OPTIMAL DOMAIN OF q-CONCAVE OPERATORS AND VECTOR MEASURE REPRESENTATION OF q-CONCAVE BANACH LATTICES

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Abstract. Given a Banach space valued q-concave linear operator T defined on a σ-order continuous quasi-Banach function space, we provide a description of the optimal domain of T preserving q-concavity, that is, the largest σ-order continuous quasi-Banach function space to which T can be extended as a q-concave operator. We show in this way the existence of maximal extensions for q-concave operators. As an application, we show a representation theorem for q-concave Banach lattices through spaces of integrable functions with respect to a vector measure. This result culminates a series of representation theorems for Banach lattices using vector measures that have been obtained in the last twenty years.

1. Introduction

Let $X(\mu)$ be a σ-order continuous quasi-Banach function space related to a positive measure μ on a measurable space (Ω, Σ) such that there exists $g \in X(\mu)$ with $g > 0$ μ-a.e. and let $T\colon X(\mu) \to E$ be a continuous linear operator with values in a Banach space E. Considering the δ-ring $\Sigma_{X(\mu)}$ of all sets $A \in \Sigma$ satisfying that $\chi_A \in X(\mu)$ and the vector measure $m_T\colon \Sigma_{X(\mu)} \to E$ given by $m_T(A) = T(\chi_A)$, it follows that the space $L^1(m_T)$ of integrable functions with respect to m_T is the optimal domain of T preserving continuity. That is, the largest σ-order continuous quasi-Banach function space to which T can be extended as a continuous operator still with values in E. Moreover, the extension of T to $L^1(m_T)$ is given by the integration operator I_{m_T}. This fact was originally proved in [8, Corollary 3.3] for Banach function spaces $X(\mu)$ with μ finite and $\chi_\Omega \in X(\mu) \subset L^1(\mu)$, in which case $\Sigma_{X(\mu)}$ coincides with the σ-algebra Σ. The extension for Banach function spaces (without extra assumptions) is deduced from [3, Proposition 4]. The jump to quasi-Banach function spaces appears in [26, Theorem 4.14] for the case when μ is finite and $\chi_\Omega \in X(\mu)$ and in [16] for the general case.

Some effort has been made in recent years to solve several versions of the following general *problem:* Suppose that the operator T has a property P. Is there an optimal domain for T preserving P? that is, is there a function space Z such that T can be extended to Z as an operator with the property P in such a way that Z is the largest

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space for which this holds? And in this case, which is the relation among Z and m_T? The answer to the first question is in general no. For example, in [25] it is proved that for compacness or weak compacness T has an optimal domain only in the case when I_{m_T} is compact or weakly compact, respectively. In the same line it is shown in [5] that T has an optimal domain for AM-compacness if and only if I_{m_T} is AM-compact. However, other properties have got positive answers to our problem, see [5] for narrow operators, [3] for order-w continuous or $Y(\eta)$-extensible operators and [14] for positive order continuous operators. Also in [5] the problem is studied for Dunford-Pettis operators, but although some partial results are shown there, the question of the existence of a maximal extension is still open.

In this paper we analyze this problem for the case of q-concave operators, obtaining a positive answer. Namely, if T is q-concave we show how to compute explicitly the largest quasi-Banach function space to which T can be extended preserving q-concavity (Corollary 4.6). Even more, we prove that this optimal domain is in fact the q-concave core of the space $L^1(m_T)$ and the maximal extension is given by the integration operator I_{m_T}. These results are obtained as a particular case of the more general Theorem 4.4 which gives the optimal domain for a class of operators (called (p,q)-power-concave) which contains the q-concave operators.

As an application we obtain an improvement in some sense of the Maurey-Rosenthal factorization of q-concave operators acting in q-convex Banach function spaces (Corollary 4.7). The reader can find information about this nowadays classical topic for example in [11], [12] and the references therein.

In the last section we provide a new representation theorem for q-concave Banach lattices in terms of a vector measure. This type of representation theorems has its origin in [7, Theorem 8], where it is proved that every order continuous Banach lattice F with a weak unit is order isometric to a space $L^1(\nu)$ of a vector measure ν defined on a σ-algebra. Later in [17, Proposition 2.4] it is shown that if moreover F is p-convex then it is order isometric to $L^p(m)$ for another vector measure m. Similar results work for F without weak unit but in this case the vector measures used in the representations of F are defined in a δ-ring, see [15, Theorem 5] and [6, Theorem 10]. Also there are representation theorems for F replacing σ-order continuity by the Fatou property, in this case through spaces of weakly integrable functions, see [9], [10], [15] and [18]. For $p,q \in [1,\infty)$, in Theorem 5.4 we obtain that every q-concave and p-convex Banach lattice is order isometric to a space $L^p(m)$ of a vector measure m defined on a δ-ring whose integration operator I_{m_T} is $\frac{q}{p}$-concave. The converse is also true. In particular, every q-concave Banach lattice is order isometric to a space $L^1(m)$ of a vector measure m having a q-concave integration operator.

2. Preliminaries

In this section we establish the notation and present the basic results on quasi-Banach function spaces (including the proof of some of them for completeness) and on vector measure integration, which will be used through the whole paper.

Let (Ω, Σ) be a fixed measurable space. For a measure $\mu\colon \Sigma \to [0,\infty]$, we denote by $L^0(\mu)$ the space of all Σ–measurable real valued functions on Ω, where functions which are equal μ–a.e. are identified.

Given two set functions $\mu, \lambda\colon \Sigma \to [0,\infty]$ we will write $\lambda \ll \mu$ if $\mu(A) = 0$ implies $\lambda(A) = 0$. If $\lambda \ll \mu$ and $\mu \ll \lambda$ we will say that μ and λ are *equivalent*. If $\mu, \lambda\colon \Sigma \to [0,\infty]$ are two measures with $\lambda \ll \mu$, then the map $[i]\colon L^0(\mu) \to L^0(\lambda)$ which takes a μ–a.e. class in $L^0(\mu)$ represented by f into the λ–a.e. class represented by the same f, is a well defined linear map. In order to simplify notation we will write $[i](f) = f$. Note that if λ and μ are equivalent then $L^0(\mu) = L^0(\lambda)$ and $[i]$ is the identity map i.

2.1. **Quasi-Banach function spaces.** Let X be a real vector space and $\|\cdot\|_X$ a *quasi-norm* on X, that is a function $\|\cdot\|_X\colon X \to [0,\infty)$ satisfying the following conditions:

(i) $\|x\|_X = 0$ if and only if $x = 0$,

(ii) $\|\alpha x\|_X = |\alpha| \cdot \|x\|_X$ for all $\alpha \in \mathbb{R}$ and $x \in X$, and

(iii) there is a constant $K \geq 1$ such that $\|x + y\|_X \leq K(\|x\|_X + \|y\|_X)$ for all $x, y \in X$.

For $0 < r \leq 1$ being such that $K = 2^{\frac{1}{r}-1}$, it follows that

$$\Big\| \sum_{j=1}^{n} x_j \Big\|_X \leq 4^{\frac{1}{r}} \Big(\sum_{j=1}^{n} \|x_j\|_X^r \Big)^{\frac{1}{r}} \tag{2.1}$$

for every finite subset $(x_j)_{j=1}^n \subset X$, see [19, Lemma 1.1]. The quasi-norm $\|\cdot\|_X$ induces a metrizable vector topology on X where a base of neighborhoods of 0 is given by sets of the form $\{x \in X : \|x\|_X \leq \frac{1}{n}\}$. So, a sequence (x_n) converges to x in X if and only if $\|x - x_n\|_X \to 0$. If such topology is complete then X is said to be a *quasi-Banach space* (*Banach space* if $K = 1$).

Having in mind the inequality (2.1), standard arguments show the next result.

Proposition 2.1. *The following statements are equivalent:*

(a) *X is complete.*

(b) *For every $0 < r' \leq r$ (r as in (2.1)) it follows that if $(x_n) \subset X$ is such that $\sum \|x_n\|_X^{r'} < \infty$ then $\sum x_n$ converges in X.*

(c) *There exists $r' > 0$ satisfying that if $(x_n) \subset X$ is such that $\sum \|x_n\|_X^{r'} < \infty$ then $\sum x_n$ converges in X.*

Note that if a series $\sum x_n$ converges in X then

$$\Big\| \sum x_n \Big\|_X \leq 4^{\frac{1}{r}} K \Big(\sum \|x_n\|_X^r \Big)^{\frac{1}{r}}, \tag{2.2}$$

where r is as in (2.1). By using the map $|||\cdot|||$ given in [19, Theorem 1.2], it is routine to check that if $x_n \to x$ in X then

$$4^{-\frac{1}{r}} \limsup \|x_n\|_X \leq \|x\|_X \leq 4^{\frac{1}{r}} \liminf \|x_n\|_X. \tag{2.3}$$

Also note that a linear map $T\colon X \to Y$ between quasi-Banach spaces is continuous if and only if there exists a constant $M > 0$ such that $\|Tx\|_Y \leq M\|x\|_X$ for all $x \in X$, see [19, p. 8].

By a *quasi-Banach function space* (briefly, quasi-B.f.s.) we mean a quasi-Banach space $X(\mu) \subset L^0(\mu)$ satisfying that if $f \in X(\mu)$ and $g \in L^0(\mu)$ with $|g| \leq |f|$ μ–a.e. then

$g \in X(\mu)$ and $\|g\|_{X(\mu)} \le \|f\|_{X(\mu)}$. If $X(\mu)$ is a Banach space we will refer it as a *Banach function space* (briefly, B.f.s.). In particular, a quasi-B.f.s. is a quasi-Banach lattice for the μ-a.e. pointwise order, in which the convergence in quasi-norm of a sequence implies the convergence μ-a.e. for some subsequence. Let us prove this important fact.

