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L^1-spaces of vector measures defined on δ-rings

By

OLVIDO DELGADO

Abstract. Given a vector measure ν defined on a δ-ring with values in a Banach space, we study the relation between the analytic properties of the measure ν and the lattice properties of the space $L^1(\nu)$ of real functions which are integrable with respect to ν.

Introduction. The classical theory of integration of scalar functions with respect to a vector measure (defined on a σ-algebra) was created by Bartle, Dunford and Schwartz for studying the vector extension of the Riesz representation theorem [1]. The corresponding space of integrable functions has been thoroughly studied and is now well understood; see [6], [7], [8], [11], [16], [17]. An application is the study of operators T between function spaces through the vector measure $\nu(A) = T(\chi_A)$ and its space $L^1(\nu)$ of integrable functions; see [1]. A crucial role in this study is played by the "good" properties of the space $L^1(\nu)$ namely, it is an order continuous Banach lattice with weak order unit.

There are, however, important operators which cannot be directly studied via this classical integration procedure. This happens, for example, with the Hilbert transform on the real line. In this case, the vector measure associated to the operator is defined only for Lebesgue measurable sets of finite measure, which do not constitute a σ-algebra. Thus, we are naturally lead to consider vector measures which are defined on structures weaker than σ-algebras. The extension of the integration theory to vector measures defined on δ-rings was done by Lewis [12] and Masani and Niemi [14], [15].

In this paper we consider a vector measure ν defined on a δ-ring $\mathcal{R}$ of sets of Ω, taking values in a Banach space. We analyse the differences with vector measures defined on σ-algebras; in particular, $L^1(\nu)$ is an order continuous Banach lattice which may not have a weak order unit if ν is only defined on a δ-ring. We study the effect on the space $L^1(\nu)$ of certain properties of ν: i.e., strong additivity and σ-finiteness. Namely, we show that $L^1(\nu)$ has a weak unit g if and only if ν is σ-finite (Theorem 3.3), and in this case $L^1(\nu)$ is order isometric to $L^1(\nu_g)$, where ν_g is the vector measure defined on the σ-algebra of sets locally in $\mathcal{R}$ by $\nu_g(A) = \int_A g\, d\nu$ (Theorem 3.5). In the case when ν is strongly additive,

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$g = \chi_\Omega$ is a weak unit of $L^1(\nu)$, ν_g is an extension of ν and $L^1(\nu)$ coincides with $L^1(\nu_g)$ (Corollary 3.2).

1. Preliminaries. Throughout this paper, $\mathcal{R}$ will be a δ-ring of subsets of a set Ω, that is, a ring of sets closed under countable intersections. We denote by $\mathcal{R}^{\text{loc}}$ the σ-algebra of subsets A of Ω such that $A \cap B \in \mathcal{R}$ for every $B \in \mathcal{R}$. The space of measurable real functions on $(\Omega, \mathcal{R}^{\text{loc}})$ is denoted by $\mathcal{M}$. The simple functions are related to $\mathcal{R}^{\text{loc}}$ and the simple functions based on $\mathcal{R}$ will be called *$\mathcal{R}$-simple functions*. The space of all $\mathcal{R}$-simple functions is denoted by $\mathcal{S}(\mathcal{R})$.

Let $\lambda : \mathcal{R} \to \mathbb{R}$, be a countably additive measure, that is $\sum \lambda(A_n)$ converges to $\lambda(\cup A_n)$ whenever (A_n) are pairwise disjoint sets in $\mathcal{R}$ such that $\cup A_n \in \mathcal{R}$. The *variation* of λ is the countably additive measure $|\lambda| : \mathcal{R}^{\text{loc}} \to [0, \infty]$ given by

$$|\lambda|(A) = \sup\left\{\sum |\lambda(A_i)| \ : \ (A_i) \text{ finite disjoint sequence in } \mathcal{R} \cap 2^A\right\};$$

see [12, Section 2], [14, Definition 2.3 and Lemma 2.4]. A function $f \in \mathcal{M}$ is *integrable with respect to* λ if $|f|_{1,\lambda} = \int |f| d|\lambda| < \infty$. Identifying functions which are equal $|\lambda|$-a.e., the space $L^1(\lambda)$ of integrable functions with respect to λ is a Banach space with norm $|\cdot|_{1,\lambda}$, in which $\mathcal{S}(\mathcal{R})$ is dense [14, Triv. 2.15]. For $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i} \in \mathcal{S}(\mathcal{R})$, $\int \varphi \, d\lambda := \sum_{i=1}^{n} a_i \lambda(A_i)$. For $f \in L^1(\lambda)$, $\int f \, d\lambda := \lim \int \varphi_n \, d\lambda \in \mathbb{R}$, where $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ converges to f in $L^1(\lambda)$. If $f \in L^1(\lambda)$, the set function $\lambda_f : \mathcal{R}^{\text{loc}} \to \mathbb{R}$ defined by $\lambda_f(A) = \int_A f \, d\lambda = \int f \chi_A \, d\lambda$ is a countably additive measure, with $|\lambda_f|(A) = \int_A |f| d|\lambda|$ for all $A \in \mathcal{R}^{\text{loc}}$; see [14, pp. 212–228].

