in other case. Since ν is positive, $\nu(A) \le \nu(\Omega)$, hence ξ takes values in the Banach space $L^1(\Delta, \Xi, \lambda)$. Then, from [13, Theorem IX.2.2], there exist a control measure μ for ξ (which can be taken to be of Rybakov type, that is, $\mu = |x^*\xi|$, for some x^* in $L^\infty(\Delta, \Xi, \lambda)$). Thus, μ is an equivalent measure for ξ. Since ξ and ν have the same null sets, it follows that μ and ν are equivalent.

For a general ν, the claim follows from X being continuously embedded in $L^0(\Delta, \Xi, \lambda)$, and a result of Talagrand [31, Theorem B], based in the ideas of Maurey in [20]: any measure with values in an space $L^0(\Delta, \Xi, \lambda)$ has an equivalent control measure.

We present conditions on a positive measure ν guaranteeing that $L^1(\nu)$ is a C–space.

Proposition 3.4. *Let X be an F–lattice which is a C–space and $\nu\colon \Sigma \to X$ a positive measure having an equivalent positive real measure. Then $L^1(\nu)$ is a C–space.*

Proof. We will prove that $L^1(\nu)$ is a C–space by showing that every C–sequence $(f_n) \subset L^1(\nu)$ tends to zero; see [26, Proposition 3.10.3]. From (3.1) it follows that, for each $A \in \Sigma$, the sequence $(\int_A f_n d\nu)$ is a C–sequence in X. Since X is a C–space, $\int_A f_n d\nu \to 0$ in X, for each $A \in \Sigma$. Consider the measures $\nu_n\colon \Sigma \to X$ defined by $\nu_n(A) := \int_A f_n d\nu$, for $A \in \Sigma$ (for more details on density measures, see Proposition 5.3 below). Let μ be a positive real measure equivalent to ν. Since $\lim_n \nu_n(A)$ exists for each $A \in \Sigma$ and $\mu(A) = 0$ implies $\nu_n(A) = 0$, from a generalized version of the Vitali–Hahn–Saks theorem, [34, 2.7.2], it follows that $\sup_n \|\nu_n(A)\|_X \to 0$ as $\mu(A) \to 0$. That is, given $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta$ implies $\sup_n \|\nu_n(A)\|_X < \varepsilon$.

Since ν and μ are equivalent, $L^1(\nu)$ is an F.f.s. over (Ω, Σ, μ). Hence, $L^1(\nu)$ is continuously included in $L^0(\Omega, \Sigma, \mu)$. Thus, (f_n) is a C–sequence in $L^0(\Omega, \Sigma, \mu)$ and so, by the Kolmogorov–Kintchine inequality, we have that $\sum |f_n|^2$ converges μ–a.e., so $f_n \to 0$ μ–a.e. Applying Egoroff's theorem, for the $\delta > 0$ given above, there exists $Z_\delta \in \Sigma$ with $\mu(Z_\delta) < \delta$ and such that $f_n \to 0$ uniformly on $\Omega \backslash Z_\delta$. Then, for every $A \in \Sigma$, we have

$$\begin{aligned}\Big\|\int_A f_n d\nu\Big\|_X &\le \Big\|\int_{A\cap Z_\delta} f_n d\nu\Big\|_X + \Big\|\int_{A\cap\Omega\setminus Z_\delta} f_n d\nu\Big\|_X \\ &\le \sup_n \Big\|\nu_n(A\cap Z_\delta)\Big\|_X + \dot\nu(f_n\chi_{\Omega\setminus Z_\delta}) \\ &\le \varepsilon + \dot\nu(\|f_n\chi_{\Omega\setminus Z_\delta}\|_\infty) \le 2\varepsilon,\end{aligned}$$

for large enough n. From (3.1), it follows that $f_n \to 0$ in $L^1(\nu)$. □

Returning to the case of stochastic measures, from Remark 3.3, Proposition 3.4, and since $L^0([0,1])$ is a C–space, we have the following conclusion.

Theorem 3.5. *Let ν be a positive stochastic measure. Then $L^1(\nu)$ is a C–space.*

In general, the bounded multiplier test is not satisfied in F–spaces. It is a remarkable fact that for stochastic measures, the space $L^1(\nu)$ always satisfies the bounded multiplier test. The following lemma, from [28], is needed to proof this result.

Lemma 3.6. *Let $(g_j)_1^m$ be measurable functions and $|b_j| \le 1$, $1 \le j \le m$. Then*

$$m\left(\left[\Big|\sum_{j=1}^m b_j g_j(t)\Big| > 8\varepsilon\right]\right) \le 8 \max_{\delta_j=\pm1} m\left(\left[\Big|\sum_{j=1}^m \delta_j g_j(t)\Big| > \varepsilon\right]\right),$$

where, if g is a measurable function, we denote $[|g| > r] := \{t \in [0,1] : |g(t)| > r\}$.

