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**Generalized perfect spaces** <sup>☆</sup>

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## ABSTRACT

Given two Banach function spaces  $X$  and  $Y$  related to a measure  $\mu$ , the  $Y$ -dual space  $X^Y$  of  $X$  is defined as the space of the multipliers from  $X$  to  $Y$ . The space  $X^Y$  is a generalization of the classical Köthe dual space of  $X$ , which is obtained by taking  $Y = L^1(\mu)$ . Under minimal conditions, we can consider the  $Y$ -bidual space  $X^{YY}$  of  $X$  (i.e. the  $Y$ -dual of  $X^Y$ ). As in the classical case, the containment  $X \subset X^{YY}$  always holds. We give conditions guaranteeing that  $X$  coincides with  $X^{YY}$ , in which case  $X$  is said to be  $Y$ -perfect. We also study when  $X$  is isometrically embedded in  $X^{YY}$ . Properties involving  $p$ -convexity,  $p$ -concavity and the order of  $X$  and  $Y$ , will have a special relevance.

## 1. INTRODUCTION

In the theory of function spaces on a measure space  $(\Omega, \Sigma, \mu)$ , the classical Köthe dual (or associate) space  $X'$  of a *Banach function space* (briefly, B.f.s.)  $X$ , plays an important role due to the fact that it is identified with the elements  $x^*$  of  $X^*$ ,

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the topological dual space of  $X$ , acting on  $X$  as an integral, i.e. there exists a measurable function  $g$  such that  $x^*(f) = \int fg d\mu$  for all  $f \in X$ . We can interpret  $X'$  in a different way, as the space of the *multipliers* from  $X$  to  $L^1(\mu)$ , that is, the measurable functions  $g$  defining a multiplication operator from  $X$  to  $L^1(\mu)$ . From this point of view, a generalization of the Köthe dual is possible by taking any B.f.s.  $Y$  in the place of  $L^1(\mu)$ . Namely, the  *$Y$ -dual space* of  $X$ , denoted by  $X^Y$ , is the space of multipliers from  $X$  to  $Y$ , which under an elementary requirement becomes a B.f.s. when endowed with the usual operator norm.

The Köthe dual of a B.f.s. takes a crucial part in the interplay between the order and the topology of  $X$ . For instance,  $X$  is order continuous if and only if  $X^*$  coincides with  $X'$ , or  $X$  satisfies the Fatou property if and only if  $X$  coincides with the Köthe bidual  $X''$  of  $X$ . We pay attention to this second case, in which  $X$  is said to be a *perfect space*. Is there an analogous result for the general case? Or in other words, denoting by  $X^{YY}$  the  *$Y$ -bidual space* of  $X$  (i.e. the  $Y$ -dual of  $X^Y$ ), when does  $X$  coincide with  $X^{YY}$ ? When it does, it is called  *$Y$ -perfect*. We will study this question, which was already posed by Maligranda and Persson in [11, p. 337]. In this item they present some properties concerning the generalized duality and provides a description for the space  $X^Y$  in some special cases, all of them of great usefulness in the development of this paper.

As in the classical case, the containment  $X \subset X^{YY}$  always holds and is continuous with  $\|f\|_{X^{YY}} \leq \|f\|_X$  for all  $f \in X$ . So, another weaker question will be dealt in this paper: When  $X$  is isometrically embedded in  $X^{YY}$ ?

In the following section we analyze in detail the difficulties of solving the above questions, by comparing with the classical case. Section 3 collects some results for the  $Y$ -dual and the  $Y$ -bidual of a B.f.s. In particular, we prove that  $X^Y$  inherits some of the order properties of  $Y$  as the Fatou property (Proposition 3.3), and that the property of being an  $Y$ -perfect space is transitive (Proposition 3.4). This last fact will be of special relevance for the proof of our main result in Section 5, where conditions on  $X$  and  $Y$ , involving  $p$ -convexity and  $q$ -concavity properties, are given for  $X$  to be an  $Y$ -perfect space (Theorem 5.7). Another important tool used for proving this result is what we call the  $p$ -power of a B.f.s. In Section 4 we study the  $Y$ -perfect property for different  $p$ -powers of the same B.f.s. Finally, in Section 6 we exhibit some couples of particular rearrangement invariant B.f.s.' (namely: Lorentz, Marcinkiewicz and Orlicz spaces) satisfying the generalized perfectness property.