Proposition 2.2. *If $f_n \to f$ in a quasi-B.f.s. $X(\mu)$, then there exists a subsequence $f_{n_j} \to f$ μ–a.e.*

Proof. Let r be as in (2.1). We can take a strictly increasing sequence $(n_j)_{j\ge1}$ such that $\|f - f_{n_j}\|_{X(\mu)} \le \frac{1}{2^j}$. For every $m \ge 1$, since

$$\sum_{j\ge m} \|f - f_{n_j}\|_{X(\mu)}^r \le \sum_{j\ge m} \frac{1}{2^{jr}} < \infty,$$

by Proposition 2.1 and (2.2), it follows that $g_m = \sum_{j\ge m} |f - f_{n_j}|$ converges in $X(\mu)$ and $\|g_m\|_{X(\mu)} \le 4^{\frac{1}{r}} K(\sum_{j\ge m} \frac{1}{2^{jr}})^{\frac{1}{r}}$. Fix $N \ge 1$ and let $A_j^N = \{\omega \in \Omega : |f(\omega) - f_{n_j}(\omega)| > \frac{1}{N}\}$. Since

$$\chi_{\cap_{m\ge1}\cup_{j\ge m} A_j^N} \le \chi_{\cup_{j\ge m} A_j^N} \le \sum_{j\ge m} \chi_{A_j^N} \le N \sum_{j\ge m} |f - f_{n_j}| = N g_m,$$

then

$$\|\chi_{\cap_{m\ge1}\cup_{j\ge m} A_j^N}\|_{X(\mu)} \le N\|g_m\|_{X(\mu)} \le 4^{\frac{1}{r}} NK\Big(\sum_{j\ge m} \frac{1}{2^{jr}}\Big)^{\frac{1}{r}}.$$

Taking $m \to \infty$ we have that $\|\chi_{\cap_{m\ge1}\cup_{j\ge m} A_j^N}\|_{X(\mu)} = 0$ and so $\mu(\cap_{m\ge1} \cup_{j\ge m} A_j^N) = 0$. Then $\mu(\cup_{N\ge1} \cap_{m\ge1} \cup_{j\ge m} A_j^N) = 0$, from which $f_{n_j} \to f$ μ-a.e. □

A quasi-B.f.s. $X(\mu)$ is *σ-order continuous* if for every $(f_n) \subset X(\mu)$ with $f_n \downarrow 0$ μ-a.e. it follows that $\|f_n\|_X \downarrow 0$. It has the *σ-Fatou property* if for every sequence $(f_n) \subset X$ such that $0 \le f_n \uparrow f$ μ-a.e. and $\sup_n \|f_n\|_X < \infty$ we have that $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$.

A similar argument to that given in [21, p. 2] for Banach lattices shows that every positive linear operator between quasi-Banach lattices is automatically continuous. In particular, all inclusions between quasi-B.f.s. are continuous.

The intersection $X(\mu)\cap Y(\mu)$ and the sum $X(\mu)+Y(\mu)$ of two quasi-B.f.s.' (B.f.s.') $X(\mu)$ and $Y(\mu)$ are quasi-B.f.s.' (B.f.s.') endowed respectively with the quasi-norms (norms)

$$\|f\|_{X(\mu)\cap Y(\mu)} = \max\big\{\|f\|_{X(\mu)}, \|f\|_{Y(\mu)}\big\}$$

and

$$\|f\|_{X(\mu)+Y(\mu)} = \inf\big(\|f_1\|_{X(\mu)} + \|f_2\|_{Y(\mu)}\big),$$

where the infimum is taken over all possible representations $f = f_1 + f_2$ μ-a.e. with $f_1 \in X(\mu)$ and $f_2 \in Y(\mu)$. The σ-order continuity is also preserved by this operations: if $X(\mu)$ and $Y(\mu)$ are σ-order continuous then $X(\mu) \cap Y(\mu)$ and $X(\mu) + Y(\mu)$ are σ-order continuous. Detailed proofs of these facts can be found in [16], see also [1, § 3, Theorem 1.3] for the standard parts.

Let $p \in (0,\infty)$. The *p-power* of a quasi-B.f.s. $X(\mu)$ is the quasi-B.f.s.

$$X(\mu)^p = \big\{f \in L^0(\mu) : |f|^p \in X(\mu)\big\}$$

endowed with the quasi-norm

$$\|f\|_{X(\mu)^p} = \|\,|f|^p\,\|_{X(\mu)}^{\frac{1}{p}}.$$

The reader can find a complete explanation of the space $X^p(\mu)$ for instance in [26, § 2.2] for the case when μ is finite and $\chi_\Omega \in X(\mu)$. The proofs given there, with the natural modifications, work in our general case. However, note that the notation is different: our p-powers here are the $\frac{1}{p}$-th powers there. This standard space can be found in different sources, unfortunately, notation is not exactly the same in all of them.

The following remark collects some results on the space $X(\mu)^p$ which will be used in the next sections. First, recall that a quasi-B.f.s. $X(\mu)$ is *p-convex* if there exists a constant $C > 0$ such that

$$\Big\|\Big(\sum_{j=1}^{n}|f_j|^p\Big)^{\frac{1}{p}}\Big\|_{X(\mu)} \leq C\Big(\sum_{j=1}^{n}\|f_j\|^p_{X(\mu)}\Big)^{\frac{1}{p}}$$

for every finite subset $(f_j)_{j=1}^n \subset X(\mu)$. The smallest constant satisfying the previous inequality is called the *p-convexity constant* of $X(\mu)$ and is denoted by $M^{(p)}[X(\mu)]$.

Remark 2.3. Let $X(\mu)$ be a quasi-B.f.s. The following statements hold:

(a) $X(\mu)^p$ is σ-order continuous if and only if $X(\mu)$ is σ-order continuous.

(b) If $\chi_\Omega \in X(\mu)$ and $0 < p \leq q < \infty$ then $X(\mu)^q \subset X(\mu)^p$.

(c) If $X(\mu)$ is a B.f.s. then $X(\mu)^p$ is p-convex.

(d) If $X(\mu)$ is a B.f.s. and $p \geq 1$ then $\|\cdot\|_{X(\mu)^p}$ is a norm and so $X(\mu)^p$ is a B.f.s.

(e) If $X(\mu)$ is $\frac{1}{p}$-convex with $M^{(\frac{1}{p})}[X(\mu)] = 1$ then $\|\cdot\|_{X(\mu)^p}$ is a norm and so $X(\mu)^p$ is a B.f.s.

Let $T\colon X(\mu) \to E$ be a linear operator defined on a quasi-B.f.s. $X(\mu)$ and with values in a quasi-Banach space E. For $q \in (0, \infty)$, the operator T is said to be *q-concave* if there exists a constant $C > 0$ such that

$$\Big(\sum_{j=1}^{n}\|T(f_j)\|_E^q\Big)^{\frac{1}{q}} \leq C\Big\|\Big(\sum_{j=1}^{n}|f_j|^q\Big)^{\frac{1}{q}}\Big\|_{X(\mu)}$$

for every finite subset $(f_j)_{j=1}^n \subset X(\mu)$. A quasi-B.f.s. $X(\mu)$ is *q-concave* if the identity map $i\colon X(\mu) \to X(\mu)$ is q-concave.

Note that if T is q-concave then it is p-concave for all $p > q$. A proof of this fact can be found in [26, Proposition 2.54.(iv)] for the case when μ is finite and $\chi_\Omega \in X(\mu)$. An adaptation of this proof to our context works.

Proposition 2.4. *If $X(\mu)$ is a q-concave quasi-B.f.s. then it is σ-order continuous.*

Proof. Since q-concavity implies p-concavity for every $q < p$, we only have to consider the case $q \geq 1$. Denote by C the q-concavity constant of $X(\mu)$ and consider $(f_n) \subset X(\mu)$ such that $f_n \downarrow 0$ μ-a.e. For every strictly increasing subsequence (n_k) we have that

$$\begin{aligned}\Big(\sum_{k=1}^{m}\|f_{n_k} - f_{n_{k+1}}\|^q_{X(\mu)}\Big)^{\frac{1}{q}} &\leq C\Big\|\Big(\sum_{k=1}^{m}|f_{n_k} - f_{n_{k+1}}|^q\Big)^{\frac{1}{q}}\Big\|_{X(\mu)} \\ &\leq C\Big\|\sum_{k=1}^{m}|f_{n_k} - f_{n_{k+1}}|\Big\|_{X(\mu)} \\ &= C\,\|f_{n_1} - f_{n_{m+1}}\|_{X(\mu)} \\ &\leq C\,\|f_{n_1}\|_{X(\mu)}\end{aligned}$$

for all $m \geq 1$. Then, (f_n) is a Cauchy sequence in $X(\mu)$, as in other case we can find $\delta > 0$ and two subsequences (n_k), (m_k) such that $n_k < m_k < n_{k+1} < m_{k+1}$ and $\delta < \|f_{n_k} - f_{m_k}\|_{X(\mu)} \leq \|f_{n_k} - f_{n_{k+1}}\|_{X(\mu)}$ for all k, which is a contradiction. Let $h \in X(\mu)$ be such that $f_n \to h$ in $X(\mu)$. From Proposition 2.2, there exists a subsequence $f_{n_j} \to h$ μ–a.e. and so $h = 0$ μ-a.e. Hence, $\|f_n\|_{X(\mu)} \downarrow 0$. □

Lemma 2.5. *Let $X(\mu)$ and $Y(\mu)$ be two quasi-B.f.s.' and consider a linear operator $T\colon X(\mu)+Y(\mu) \to E$ with values in a quasi-Banach space E. The operator T is q-concave if and only if both $T\colon X(\mu) \to E$ and $T\colon Y(\mu) \to E$ are q-concave.*

Proof. If $T\colon X(\mu) + Y(\mu) \to E$ is q-concave, since $X(\mu) \subset X(\mu) + Y(\mu)$ continuously, it follows that $T\colon X(\mu) \to E$ is q-concave. Similarly, $T\colon Y(\mu) \to E$ is q-concave.

Suppose that $T\colon X(\mu) \to E$ and $T\colon Y(\mu) \to E$ are q-concave and denote by C_X and C_Y their respective q-concavity constants. Write K for the constant satisfying the property (iii) of the quasi-norm $\|\cdot\|_E$. We will use the inequality:

$$(a+b)^t \leq \max\{1, 2^{t-1}\}(a^t + b^t) \tag{2.4}$$

where $0 \leq a, b < \infty$ and $0 < t < \infty$. Let $(f_j)_{j=1}^n \subset X(\mu)+Y(\mu)$. For $h = \big(\sum_{j=1}^n |f_j|^q\big)^{\frac{1}{q}} \in X(\mu)+Y(\mu)$, consider $h_1 \in X(\mu)$ and $h_2 \in Y(\mu)$ such that $h = h_1 + h_2$ μ-a.e. Taking the set $A = \big\{\omega \in \Omega :\ h(\omega) \leq 2|h_1(\omega)|\big\}$, $\alpha_q = \max\{1, 2^{q-1}\}$ and using (2.4), we have that

$$\begin{aligned}\sum_{j=1}^n \|T(f_j)\|_E^q &\leq K^q \sum_{j=1}^n \Big(\|T(f_j\chi_A)\|_E + \|T(f_j\chi_{\Omega\setminus A})\|_E\Big)^q \\ &\leq K^q\,\alpha_q \left(\sum_{j=1}^n \|T(f_j\chi_A)\|_E^q + \sum_{j=1}^n \|T(f_j\chi_{\Omega\setminus A})\|_E^q\right).\end{aligned}$$