Theorem 3.7. *Let ν be an stochastic measure. Then $L^1(\nu)$ satisfies the bounded multiplier test.*

Proof. Let $\sum f_n$ be an unconditionally convergent series in $L^1(\nu)$. Given $\varepsilon > 0$, there exists n_ε such that for every finite set $M \subset \mathbb{N}$ with $\min M > n_\varepsilon$ we have

$$\dot\nu\left(\sum_{n\in M} f_n\right) \le \frac{\varepsilon^2}{2(1+\varepsilon)}.$$

Given (δ_j) with $\delta_j = \pm1$ and $m > n > n_\varepsilon$, set $I^+ = \{j : n \le j \le m, \delta_j = 1\}$ and $I^- = \{j : n \le j \le m, \delta_j = -1\}$. Let φ be a simple function with $|\varphi| \le 1$, then

$$\Big\|\sum_{j=n}^m \delta_j \int \varphi f_j \, d\nu\Big\|_0 \le \dot\nu\left(\sum_{j=n}^m \delta_j f_j\right) \le \dot\nu\left(\sum_{j\in I^+} f_j\right) + \dot\nu\left(\sum_{j\in I^-} f_j\right) \le \frac{\varepsilon^2}{1+\varepsilon}.$$

Then, we have

$$\max_{\delta_j=\pm1} m\left[\Big|\sum_{j=n}^m \delta_j \int \varphi f_j \, d\nu\Big| > \varepsilon\right] \le \max_{\delta_j=\pm1} \frac{1+\varepsilon}{\varepsilon}\Big\|\sum_{j=n}^m \delta_j \int \varphi f_j \, d\nu\Big\|_0 \le \varepsilon.$$

Applying Lemma 3.6 to our setting, we have, for (b_j) with $|b_j| \leq 1$,

$$m\left[\Big|\sum_{j=n}^{m} b_j \int \varphi f_j \, d\nu\Big| > 8\varepsilon\right] \leq 8 \max_{\delta_j=\pm 1} m\left[\Big|\sum_{j=n}^{m} \delta_j \int \varphi f_j \, d\nu\Big| > \varepsilon\right] \leq 8\varepsilon.$$

Then,

$$\Big\|\sum_{j=n}^{m} b_j \int \varphi f_j \, d\nu\Big\|_0 \leq m\left[\Big|\sum_{j=n}^{m} b_j \int \varphi f_j \, d\nu\Big| > 8\varepsilon\right] + \frac{8\varepsilon}{1+8\varepsilon} \leq 16\varepsilon,$$

for all $m > n > n_\varepsilon$, (b_j) with $|b_j| \leq 1$ and φ with $|\varphi| \leq 1$. Noting that the semivariation of ν for any function $g \in L^1(\nu)$ can be written as

$$\dot{\nu}(g) = \sup\left\{\Big\|\int \varphi g d\nu\Big\|_X : \varphi \text{ simple}, |\varphi| \leq 1\right\}, \tag{3.2}$$

it follows that

$$\dot{\nu}\left(\sum_{j=n}^{m} b_j f_j\right) = \sup\left\{\Big\|\sum_{j=n}^{m} b_j \int \varphi f_j \, d\nu\Big\|_0 : \varphi \text{ simple}, |\varphi| \leq 1\right\} \leq 16\varepsilon,$$

for all $m > n > n_\varepsilon$ and (b_j) with $|b_j| \leq 1$. Hence, $\sum b_n f_n$ converges in $L^1(\nu)$ for every $(b_n) \in \ell^\infty$. □

An important question is the following: What spaces arise as L^1 of an stochastic measure? A first simple example is the space $L^0([0,1])$ itself. We have $L^1(\nu) = L^0([0,1])$ for the measure ν defined by

$$A \in \mathcal{M} \mapsto \nu(A) := \chi_A \in L^0([0,1]).$$

The next result characterizes the class of the spaces arising as $L^1(\nu)$ for an F–space valued measure ν having an equivalent positive real measure.

Theorem 3.8. *Let E be an order continuous F.f.s. over a finite measure space $(\Omega, \Sigma, \lambda)$. Then there exists an F–space valued measure ν such that E and $L^1(\nu)$ are linear and order isomorphic, and isometric. Moreover, the measures ν and λ are equivalent.*

Proof. Consider the set function $\nu\colon \Sigma \to E$ defined by $\nu(A) := \chi_A$ for $A \in \Sigma$. It is clearly additive and, since E is order continuous, is countably additive. Note that ν is positive, so it is L^∞–bounded.