## 2. PRELIMINARIES

Let  $(\Omega, \Sigma, \mu)$  be a  $\sigma$ -finite measure space. We denote by  $L^0(\mu)$  the space of all measurable finite real functions on  $\Omega$ , where functions which are equal  $\mu$ -a.e. are identified. A *Banach function space* related to  $\mu$  (briefly B.f.s.) is a Banach space  $X \subseteq L^0(\mu)$  with norm  $\|\cdot\|_X$ , such that if  $f \in L^0(\mu)$ ,  $g \in X$  and  $|f| \leq |g|$   $\mu$ -a.e. then  $f \in X$  and  $\|f\|_X \leq \|g\|_X$ . In particular,  $X$  is a Banach lattice with the pointwise  $\mu$ -a.e. order. A B.f.s.  $X$  has the *Fatou property* if for every sequence  $(f_n) \subseteq X$  such that  $0 \leq f_n \uparrow f$   $\mu$ -a.e. and  $\sup_n \|f_n\|_X < \infty$ , we have that  $f \in X$  and  $\|f_n\|_X \uparrow \|f\|_X$ . A B.f.s.  $X$  is *order semi-continuous* if for every  $f, f_n \in X$ , such

that  $0 \leq f_n \uparrow f$   $\mu$ -a.e., we have that  $\|f_n\|_X \uparrow \|f\|_X$ . Note that if a B.f.s.  $X$  has the Fatou property, then  $X$  is order semi-continuous. For issues related to B.f.s.'s, see [17, Ch. 15] considering the function norm  $\rho$  defined as  $\rho(f) = \|f\|_X$  if  $f \in X$  and  $\rho(f) = \infty$  in other case.

Given two B.f.s.'s  $X$  and  $Y$ , the  $Y$ -dual space of  $X$  is defined as

$$X^Y := \{h \in L^0(\mu) : hf \in Y \text{ for all } f \in X\},$$

i.e. the space of multipliers from  $X$  to  $Y$ . The map  $\|\cdot\|_{X^Y}$  given by

$$\|h\|_{X^Y} := \sup_{f \in B_X} \|hf\|_Y, \quad \text{for } h \in X^Y,$$

defines a natural seminorm on  $X^Y$ . Note that the supremum above is finite. Indeed, if  $0 \leq h \in X^Y$ , then it defines a positive multiplication operator between two Banach lattices and so it is continuous, see [9, p. 2]. The same holds for a general  $h \in X^Y$  by taking positive and negative parts. In order to  $\|\cdot\|_{X^Y}$  be a norm, it is necessary and sufficient to require that  $X$  to be *saturated*, that is, there exists no  $A \in \Sigma$  with  $\mu(A) > 0$  such that  $f\chi_A = 0$   $\mu$ -a.e. for all  $f \in X$ . Note that  $X$  is saturated if and only if  $X$  has a *weak unit* (i.e.  $g \in X$  such that  $g > 0$   $\mu$ -a.e.). In particular,  $X \neq \{0\}$ .

Let  $X, Y$  be B.f.s.'s with  $X$  saturated. Then,  $X^Y$  is a B.f.s. endowed with the norm  $\|\cdot\|_{X^Y}$ , see [11, Proposition 2]. The space  $X^Y$  generalizes the classical Köthe dual space  $X'$  of a B.f.s.  $X$ , which is obtained taking  $Y$  as the space  $L^1(\mu)$ . In this classical case,  $X$  saturated always implies that  $X'$  is saturated, [17, Ch. 15, §71, Theorem 4]. This fails for the general case. In order to obtain a second  $Y$ -dual space of  $X$  with structure of B.f.s.,  $X^Y$  is needed to be saturated. This fact justifies the following comments about saturation for this space. First of all, we note that  $X^Y$  may be trivial.

**Example 2.1.** Suppose  $(\Omega, \Sigma, \mu)$  is non-atomic. Then,  $L^p(\mu)^{L^q(\mu)} = \{0\}$  whenever  $1 \leq p < q \leq \infty$ . See [11, Theorem 2].

Also, even if  $X^Y$  is non-trivial, it may be non-saturated. For instance, every B.f.s.  $Y$  satisfies

$$(1) \quad L^\infty(\mu)^Y \equiv Y,$$

that is, both spaces coincide with equal norms. This fact, mentioned in [11, §2.(f)], can be directly proved. Then, when  $Y$  is non-trivial and non-saturated,  $L^\infty(\mu)^Y$  is so.