Note that $(f_j\chi_A)_{j=1}^n \subset X(\mu)$ as $|f_j|\chi_A \leq h\chi_A \leq 2|h_1|$ for all j. Then,

$$\begin{aligned}\sum_{j=1}^n \|T(f_j\chi_A)\|_E^q &\leq C_X^q \Big\|\Big(\sum_{j=1}^n |f_j|^q\Big)^{\frac{1}{q}}\chi_A\Big\|_{X(\mu)}^q \\ &= C_X^q\,\|h\chi_A\|_{X(\mu)}^q \leq 2^q C_X^q\,\|h_1\|_{X(\mu)}^q.\end{aligned}$$

Similarly, $(f_j\chi_{\Omega\setminus A})_{j=1}^n \subset Y(\mu)$ as $|f_j|\chi_{\Omega\setminus A} \leq h\chi_{\Omega\setminus A} \leq 2|h_2|$ μ-a.e. for all j and so

$$\sum_{j=1}^n \|T(f_j\chi_{\Omega\setminus A})\|_E^q \leq 2^q C_Y^q\,\|h_2\|_{Y(\mu)}^q.$$

Denoting $C = \max\{C_X, C_Y\}$ and using again (2.4), it follows that

$$\begin{aligned}\Big(\sum_{j=1}^n \|T(f_j)\|_E^q\Big)^{\frac{1}{q}} &\leq 2KC\alpha_q^{\frac{1}{q}}\left(\|h_1\|_{X(\mu)}^q + \|h_2\|_{Y(\mu)}^q\right)^{\frac{1}{q}} \\ &\leq 2^{1+|1-\frac{1}{q}|}KC\left(\|h_1\|_{X(\mu)} + \|h_2\|_{Y(\mu)}\right).\end{aligned}$$

Taking infimum over all representations $\big(\sum_{j=1}^n |f_j|^q\big)^{\frac{1}{q}} = h_1 + h_2$ μ-a.e. with $h_1 \in X(\mu)$ and $h_2 \in Y(\mu)$, we have that

$$\Big(\sum_{j=1}^n \|T(f_j)\|_E^q\Big)^{\frac{1}{q}} \leq 2^{1+|1-\frac{1}{q}|}KC\,\Big\|\Big(\sum_{j=1}^n |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{X(\mu)+Y(\mu)}.$$

□

Further information on Banach lattices and function spaces can be found for instance in [1, 19, 21, 22, 26] and [28].

2.2. **Integration with respect to a vector measure defined on a δ-ring.** Let $\mathcal{R}$ be a δ*–ring* of subsets of Ω (i.e. a ring closed under countable intersections) and let $\mathcal{R}^{loc}$ be the σ–algebra of all subsets A of Ω such that $A \cap B \in \mathcal{R}$ for all $B \in \mathcal{R}$. Note that $\mathcal{R}^{loc} = \mathcal{R}$ whenever $\mathcal{R}$ is a σ-algebra. Write $\mathcal{S}(\mathcal{R})$ for the space of all $\mathcal{R}$*–simple functions* (i.e. simple functions supported in $\mathcal{R}$).

A Banach space valued set function $m\colon \mathcal{R} \to E$ is a *vector measure* (*real measure* when $E = \mathbb{R}$) if $\sum m(A_n)$ converges to $m(\cup A_n)$ in E for each sequence $(A_n) \subset \mathcal{R}$ of pairwise disjoint sets with $\cup A_n \in \mathcal{R}$.

The *variation* of a real measure $\lambda\colon \mathcal{R} \to \mathbb{R}$ is the measure $|\lambda|\colon \mathcal{R}^{loc} \to [0, \infty]$ given by

$$|\lambda|(A) = \sup\Big\{ \sum |\lambda(A_j)| : (A_j) \text{ finite disjoint sequence in } \mathcal{R} \cap 2^A \Big\}.$$

The variation $|\lambda|$ is finite on $\mathcal{R}$. The space $L^1(\lambda)$ of *integrable functions with respect to* λ is defined as the classical space $L^1(|\lambda|)$ with the usual norm $|f|_\lambda = \int_\Omega |f|\, d|\lambda|$. The integral of an $\mathcal{R}$–simple function $\varphi = \sum_{j=1}^n a_j \chi_{A_j}$ over $A \in \mathcal{R}^{loc}$ is defined in the natural way by $\int_A \varphi\, d\lambda = \sum_{j=1}^n a_j \lambda(A_j \cap A)$. The space $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\lambda)$. This allows to define the integral of a function $f \in L^1(\lambda)$ over $A \in \mathcal{R}^{loc}$ as $\int_A f\, d\lambda = \lim \int_A \varphi_n\, d\lambda$ for any sequence $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ converging to f in $L^1(\lambda)$.

The *semivariation* of a vector measure $m\colon \mathcal{R} \to E$ is the function $\|m\|\colon \mathcal{R}^{loc} \to [0, \infty]$ defined by

$$\|m\|(A) = \sup_{x^* \in B_{E^*}} |x^* m|(A),$$

where B_{E^*} is the closed unit ball of the topological dual E^* of E and $|x^* m|$ is the variation of the real measure $x^* m$ given by the composition of m with x^*. The semivariation $\|m\|$ is finite on $\mathcal{R}$.

A set $A \in \mathcal{R}^{loc}$ is said to be m*–null* if $m(B) = 0$ for every $B \in \mathcal{R} \cap 2^A$. This is equivalent to $\|m\|(A) = 0$. It is known that there exists a measure $\eta\colon \mathcal{R}^{loc} \to [0, \infty]$ equivalent to $\|m\|$ (see [2, Theorem 3.2]). Denote $L^0(m) = L^0(\eta)$.

The space $L^1_w(m)$ of *weakly integrable* functions with respect to m is defined as the space of (m-a.e. equal) functions $f \in L^0(m)$ such that $f \in L^1(x^* m)$ for every $x^* \in E^*$. The space $L^1(m)$ of *integrable* functions with respect to m consists in all functions $f \in L^1_w(m)$ satisfying that for each $A \in \mathcal{R}^{loc}$ there exists $x_A \in E$, which is denoted by $\int_A f\, dm$, such that

$$x^*(x_A) = \int_A f\, dx^* m, \quad \text{for all } x^* \in E^*.$$

The spaces $L^1(m)$ and $L^1_w(m)$ are B.f.s.' related to the measure space $(\Omega, \mathcal{R}^{loc}, \eta)$, and the expression

$$\|f\|_m = \sup_{x^* \in B_{E^*}} \int_\Omega |f|\, d|x^* m|$$

gives a norm for both spaces. The norm of $f \in L^1(m)$ can also be computed by means of the formula

$$\|f\|_m = \sup\left\{ \left\| \int_\Omega f \varphi\, dm \right\|_E : \varphi \in \mathcal{S}(\mathcal{R}),\, |\varphi| \le 1 \right\}. \tag{2.5}$$

Moreover, $L^1(m)$ is σ-order continuous and contains $\mathcal{S}(\mathcal{R})$ as a dense subset and $L^1_w(m)$ has the σ-Fatou property. For every $\mathcal{R}$-simple function $\varphi = \sum_{j=1}^n \alpha_j \chi_{A_i}$ it follows that $\int_A \varphi\, dm = \sum_{j=1}^n \alpha_i m(A_j \cap A)$ for all $A \in \mathcal{R}^{loc}$.

The *integration operator* $I_m : L^1(m) \to E$ given by $I_m(f) = \int_\Omega f\, dm$, is a continuous linear operator with $\|I_m(f)\|_E \le \|f\|_m$. If m is *positive*, that is $m(A) \ge 0$ for all $A \in \mathcal{R}$, then $\|f\|_m = \|I_m(|f|)\|_E$ for all $f \in L^1(m)$.

For every $g \in L^1(m)$, the set function $m_g : \mathcal{R}^{loc} \to E$ given by $m_g(A) = I_m(g\chi_A)$ is a vector measure. Moreover, $f \in L^1(m_g)$ if and only if $fg \in L^1(m)$, and in this case $\|f\|_{L^1(m_g)} = \|fg\|_{L^1(m)}$.

For definitions and general results regarding integration with respect to a vector measure defined on a δ-ring we refer to [4, 13, 20, 23, 24].

Let $p \in (0, \infty)$. We denote by $L^p(m)$ the p-power of $L^1(m)$, that is,

$$L^p(m) = \big\{ f \in L^0(m) : |f|^p \in L^1(m) \big\}.$$

As noted in Remark 2.3, the space $L^p(m)$ is a σ-order continuous quasi-B.f.s. with the quasi-norm $\|f\|_{L^p(m)} = \|\, |f|^p\, \|^{1/p}_{L^1(m)}$. Moreover, if $p \ge 1$ then $\|\cdot\|_{L^p(m)}$ is a norm and so $L^p(m)$ is a B.f.s. Direct proofs of these facts and some general results on the spaces $L^p(m)$ can be found in [6].

3. The q-concave core of a σ-order continuous quasi-B.f.s

Let $X(\mu)$ be a σ-order continuous quasi-B.f.s. and $q \in (0, \infty)$. We define the space $qX(\mu)$ to be the set of functions $f \in X(\mu)$ such that

$$\|f\|_{qX(\mu)} = \sup \Big(\sum_{j=1}^n \|f_j\|^q_{X(\mu)} \Big)^{\frac{1}{q}} < \infty,$$

where the supremum is taken over all finite set $(f_j)_{j=1}^n \subset X(\mu)$ satisfying $|f| = \big(\sum_{j=1}^n |f_j|^q \big)^{\frac{1}{q}}$ μ-a.e. Note that $\|f\|_{X(\mu)} \le \|f\|_{qX(\mu)}$.

Proposition 3.1. *The space $qX(\mu)$ is a quasi-B.f.s. with quasi-norm $\|\cdot\|_{qX(\mu)}$.*

Proof. First let us see that if $f \in qX(\mu)$ and $g \in L^0(\mu)$ with $|g| \le |f|$ μ–a.e. then $g \in qX(\mu)$ and $\|g\|_{qX(\mu)} \le \|f\|_{qX(\mu)}$. Note that $g \in X(\mu)$ as $f \in X(\mu)$. Let $(g_j)_{j=1}^n \subset X(\mu)$ be such that $|g| = \big(\sum_{j=1}^n |g_j|^q \big)^{\frac{1}{q}}$ μ-a.e. and take $h = \big|\, |f|^q - |g|^q \,\big|^{\frac{1}{q}} \in X(\mu)$. Since $|f| = \big(\sum_{j=1}^n |g_j|^q + |h|^q \big)^{\frac{1}{q}}$ μ-a.e., we have that

$$\Big(\sum_{j=1}^n \|g_j\|^q_{X(\mu)} \Big)^{\frac{1}{q}} \le \Big(\sum_{j=1}^n \|g_j\|^q_{X(\mu)} + \|h\|^q_{X(\mu)} \Big)^{\frac{1}{q}} \le \|f\|_{qX(\mu)}.$$

Taking supremum over all $(g_j)_{j=1}^n \subset X$ with $|g| = \big(\sum_{j=1}^n |g_j|^q \big)^{\frac{1}{q}}$ μ-a.e., we have that $g \in qX(\mu)$ with $\|g\|_{qX(\mu)} \le \|f\|_{qX(\mu)}$.