Any simple function φ satisfies $\int \varphi d\nu = \varphi$. Thus, $\dot{\nu}(\varphi) = \|\varphi\|_E$. In particular, $\dot{\nu}(A) = \|\chi_A\|_E$ so, λ and ν have the same null sets. Since the simple functions are dense in both $L^1(\nu)$ and E (by order continuity of both spaces), the spaces $L^1(\nu)$ and E coincide and $\dot{\nu}(f) = \|f\|_E$ for all $f \in E$. □

Remark 3.9. In the context of Banach space valued measures, the result corresponding to Theorem 3.8 is [4, Theorem 8].

Proposition 3.10. *Let X be an order continuous F.f.s. over $[0,1]$ which is isomorphic to a subspace of $L^0([0,1])$. Then there exists an stochastic measure ν such that $L^1(\nu)$ is linear and order isomorphic to X.*

Proof. Consider the measure $\nu\colon \mathcal{M} \to X$ defined by $\nu(A) := \chi_A$ for $A \in \mathcal{M}$. From Theorem 3.8 if follows that $L^1(\nu) = X$. Let Φ be an isomorphism between X and a subspace $Z \subset L^0([0,1])$. The measure $\eta := \Phi \circ \nu$ is $L^0([0,1])$–valued, countably additive and $L^1(\eta)$ is linear (and order) isomorphic to X. □

Remark 3.11. Consider the Rademacher functions defined by $r_n(t) := \operatorname{sign}\sin(2^n \pi t)$, for $t \in [0,1]$ and $n \geq 1$. The closed linear subspace generated by (r_n) in $L^0([0,1])$ is isomorphic to ℓ^2 (since $(a_n) \in \ell^2$ if and only if $\sum a_n r_n$ converges in measure, due to the independence of (r_n)). Combining this fact with the measure $A \in \mathcal{M} \mapsto \nu(A) := \chi_A \in L^2([0,1])$, for which $L^1(\nu) = L^2([0,1])$, we obtain an stochastic measure η such that $L^1(\eta) = L^2([0,1])$.

4. Optimal domain for $L^0([0,1])$–valued operators

Let T be a linear operator defined on a F.f.s. E and with values in an F–space X. We look for conditions which allow us to extended T to a domain larger than E. The main tool for this will be the X–valued measure canonically associated with T.

Proposition 4.1. *Let E be an F.f.s. over a finite measure space $(\Omega, \Sigma, \lambda)$, X an F–space and $T\colon E \to X$ a linear operator. Suppose that T satisfies*

(i) *$T(f_n) \to T(f)$ in X, whenever $0 \leq f_n \uparrow f\,\lambda$–a.e., $f_n, f \in E$.*
(ii) *The restriction of T to $L^\infty(\lambda)$ is continuous for the topology of uniform convergence.*
(iii) *$\lambda(A) = 0$ whenever $T(\chi_B) = 0$ for every $B \subset A$.*

Then $\nu\colon \Sigma \to X$ defined by

$$\nu(A) := T(\chi_A),\ A \in \Sigma,$$

is a countably additive, L^∞–bounded measure, which is equivalent to λ. The space E is continuously included in $L^1(\nu)$, and the integration operator with respect to ν extends T to $L^1(\nu)$.

Proof. The set function ν is additive due to the linearity of T. Countable additivity of ν follows from condition (i) applied to $\chi_{A_n} \uparrow \chi_{\cup A_n}$, when $(A_n) \subset \Sigma$ is an increasing sequence. Since $T\varphi = \int \varphi\, d\nu$, for φ simple, condition (ii) implies that ν is L^∞–bounded. By definition of ν, every λ–null set is ν–null. Condition (iii) implies that every ν–null set is λ–null. Hence, ν and λ are equivalent. Since T and the integration operator coincide on the simple functions, they agree also on $L^\infty(\nu) = L^\infty(\lambda)$.

Let $f \in E$. Since E is a lattice, it suffices to consider the case $f \geq 0$. Let (φ_n) be a sequence of simple functions such that $0 \leq \varphi_n \uparrow f$. Let $h \in L^\infty(\nu)$, with $h \geq 0$. Since $0 \leq \varphi_n h \uparrow fh$, by (i) we have $\int \varphi_n h \, d\nu = T(\varphi_n h) \to T(fh)$ in X. Decomposing $h \in L^\infty(\nu)$ as $h = h^+ - h^-$, we deduce that $f \in L^1(\nu)$, and $\int f d\nu = \lim \int \varphi_n d\nu = T(f)$. Hence, E is contained in $L^1(\nu)$ and the integration operator with respect to ν extends T to $L^1(\nu)$.

Note that the inclusion of E in $L^1(\nu)$ is injective since ν and λ have the same null sets. This inclusion is continuous since it is a positive linear operator between F–lattices. □

Remark 4.2. In Proposition 4.1 we have the following:

(a) If $E = L^\infty(\lambda)$, condition (i) can be replaced by the weaker condition: $T(\chi_{A_n}) \to T(\chi_{\cup A_n})$ in X, for every increasing sequence $(A_n) \subset \Sigma$.
(b) If E is order continuous and $T \colon E \to X$ is continuous then conditions (i) and (ii) are satisfied.