Note that  $X^Y$  saturated implies that  $Y$  is saturated. Indeed, if not, then there exists  $A \in \Sigma$  such that  $\mu(A) > 0$  and  $g\chi_A = 0$   $\mu$ -a.e. for all  $g \in Y$ . Since  $X$  is saturated, we can take  $f \in X$  such that  $f > 0$   $\mu$ -a.e. Then, for every  $h \in X^Y$ , we have that  $hf \in Y$  and so  $hf\chi_A = 0$   $\mu$ -a.e. Hence,  $h\chi_A = 0$   $\mu$ -a.e., contradicting the fact that  $X^Y$  is saturated. The converse does not hold, that is, there exist saturated B.f.s.'s  $X$  and  $Y$  such that  $X^Y$  is non-trivial and non-saturated.

**Example 2.2.** Consider the measure space  $([0, 2], \mathcal{B}([0, 2]), \lambda)$ , where  $\mathcal{B}([0, 2])$  is the  $\sigma$ -algebra of all Borel sets of  $[0, 2]$  and  $\lambda$  is the Lebesgue measure on  $[0, 2]$ . Let us define the saturated B.f.s.’

$$X := \{f \in L^0(\lambda): f\chi_{[0,1]} \in L^1(\lambda) \text{ and } f\chi_{[1,2]} \in L^2(\lambda)\}$$

with norm  $\|f\|_X := \|f\chi_{[0,1]}\|_1 + \|f\chi_{[1,2]}\|_2$ , and let  $Y = L^2(\lambda)$ . Note that  $X^Y$  is non-trivial, since for instance  $\chi_{[1,2]} \in X^Y$ . Let us see that the space  $X^Y$  is non-saturated. Let  $h \in X^Y$ . For every  $g \in L^1(\lambda)$ , we have that  $g\chi_{[0,1]} \in X$  and so  $hg\chi_{[0,1]} \in Y = L^2(\lambda)$ . Then,  $h\chi_{[0,1]} \in L^1(\lambda)^{L^2(\lambda)} = \{0\}$  (see Example 2.1), that is,  $h\chi_{[0,1]} = 0$   $\lambda$ -a.e.

**Remark 2.3.** Let  $X, Y$  be B.f.s.’ with  $X$  saturated. A condition guaranteeing that  $X^Y$  is saturated, is that  $X \subseteq Y$ . Indeed, this containment holds if and only if  $L^\infty(\mu) \subseteq X^Y$  and, since  $L^\infty(\mu)$  is saturated,  $X^Y$  is so.

As an immediate consequence of Remark 2.3,  $Y^Y$  is saturated for every saturated B.f.s.  $Y$ . Moreover, from [11, Theorem 1], we have that

$$(2) \quad Y^Y \equiv L^\infty(\mu).$$

Let  $X, Y$  be B.f.s.’ with  $X$  saturated. Whenever  $X^Y$  is saturated, we can consider the  $Y$ -bidual space of  $X$ , that is, the  $Y$ -dual space of  $X^Y$ , which will be denoted by  $X^{YY}$ . Then,

$$X^{YY} := \{\xi \in L^0(\mu): \xi h \in Y \text{ for all } h \in X^Y\},$$

that is the space of multipliers from  $X^Y$  to  $Y$ . The space  $X^{YY}$  is a B.f.s. endowed with the norm

$$\|\xi\|_{X^{YY}} := \sup_{h \in B_{X^Y}} \|\xi h\|_Y.$$

Taking  $Y$  as the space  $L^1(\mu)$ , we obtain that  $X^{YY}$  is just the classical Köthe bidual space  $X''$  of  $X$ . Analogously to  $X''$ , the space  $X^{YY}$  always contains  $X$  and

$$(3) \quad \|f\|_{X^{YY}} \leq \|f\|_X \quad \text{for all } f \in X.$$

In particular,  $X^{YY}$  is saturated. At this point natural questions arise:

#### Questions 2.4.

- (i) When is  $X$  isometrically embedded in  $X^{YY}$  (i.e. (3) is an equality)?
- (ii) When is  $X$  a  $Y$ -perfect space (i.e.  $X \equiv X^{YY}$ )?