It is direct to check that $\|\cdot\|_{qX(\mu)}$ satisfies the properties (i) and (ii) of a quasi-norm. Let K be the constant satisfying the property (iii) of a quasi-norm for $\|\cdot\|_{X(\mu)}$. Given $f, g \in qX(\mu)$ and $(h_j)_{j=1}^n \subset X$ such that $|f+g| = \big(\sum_{j=1}^n |h_j|^q \big)^{\frac{1}{q}}$ μ-a.e., by taking

$A=\{\omega\in\Omega:\ |f(\omega)+g(\omega)|\le 2|f(\omega)|\}$, $\alpha_q=\max\{1,2^{q-1}\}$ and using (2.4), we have that

$$\begin{aligned}\sum_{j=1}^n\|h_j\|_{X(\mu)}^q &\le K^q\sum_{j=1}^n\big(\|h_j\chi_A\|_{X(\mu)}+\|h_j\chi_{\Omega\setminus A}\|_{X(\mu)}\big)^q\\ &\le K^q\alpha_q\Big(\sum_{j=1}^n\|h_j\chi_A\|_{X(\mu)}^q+\sum_{j=1}^n\|h_j\chi_{\Omega\setminus A}\|_{X(\mu)}^q\Big).\end{aligned}$$

Note that $|f+g|\chi_A, |f+g|\chi_{\Omega\setminus A}\in qX(\mu)$ as $|f+g|\chi_A\le 2|f|$ and $|f+g|\chi_{\Omega\setminus A}\le 2|g|$. Then,

$$\begin{aligned}\sum_{j=1}^n\|h_j\|_{X(\mu)}^q &\le K^q\alpha_q\Big(\big\||f+g|\chi_A\big\|_{qX(\mu)}^q+\big\||f+g|\chi_{\Omega\setminus A}\big\|_{qX(\mu)}^q\Big)\\ &\le 2^qK^q\alpha_q\big(\|f\|_{qX(\mu)}^q+\|g\|_{qX(\mu)}^q\big).\end{aligned}$$

By using again (2.4), we have that

$$\begin{aligned}\Big(\sum_{j=1}^n\|h_j\|_{X(\mu)}^q\Big)^{\frac1q} &\le 2K\alpha_q^{\frac1q}\big(\|f\|_{qX(\mu)}^q+\|g\|_{qX(\mu)}^q\big)^{\frac1q}\\ &\le 2^{1+|1-\frac1q|}K\big(\|f\|_{qX(\mu)}+\|g\|_{qX(\mu)}\big).\end{aligned}$$

Taking supremum over all $(h_j)_{j=1}^n\subset X$ with $|f+g|=\big(\sum_{j=1}^n|h_j|^q\big)^{\frac1q}$ μ-a.e., we have that

$$\|f+g\|_{qX(\mu)}\le 2^{1+|1-\frac1q|}K\big(\|f\|_{qX(\mu)}+\|g\|_{qX(\mu)}\big). \tag{3.1}$$

Finally, let us prove that $qX(\mu)$ is complete. Denote by r and r' the constants satisfying (2.1) for $X(\mu)$ and $qX(\mu)$ respectively. Note that $r'<r$ as $2^{1+|1-\frac1q|}K>K$. Let $(f_n)\subset qX(\mu)$ be such that $\sum\|f_n\|_{qX(\mu)}^{r'}<\infty$. Since $\|\cdot\|_{X(\mu)}\le\|\cdot\|_{qX(\mu)}$, from Proposition 2.1, we have that $\sum_{j=1}^k f_j\to g$ and $\sum_{j=1}^k|f_j|\to\tilde g$ in $X(\mu)$. From Proposition 2.2, it follows that $\sum_{j=1}^k f_j\to g$ and $\sum_{j=1}^k|f_j|\to\tilde g$ pointwise except on a μ-null set Z. Fix any $\gamma>1$ and consider the sets $A_k=\big\{\omega\in\Omega:\ |g(\omega)|\le\gamma\sum_{j=1}^k|f_j(\omega)|\big\}$. Note that $\Omega\setminus\cup A_k\subset Z$ and so it is μ-null. Indeed, if $\omega\notin Z$ and $|g(\omega)|>\gamma\sum_{j=1}^k|f_j(\omega)|$ for all k (in particular $\sum|f_n(\omega)|\neq 0$), then $\gamma\sum|f_n(\omega)|\le|g(\omega)|\le\sum|f_n(\omega)|<\infty$, which is a contradiction. Also note that $g\chi_{A_k}\in qX(\mu)$ as $|g|\chi_{A_k}\le\gamma\sum_{j=1}^k|f_j|$. Given $(h_j)_{j=1}^n\subset X$ with $|g|=\big(\sum_{j=1}^n|h_j|^q\big)^{\frac1q}$ μ-a.e., we have that

$$\begin{aligned}\Big(\sum_{j=1}^n\|h_j\chi_{A_k}\|_{X(\mu)}^q\Big)^{\frac1q} &\le \|g\chi_{A_k}\|_{qX(\mu)}\le\gamma\Big\|\sum_{j=1}^k|f_j|\Big\|_{qX(\mu)}\\ &\le 4^{\frac1{r'}}\gamma\Big(\sum_{j=1}^k\|f_j\|_{qX(\mu)}^{r'}\Big)^{\frac1{r'}}\\ &\le 4^{\frac1{r'}}\gamma\Big(\sum\|f_n\|_{qX(\mu)}^{r'}\Big)^{\frac1{r'}}.\end{aligned}$$

On other hand, since $X(\mu)$ is σ-order continuous and $|h_j|\chi_{A_k}\uparrow|h_j|$ μ-a.e. as $k\to\infty$, we have that $h_j\chi_{A_k}\to h_j$ in $X(\mu)$ as $k\to\infty$. Taking limit as $k\to\infty$ in the above inequality and applying (2.3), we obtain that

$$\Big(\sum_{j=1}^n\|h_j\|_{X(\mu)}^q\Big)^{\frac1q}\le 4^{\frac1r+\frac1{r'}}\gamma\Big(\sum\|f_n\|_{qX(\mu)}^{r'}\Big)^{\frac1{r'}}.$$

Now, taking supremum over all $(h_j)_{j=1}^n \subset X$ with $|g| = \big(\sum_{j=1}^n |h_j|^q\big)^{\frac{1}{q}}$ μ-a.e., it follows that $g \in qX(\mu)$ with $\|g\|_{qX(\mu)} \leq 4^{\frac{1}{r}+\frac{1}{r'}}\gamma\big(\sum \|f_n\|_{qX(\mu)}^{r'}\big)^{\frac{1}{r'}}$. Even more, since γ is arbitrary, taking $\gamma \to 1$ we have that

$$\Big\|\sum f_n\Big\|_{qX(\mu)} \leq 4^{\frac{1}{r}+\frac{1}{r'}}\Big(\sum \|f_n\|_{qX(\mu)}^{r'}\Big)^{\frac{1}{r'}}.$$

Of course $\sum_{j=1}^n f_j \to g$ in $qX(\mu)$ as

$$\Big\|g - \sum_{j=1}^n f_j\Big\|_{qX(\mu)} = \Big\|\sum_{j>n} f_j\Big\|_{qX(\mu)} \leq 4^{\frac{1}{r}+\frac{1}{r'}}\Big(\sum_{j>n} \|f_j\|_{qX(\mu)}^{r'}\Big)^{\frac{1}{r'}} \to 0.$$

Therefore, from Proposition 2.1 it follows that $qX(\mu)$ is complete. □

Proposition 3.2. *The space $qX(\mu)$ is q-concave. In consequence, it is also σ-order continuous.*

Proof. Let $(f_j)_{j=1}^n \subset qX(\mu)$ and consider $(h_k^j)_{k=1}^{m_j} \subset X(\mu)$ with $|f_j| = \big(\sum_{k=1}^{m_j} |h_k^j|^q\big)^{\frac{1}{q}}$ μ-a.e. for each j. Since $\big(\sum_{j=1}^n |f_j|^q\big)^{\frac{1}{q}} = \big(\sum_{j=1}^n \sum_{k=1}^{m_j} |h_k^j|^q\big)^{\frac{1}{q}}$ μ-a.e., it follows that

$$\sum_{j=1}^n \sum_{k=1}^{m_j} \|h_k^j\|_{X(\mu)}^q \leq \Big\|\Big(\sum_{j=1}^n |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{qX(\mu)}^q.$$

Taking supremum for each $j = 1, ..., n$ over all $(h_k^j)_{k=1}^{m_j} \subset X(\mu)$ with $|f_j| = \big(\sum_{k=1}^{m_j} |h_k^j|^q\big)^{\frac{1}{q}}$ μ-a.e., we have that

$$\sum_{j=1}^n \|f_j\|_{qX(\mu)}^q \leq \Big\|\Big(\sum_{j=1}^n |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{qX(\mu)}^q$$

and so $qX(\mu)$ is q-concave. The σ-order continuity is given by Proposition 2.4. □

Even more, the following proposition shows that $qX(\mu)$ is in fact the *q-concave core* of $X(\mu)$, that is, the largest q-concave quasi-B.f.s. related to μ contained in $X(\mu)$. In particular, $qX(\mu) = X(\mu)$ whenever $X(\mu)$ is q-concave.

Proposition 3.3. *Let $Z(\xi)$ be a quasi-B.f.s. with $\mu \ll \xi$. The following statements are equivalent:*

(a) *$[i]\colon Z(\xi) \to X(\mu)$ is well defined and q-concave.*

(b) *$[i]\colon Z(\xi) \to qX(\mu)$ is well defined.*

In particular, $qX(\mu)$ is the q-concave core of $X(\mu)$.

Proof. (a) ⇒ (b) Denote by C the q-concavity constant of the operator $[i]\colon Z(\xi) \to X(\mu)$. Let $f \in Z(\xi)$ (so $f \in X(\mu)$) and $(f_j)_{j=1}^n \subset X(\mu)$ with $|f| = \big(\sum_{j=1}^n |f_j|^q\big)^{\frac{1}{q}}$ except on a μ-null set N. Since $|f_j|\chi_{\Omega\setminus N} \leq |f|$ pointwise (so ξ-a.e.), then $f_j\chi_{\Omega\setminus N} \in Z(\xi)$. Noting that $f_j = f_j\chi_{\Omega\setminus N}$ μ-a.e., it follows that

$$\begin{aligned}\Big(\sum_{j=1}^n \|f_j\|_{X(\mu)}^q\Big)^{\frac{1}{q}} &= \Big(\sum_{j=1}^n \|f_j\chi_{\Omega\setminus N}\|_{X(\mu)}^q\Big)^{\frac{1}{q}} \\ &\leq C\,\Big\|\Big(\sum_{j=1}^n |f_j|^q\Big)^{\frac{1}{q}}\chi_{\Omega\setminus N}\Big\|_{Z(\xi)} \leq C\,\|f\|_{Z(\xi)}.\end{aligned}$$

Hence $f \in qX(\mu)$ with $\|f\|_{qX(\mu)} \leq C\,\|f\|_{Z(\xi)}$.