The previous results allow us to establish an *optimality* property of the spaces $L^1(\nu)$, which is adequately explained in the statement and proof of the next result.

Theorem 4.3. *Let $(\Omega, \Sigma, \lambda)$ be a finite measure space, X an F–space and $T\colon L^\infty(\lambda) \to X$ a continuous linear operator satisfying*

(i) $T(\chi_{A_n}) \to T(\chi_{\cup A_n})$ *in X, for every increasing sequence $(A_n) \subset \Sigma$.*
(ii) $\lambda(A) = 0$ *whenever $T(\chi_B) = 0$ for every $B \subset A$.*

Then, for ν the L^∞–bounded measure defined by $\nu(A) := T(\chi_A)$, for $A \in \Sigma$, the space $L^1(\nu)$ is the largest order continuous F.f.s. over $(\Omega, \Sigma, \lambda)$ to which T can be extended still taking values in X.

Proof. From Proposition 4.1 and Remark 4.2.(a), it follows that the integration operator with respect to ν extends T to $L^1(\nu)$. Note that $L^1(\nu)$ is order continuous and, due to (ii), it is an F.f.s. over $(\Omega, \Sigma, \lambda)$.

Let E be an order continuous F.f.s. over $(\Omega, \Sigma, \lambda)$, and $\widetilde{T}\colon E \to X$ a continuous linear operator such that when restricted to $L^\infty(\lambda)$ coincides with T. Then, from Proposition 4.1 and Remark 4.2.(b), and noting that the measure $\tilde{\nu}$ given by $\tilde{\nu}(A) := \widetilde{T}(\chi_A)$, for $A \in \Sigma$, coincides with ν, it follows that E is continuously included in $L^1(\nu)$ and the integration operator with respect to ν extends $\widetilde{T}$ to $L^1(\nu)$. □

Corollary 4.4. *Let E be an order continuous F.f.s. over a finite measure space $(\Omega, \Sigma, \lambda)$, X an F–space and $T\colon E \to X$ a continuous linear operator such that $\lambda(A) = 0$ whenever $T(\chi_B) = 0$ for every $B \subset A$. Then the conclusions of Theorem* 4.3 *hold.*

More can be said about the optimal domain of an operator T, in the case when T is given by a positive kernel. Let $K\colon [0,1] \times [0,1] \to [0, \infty)$ be a measurable function and T the linear operator defined, for a measurable function f, by

$$T(f)(x) = \int_0^1 f(y) K(x,y) dy \quad \text{for } x \in [0,1],$$

provided the integral exists for m–a.e. $x \in [0,1]$. In order for T to be defined for characteristic functions (and so, for simple functions) is needed that, for m–a.e. $x \in [0,1]$, the function K_x, defined by $K_x(y) := K(x,y)$, $y \in [0,1]$, is in $L^1([0,1])$. In this case, the set function $\nu\colon \mathcal{M} \to L^0([0,1])$ given by $\nu(A) := T(\chi_A)$, for $A \in \mathcal{M}$, is well defined and additive. Clearly, every m–null set is ν–null. For m and ν being equivalent is necessary and sufficient (via Fubini's theorem) that K satisfies

$$\int_0^1 K(x,y)dx > 0 \quad m\text{–a.e. } y \in [0,1]\,; \tag{4.1}$$

this occurs if and only if $T|f| = 0$ m–a.e. implies $f = 0$ m–a.e. A kernel satisfying these conditions will be called an *admissible kernel*.

Definition 4.5. Given an operator T defined by an admissible kernel K, and given an F.f.s. X over $[0,1]$, we call the *X–proper domain* of T the space

$$[T,X] := \{f \in L^0([0,1]) : T|f| \in X\}\,. \tag{4.2}$$

Note that $[T,X]$ endowed with the F–norm $\|f\|_{[T,X]} := \|T|f|\,\|_X$, $f \in [T,X]$, and the m–a.e. order, is an F–lattice satisfying that if $|f| \le |g|$ m–a.e. with $g \in [T,X]$, then $f \in [T,X]$ and $\|f\|_{[T,X]} \le \|g\|_{[T,X]}$. In fact, $[T,X]$ is the largest F–lattice contained in $L^0([0,1])$ with this last property, on which T is defined and takes values in X. Also, since $|Tf| \le T|f|$, we have that $T\colon [T,X] \to X$ is well defined and continuous with $\|Tf\|_X \le \|f\|_{[T,X]}$. See [7], [8], [9], for various aspects of the spaces $[T,X]$ in the case when X is a Banach space.

Remark 4.6. In the particular case when $X = L^0([0,1])$, the space $[T, L^0([0,1])]$ coincides (with equivalent F–norm) with the *proper domain* for T studied by Aronszajn and Szeptycki in [2] and [30].