Let X be a real Banach space, with dual space X^* and B_{X^*} be the unit ball of X^*. Consider a *vector measure* $\nu : \mathcal{R} \to X$, that is, $\sum \nu(A_n)$ converges to $\nu(\cup A_n)$ in X for every sequence (A_n) of pairwise disjoint sets in $\mathcal{R}$ with $\cup A_n \in \mathcal{R}$. The *semivariation* of ν is the set function defined on $\mathcal{R}^{\text{loc}}$ by $\|\nu\|(A) = \sup\{|x^*\nu|(A) \ : \ x^* \in B_{X^*}\}$, where $|x^*\nu|$ is the variation of the measure $x^*\nu : \mathcal{R} \to \mathbb{R}$. The semivariation of ν is finite on $\mathcal{R}$ and satisfies

$$(1) \qquad \frac{\|\nu\|(A)}{2} \leqq \sup\{\|\nu(B)\| : B \in \mathcal{R} \cap 2^A\} \leqq \|\nu\|(A), \quad A \in \mathcal{R}^{\text{loc}};$$

see [12, Section 2], [15, Lemma 3.4 and Corollary 3.5]. A set $B \in \mathcal{R}^{\text{loc}}$ is ν-null if $\|\nu\|(B) = 0$. A property holds ν-*almost everywhere* (ν-a.e.) if it holds except on a ν-null set.

Let E be a Banach lattice with norm $\|\cdot\|$ and order $\leqq$. An *ideal* of E is a closed subspace F such that $y \in F$ whenever $y \in E$ with $|y| \leqq |x|$ for some $x \in F$. A *weak unit* of E is an element $0 < e \in E$ with the property that $\inf\{x, e\} = 0$ implies $x = 0$. A Banach lattice is *order continuous* if order bounded increasing sequences are norm convergent. A *Banach function space* with respect to a measure space $(\Omega, \Sigma, \lambda)$ is a Banach space E of (equivalence classes of) measurable functions which are integrable with respect to λ

over sets of finite measure, contains all simple functions whose support has finite measure, and satisfies $f \in E$ with $\|f\| \leqq \|g\|$ whenever $|f| \leqq |g|$ with $g \in E$. For issues related to Banach lattices see [13].

2. Integration with respect to vector measures on δ-rings. We recall the integration theory with respect to vector measures defined on δ-rings, due to Lewis [12] and Masani and Niemi [14], [15]. Let $\nu : \mathcal{R} \to X$ be a vector measure. We denote by $L^1_w(\nu)$ the space of functions in $\mathcal{M}$ which are integrable with respect to $x^*\nu$ for all $x^* \in X^*$. Functions which are equal ν-a.e. are identified. The space $L^1_w(\nu)$ is a Banach space, endowed with the norm

$$\|f\|_\nu = \sup\left\{\int |f| d\,|x^*\nu| \;:\; x^* \in B_{X^*}\right\},$$

in which convergence in norm of a sequence implies ν-a.e. convergence of some subsequence, [15, Lemma 3.13]. The space $L^1_w(\nu)$ is a Banach lattice for the ν-a.e. order and an ideal of measurable functions, that is, if $|f| \leqq |g|$ ν-a.e. with $f \in \mathcal{M}$ and $g \in L^1_w(\nu)$, then $f \in L^1_w(\nu)$. A function $f \in L^1_w(\nu)$ is *integrable with respect to* ν if for each $A \in \mathcal{R}^{\text{loc}}$ there is a vector denoted by $\int_A f\, d\nu \in X$, such that

$$x^*\left(\int_A f\, d\nu\right) = \int_A f\, dx^*\nu \quad \text{for all } x^* \in X^*.$$

When $A = \Omega$, we simply write $\int f\, d\nu$ for $\int_\Omega f\, d\nu$. We denote by $L^1(\nu)$ the space of integrable functions with respect to ν. If $\varphi = \sum_{i=1}^n a_i \chi_{A_i} \in \mathcal{S}(\mathcal{R})$ then $\varphi \in L^1(\nu)$ with $\int_A \varphi\, d\nu = \sum_{i=1}^n a_i \nu(A_i \cap A)$, for $A \in \mathcal{R}^{\text{loc}}$. Furthermore, $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\nu)$, [12, Theorem 3.5]. If X does not contain a copy of c_0, then $L^1_w(\nu) = L^1(\nu)$, [12, Theorem 5.1]. The integration operator $f \in L^1(\nu) \to \int f\, d\nu \in X$ is linear and continuous with $\|\int f\, d\nu\| \leqq \|f\|_\nu$. Given $f \in L^1(\nu)$, the set function

$$A \in \mathcal{R}^{\text{loc}} \to \nu_f(A) = \int_A f\, d\nu \in X \tag{2}$$

is a vector measure with semivariation $\|\nu_f\|(A) = \|f\chi_A\|_\nu$; see [12, Theorem 3.2], [15, Theorem 4.4]. For $f \in L^1(\nu)$, applying (1) to the vector measure ν_f we have

$$\frac{\|f\|_\nu}{2} \leqq \sup\left\{\left\|\int_A f\, d\nu\right\| \;:\; A \in \mathcal{R}\right\} \leqq \|f\|_\nu, \tag{3}$$

which then gives an equivalent norm in $L^1(\nu)$. The space $L^1(\nu)$ is a Banach lattice for the order structure of $L^1_w(\nu)$. Moreover, it is an ideal within the measurable functions and also in $L^1_w(\nu)$; see [15, Theorem 4.10]. From the dominated convergence theorem proved by Lewis in this setting, [12, Theorem 3.3], it follows that $L^1(\nu)$ is order continuous.

Each vector measure ν defined on a σ-algebra satisfies $\chi_\Omega \in L^1(\nu)$ and so $\|\nu\|(\Omega) = \|\chi_\Omega\|_\nu < \infty$, that is, ν is bounded by (1). This does not hold in general for vector measures defined on δ-rings, as the following example shows.