(b) $\Rightarrow$ (a) Clearly $[i]\colon Z(\xi) \to X(\mu)$ is well defined as $qX(\mu) \subset X(\mu)$. Denote by M the continuity constant of $[i]\colon Z(\xi) \to qX(\mu)$ (recall that every positive operator between quasi-B.f.s.' is continuous). For every $(f_j)_{j=1}^n \subset Z(\xi)$ we have that $\big(\sum_{j=1}^n |f_j|^q\big)^{\frac{1}{q}}$ is in $qX(\mu)$ as it is in $Z(\xi)$, and so

$$\Big(\sum_{j=1}^n \|f_j\|_{X(\mu)}^q\Big)^{\frac{1}{q}} \leq \Big\|\Big(\sum_{j=1}^n |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{qX(\mu)} \leq M\Big\|\Big(\sum_{j=1}^n |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{Z(\xi)}.$$

Hence, $[i]\colon Z(\xi) \to X(\mu)$ is q-concave.

In particular, if $Z(\mu)$ is a q-concave quasi-B.f.s. such that $Z(\mu) \subset X(\mu)$, we have that $i\colon Z(\mu) \to X(\mu)$ is well defined, continuous and so q-concave. Then, from (a) $\Rightarrow$ (b) we have that $Z(\mu) \subset qX(\mu)$. $\square$

For $p \in (0, \infty)$, the p-power of $qX(\mu)$ can be described in terms of the p-power of $X(\mu)$.

Proposition 3.4. *The equality* $\big(qX(\mu)\big)^p = qpX(\mu)^p$ *holds with equal norms.*

Proof. Let $f \in \big(qX(\mu)\big)^p$. Since $|f|^p \in qX(\mu)$, in particular $|f|^p \in X(\mu)$ and so $f \in X(\mu)^p$. Consider $(f_j)_{j=1}^n \subset X(\mu)^p$ satisfying that $|f| = \big(\sum_{j=1}^n |f_j|^{qp}\big)^{\frac{1}{qp}}$ μ-a.e. Noting that $(|f_j|^p)_{j=1}^n \subset X(\mu)$ and $|f|^p = \big(\sum_{j=1}^n (|f_j|^p)^q\big)^{\frac{1}{q}}$ μ-a.e., we have that

$$\Big(\sum_{j=1}^n \|f_j\|_{X(\mu)^p}^{qp}\Big)^{\frac{1}{qp}} = \Big(\sum_{j=1}^n \|\,|f_j|^p\,\|_{X(\mu)}^q\Big)^{\frac{1}{qp}} \leq \|\,|f|^p\,\|_{qX(\mu)}^{\frac{1}{p}} = \|f\|_{(qX(\mu))^p}.$$

Then, $f \in qpX(\mu)^p$ and $\|f\|_{qpX(\mu)^p} \leq \|f\|_{(qX(\mu))^p}$.

Let now $f \in qpX(\mu)^p$. In particular $f \in X(\mu)^p$ and so $|f|^p \in X(\mu)$. Consider $(f_j)_{j=1}^n \subset X(\mu)$ satisfying that $|f|^p = \big(\sum_{j=1}^n |f_j|^q\big)^{\frac{1}{q}}$ μ-a.e. Noting that $(|f_j|^{\frac{1}{p}})_{j=1}^n \subset X(\mu)^p$ and $|f| = \big(\sum_{j=1}^n (|f_j|^{\frac{1}{p}})^{qp}\big)^{\frac{1}{qp}}$ μ-a.e., we have that

$$\Big(\sum_{j=1}^n \|f_j\|_{X(\mu)}^q\Big)^{\frac{1}{q}} = \Big(\sum_{j=1}^n \|\,|f_j|^{\frac{1}{p}}\,\|_{X(\mu)^p}^{qp}\Big)^{\frac{1}{q}} \leq \|f\|_{qpX(\mu)^p}^p.$$

Then, $|f|^p \in qX(\mu)$ and $\|\,|f|^p\,\|_{qX(\mu)} \leq \|f\|_{qpX(\mu)^p}^p$. Hence, $f \in \big(qX(\mu)\big)^p$ and $\|f\|_{(qX(\mu))^p} = \|\,|f|^p\,\|_{qX(\mu)}^{\frac{1}{p}} \leq \|f\|_{qpX(\mu)^p}$. $\square$

4. Optimal domain for (p, q)-power-concave operators

Let $X(\mu)$ be a σ-order continuous quasi-B.f.s. satisfying what we call the *σ-property*:

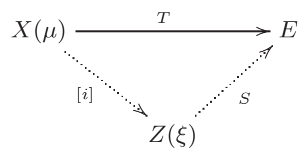
$$\Omega = \cup\Omega_n \text{ with } \chi_{\Omega_n} \in X(\mu) \text{ for all } n,$$

and let $T\colon X(\mu) \to E$ be a continuous linear operator with values in a Banach space E. We consider the δ-ring

$$\Sigma_{X(\mu)} = \big\{A \in \Sigma : \chi_A \in X(\mu)\big\}$$

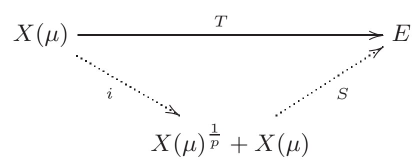
and the vector measure $m_T\colon \Sigma_{X(\mu)} \to E$ given by $m_T(A) = T(\chi_A)$. Note that the σ-property of $X(\mu)$ guarantees that $\Sigma_{X(\mu)}^{loc} = \Sigma$ and since $\|m_T\| \ll \mu$ we have that $[i]\colon L^0(\mu) \to L^0(m_T)$ is well defined. Also note that a quasi-B.f.s. has the σ-property if and only if it contains a function $g > 0$ μ-a.e.

As an extension of [3, § 3] to quasi-B.f.s.', in [16] it is proved that $[i]\colon X(\mu) \to L^1(m_T)$ is well defined and $T = I_{m_T} \circ [i]$. Even more, $L^1(m_T)$ is the largest σ-order continuous quasi-B.f.s. with this property. That means, if $Z(\xi)$ is a σ-order continuous quasi-B.f.s. with $\xi \ll \mu$ and T factors as



with S being a continuous linear operator, then $[i]\colon Z(\xi) \to L^1(m_T)$ is well defined and $S = I_{m_T} \circ [i]$. In other words, $L^1(m_T)$ is the optimal domain to which T can be extended preserving continuity.

In this section we present the main results of the paper, including a description of the optimal domain for T (when T is q-concave) preserving q-concavity. First, we have to provide a natural non-finite measure version of the so called p-th power factorable operators, which were developed for the first time in [26, § 5.1] for the case of finite measures. For $p \in (0, \infty)$, we say that T is a *p-th power factorable operator with a continuous extension* if there is a continuous linear extension of T to $X(\mu)^{\frac{1}{p}} + X(\mu)$, i.e. T factors as



for a continuous linear operator S.

Regarding this definition and having in mind Remark 2.3.(b), two standard cases must be considered whenever $\chi_\Omega \in X(\mu)$. If $1 < p$ we have that $X(\mu)^{\frac{1}{p}} + X(\mu) = X(\mu)^{\frac{1}{p}}$, and then the definition of p-th power factorable operator with a continuous extension coincides with the one given in [26, Definition 5.1]. However, if $p \leq 1$ we have that $X(\mu)^{\frac{1}{p}} + X(\mu) = X(\mu)$ and so p-th power factorable operators with continuous extensions are just continuous operators.

The following result, which is proved in [16] in order to find the optimal domain for p-th power factorable operators, will be the starting point of our work in this section. The proof is an adaptation to our setting of the proof given in [26, Theorem 5.7] for the case when μ is finite, $\chi_\Omega \in X(\mu)$ and $p \geq 1$.

Theorem 4.1. *The following statements are equivalent.*

(a) *T is p-th power factorable with a continuous extension.*

(b) *$[i]\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to L^1(m_T)$ is well defined.*

(c) *$[i]\colon X(\mu) \to L^p(m_T) \cap L^1(m_T)$ is well defined.*

(d) *There exists $M > 0$ such that $\|Tf\|_E \leq M\|f\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}$ for all $f \in X(\mu)$.*

Moreover, if (a)-(d) holds, the extension of T to $X(\mu)^{\frac{1}{p}} + X(\mu)$ coincides with the integration operator $I_{m_T} \circ [i]$.

In a brief overview, (a) implies (b) and the fact that the extension of T to $X(\mu)^{\frac{1}{p}} + X(\mu)$ is just $I_{m_T} \circ [i]$ follow from the optimality of $L^1(m_T)$. Note that $X(\mu)^{\frac{1}{p}} + X(\mu)$ is σ-order continuous as $X(\mu)$ is so. The equivalence between (b) and (c) is a direct check. Statement (b) implies (d) since $[i]\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to L^1(m_T)$ is continuous (as it is positive) and $T = I_{m_T} \circ [i]$. Finally, (d) implies (a) is based on a standard argument which use the approximation of a measurable function through functions in $X(\mu)$ (possible by the σ-property) to construct an extension of T to $X(\mu)^{\frac{1}{p}} + X(\mu)$. For a detailed proof of Theorem 4.1 see [16], where moreover it is proved that if T is p-th power factorable with a continuous extension then $L^p(m_T) \cap L^1(m_T)$ is the optimal domain to which T can be extended preserving this property.

Now, let us go to the new results on optimal domains. We consider the following property stronger than p-th power factorable and look for its optimal domain.

For $p, q \in (0, \infty)$, we say that T is (p,q)-*power-concave* if there exists a constant $C > 0$ such that

$$\Big(\sum_{j=1}^{n} \|T(f_j)\|_E^{\frac{q}{p}} \Big)^{\frac{p}{q}} \leq C \Big\| \Big(\sum_{j=1}^{n} |f_j|^{\frac{q}{p}} \Big)^{\frac{p}{q}} \Big\|_{X(\mu)^{\frac{1}{p}} + X(\mu)}$$

for every finite subset $(f_j)_{j=1}^n \subset X(\mu)$. If $\chi_\Omega \in X(\mu)$ and $p \geq 1$ we have that $X(\mu)^{\frac{1}{p}} + X(\mu) = X(\mu)^{\frac{1}{p}}$, and then our definition of (p,q)-power-concave operator coincides with the one given in [26, Definition 6.1].