Note that in the expression “ $X$  is  $Y$ -perfect” or when  $X^{YY}$  appears without any specification, it must be understood that the minimal requirements which allow to consider the  $Y$ -bidual space of  $X$  are satisfied, namely  $X$  and  $X^Y$  saturated.

For a saturated B.f.s.  $Y$ , simple examples of  $Y$ -perfect spaces follow from (1) and (2). Namely,  $L^\infty(\mu)$  and  $Y$  are always  $Y$ -perfect.

Questions 2.4 are solved for the classical case  $X''$ . In this case, (3) is an equality (i.e.  $X'$  is a norming subspace of  $X^*$ ) if and only if  $X$  is order semi-continuous. Moreover,  $X \equiv X''$  if and only if  $X$  has the Fatou property (see for instance [9, Proposition 1.b.18] and [17, Ch. 15, §71, Theorem 1]).

In the general case,  $X$  having the Fatou property is neither a necessary nor sufficient condition for  $X$  to be  $Y$ -perfect. Indeed,  $X$  is  $X$ -perfect, even if  $X$  does not have the Fatou property. For the converse implication, we can consider the following counterexample.

**Example 2.5.** If  $1 \leq p < q \leq \infty$ , then  $(\ell^p)^{\ell^q} \equiv \ell^\infty$  (see [11, Theorem 2]). So,  $\ell^p$  has the Fatou property but is not  $\ell^q$ -perfect, since

$$(\ell^p)^{\ell^q \ell^q} \equiv (\ell^\infty)^{\ell^q} \equiv \ell^q \not\supseteq \ell^p.$$

Similarly, the order semi-continuity of  $X$  is neither a necessary nor sufficient condition for (3) to be an equality. Indeed, (3) is an equality whenever  $X$  and  $Y$  coincide, even if  $X$  is not order semi-continuous. Conversely, in Example 2.5,  $\ell^p$  is order semi-continuous but (3) is not an equality for  $Y = \ell^q$ .

Trying to solve Questions 2.4 in the general case by giving equivalent conditions turns out to be a very difficult (if not impossible) task, due to the fact already shown that the general duality includes plenty of cases totally different to the classical one. We only need to notice that every saturated B.f.s.  $Y$  is  $Y$ -perfect, without any kind of requirement on  $Y$ . Therefore, we will not look for equivalent conditions but conditions guaranteeing that  $X$  is  $Y$  perfect or that  $X$  is isometrically embedded in  $X^{YY}$ . Before that, we will exhibit some properties of the generalized dual spaces which will be used throughout this paper.

### 3. SOME PROPERTIES RELATED TO GENERALIZED DUALITY

Given two B.f.s.'  $X$  and  $Y$ , we will use the expression " $X \hookrightarrow_c Y$ " to mean that  $X$  is continuously contained in  $Y$  with  $\|f\|_Y \leq c \|f\|_X$  for all  $f \in X$ . The expression " $X \hookrightarrow_i Y$ " will mean that  $X$  is continuously contained in  $Y$  with  $\|f\|_X = \|f\|_Y$  for all  $f \in X$ . In the case when  $X = Y$  as sets, the identity map between  $X$  and  $Y$  is an order isomorphism, since it is a positive linear operator between Banach lattices, see [9, p. 2]. As in the previous section, we will write  $X \equiv Y$  whenever the isomorphism is an isometry, that is, the norms coincide.

**Lemma 3.1.** *Let  $X, Y, Z$  be B.f.s.' with  $X$  saturated. Then,*

- (a)  $Y \hookrightarrow_{c(i)} Z \Rightarrow X^Y \hookrightarrow_{c(i)} X^Z$ ,
- (b)  $X \hookrightarrow_c Z \Rightarrow Z^Y \hookrightarrow_c X^Y$ .

*Moreover, if  $Y$  is order semi-continuous,*

(c)  $X \hookrightarrow_i Z \Rightarrow Z^Y \hookrightarrow_i X^Y$ .