Example 2.1. Let $\mathcal{R}$ be the δ-ring of Borel subsets of $\mathbb{R}$ having finite Lebesgue measure, which is denoted by m. Then, $\mathcal{R}^{\text{loc}}$ is the σ-algebra of all Borel subsets of $\mathbb{R}$. For $1 \leqq p < \infty$, the vector measure $\nu : \mathcal{R} \to L^p(\mathbb{R})$ defined by $\nu(A) = \chi_A$ is not bounded, since $\|\nu(A)\|_p = m(A)^{1/p}$, for all $A \in \mathcal{R}$. Moreover, $\|\varphi\|_\nu = \|\varphi\|_p$ for every $\varphi \in \mathcal{S}(\mathcal{R})$ with $\mathcal{S}(\mathcal{R})$ dense in both $L^1(\nu)$ and $L^p(\mathbb{R})$. So, $L^1(\nu) = L^p(\mathbb{R})$. The space $L^p(\mathbb{R})$ does not contain a copy of c_0, and hence, $L^1_w(\nu) = L^1(\nu)$.

Since the space $L^1(\nu)$ of a vector measure ν defined on a δ-ring is an ideal of measurable functions, the space of bounded measurable functions is included in $L^1(\nu)$ if and only if $\chi_\Omega \in L^1(\nu)$. This fails to hold if ν is not bounded, in which case (by (1)) $\chi_\Omega \notin L^1_w(\nu)$. A condition which guarantees the integrability with respect to ν of a bounded measurable function f is that its support satisfies $\chi_{\text{supp}(f)} \in L^1(\nu)$. In this case, $\|f\|_\nu \leqq \|f\|_\infty \cdot \|\nu\|(\text{supp}(f))$.

Curbera showed that the class of order continuous Banach lattices having a weak unit coincides with the class of spaces L^1 for vector measures defined on σ-algebras, [6, Theorem 8]. The space $L^1(\nu)$ for a vector measure ν defined on a δ-ring, may not have a weak unit; see Example 2.2. In this setting the above characterization still holds without a weak unit, since Curbera proved the following: If E is an order continuous Banach lattice, then there exists a vector measure ν defined on some δ-ring of sets such that E is order isometric to $L^1(\nu)$; [5, pp. 22–23].

Example 2.2. Let Γ be an abstract set and $\mathcal{R}$ be the δ-ring of finite subsets of Γ. Then $\mathcal{R}^{\text{loc}} = 2^\Gamma$. Given $p \in [1, \infty]$, set $X_p = \ell^p(\Gamma)$ for $p < \infty$ and $X_\infty = c_0(\Gamma)$. Consider the vector measure $\nu : \mathcal{R} \to X_p$ defined by $\nu(A) = \sum_{\gamma \in A} e_\gamma$, where e_γ is the characteristic function of the point $\gamma \in \Gamma$. This vector measure was considered in [5, p. 23] for $p = 1$. The only ν-null set is the empty set. Note that $\int \varphi \, d\nu = \sum \varphi(\gamma) e_\gamma = \varphi$ for all $\varphi \in \mathcal{S}(\mathcal{R})$. Each $x^* \in X_p^*$ is identified with some $(x_\gamma)_{\gamma\in\Gamma} \in \ell^q(\Gamma)$, where $1/p + 1/q = 1$. So, $x^*\nu(A) = \sum_{\gamma \in A} x_\gamma$ and $|x^*\nu|(A) = \sum_{\gamma \in A} |x_\gamma|$ for all $A \in \mathcal{R}$. Suppose $p < \infty$. In this case, X_p does not contain a copy of c_0, so $L^1_w(\nu) = L^1(\nu)$. Given $\varphi \in \mathcal{S}(\mathcal{R})$, for each $x^* = (x_\gamma)_{\gamma\in\Gamma} \in X_p^*$ we have $\int |\varphi| d|x^*\nu| = \sum |\varphi(\gamma)||x_\gamma| \leqq \|\varphi\|_p \|x^*\|_q$ and so $\|\varphi\|_\nu \leqq \|\varphi\|_p$. From (3), $\|\varphi\|_p = \|\varphi\|_\nu$. Since $\mathcal{S}(\mathcal{R})$ is dense in X_p, we have $L^1(\nu) = \ell^p(\Gamma)$. Suppose $p = \infty$. Given $f \in \mathcal{M}$, since each $x^* = (x_\gamma)_{\gamma\in\Gamma} \in \ell^1(\Gamma)$ has countable support, we have $\int |f| d|x^*\nu| = \sum |f(\gamma)||x_\gamma| \leqq \|f\|_\infty \|x^*\|_1$. Then, $|f(\gamma)| = \int |f| d|e_\gamma \nu| \leqq \|f\|_\nu \leqq \|f\|_\infty$ for all $\gamma \in \Gamma$, and so $\|f\|_\nu = \|f\|_\infty$. Hence,

$L^1_w(\nu) = \ell^\infty(\Gamma)$. Moreover, $L^1(\nu) = c_0(\Gamma)$, since $\mathcal{S}(\mathcal{R})$ is dense in $c_0(\Gamma)$. In particular, $L^1(\nu)$ is a proper closed subspace of $L^1_w(\nu)$. For any $p \in [1, \infty]$, since each element of X_p has countable support, $L^1(\nu)$ has a weak unit if and only if Γ is countable.

In the following result we characterize integrability by extending to the setting of δ-rings the definition of integrable function given in [1].

Proposition 2.3. *A function $f \in \mathcal{M}$ belongs to $L^1(\nu)$ if and only if there is a sequence $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ such that*

a) *(φ_n) converges to f ν-a.e.*
b) *$(\int_A \varphi_n \, d\nu)$ converges in norm of X for all set $A \in \mathcal{R}^{\text{loc}}$.*

Proof. Necessity is proved by taking a sequence in $\mathcal{S}(\mathcal{R})$ converging to f in the norm of $L^1(\nu)$ and ν-a.e. and using continuity of the integration operator.

Suppose $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ satisfying a) and b). It suffices to prove that (φ_n) is Cauchy in $L^1(\nu)$. For, in this case, $\varphi_n \to g$ in $L^1(\nu)$ for some $g \in L^1(\nu)$ and hence, some subsequence of (φ_n) converges to g ν-a.e. Then a) implies $f = g \in L^1(\nu)$.