Remark 4.2. The following statements hold:

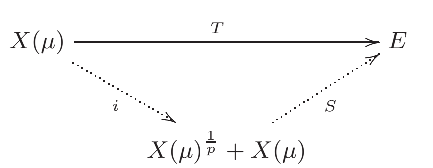
(i) A $(1,q)$-power-concave operator is just a q-concave operator.

(ii) If T is (p,q)-power-concave then T is $\frac{q}{p}$-concave, as $X(\mu) \subset X(\mu)^{\frac{1}{p}} + X(\mu)$ continuously.

(iii) If $\chi_\Omega \in X(\mu)$ and $p < 1$, since $X(\mu)^{\frac{1}{p}} + X(\mu) = X(\mu)$, we have that (p,q)-power-concavity coincides with $\frac{q}{p}$-concavity.

(iv) If T is (p,q)-power-concave then T is p-th power factorable with a continuous extension. Indeed, the (p,q)-power-concave inequality applied to an unique function is just the item (d) of Theorem 4.1

As we will see in the next result, (p,q)-power-concavity is close related to the following property. We say that T is *p-th power factorable with a q-concave extension* if there exists a q-concave linear extension of T to $X(\mu)^{\frac{1}{p}} + X(\mu)$, i.e. T factors as



with S being a q-concave linear operator. In this case, it is direct to check that T is q-concave.

Theorem 4.3. *The following statements are equivalent:*

(a) *T is (p,q)-power-concave.*

(b) *T is p-th power factorable with a $\frac{q}{p}$-concave extension.*

(c) $[i]\colon X(\mu)^{\frac{1}{p}}+X(\mu)\to L^1(m_T)$ *is well defined and* $\frac{q}{p}$*-concave.*

(d) $[i]\colon X(\mu)\to L^1(m_T)$ *is well defined and* $\frac{q}{p}$*-concave, and* $[i]\colon X(\mu)\to L^p(m_T)$ *is well defined and* q*-concave.*

(e) $[i]\colon X(\mu)\to \frac{q}{p}L^1(m_T)\cap qL^p(m_T)$ *is well defined.*

Moreover, if (a)-(e) holds, the extension of T *to* $X(\mu)^{\frac{1}{p}}+X(\mu)$ *coincides with the integration operator* $I_{m_T}\circ[i]$.

Proof. First note that $\frac{q}{p}L^1(m_T)\cap qL^p(m_T)$ is σ-order continuous as a consequence of Proposition 2.4.

(a) $\Rightarrow$ (b) From Remark 4.2.(iv) we have that T is p-th power factorable with a continuous extension. Let $S\colon X(\mu)^{\frac{1}{p}}+X(\mu)\to E$ be a continuous linear operator extending T. We are going to see that S is $\frac{q}{p}$-concave. Since T is (p,q)-power-concave and $S=T$ on $X(\mu)$, there exists $C>0$ such that

$$\Big(\sum_{j=1}^n\|S(f_j)\|_E^{\frac{q}{p}}\Big)^{\frac{p}{q}}\le C\,\Big\|\Big(\sum_{j=1}^n|f_j|^{\frac{q}{p}}\Big)^{\frac{p}{q}}\Big\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}$$

for all finite subset $(f_j)_{j=1}^n\subset X(\mu)$. Consider $(f_j)_{j=1}^n\subset X(\mu)^{\frac{1}{p}}+X(\mu)$ with $f_j\ge 0$ μ-a.e. for all j. The σ-property of $X(\mu)$ allows to find for each $j=1,...,n$ a sequence $(h_k^j)\subset X(\mu)$ such that $0\le h_k^j\uparrow f_j$ μ-a.e. as $k\to\infty$ (see [16] for the details). For every k, we have that

$$\begin{aligned}\Big(\sum_{j=1}^n\|S(h_k^j)\|_E^{\frac{q}{p}}\Big)^{\frac{p}{q}} &\le C\,\Big\|\Big(\sum_{j=1}^n|h_k^j|^{\frac{q}{p}}\Big)^{\frac{p}{q}}\Big\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}\\ &\le C\,\Big\|\Big(\sum_{j=1}^n|f_j|^{\frac{q}{p}}\Big)^{\frac{p}{q}}\Big\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}.\end{aligned}$$

On other hand, since $X(\mu)^{\frac{1}{p}}+X(\mu)$ is σ-order continuous, it follows that $h_k^j\to f_j$ in $X(\mu)^{\frac{1}{p}}+X(\mu)$ as $k\to\infty$, and so $S(h_k^j)\to S(f_j)$ in E as $k\to\infty$. Hence, taking limit as $k\to\infty$ in the above inequality, it follows that

$$\Big(\sum_{j=1}^n\|S(f_j)\|_E^{\frac{q}{p}}\Big)^{\frac{p}{q}}\le C\,\Big\|\Big(\sum_{j=1}^n|f_j|^{\frac{q}{p}}\Big)^{\frac{p}{q}}\Big\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}.$$

For a general $(f_j)_{j=1}^n\subset X(\mu)^{\frac{1}{p}}+X(\mu)$, write $f_j=f_j^+-f_j^-$ where f_j^+ and f_j^- are the positive and negative parts respectively of each f_j. By using inequality (2.4) and denoting $\alpha_{p,q}=\max\{1,2^{1-\frac{p}{q}}\}$, we have that

$$\begin{aligned}\Big(\sum_{j=1}^n\|S(f_j)\|_E^{\frac{q}{p}}\Big)^{\frac{p}{q}} &\le \Big(\sum_{j=1}^n\big(\|S(f_j^+)\|_E+\|S(f_j^-)\|_E\big)^{\frac{q}{p}}\Big)^{\frac{p}{q}}\\ &\le \alpha_{p,q}\Big(\sum_{j=1}^n\|S(f_j^+)\|_E^{\frac{q}{p}}+\sum_{j=1}^n\|S(f_j^-)\|_E^{\frac{q}{p}}\Big)^{\frac{p}{q}}\\ &\le \alpha_{p,q}\,C\,\Big\|\Big(\sum_{j=1}^n|f_j^+|^{\frac{q}{p}}+\sum_{j=1}^n|f_j^-|^{\frac{q}{p}}\Big)^{\frac{p}{q}}\Big\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}\\ &= \alpha_{p,q}\,C\,\Big\|\Big(\sum_{j=1}^n|f_j|^{\frac{q}{p}}\Big)^{\frac{p}{q}}\Big\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}\end{aligned}$$

(for the last equality note that $|f_j|^{\frac{q}{p}} = |f_j^+|^{\frac{q}{p}} + |f_j^-|^{\frac{q}{p}}$ as f_j^+ and f_j^- have disjoint support).

(b) $\Rightarrow$ (c) Since T is p-th power factorable with a $\frac{q}{p}$-concave (and so continuous) extension, from Theorem 4.1, the map $[i]\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to L^1(m_T)$ is well defined. Let $S\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to E$ be a $\frac{q}{p}$-concave linear operator extending T. Note that $S = I_{m_T} \circ [i]$ (Theorem 4.1). Denote by C the $\frac{q}{p}$-concavity constant of S. Consider $(f_j)_{j=1}^n \subset X(\mu)^{\frac{1}{p}} + X(\mu)$ and fix $\varepsilon > 0$. For each j, by (2.5), we can take $\varphi_j \in \mathcal{S}(\Sigma_{X(\mu)})$ such that $|\varphi_j| \le 1$ and

$$\|f_j\|_{L^1(m_T)} \le \Big(\frac{\varepsilon}{2^j}\Big)^{\frac{p}{q}} + \|I_{m_T}(f_j\varphi_j)\|_E.$$

Since $f_j\varphi_j \in X(\mu)^{\frac{1}{p}} + X(\mu)$ as $|f_j\varphi_j| \le |f_j|$, then $I_{m_T}(f_j\varphi_j) = S(f_j\varphi_j)$. So, by using inequality (2.4) and the $\frac{q}{p}$-concavity of S, we have that

$$\begin{aligned}
\sum_{j=1}^n \|f_j\|_{L^1(m_T)}^{\frac{q}{p}} &\le \sum_{j=1}^n \bigg(\Big(\frac{\varepsilon}{2^j}\Big)^{\frac{p}{q}} + \|S(f_j\varphi_j)\|_E\bigg)^{\frac{q}{p}} \\
&\le \max\{1, 2^{\frac{q}{p}-1}\}\bigg(\sum_{j=1}^n \frac{\varepsilon}{2^j} + \sum_{j=1}^n \|S(f_j\varphi_j)\|_E^{\frac{q}{p}}\bigg) \\
&\le \max\{1, 2^{\frac{q}{p}-1}\}\bigg(\varepsilon + C^{\frac{q}{p}} \Big\|\Big(\sum_{j=1}^n |f_j\varphi_j|^{\frac{q}{p}}\Big)^{\frac{p}{q}}\Big\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}^{\frac{q}{p}}\bigg) \\
&\le \max\{1, 2^{\frac{q}{p}-1}\}\bigg(\varepsilon + C^{\frac{q}{p}} \Big\|\Big(\sum_{j=1}^n |f_j|^{\frac{q}{p}}\Big)^{\frac{p}{q}}\Big\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}^{\frac{q}{p}}\bigg).
\end{aligned}$$

Taking limit as $\varepsilon \to 0$, we obtain

$$\sum_{j=1}^n \|f_j\|_{L^1(m_T)}^{\frac{q}{p}} \le C^{\frac{q}{p}}\ \max\{1, 2^{\frac{q}{p}-1}\}\Big\|\Big(\sum_{j=1}^n |f_j|^{\frac{q}{p}}\Big)^{\frac{p}{q}}\Big\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}^{\frac{q}{p}}$$

and so

$$\Big(\sum_{j=1}^n \|f_j\|_{L^1(m_T)}^{\frac{q}{p}}\Big)^{\frac{p}{q}} \le C\ \max\{1, 2^{1-\frac{p}{q}}\}\Big\|\Big(\sum_{j=1}^n |f_j|^{\frac{q}{p}}\Big)^{\frac{p}{q}}\Big\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}.$$

Hence, $[i]\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to L^1(m_T)$ is $\frac{q}{p}$-concave.

(c) $\Leftrightarrow$ (d) From Theorem 4.1, we have that $[i]\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to L^1(m_T)$ is well defined if and only if $[i]\colon X(\mu) \to L^p(m_T) \cap L^1(m_T)$ is well defined, which is equivalent to $[i]\colon X(\mu) \to L^1(m_T)$ and $[i]\colon X(\mu) \to L^p(m_T)$ well defined. By Lemma 2.5 we have that $[i]\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to L^1(m_T)$ is $\frac{q}{p}$-concave if and only if $[i]\colon X(\mu)^{\frac{1}{p}} \to L^1(m_T)$ and $[i]\colon X(\mu) \to L^1(m_T)$ are $\frac{q}{p}$-concave. On other hand, it is straightforward to verify that $[i]\colon X(\mu)^{\frac{1}{p}} \to L^1(m_T)$ is $\frac{q}{p}$-concave if and only if $[i]\colon X(\mu) \to L^p(m_T)$ is q-concave.