The following theorem gives conditions allowing to identify the X–proper domain of T with the space of integrable functions with respect to the X–valued measure associated to T.

Theorem 4.7. *Let X be an F.f.s. over $[0,1]$ and K an admissible kernel. Suppose that the operator T associated with K satisfies*

(i) $T(\chi_{[0,1]}) \in X$.
(ii) $T(\chi_{A_n}) \to T(\chi_{\cup A_n})$ *in X, for every increasing sequence $(A_n) \subset \mathcal{M}$.*

Then,

(a) *The measure $\nu_X : \mathcal{M} \to X$, defined by $\nu_X(A) := T(\chi_A)$, for $A \in \mathcal{M}$, is countably additive and L^∞–bounded.*
(b) $L^1(\nu_X) \subset [T,X] \subset L^1_b(\nu_X)$, *the inclusions being continuous.*
(c) *If X is order continuous, then $[T,X]$ is order continuous and* $[T,X] = L^1(\nu_X)$.
(d) *If X is a C–space, then $[T,X]$ is a C–space and $[T,X] = L^1(\nu_X)$.*

Proof. (a) Condition (i) implies ν_X is well defined, and from condition (ii) it follows that ν_X is countably additive. Moreover, ν_X is L^∞–bounded since it is positive.

(b) Let $0 \le f \in L^1(\nu_X)$. Consider a sequence (φ_n) of simple functions with $0 \le \varphi_n \uparrow f$. Then, by the Monotone Convergence theorem, $T\varphi_n \uparrow Tf$. Since $L^1(\nu_X)$ is order continuous, $\varphi_n \to f$ in $L^1(\nu_X)$ and so, $T\varphi_n = \int \varphi_n d\nu_X \to \int f d\nu_X$ in X. Thus, $Tf = \int f d\nu_X \in X$, that is, $f \in [T, X]$. Then $L^1(\nu_X)$ is continuously contained in $[T, X]$. Moreover, since ν_X is positive, from (3.1) we have $\dot{\nu}_X(f) = \|\int |f| d\nu_X\|_X = \|T|f|\,\|_X = \|f\|_{[T,X]}$, for all $f \in L^1(\nu_X)$.

Given $f \in [T, X]$ and a simple function φ with $|\varphi| \le |f|$, it follows that $|\int \varphi d\nu_X| = |T\varphi| \le T|f| \in X$ and so $\|\int \varphi d\nu\|_X \le \|T|f|\,\|_X$. Then, $\dot{\nu}_X(f) \le \|T|f|\,\|_X < \infty$, and so $[T, X]$ is contained in $L^1_w(\nu_X)$. Applying the same argument to αf, we have $\dot{\nu}_X(\alpha f) \le \|\,|\alpha| T|f|\,\|_X \to 0$ whenever $\alpha \to 0$, and so $[T, X]$ is contained in $L^1_b(\nu_X)$. Hence, $[T, X]$ is continuously contained in $L^1_b(\nu_X)$, since the inclusion is a positive map.

(c) Let $0 \le f_n \uparrow f \in [T, X]$, then $0 \le Tf_n \uparrow Tf \in X$. Since X is order continuous, $Tf_n \to Tf$ in X and so, $f_n \to f$ in $[T, X]$. Hence, $[T, X]$ is order continuous. From (b), Proposition 4.1, and Remark 4.2.(b), it follows that $L^1(\nu_X) = [T, X]$.

(d) When X is a C–space, we have $L^1(\nu_X) = L^1_b(\nu_X)$ (see [32, Theorem 7.4]) and so, from (b) it follows that $L^1(\nu_X) = [T, X]$. From Proposition 3.4, we have that $[T, X] = L^1(\nu_X)$ is C–space. □

Remark 4.8. Condition (i) in Theorem 4.7 implies that the simple functions are contained in $[T, X]$, and so $[T, X]$ is a F.f.s. over $[0, 1]$. Also, note that under the conditions of Theorem 4.7, $T \colon L^\infty([0, 1]) \to X$ satisfies the hypothesis of Theorem 4.3.

Remark 4.9. Note that, in the case when $X = L^0([0, 1])$, from (4.2), we have that $[T, L^0([0, 1])]$ is the space

$$\big\{f \in L^0([0, 1]) : fK_x \in L^1([0, 1]),\ m\text{–a.e. } x \in [0, 1]\big\}.$$

Moreover, in this case the conditions of Theorem 4.7 are satisfied and, hence, all its conclusions hold. In particular, since $L^0([0, 1])$ is order continuous, we have $L^1(\nu) = [T, L^0([0, 1])]$, where ν is the measure associated to T considered with values in $L^0([0, 1])$. From this it follows that the space $[T, L^0([0, 1])]$ (the proper domain for T of Aronszajn and Szeptycki), satisfies the properties of $L^1(\nu)$ when ν is a positive stochastic measure. Hence, we have the following result.