For each n, consider the vector measure $\nu_n : \mathcal{R}^{\text{loc}} \to X$ defined by $\nu_n(A) = \int_A \varphi_n \, d\nu$. Let $\lambda_n = |x_n^* \nu_n|$, for $x_n^* \in B_{X^*}$, be a Rybakov control measure for ν_n; see [9, Theorem IX.2.2]. The non-negative measure $\mu = \sum_{n \geqq 1} \frac{\lambda_n}{2^n(\lambda_n(\Omega)+1)}$ satisfies $\lim_{\mu(A)\to 0} \|\nu_n(A)\| = 0$, for every n. This together with b), are the hypothesis of Vitali-Hahn-Saks's theorem; see [9, Corollary I.5.6]. So, the previous limit is uniform in n. Hence, given $\varepsilon > 0$ there is $\delta > 0$ such that, for all $n \geqq 1$ and all $A \in \mathcal{R}^{\text{loc}}$ with $\mu(A) < \delta$, we have

$$\left\| \int_A \varphi_n \, d\nu \right\| = \|\nu_n(A)\| < \varepsilon. \tag{4}$$

Set $B_m = \bigcap_{j=1}^{m} \varphi_j^{-1}(\{0\})$ and $B = \bigcap_{m \geqq 1} B_m$. Since $\mu(B_m \backslash B) \to 0$ as $m \to \infty$, there is m_δ such that $\mu(B_{m_\delta} \backslash B) < \delta/2$.

Since ν-null sets are μ-null, a) implies that $\varphi_n \to f$ μ-a.e. For the above $\delta > 0$, Egoroff's Theorem ensures the existence of a set $Z_\delta \in \mathcal{R}^{\text{loc}}$ such that $\mu(Z_\delta) < \delta/2$ and $\varphi_n \to f$ uniformly on Z_δ^c. Noting that $\|\nu\|(B_{m_\delta}^c) < \infty$ (as $B_{m_\delta}^c \in \mathcal{R}$), it is possible to choose $n_{\varepsilon,\delta}$ such that

$$\|(\varphi_n - f)\chi_{Z_\delta^c}\|_\infty \leqq \frac{\varepsilon}{2(1 + \|\nu\|(B_{m_\delta}^c))}, \qquad \text{for all } n \geqq n_{\varepsilon,\delta}.$$

So, for all $D \in \mathcal{R}^{\text{loc}}$ and all $m, n \geqq n_{\varepsilon,\delta}$ it follows

$$\begin{aligned} \left\| \int_{D \cap Z_\delta^c \cap B_{m_\delta}^c} (\varphi_n - \varphi_m) \, d\nu \right\| &\leqq \|(\varphi_n - \varphi_m)\chi_{Z_\delta^c \cap B_{m_\delta}^c}\|_\nu \\ &\leqq \|(\varphi_n - \varphi_m)\chi_{Z_\delta^c}\|_\infty \cdot \|\nu\|(B_{m_\delta}^c) \leqq \varepsilon. \end{aligned}$$

Set $H_\delta = Z_\delta \cup B_{m_\delta}$. Since $\mu(H_\delta \backslash B) < \delta$ and $\varphi_n \chi_B = 0$, for all $D \in \mathcal{R}^{\text{loc}}$ and all n, it follows from (4) that

$$\left\| \int\limits_{D \cap H_\delta} (\varphi_n - \varphi_m)\, d\nu \right\| = \left\| \int\limits_{D \cap (H_\delta \backslash B)} (\varphi_n - \varphi_m)\, d\nu \right\| \leqq 2\varepsilon.$$

Hence, $\| \int_D (\varphi_n - \varphi_m)\, d\nu \| \leqq 3\varepsilon$, for all $D \in \mathcal{R}^{\text{loc}}$ and all $m, n \geqq n_{\varepsilon,\delta}$. From (3), $\|\varphi_n - \varphi_m\|_\nu \leqq 6\varepsilon$ for all $n, m \geqq n_{\varepsilon,\delta}$. Thus $f \in L^1(\nu)$. $\square$

3. Analytic properties of ν and lattice properties of $L^1(\nu)$. Let $\nu : \mathcal{R} \to X$ be a vector measure on a δ-ring $\mathcal{R}$. The vector measure ν is called *strongly additive* if $\nu(A_n) \to 0$ whenever (A_n) is a disjoint sequence in $\mathcal{R}$. We say that ν is *σ-finite* if there exists a sequence $(A_n) \subset \mathcal{R}$ and a ν-null set $N \in \mathcal{R}^{\text{loc}}$ such that $\Omega = (\cup A_n) \cup N$. It is obvious that any vector measure defined on a σ-algebra is strongly additive and σ-finite.

Strongly additive measures are σ-finite; see [4, Lemma 1.1]. The converse does not hold as Example 2.1 shows.

A countably additive measure $\lambda : \mathcal{R} \to [0, \infty]$ is a *control measure* for ν if it satisfies: 1) $\lim\limits_{\lambda(A)\to 0} \|\nu(A)\| = 0$, and 2) every ν-null set of $\mathcal{R}^{\text{loc}}$ is λ-null. The measure λ is a *local control measure* for ν if 1) is replaced by 1') for every $B \in \mathcal{R}$, $\lim\limits_{\substack{\lambda(A)\to 0 \\ A \in \mathcal{R} \cap 2^B}} \|\nu(A)\| = 0$. Condition 1') is just $\nu \ll \lambda$ on $\mathcal{R}$ in [14, Definition 2.36]. From [15, Proposition 3.6], 1') is equivalent to $\nu(A) = 0$ whenever $A \in \mathcal{R}$ with $\lambda(A) = 0$, and this happens (by (1)) if and only if every λ-null set of $\mathcal{R}^{\text{loc}}$ is ν-null. Conditions 1) and 1') coincide if ν is defined on a σ-algebra. For the reason of introducing the concept of local control measure see [14, pp. 231–232].