(d) $\Leftrightarrow$ (e) follows from Proposition 3.3.

(c) $\Rightarrow$ (a) Denote by C the $\frac{q}{p}$-concavity constant of $[i]\colon X(\mu)^{\frac{1}{p}} + X(\mu) \to L^1(m_T)$. Consider $(f_j)_{j=1}^n \subset X(\mu)$ and note that $f_j \in L^1(m_T)$ with $I_{m_T}(f_j) = T(f_j)$ for all j.

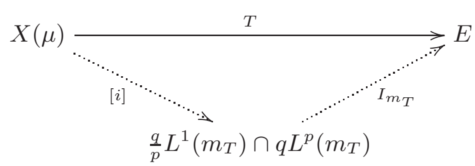
Then,

$$\begin{aligned}\Big(\sum_{j=1}^{n}\|T(f_j)\|_E^{\frac{q}{p}}\Big)^{\frac{p}{q}} &= \Big(\sum_{j=1}^{n}\|I_{m_T}(f_j)\|_E^{\frac{q}{p}}\Big)^{\frac{p}{q}} \\ &\le \Big(\sum_{j=1}^{n}\|f_j\|_{L^1(m_T)}^{\frac{q}{p}}\Big)^{\frac{p}{q}} \\ &\le C\Big\|\Big(\sum_{j=1}^{n}|f_j|^{\frac{q}{p}}\Big)^{\frac{p}{q}}\Big\|_{X(\mu)^{\frac{1}{p}}+X(\mu)}.\end{aligned}$$

□

Note that $qL^p(m_T) = \big(\frac{q}{p}L^1(m_T)\big)^p$ (see Proposition 3.4). In particular, in the case when T is (p,q)-power-concave and $\chi_\Omega \in X(\mu)$ (so $\chi_\Omega \in \frac{q}{p}L^1(m_T)$), from Remark 2.3.(b) it follows that $\frac{q}{p}L^1(m_T)\cap qL^p(m_T) = qL^p(m_T)$ if $p \ge 1$ and $\frac{q}{p}L^1(m_T)\cap qL^p(m_T) = \frac{q}{p}L^1(m_T)$ if $p < 1$.

Theorem 4.4. *Suppose that T is (p,q)-power-concave. Then, T factors as*

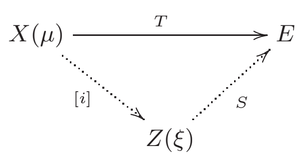


(4.1)

with I_{m_T} being (p,q)-power-concave. Moreover, the factorization is optimal *in the sense:*

If $Z(\xi)$ is a σ-order continuous quasi-B.f.s. such that $\xi \ll \mu$ and

(4.2)



with S being a (p,q)-power-concave linear operator $\Bigg\}\implies$ *$[i]\colon Z(\xi) \to \frac{q}{p}L^1(m_T)\cap qL^p(m_T)$ is well defined and $S = I_{m_T}\circ[i]$.*

Proof. The factorization (4.1) follows from Theorem 4.3. The space $\frac{q}{p}L^1(m_T)\cap qL^p(m_T)$ is σ-order continuous as noted before and satisfies the σ-property as $X(\mu)$ does. Since $I_{m_T}\colon \frac{q}{p}L^1(m_T)\cap qL^p(m_T) \to E$ is continuous (as $I_{m_T}\colon L^1(m_T)\to E$ is so), we can apply Theorem 4.3 to see that it is (p,q)-power-concave. Note that $\Sigma_{X(\mu)} \subset \Sigma_{\frac{q}{p}L^1(m_T)\cap qL^p(m_T)}$ and $m_{I_{m_T}}(A) = I_{m_T}(\chi_A) = T(\chi_A) = m_T(A)$ for all $A \in \Sigma_{X(\mu)}$. That is, m_T is the restriction of $m_{I_{m_T}}\colon \Sigma_{\frac{q}{p}L^1(m_T)\cap qL^p(m_T)} \to E$ to $\Sigma_{X(\mu)}$. From [3, Lemma 3], it follows that $L^1(m_{I_{m_T}}) = L^1(m_T)$. Then,

$$[i]\colon \tfrac{q}{p}L^1(m_T)\cap qL^p(m_T) \to \tfrac{q}{p}L^1(m_{I_{m_T}})\cap qL^p(m_{I_{m_T}})$$

is well defined as $\frac{q}{p}L^1(m_{I_{m_T}})\cap qL^p(m_{I_{m_T}}) = \frac{q}{p}L^1(m_T)\cap qL^p(m_T)$.

Let $Z(\xi)$ satisfy (4.2). In particular, $Z(\xi)$ has the σ-property. From Theorem 4.3 applied to the operator $S\colon Z(\xi)\to E$, we have that $[i]\colon Z(\xi)\to \frac{q}{p}L^1(m_S)\cap qL^p(m_S)$ is well defined and $S = I_{m_S}\circ[i]$. Since $\Sigma_{X(\mu)} \subset \Sigma_{Z(\xi)}$ and $m_S(A) = S(\chi_A) = T(\chi_A) = m_T(A)$

for all $A \in \Sigma_{X(\mu)}$ (i.e. m_T is the restriction of $m_S\colon \Sigma_{Z(\xi)} \to E$ to $\Sigma_{X(\mu)}$), from [3, Lemma 3], it follows that $L^1(m_S) = L^1(m_T)$ and $I_{m_S} = I_{m_T}$. Therefore,

$$[i]\colon Z(\xi) \to \tfrac{q}{p}L^1(m_S) \cap qL^p(m_S) = \tfrac{q}{p}L^1(m_T) \cap qL^p(m_T)$$

is well defined and $S = I_{m_S} \circ [i] = I_{m_T} \circ [i]$. □

We can rewrite Theorem 4.4 in terms of optimal domains.

Corollary 4.5. *Suppose that T is (p,q)-power-concave. Then $\frac{q}{p}L^1(m_T) \cap qL^p(m_T)$ is the largest σ-order continuous quasi-B.f.s. to which T can be extended as a (p,q)-power-concave operator still with values in E. Moreover, the extension of T to the space $\frac{q}{p}L^1(m_T) \cap qL^p(m_T)$ is given by the integration operator I_{m_T}.*

Recalling that the $(1,q)$-power-concave operators coincide with the q-concave operators, we obtain our main result.

Corollary 4.6. *Suppose that T is q-concave. Then $qL^1(m_T)$ is the largest σ-order continuous quasi-B.f.s. to which T can be extended as a q-concave operator still with values in E. Moreover, the extension of T to $qL^1(m_T)$ is given by the integration operator I_{m_T}.*

Let us give now a direct application related to the Maurey-Rosenthal factorization of q-concave operators defined on a q-convex quasi-B.f.s. In the case when T is q-concave, by Corollary 4.6, the integration operator I_{m_T} extends T to the space $qL^1(m_T)$. Note that the map $[i]\colon X(\mu) \to qL^1(m_T)$ is q-concave as it is continuous and $qL^1(m_T)$ is q-concave. From a variant of the Maurey-Rosenthal theorem proved in [11, Corollary 5], under some extra conditions, if $X(\mu)$ is q-convex then $[i]\colon X(\mu) \to qL^1(m_T)$ factors through the space $L^q(\mu)$. So, we obtain the following improvement of the usual factorization of q-concave operators on q-convex quasi-B.f.s.'.

Corollary 4.7. *Let $1 \le q < \infty$. Assume that μ is σ-finite and that $X(\mu)$ is q-convex and has the σ-Fatou property. If T is q-concave then it can be factored as*

$$\begin{array}{ccc} X(\mu) & \xrightarrow{\quad T \quad} & E \\ {\scriptstyle M_g}\downarrow & & \uparrow{\scriptstyle I_{m_T}} \\ L^q(\mu) & \xrightarrow{\quad M_{g^{-1}} \quad} & qL^1(m_T) \end{array}$$

for positive multiplication operators M_g and $M_{g^{-1}}$. The converse is also true.

5. Vector measure representation of q-concave Banach lattices

In this last section we look for a characterization of the class of Banach lattices which are p-convex and q-concave in terms of spaces of integrable functions with respect to a vector measure. For $1 < p$, it is known that order continuous p-convex Banach lattices can be order isometrically represented as spaces L^p of a vector measure defined on a δ-ring (see [6, Theorem 10]). We will see that the addition of the q-concavity property to the represented Banach lattice translates to adding some concavity property to the corresponding integration map.

First let us show two results concerning concavity for the integration operator of a vector measure which will be needed later.

Let $m\colon \mathcal{R} \to E$ be a vector measure defined on a δ–ring $\mathcal{R}$ of subsets of Ω and with values in a Banach space E.

Proposition 5.1. *The integration operator $I_m\colon L^1(m) \to E$ is q-concave if and only if $L^1(m)$ is q-concave.*

Proof. Suppose that $I_m\colon L^1(m) \to E$ is q-concave and denote by C its q-concavity constant. Take $(f_j)_{j=1}^n \subset L^1(m)$ and $(\varphi_j)_{j=1}^n \subset \mathcal{S}(\mathcal{R})$ with $|\varphi_j| \leq 1$ for all j. Since $(f_j\varphi_j)_{j=1}^n \subset L^1(m)$, as $|f_j\varphi_j| \leq |f_j|$ for all j, we have that

$$\Big(\sum_{j=1}^n \|I_m(f_j\varphi_j)\|_E^q\Big)^{\frac{1}{q}} \leq C\,\Big\|\Big(\sum_{j=1}^n |f_j\varphi_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^1(m)} \leq C\,\Big\|\Big(\sum_{j=1}^n |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^1(m)}.$$

Taking supremum for each $j = 1,..,n$ over all $\varphi_j \in \mathcal{S}(\mathcal{R})$ with $|\varphi_j| \leq 1$, from (2.5), if follows that

$$\Big(\sum_{j=1}^n \|f_j\|_{L^1(m)}^q\Big)^{\frac{1}{q}} \leq C\,\Big\|\Big(\sum_{j=1}^n |f_j|^q\Big)^{\frac{1}{q}}\Big\|_{L^1(m)}.$$

The converse is obvious as I_m is continuous. □

Direct useful consequences can be deduced of the fact that the integration map I_m : $L^1(m) \to E$ is q-concave. Assume that m is defined on a σ-algebra and note that q-concavity for $q \geq 1$ always implies $(q,1)$-concavity (see the definition for instance in [27, p. 61]). Thus, by [27, Proposition 7.9], if I_m is q-concave for $q \geq 1$ then it is *weakly completely continuous* (i.e. it maps weak Cauchy sequences into weakly convergent sequences). Moreover, this implies that $L^1(m)$ coincides with the space $L^1_w(m)$ and so it has the σ-Fatou property.