Proposition 4.10. *The space $[T, L^0([0, 1])]$ is an order continuous C–space, satisfying the bounded multiplier test.*

5. Examples

We have already seen examples of stochastic measures such that the corresponding space $L^1(\nu)$ is a Banach space (Remark 3.11), or an F–space (namely, $L^0([0, 1])$). Next we exhibit an example where the space $L^1(\nu)$ is a Frèchet not Banach space.

Example 5.1. Consider the (admissible) kernel $K(x,y) := 2\cdot\chi_{[\frac{x}{2},\frac{x+1}{2}]}(y)$ for $x,y\in[0,1]$. The associated measure is

$$\nu(A)(x) = \int_0^1 \chi_A\Big(\frac{x+y}{2}\Big)\,dy\,.$$

Considering ν as an $L^0([0,1])$–valued measure, from Remark 4.9, we have that $L^1(\nu)$ coincides with the space $L^1_{\text{loc}}\big((0,1)\big)$ of locally integrable functions on $(0,1)$. Since an F–lattice has an unique complete topology (due to the continuity of positive maps), it follows that $L^1(\nu)$ is order isomorphic (via identity operator) to $L^1_{\text{loc}}\big((0,1)\big)$.

The following is an example, different from Remark 3.11, of a measure for which $L^1(\nu)$ is a Banach space. In this case, the measure arises from a classical kernel.

Example 5.2. Given $0<\alpha<1$, consider the (admissible) kernel of the fractional integral $K(x,y) := |x-y|^{\alpha-1}$, for $x,y\in[0,1]$ with $x\neq y$, and K equal to zero in other case. Let ν be the associated measure considered as $L^0([0,1])$–valued. Let us see, using the semivariation of ν, that the spaces $L^1(\nu)$ and $L^1[0,1]$ are order isomorphic (via identity operator). For a simple function φ, we have, for every $x\in[0,1]$,

$$\Big(\int |\varphi|d\nu\Big)(x) = \int_0^1 \frac{|\varphi(y)|}{|x-y|^{1-\alpha}}\,dy \geq \|\varphi\|_1\,.$$

Thus, from (3.1) we have

$$\dot{\nu}(\varphi) = \Big\|\int |\varphi|d\nu\Big\|_0 \geq \frac{\|\varphi\|_1}{1+\|\varphi\|_1}\,.$$

On the other hand,

$$\begin{aligned}\dot{\nu}(\varphi) \;&\leq\; \Big\|\int |\varphi|d\nu\Big\|_1 = \int_0^1\int_0^1 \frac{|\varphi(y)|}{|x-y|^{1-\alpha}}\,dy\,dx\\ &=\; \int_0^1 |\varphi(y)|\int_0^1 \frac{1}{|x-y|^{1-\alpha}}\,dx\,dy \leq \frac{2}{\alpha}\,\|\varphi\|_1\,.\end{aligned}$$

The previous examples can be extended by considering vector measures having a density with respect to a given vector measure.

Proposition 5.3. *Let X be an F–space and $\nu\colon\Sigma\to X$ a countably additive L^∞–bounded measure. For $h\in L^1(\nu)$, consider $\nu_h\colon\Sigma\to X$ defined by $\nu_h(A) := \int_A h\,d\nu$, for $A\in\Sigma$. Then, ν_h is countably additive and L^∞–bounded; and $f\in L^1(\nu_h)$ if and only if $fh\in L^1(\nu)$. Moreover, $\int f d\nu_h = \int fh\,d\nu$, and $\dot{\nu}_h(f) = \dot{\nu}(fh)$, for $f\in L^1(\nu_h)$.*

Proof. Since $L^1(\nu)$ is an ideal of measurable functions, $h \in L^1(\nu)$ implies $h\chi_A \in L^1(\nu)$, for every $A \in \Sigma$. So, ν_h is well defined. Given a sequence $(A_n) \subset \Sigma$ of disjoint sets, since $L^1(\nu)$ is order continuous, we have $\dot{\nu}(h\chi_{\cup_{j>n}A_j}) \to 0$. Thus, $\|\nu_h(\cup_{j>n}A_j)\|_X \to 0$ and so, ν_h is countably additive. For a simple function φ, we have

$$\Big\| \int \varphi \, d\nu_h \Big\|_X = \Big\| \int \varphi h \, d\nu \Big\|_X \leq \dot{\nu}(\varphi h) \leq \dot{\nu}(\|\varphi\|_\infty h) \to 0$$

whenever $\|\varphi\|_\infty \to 0$. Thus, ν_h is L^∞–bounded. Writing the semivariation as in (3.2) it follows that $\dot{\nu}_h(\varphi) = \dot{\nu}(\varphi h)$, for every simple function φ. From this equality, the proposition follows in a standard way. □

Remark 5.4. For a measurable function h, we denote by m_h the measure with density h with respect to the Lebesgue measure. From Proposition 5.3, we have

(a) If η is the stochastic measure in Remark 3.11, and $h \in L^2([0,1]) = L^1(\eta)$, then $L^1(\eta_h) = L^2([0,1], m_{h^2})$.
(b) If ν is the measure in Example 5.1, and $h \in L^1_{\text{loc}}\big((0,1)\big) = L^1(\nu)$, then $L^1(\nu_h) = L^1_{\text{loc}}((0,1), m_{|h|})$.
(c) If ν is the measure in Example 5.2, and $h \in L^1([0,1]) = L^1(\nu)$, then $L^1(\nu_h) = L^1([0,1], m_{|h|})$.