The next result exhibits conditions equivalent to the strong additivity of a vector measure defined on a ring of sets, which is not assumed to be a δ-ring.

Theorem 3.1. *Let X be a Banach space, $\mathcal{R}$ a ring of subsets of Ω and $\nu : \mathcal{R} \to X$ a vector measure. The following are equivalent:*

a) *The measure ν is strongly additive.*
b) *There exists a σ-algebra Σ containing $\mathcal{R}$ and a vector measure $\hat{\nu} : \Sigma \to X$ such that $\hat{\nu}(A) = \nu(A)$ for all $A \in \mathcal{R}$ (i.e., $\hat{\nu}$ extends ν).*
c) *There exists a bounded control measure for ν.*

If these conditions hold, then we can take $|x_0^* \nu|$ as a bounded control measure for a certain $x_0^* \in B_{X^*}$.

P r o o f. For the equivalence between a) and c), see [3, Theorem 2].

Condition a) is equivalent to the existence of a vector measure $\tilde{\nu} : \sigma(\mathcal{R}) \to X$ which extends ν, where $\sigma(\mathcal{R})$ is the σ-ring generated by $\mathcal{R}$; see [10, Theorem on Extension].

In this case, by [4, Lemma 1.1], there exists $(A_n) \subset \mathcal{R}$ such that $N = \Omega\backslash(\cup A_n)$ is a ν-null set, in the sense that $\nu(A) = 0$ for all $A \in \mathcal{R} \cap 2^N$. Consider the σ-algebra $\Sigma = \{A \cup B : A \in \sigma(\mathcal{R})$ and $B \subset N\}$. Noting that $\tilde{\nu}(A) = 0$ for all ν-null sets $A \in \sigma(\mathcal{R})$, it follows that $\hat{\nu} : \Sigma \to X$ defined by $\hat{\nu}(A \cup B) = \tilde{\nu}(A)$ is a vector measure which extends ν. So a) and b) are equivalent.

Suppose a)-c) hold. Let $|x_0^*\hat{\nu}|$, for $x_0^* \in B_{X^*}$, be a Rybakov control measure for the vector measure $\hat{\nu}$ given in b). Since $|x_0^*\nu| \leqq |x_0^*\hat{\nu}|$ with equality on $\mathcal{R}$, then $|x_0^*\nu|$ is a bounded control measure for ν. □

Applying Theorem 3.1 to the case of δ-rings we obtain the following result.

Corollary 3.2. *Let X be a Banach space, $\mathcal{R}$ a δ-ring of sets of Ω and $\nu : \mathcal{R} \to X$ a vector measure.*

a) *If ν is strongly additive, then $L^1(\nu)$ coincides with $L^1(\hat{\nu})$, where $\hat{\nu} : \mathcal{R}^{\text{loc}} \to X$ is a vector measure which extends ν.*
b) *The vector measure ν is strongly additive if and only if $\chi_\Omega \in L^1(\nu)$.*
c) *If ν is strongly additive, then $L^1(\nu)$ is an order continuous Banach function space with respect to $(\Omega, \mathcal{R}^{\text{loc}}, |x_0^*\nu|)$, where $|x_0^*\nu|$ is a bounded control measure for a certain $x_0^* \in B_{X^*}$.*

Proof. Suppose ν is strongly additive. In the proof of Theorem 3.1, since $\mathcal{R}$ is a δ-ring, we have $N \in \mathcal{R}^{\text{loc}}$, $\sigma(\mathcal{R}) \subset \mathcal{R}^{\text{loc}} \subset \Sigma$ and $\hat{\nu} : \mathcal{R}^{\text{loc}} \to X$ is a vector measure extending ν. Note that the ν-null and $\hat{\nu}$-null sets coincide. For all $\varphi \in \mathcal{S}(\mathcal{R})$, we have $\|\varphi\|_\nu = \|\varphi\|_{\hat{\nu}}$. Furthermore, $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\hat{\nu})$. This follows from the order continuity of $L^1(\hat{\nu})$ together with the fact that ν is σ-finite. Thus, $L^1(\nu)$ coincides with $L^1(\hat{\nu})$. Hence a) holds.

Let us verify b). If ν is strongly additive, then a) implies $\chi_\Omega \in L^1(\hat{\nu}) = L^1(\nu)$. Conversely, if $\chi_\Omega \in L^1(\nu)$, by (2), the set function $\hat{\nu} : \mathcal{R}^{\text{loc}} \to X$ defined by $\hat{\nu}(A) = \int_A \chi_\Omega d\nu$ is a vector measure which extends ν. Thus, from Theorem 3.1, it follows that ν is strongly additive.

For c) just observe that, by Theorem 3.1, there exists such a control measure $|x_0^*\nu|$ which is defined on $\mathcal{R}^{\text{loc}}$, since it is the variation of $x_0^*\nu : \mathcal{R} \to \mathbb{R}$. □

Every strongly additive vector measure ν defined on a δ-ring is bounded, since if χ_Ω is in $L^1(\nu)$, then $\|\nu\|(\Omega) = \|\chi_\Omega\|_\nu < \infty$. For $p = \infty$, Example 2.2 exhibits a bounded measure which is not strongly additive (since $\|\nu(A)\|_\infty = 1$ for every non empty set $A \in \mathcal{R}$).

The vector measure ν given in Example 2.2 (for any $p \in [1, \infty]$) is σ-finite if and only if Γ is countable, that is, if a weak unit exists for $L^1(\nu)$. For $p = \infty$, ν is bounded. For $p < \infty$ and Γ infinite, ν is not bounded. This shows that there is no relation between boundedness and σ-finiteness.