In the case when $\chi_\Omega \in L^1(m)$ (for instance if m is defined on a σ-algebra), we obtain a further result regarding (p,q)-power-concave operators.

Proposition 5.2. *Suppose that $\chi_\Omega \in L^1(m)$ and $p \geq 1$. The integration operator $I_m\colon L^p(m) \to E$ is (p,q)-power-concave if and only if $L^p(m)$ is q-concave.*

Proof. First note that under the hypothesis it follows that $L^p(m)$ has the σ-property (in fact $\chi_\Omega \in L^p(m)$) and $L^p(m) \subset L^1(m)$. So, $I_m\colon L^p(m) \to E$ is well defined and continuous.

Suppose that $I_m\colon L^p(m) \to E$ is (p,q)-power-concave. From Theorem 4.3, we have that $[i]\colon L^p(m) \to \frac{q}{p}L^1(m_{I_m}) \cap qL^p(m_{I_m})$ is well defined. Note that $(\mathcal{R}^{loc})_{L^p(m)} = \mathcal{R}^{loc}$ and so m_{I_m} coincides with m_{χ_Ω} (see Preliminaries). Then, $L^1(m_{I_m}) = L^1(m)$ and so $L^p(m) \subset \frac{q}{p}L^1(m) \cap qL^p(m) \subset qL^p(m)$. Hence, $L^p(m)$ is q-concave as $L^p(m) = qL^p(m)$.

Suppose now that $L^p(m)$ is q-concave. Then, it is direct to check that $L^1(m)$ is $\frac{q}{p}$-concave. Since $L^p(m) \subset L^1(m)$, the integration operator $I_m\colon L^1(m) \to E$ is continuous and $\big(L^p(m)\big)^{\frac{1}{p}} + L^p(m) = L^1(m)$, it follows that $I_m\colon L^p(m) \to E$ satisfies the inequality of the definition of (p,q)-power-concave operator. □

Let us go now to the representation of q-concave Banach lattices as spaces of integrable functions. We begin by considering B.f.s.'.

Proposition 5.3. *Let $p,q \in (0,\infty)$ and let $Z(\xi)$ be a q-concave B.f.s. which is also p-convex in the case when $p > 1$. Then, $Z(\xi)$ coincides with the space $L^p(m)$ of a Banach* *space valued vector measure $m\colon \mathcal{R} \to E$ defined on a δ-ring whose integration operator $I_m\colon L^1(m) \to E$ is $\frac{q}{p}$-concave. Moreover, if $\chi_\Omega \in Z(\xi)$, the vector measure m is defined on a σ-algebra.*

Proof. Note that if $p \le 1$ then $Z(\xi)^{\frac{1}{p}}$ is a B.f.s. (see Remark 2.3.(d)). In the case when $p > 1$, renorming $Z(\xi)$ if it is necessary, we can assume that the p-convexity constant of $Z(\xi)$ is equal to 1 (see [21, Proposition 1.d.8]), and so $Z(\xi)^{\frac{1}{p}}$ is a B.f.s. (see Remark 2.3.(e)). Consider the δ-ring $\Sigma_{Z(\xi)} = \big\{A \in \Sigma : \chi_A \in Z(\xi)\big\}$ and the finitely additive set function $m\colon \Sigma_{Z(\xi)} \to Z(\xi)^{\frac{1}{p}}$ given by $m(A) = \chi_A$. Since $Z(\xi)^{\frac{1}{p}}$ is σ-order continuous, as $Z(\xi)$ is so by Proposition 2.4, it follows that m is a vector measure. Let us see that $L^1(m) = Z(\xi)^{\frac{1}{p}}$ with equal norms and so we will have that $Z(\xi)$ coincides with $L^p(m)$. For $\varphi \in \mathcal{S}(\Sigma_{Z(\xi)})$ we have that $\varphi \in Z(\xi)^{\frac{1}{p}}$ and $I_m(\varphi) = \varphi$. Moreover, since m is positive,

$$\|\varphi\|_{L^1(m)} = \|I_m(|\varphi|)\|_{Z(\xi)^{\frac{1}{p}}} = \|\varphi\|_{Z(\xi)^{\frac{1}{p}}}. \tag{5.1}$$

In particular, by taking $\varphi = \chi_A$, we obtain that $\|m\|$ is equivalent to ξ. Given $f \in L^1(m)$, since $\mathcal{S}(\Sigma_{Z(\xi)})$ is dense in $L^1(m)$, we can take $(\varphi_n) \subset \mathcal{S}(\Sigma_{Z(\xi)})$ such that $\varphi_n \to f$ in $L^1(m)$ and m-a.e. From (5.1), we have that (φ_n) is a Cauchy sequence in $Z(\xi)^{\frac{1}{p}}$ and so there is $h \in Z(\xi)^{\frac{1}{p}}$ such that $\varphi_n \to h$ in $Z(\xi)^{\frac{1}{p}}$. Taking a subsequence $\varphi_{n_j} \to h$ ξ-a.e. we see that $f = h \in Z(\xi)^{\frac{1}{p}}$ and

$$\|f\|_{Z(\xi)^{\frac{1}{p}}} = \lim \|\varphi_n\|_{Z(\xi)^{\frac{1}{p}}} = \lim \|\varphi_n\|_{L^1(m)} = \|f\|_{L^1(m)}.$$

Let now $f \in Z(\xi)^{\frac{1}{p}}$ and take $(\varphi_n) \subset \mathcal{S}(\Sigma)$ such that $0 \le \varphi_n \uparrow |f|$. For any n, writing $\varphi_n = \sum_{j=1}^m \alpha_j \chi_{A_j}$ with $(A_j)_{j=1}^m$ being pairwise disjoint and $\alpha_j > 0$ for all j, we see that $\chi_{A_j} \le \alpha_j^{-1/p}|f|^{1/p}$ and so $\varphi_n \in \mathcal{S}(\Sigma_{Z(\xi)})$. On other hand, since $Z(\xi)^{\frac{1}{p}}$ is σ-order continuous, we have that $\varphi_n \to f$ in $Z(\xi)^{\frac{1}{p}}$. From (5.1), we have that (φ_n) is a Cauchy sequence in $L^1(m)$ and so there is $h \in L^1(m)$ such that $\varphi_n \to h$ in $L^1(m)$. Taking a subsequence $\varphi_{n_j} \to h$ m-a.e. we see that $f = h \in L^1(m)$.

Hence, $L^1(m) = Z(\xi)^{\frac{1}{p}}$ with equal norms and, since $Z(\xi)$ is q-concave, it follows that $L^1(m)$ is $\frac{q}{p}$-concave. From Proposition 5.1, the integration operator $I_m\colon L^1(m) \to E$ is $\frac{q}{p}$-concave.

Note that if $\chi_\Omega \in Z(\xi)$, then $\Sigma_{Z(\xi)} = \Sigma$ and so m is defined on a σ-algebra. □

For the final result we need some concepts related to Banach lattices. The definitions of p-convexity, q-concavity and σ-order continuity for Banach lattices are the same that for B.f.s.'. A Banach lattice F is said to be *order continuous* if for every downwards directed system $(x_\tau) \subset F$ with $x_\tau \downarrow 0$ it follows that $\|x_\tau\|_F \downarrow 0$ and is said to be *σ-complete* if every order bounded sequence in F has a supremum. A Banach lattice which is σ-order continuous and σ-complete at the same time is order continuous, see [21, Proposition 1.a.8]. A *weak unit* of a Banach lattice F is an element $0 \le e \in F$ such that $\inf\{x, e\} = 0$ implies $x = 0$. An operator $T\colon F_1 \to F_2$ between Banach lattices is said to be an *order isometry* if it is linear, one to one, onto, $\|Tx\|_{F_2} = \|x\|_{F_1}$ for all $x \in F_1$ and $T(\inf\{x, y\}) = \inf\{Tx, Ty\}$ for all $x, y \in F_1$. In particular, an order isometry is a positive operator. So, by using [21, Proposition 1.d.9], it is direct to check that every order isometry preserves p-convexity and q-concavity whenever $p, q \ge 1$.

Theorem 5.4. *Let $p, q \in [1, \infty)$ and let F be a Banach lattice. The following statements are equivalent:*

(a) *F is q-concave and p-convex.*

(b) *F is order isometric to a space $L^p(m)$ of a Banach space valued vector measure $m\colon \mathcal{R} \to E$ defined on a δ-ring whose integration operator $I_m\colon L^1(m) \to E$ is $\frac{q}{p}$-concave.*

Moreover, (a) holds with F having a weak unit if and only if (b) holds with m defined on a σ-algebra. In this last case $I_m\colon L^p(m) \to E$ is (p, q)-power-concave.

Proof. (a) $\Rightarrow$ (b) Since F is q-concave, it satisfies a lower q-estimate (see [21, Definition 1.f.4]) and then it is σ-complete and σ-order continuous (see the proof of [21, Proposition 1.f.5]). So, F is order continuous. From [15, Theorem 5] we have that F is order isometric to a space $L^1(\nu)$ of a Banach space valued vector measure ν defined on a δ-ring. Then, $L^1(\nu)$ is a B.f.s. satisfying the conditions of Proposition 5.3 and so $L^1(\nu) = L^p(m)$ with $m\colon \mathcal{R} \to E$ being a vector measure defined on a δ-ring $\mathcal{R}$ and with values in a Banach space E, whose integration operator $I_m\colon L^1(m) \to E$ is $\frac{q}{p}$-concave.

(b) $\Rightarrow$ (a) Since $L^p(m)$ is p-convex (Remark **??**.(c)) and q-concave (as $L^1(m)$ is $\frac{q}{p}$-concave by Proposition 5.1), F also is.

Now suppose that (a) holds with F having a weak unit. From [7, Theorem 8] we have that F is order isometric to a space $L^1(\nu)$ of a Banach space valued vector measure ν defined on a σ-algebra. Since $\chi_\Omega \in L^1(\nu)$, from Proposition 5.3 we have that (b) holds with m defined on a σ-algebra.

Conversely, if (b) holds with m defined on a σ-algebra then $\chi_\Omega \in L^p(m)$ (as $\chi_\Omega \in L^1(m)$). So, the image of χ_Ω by the order isometry is a weak unit in F. Moreover, from Proposition 5.2 it follows that $I_m\colon L^p(m) \to E$ is (p, q)-power-concave. □

In particular, from Theorem 5.4 we obtain that a Banach lattice is q-concave (with $q \geq 1$) if and only if it is order isometric to a space $L^1(m)$ of a vector measure m with a q-concave integration operator.

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