Note that the identities between the above spaces are order isomorphisms.

Remark 5.5. Let ν be an stochastic measure defined by an admissible kernel K and denote $h(y) := \int_0^1 K(s,y)\, ds$, for $y \in [0,1]$. If $f \in L^1([0,1], m_h)$, then

$$\begin{aligned} \int_0^1 \int_0^1 |f(y)| K_x(y)\, dy\, dx &= \int_0^1 |f(y)| \int_0^1 K(x,y)\, dx\, dy \\ &= \int_0^1 |f(y)| h(y)\, dy < \infty \,. \end{aligned}$$

Thus, $\int_0^1 |f(y)| K_x(y)\, dy < \infty$, for m–a.e. $x \in [0,1]$. From Remark 4.9, it follows that $f \in L^1(\nu)$. Hence, we always have the continuous embedding $L^1([0,1], m_h) \hookrightarrow L^1(\nu)$.

Suppose K satisfies that there exists a constant $C > 0$ and $A \in \mathcal{M}$ with $m(A) > 0$, such that for every $x \in A$ we have

$$K(x,y) \geq C \int_0^1 K(s,y) ds \quad m\text{–a.e. } y \in [0,1]\,. \tag{5.1}$$

Then, if $f \in L^1(\nu)$, there exists $x \in A$ such that:

$$\infty > \int_0^1 |f(y)| K_x(y)\, dy \geq C \int_0^1 |f(y)| h(y)\, dy \,,$$

and so $f \in L^1([0,1], m_h)$. Hence, $L^1(\nu)$ is order isomorphic to the space $L^1([0,1], m_h)$. Note that $h > 0$ m–a.e. and $h \in L^1([0,1])$, since K is an admissible kernel.

Conversely, if $h \in L^1([0,1])$ and $h > 0$ m–a.e., from (c) in Remark 5.4 we have that $L^1([0,1], m_h) = L^1(\nu_h)$, where ν is the vector measure in Example 5.2. The stochastic measure ν_h is associated with the admissible kernel $\widetilde{K}(x,y) = h(y)|x-y|^{\alpha-1}$, for $x, y \in [0,1]$ with $x \neq y$, and $\widetilde{K}$ equal to zero in other case. The kernel $\widetilde{K}$ satisfies condition (5.1): for every $x \in [0,1]$ we have

$$\begin{aligned}\int_0^1 \widetilde{K}(s,y)ds &= h(y)\int_0^1 |s-y|^{\alpha-1}ds \leq \frac{2}{\alpha}\,h(y) \\ &\leq \frac{2}{\alpha}\,h(y)|x-y|^{\alpha-1} = \frac{2}{\alpha}\,\widetilde{K}(x,y)\end{aligned}$$

for all $y \in [0,1], y \neq x$.

Therefore, the class of the spaces $L^1(\nu)$ with ν an stochastic measure defined by an admissible kernel K satisfying (5.1) coincides with the class of spaces $L^1([0,1], m_h)$ with $h \in L^1[0,1]$ and $h > 0$ m–a.e.

Remark 5.6. Let K be an admissible kernel and ν the associated stochastic measure. For any measurable function $g\colon [0,1] \to [0,\infty)$ such that $\int_0^1 g(x)K(x,y)\,dx > 0$ m–a.e. $y \in [0,1]$, we have that the kernel $\widetilde{K}(x,y) := g(x)K(x,y)$, for $x, y \in [0,1]$, is admissible, and its associated stochastic measure $\tilde{\nu}$ satisfies $L^1(\tilde{\nu}) = L^1(\nu)$.

References

[1] C. Aliprantis, O. Burkinshaw, *Locally solid Riesz spaces*, Academic Press, New York-London (1978)

[2] N. Aronszajn, P. Szeptycki, *On general integral transformations*, Math. Ann. **163** (1966), 127–154.

[3] R.G. Bartle, N. Dunford, J. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. **7** (1955), 289–305.

[4] G. P. Curbera, *Operators into L^1 of a vector measure and applications to Banach lattices*, Math. Ann. **293** (1992), 317–330.

[5] G. P. Curbera, *When L^1 of a vector measure is an AL–space*, Pacific J. Math. **162** (1994), 287–303.