The next result gives equivalent conditions to the σ-finiteness of ν. In this case, only the existence of a *local* control measure for ν can be guaranteed.

Theorem 3.3. *Let X be a Banach space, $\mathcal{R}$ a δ-ring of sets of Ω and $\nu : \mathcal{R} \to X$ a vector measure. The following are equivalent:*

a) *The measure ν is σ-finite.*
b) *The space $L^1(\nu)$ has a weak unit.*
c) *There exists a bounded local control measure for ν.*

Proof. If ν is σ-finite, then $\Omega = (\cup A_n) \cup N$ with $A_n \in \mathcal{R}$ and N a ν-null set in $\mathcal{R}^{\text{loc}}$. The series $\sum (2^n(\|\nu\|(A_n) + 1))^{-1} \chi_{A_n}$ converges absolutely in $L^1(\nu)$. Moreover, its sum g is a weak unit in $L^1(\nu)$, since $\chi_{g^{-1}(\{0\})} = 0$ in $L^1(\nu)$. Hence b) holds.

Let g be a weak unit in $L^1(\nu)$ and $\nu_g : \mathcal{R}^{\text{loc}} \to X$ the vector measure defined by $\nu_g(A) = \int_A g \, d\nu$; see (2). Let $\lambda = |x_0^* \nu_g|$ be a Rybakov control measure for ν_g, where $x_0^* \in B_{X^*}$. A set $A \in \mathcal{R}^{\text{loc}}$ is λ-null if and only if $\|g\chi_A\|_\nu = \|\nu_g\|(A) = 0$, that is, $g\chi_A = 0$ ν-a.e. or equivalently A is ν-null. Therefore, λ is a bounded local control measure for ν. Thus b) implies c).

If c) holds for some bounded local control measure $\lambda : \mathcal{R} \to [0, \infty)$, then $|\lambda|(\Omega) = \sup_{B \in \mathcal{R}} |\lambda|(B) < \infty$. Thus, there is $(B_n) \subset \mathcal{R}$ with $|\lambda|(\Omega \backslash B_n) < 1/n$ and so $\Omega = (\cup B_n) \cup N$ with $N = \Omega \backslash (\cup B_n)$ being λ-null, that is, ν-null. So a) holds. $\square$

Remark 3.4. In the proof of Theorem 3.3, $|x_0^* \nu|$ is also a local control measure for ν, although in this case we cannot guarantee that it is bounded. In this case, the space $L^1(\nu)$ will be a Banach function space with respect to $(\Omega, \mathcal{R}^{\text{loc}}, |x_0^* \nu|)$ provided $\chi_A \in L^1(\nu)$ whenever $|x_0^* \nu|(A) < \infty$. This fact does not hold in general. For $p = \infty$ and Γ countable, Example 2.2 exhibits a measure ν such that $|x^* \nu|(A) < \infty$ for all $x^* \in X_\infty^*$ and all $A \in \mathcal{R}^{\text{loc}}$, whereas $\chi_\Gamma \notin L^1(\nu)$. For $p > 1$ and Γ uncountable (i.e., ν is non σ-finite), there are no Rybakov local control measures for ν, since each element $x^* \in X_p^*$ has countable support and so $|x^* \nu|$ has plenty of non-trivial null sets. However, for any Γ and $p \in [1, \infty]$, counting measure $\lambda : \mathcal{R} \to [0, \infty]$ is a control measure such that $L^1(\nu)$ is a Banach function space with respect to $(\Gamma, \mathcal{R}^{\text{loc}}, |\lambda|)$.

If ν is σ-finite (and so $L^1(\nu)$ has a weak unit by Theorem 3.3), then $L^1(\nu)$ is order isometric to some Banach function space; see [13, Theorem 1.b.14]. Moreover, Theorem 8 of [6] ensures that $L^1(\nu)$ is order isometric to $L^1(\mu)$ for some vector measure μ defined on a σ-algebra of sets. The following result gives a more concrete description of μ.

Theorem 3.5. *Under the conditions of Theorem 3.3, if ν is σ-finite, then $L^1(\nu)$ is order isometric to $L^1(\nu_g)$, where $\nu_g : \mathcal{R}^{\text{loc}} \to X$ is the vector measure defined by $\nu_g(A) = \int_A g \, d\nu$ and g is a weak unit in $L^1(\nu)$.*

Proof. For $\varphi \in \mathcal{S}(\mathcal{R}^{\text{loc}})$ we have $g\varphi \in L^1(\nu)$ and $\int |\varphi| d|x^* \nu_g| = \int g|\varphi| d|x^* \nu|$ for all $x^* \in X^*$, so $\|\varphi\|_{\nu_g} = \|g\varphi\|_\nu$.

Let $f \in L^1(\nu_g)$ and $(\varphi_n) \subset \mathcal{S}(\mathcal{R}^{\text{loc}})$ be such that $\varphi_n \to f$ in the norm of $L^1(\nu_g)$ and ν_g-a.e. Then $(g\varphi_n)$ is a Cauchy sequence in $L^1(\nu)$ and so it converges in norm to some

$h \in L^1(\nu)$. By taking a subsequence $(g\varphi_{n_k})$ that converges to h ν-a.e., equivalently ν_g-a.e., we see that $gf = h \in L^1(\nu)$ and $\|gf\|_\nu = \|f\|_{\nu_g}$.

Let us show that if $h \in L^1(\nu)$, then $h/g \in L^1(\nu_g)$. Suppose $h \geqq 0$ and let $(\varphi_n) \subset \mathcal{S}(\mathcal{R}^{\mathrm{loc}})$ satisfy $0 \leqq \varphi_n \uparrow h/g$. Then $g\varphi_n \uparrow h$ and, by order continuity, $(g\varphi_n)$ converges to h in $L^1(\nu)$. This implies that (φ_n) is Cauchy in $L^1(\nu_g)$, so it converges in norm to $f \in L^1(\nu_g)$ and there is a subsequence (φ_{n_k}) which converges ν_g-a.e. to f. Thus $f = h/g \in L^1(\nu_g)$. So, the operator $T : L^1(\nu_g) \to L^1(\nu)$ given by $T(f) = gf$ is an order isometry. $\square$

For a vector measure $\nu : \mathcal{R} \to X$ with no further properties, a result of Brooks and Dinculeanu shows that ν has a local control measure $\lambda : \mathcal{R} \to [0, \infty]$, [4, Theorem 3.2]. By Theorem 3.3 if ν is not σ-finite, then λ must be unbounded. In this case, it is known that the space $L^1(\nu)$ can be represented as an unconditional direct sum of a family of disjoint ideals, each one having a weak unit; see [13, Proposition 1.a.9]. According to [6, Theorem 8], each ideal is then the L^1-space of some vector measure. The next result gives a concrete representation of such a decomposition.

Theorem 3.6. *Let X be a Banach space, $\mathcal{R}$ a δ-ring and $\nu : \mathcal{R} \to X$ a vector measure. The space $L^1(\nu)$ can be decomposed into an unconditional direct sum of a family of spaces, each of which is order isometric to some $L^1(\mu)$, where μ is the vector measure ν restricted to a σ-algebra of the kind $A \cap \mathcal{R}$ for some fixed $A \in \mathcal{R}$.*

Proof. In the proof of [4, Theorem 3.1] it is shown that there exists a maximal family $\{A_\alpha : \alpha \in \Delta\}$ of non ν-null sets in $\mathcal{R}$ with $A_\alpha \cap A_\beta$ ν-null for $\alpha \neq \beta$. Let $\nu_\alpha : \mathcal{R}_\alpha \to X$ be the restriction of ν to the σ-algebra $\mathcal{R}_\alpha = A_\alpha \cap \mathcal{R} = \{B \in \mathcal{R} : B \subset A_\alpha\}$ and $\lambda_\alpha = |x_\alpha^* \nu_\alpha|$ be a Rybakov control measure for ν_α. If $B \in \mathcal{R}$, it is proved that $\lambda_\alpha(B \cap A_\alpha) = 0$ for all $\alpha \in \Delta$ except in a countable set. Then a local control measure λ is defined by $\lambda(A) = \sum_{\alpha \in \Delta} \lambda_\alpha(A \cap A_\alpha)$, for $A \in \mathcal{R}$.

Consider the bounded linear projections $P_\alpha : L^1(\nu) \to L^1(\nu)$ given by $P_\alpha(f) = f\chi_{A_\alpha}$. Note that $\big(P_\alpha(L^1(\nu))\big)_{\alpha \in \Delta}$ are disjoint closed ideals of $L^1(\nu)$. Let $f \in L^1(\nu)$ and (φ_n) be a sequence in $\mathcal{S}(\mathcal{R})$ converging to f in norm of $L^1(\nu)$ and ν-a.e. For each n, let $I_n \subset \Delta$ be a countable set with $\varphi_n \chi_{A_\alpha} = 0$ ν-a.e. for all $\alpha \notin I_n$. Then $f\chi_{A_\alpha} = 0$ ν-a.e. for all $\alpha \notin I$, where $I = \cup_n I_n$ is countable. Hence $f = \sum_{\alpha \in I} f\chi_{A_\alpha}$ ν-a.e. and the sum converges unconditionally in $L^1(\nu)$ by the order continuity of $L^1(\nu)$. Thus f is uniquely represented as an unconditional direct sum of elements of $(P_\alpha(L^1(\nu)))_{\alpha \in \Delta}$. Every space $P_\alpha(L^1(\nu))$ is order isometric to $L^1(\nu_\alpha)$, via restriction to the set A_α. $\square$

4. Examples. We end the paper by considering some relevant examples.

Example 4.1. Let $p \in [1, \infty)$ and $T : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be a linear isomorphism. Let $\mathcal{R}$ be the δ-ring of all Borel subsets of $\mathbb{R}$ having finite Lebesgue measure, in which case $\mathcal{R}^{\mathrm{loc}}$ is the σ-algebra of all Borel subsets of $\mathbb{R}$. The set function $\nu_T : \mathcal{R} \to L^p(\mathbb{R})$ defined by

$\nu_T(A) = T(\chi_A)$, is a σ-finite vector measure. Since T is an isomorphism, for all $\varphi \in \mathcal{S}(\mathcal{R})$ we have

$$\frac{1}{\|T^{-1}\|}\|\varphi\|_p \leqq \|T(\varphi)\|_p = \left\| \int \varphi \, d\nu_T \right\|_p \leqq \|T\|\|\varphi\|_p. \tag{5}$$

Then (3) implies that $\|T^{-1}\|^{-1}\|\varphi\|_p \leqq \|\varphi\|_{\nu_T} \leqq 2\|T\|\|\varphi\|_p$. In particular, the ν_T-null sets coincide with the Lebesgue measure null sets. Hence, by the density of $\mathcal{S}(\mathcal{R})$ in both $L^1(\nu_T)$ and $L^p(\mathbb{R})$, it follows that $L^1(\nu_T)$ is order isomorphic to $L^p(\mathbb{R})$. Moreover, from (5) we have $\|T^{-1}\|^{-1}\|\chi_A\|_p \leqq \|\nu_T(A)\|_p$ for all $A \in \mathcal{R}$. So, ν_T is not bounded and hence, not strongly additive.

Some interesting examples of such operators $T : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ are:

(I) The multiplication operator by some function $\varphi \in L^\infty(\mathbb{R})$ for which also $1/\varphi \in L^\infty(\mathbb{R})$.