[6] G. P. Curbera, *Banach space properties of L^1 of a vector measure*, Proc. Amer. Math. Soc. **123** (1995), 3797–3806.

[7] G. P. Curbera, W. J. Ricker, *Optimal domains for kernel operators via interpolation*, Math. Nachr. **244** (2002), 47–63.

[8] G. P. Curbera, W. J. Ricker, *Optimal domains for the kernel operator associated with Sobolev's inequality*, Studia Math. **158** (2003) 131–152 and **170** (2005), 217–218.

[9] G. P. Curbera, W. J. Ricker, *Banach lattices with the Fatou property and optimal domains of kernel operators*, Indag. Math. (N.S.), **17** (2006), 187–204.

[10] G. P. Curbera, W. J. Ricker, *Compactness properties of Sobolev imbeddings for rearrangement invariant norms*, Trans. Amer. Math. Soc., **359** (2007), 287–294.

[11] O. Delgado, *Banach function subspaces of L^1 of a vector measure and related Orlicz spaces*, Indag. Math. (N.S.) **15** (2004), 485–495.

[12] O. Delgado, *Optimal domains for kernel operators on* $[0,\infty)\times[0,\infty)$, Studia Math., to appear.

[13] J. Diestel, J. J. Jr. Uhl, *Vector measures*, Math. Surveys 15, Amer. Math. Soc., Providence, R.I., (1977)

[14] N. J. Kalton, N. T. Peck, J. W. Roberts, *L_0-valued vector measures are bounded*, Proc. Amer. Math. Soc. **85** (1982), 575–582.

[15] N. J. Kalton, N. T. Peck, J. W. Roberts, *An F-space sampler*, Cambridge University Press, Cambridge (1984).

[16] D. R. Lewis, *Integration with respect to vector measures*, Pacific J. Math. **33** (1970), 157–165.

[17] D. R. Lewis, *On integrability and summability in vector spaces*, Illinois J. Math. **16** (1972), 294–307.

[18] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces Vol. II*, Springer-Verlag, Berlin, (1979).

[19] W. A. Luxemburg, A. C. Zaanen, *Riesz spaces Vol. I*, North-Holland, Amsterdam (1971).

[20] B. Maurey, *Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces L^p*, Astérisque, No. 11, Société Mathématique de France, Paris (1974)

[21] B. Maurey, G. Pisier, *Un théorème d'extrapolation et ses conséquences*, C. R. Acad. Sci. Paris Sér. A-B **277** (1973), A39–A42.

[22] S. Okada, *The dual space of $\mathcal{L}^1(\mu)$ for a vector measure μ*, J. Math. Anal. Appl. **177** (1993), 583–599.

[23] S. Okada, W. J. Ricker, *Optimal domains and integral representations of convolution operators in $L^p(G)$*, Integral Equations Oper. Theory **48** (2004), 525–546.

[24] W. J. Ricker, *Compactness properties of extended Volterra operators in $L^p([0,1])$ for* $1\le p\le\infty$, Arch. Math. **66** (1996), 132–140.

[25] S. Rolewicz, *Metric linear spaces*, Monografie Matematyczne 56, PWN–Polish Scientific Publishers, Warsaw (1972).

[26] S. Rolewicz, *Metric linear spaces*, 2d ed., PWN–Polish Scientific Publishers, Warsaw (1984).

[27] S. Rolewicz, C. Ryll-Nardzewski, *On unconditional convergence in linear metric spaces*, Colloq. Math. **17** (1967), 327–331.

[28] C. Ryll-Nardzewski, W. A. Woyczyński, *Bounded multiplier convergence in measure of random vector series*, Proc. Amer. Math. Soc. **53** (1975), 96–98.

[29] L. Schwartz, *Un théorème de convergence dans les L^p,* $0\le p\le+\infty$, C. R. Acad. Sci. Paris Sér. A-B **268** (1969), A704–A706.

[30] P. Szeptycki, *Domains of integral transformations on general measure spaces*, Math. Ann. **242** (1979), 267–271.

[31] M. Talagrand, *Les mesures vectorielles à valeurs dans L^0 sont bornées*, Ann. Sci. École Norm. Sup. (4) **14** (1981), 445–452.

[32] E. G. F. Thomas, *Vector integration*, unpublished.

[33] Ph. Turpin, *Une mesure vectorielle non bornée*, C. R. Acad. Sci. Paris Sér. A-B **280** (1975), A509–A511.

[34] Ph. Turpin, *Intégration par rapport à une mesure à valeurs dans un espace vectoriel topologique non supposé localement convexe*, Intégration vectorielle et multivoque (Colloq., Univ. Caen, Caen, 1975), Exp. No. 8, Dép. Math., U. E. R. Sci., Univ. Caen, Caen, (1975).

[35] Ph. Turpin, *Convexités dans les espaces vectoriels topologiques généraux*, Diss. Math. **131** (1976).