(II) The dilation operator by some factor $\alpha > 0$. That is, $Tf(x) = f(\alpha x)$ for $x \in \mathbb{R}$. Of course, other composition operators are also possible.

(III) A Fourier p-multiplier operator in $L^p(\mathbb{R})$, corresponding to some (unique) p-multiplier $\psi \in L^\infty(\mathbb{R})$, so that $\widehat{Tf} = \psi \hat{f}$ for $f \in (L^2 \cap L^p)(\mathbb{R})$ is continuously extendable to $L^p(\mathbb{R})$, with $\widehat{\cdot}$ denoting the Fourier transform. In order to ensure that T is an isomorphism we require that $0 \notin \sigma(T)$, the spectrum of T. This class includes all translation operators and other classical operators. For instance, if $\psi = i \cdot sgn$ (with sgn denoting the signum function on $\mathbb{R}$) and $p > 1$, then the corresponding Fourier p-multiplier operator is the Hilbert transform. This operator is also familiar as the singular integral operator given by the principal-value integral

$$T(f)(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{x - t} \, dt, \quad x \in \mathbb{R}.$$

Example 4.2. Let $\phi\colon [0, \infty) \to [0, \infty)$ be a measurable function. Let $\mathcal{R}$ be the δ-ring of all Borel subsets of $[0, \infty)$ having finite Lebesgue measure (denoted by m), and $\mathcal{V}$ be the Volterra convolution operator

$$\mathcal{V}(f)(x) = \int_0^x \phi(x - y) f(y) \, dy,$$

defined for all $f \in \mathcal{M}$ for which $\mathcal{V}(f)(x)$ exists for m-a.e. $x \in [0, \infty)$. Suppose ϕ is integrable. Then

$$\|\nu(A)\|_\infty = \sup_{x \geqq 0} \int_0^x \phi(x - y)\chi_A(y) \, dy = \sup_{x \geqq 0} \int_{(x-A)\cap[0,x]} \phi(s) \, ds \to 0$$

as $m(A) \to 0$, since $m((x - A) \cap [0, x]) \leqq m(A)$ for all $x \geqq 0$. From this it follows that $\nu : \mathcal{R} \to (L^1 \cap L^\infty)[0, \infty)$, defined by $\nu(A) = \mathcal{V}(\chi_A)$ is a vector measure. Let X be a rearrangement invariant Banach function space on $[0, \infty)$. Since $(L^1 \cap L^\infty)[0, \infty)$ is continuously embedded in X (see [2, Theorem II.6.6]), we have that $\nu : \mathcal{R} \to X$ is also a vector measure. This measure is σ-finite.

The boundedness and the strong additivity of ν depend on X. Indeed, for $X = (L^1 \cap L^\infty)[0, \infty)$, ν is not bounded since $\|\nu(A)\|_{L^1 \cap L^\infty} \geqq \|\nu(A)\|_{L^1} = \|\phi\|_{L^1} m(A)$ for all $A \in \mathcal{R}$. For $X = L^\infty[0, \infty)$, ν is bounded since $\|\nu(A)\|_{L^\infty} \leqq \|\phi\|_{L^1}$ for all $A \in \mathcal{R}$. In both cases, ν is not strongly additive. The first case is clear. If $X = L^\infty[0, \infty)$, for all n and for large enough $a > 0$ we have

$$\|\nu(na, (n+1)a)\|_{L^\infty} \geqq \int\limits_{na}^{(n+1)a} \phi((n+1)a - y)\, dy = \int\limits_0^a \phi(s)\, ds > 0.$$

For $X = (L^1 + L^\infty)[0, \infty)$, we now show that ν is strongly additive. Since $\|\nu(A)\|_{L^1 + L^\infty} \leqq \|\nu(A)\|_{L^\infty} \leqq \|\phi\|_{L^1}$ for all $A \in \mathcal{R}$, we see that ν has bounded range on $\mathcal{R}$. By (1), $\|\chi_{[0,\infty)}\|_\nu = \|\nu\|([0, \infty)) < \infty$, and so $\chi_{[0,\infty)} \in L^1_w(\nu)$. Then, $\chi_{[0,\infty)} \in L^1(\nu)$ if for each $B \in \mathcal{R}^{\mathrm{loc}}$ (the Borel σ-algebra of $[0, \infty)$), there exists $h_B \in X$ such that $x^*(h_B) = \int\limits_B \chi_{[0,\infty)}\, dx^*\nu$ for all $x^* \in X^*$. Since X is order continuous, X^* coincides with the associate space $X' = (L^1 \cap L^\infty)[0, \infty)$; see [2, Corollary I.4.3 and Theorem II.6.4]. So, $x^* \in X^*$ corresponds to a function $g \in X'$ and $x^*(f) = \int\limits_0^\infty g(x) f(x)\, dx$, for $f \in X$. Given $B \in \mathcal{R}^{\mathrm{loc}}$, set $h_B = \mathcal{V}(\chi_B)$. Then h_B is bounded by $\|\phi\|_{L^1}$ and so $h_B \in X$. Then

$$\begin{aligned} x^*(h_B) &= \int\limits_0^\infty g(x) \int\limits_0^x \phi(x - y) \chi_B(y)\, dy\, dx \\ &= \lim_{n\to\infty} \int\limits_0^\infty g(x) \int\limits_0^x \phi(x - y) \chi_{B \cap [0,n]}(y)\, dy\, dx \\ &= \lim_{n\to\infty} \int \chi_{B \cap [0,n]}\, dx^*\nu = \int\limits_B \chi_{[0,\infty)}\, dx^*\nu. \end{aligned}$$

Hence, $\chi_{[0,\infty)} \in L^1(\nu)$ and, by Theorem 3.1, ν is strongly additive.

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O. Delgado
C. Calpe 2
2 Escalera 6C
E-21006, Huelva
Spain
olvido@us.es