**Proof.** Parts (a) and (b) can be directly proved. Suppose  $Y$  is order semi-continuous. If  $X \hookrightarrow_i Z$ , from (b) we have that  $Z^Y \hookrightarrow_1 X^Y$ . Let  $h \in Z^Y$ . Given  $f \in B_Z$ , consider a sequence of simple functions  $(\psi_n)$  satisfying that  $0 \leq \psi_n \uparrow |f|$ . Note that since  $X$  is saturated, it has a weak unit, which is equivalent to the existence of a sequence  $(A_n) \subset \Sigma$  such that  $\Omega = \bigcup_n A_n$  and  $\chi_{A_n} \in X$ . Taking  $\varphi_n := \psi_n \chi_{\bigcup_1^n A_j} \in X$ , we have that  $0 \leq \varphi_n \uparrow |f|$ . Then,  $0 \leq \varphi_n |h| \uparrow |fh| \in Y$ . Since  $Y$  is order semi-continuous,

$$\begin{aligned} \|fh\|_Y &= \lim_{n \rightarrow \infty} \|\varphi_n h\|_Y \leq \|h\|_{X^Y} \lim_{n \rightarrow \infty} \|\varphi_n\|_X \\ &= \|h\|_{X^Y} \lim_{n \rightarrow \infty} \|\varphi_n\|_Z \leq \|h\|_{X^Y} \|f\|_Z \leq \|h\|_{X^Y}. \end{aligned}$$

Then,  $\|h\|_{Z^Y} \leq \|h\|_{X^Y}$ . So, (c) holds.  $\square$

Lemma 3.1(b) and other properties related to generalized duality can be found in [11, §2].

**Remark 3.2.** As a consequence of (3) and Lemma 3.1(b), we obtain that  $X^Y$  is always  $Y$ -perfect, for each couple of B.f.s.'  $X, Y$  with  $X$  and  $X^Y$  being saturated.

An special feature of the Köthe dual space  $X'$  of a saturated B.f.s.  $X$  follows by taking  $Y = L^1(\mu)$  in Remark 3.2:  $X'$  always has the Fatou property (see also [9, p. 30]). This does not hold in the general case. For instance, we can take a B.f.s.  $Y$  which does not satisfy the Fatou property and then  $L^\infty(\mu)^Y \equiv Y$  fails in having this property. However, if  $Y$  has the Fatou property (as  $L^1(\mu)$ ), this property is transferred to  $X^Y$  for every saturated B.f.s.  $X$ . The analogous for  $Y$  being order semi-continuous also holds.

**Proposition 3.3.** *Let  $X, Y$  be B.f.s.' with  $X$  being saturated.*

- (a) *If  $Y$  has the Fatou property then  $X^Y$  also does.*  
 (b) *If  $Y$  is order semi-continuous then  $X^Y$  is so.*

**Proof.** Let us prove (a), and a similar argument works for (b). Let  $(h_n)$  be a sequence in  $X^Y$  such that  $0 \leq h_n \uparrow h$   $\mu$ -a.e. and  $\sup_n \|h_n\|_{X^Y} < \infty$ . Then, for every  $f \in X$ , we have that  $0 \leq |f|h_n \uparrow |f|h$   $\mu$ -a.e., where  $(fh_n) \subset Y$  with

$$\sup_n \|fh_n\|_Y \leq \|f\|_X \cdot \sup_n \|h_n\|_{X^Y} < \infty.$$

Hence, if  $Y$  has the Fatou property, it follows that  $fh \in Y$  and

$$\|fh\|_Y = \lim_n \|fh_n\|_Y \leq \|f\|_X \cdot \lim_n \|h_n\|_{X^Y}.$$

Therefore,  $h \in X^Y$  and  $\|h\|_{X^Y} \leq \lim_n \|h_n\|_{X^Y}$ . This fact, together with  $\|h_n\|_{X^Y} \leq \|h\|_{X^Y}$  for all  $n$ , implies that  $\|h_n\|_{X^Y} \uparrow \|h\|_{X^Y}$ . So,  $X^Y$  has the Fatou property.  $\square$

The converse of Proposition 3.3 does not hold. For instance, consider a saturated B.f.s.  $Y$  which does not satisfy the Fatou property (or is not order semi-continuous) and take the space  $Y^Y \equiv L^\infty(\mu)$  which has the Fatou property.

Note that in the case when  $X^Y$  is saturated, Proposition 3.3 can be obtained as a particular case of the following more general result by taking  $Z = L^1(\mu)$ .

**Proposition 3.4.** *Let  $X, Y, Z$  be B.f.s.' with  $X, X^Y$  and  $X^{YZ}$  saturated.*

- (a) *If  $Y$  is  $Z$ -perfect then  $X^Y$  is so.*
- (b) *If  $Y \hookrightarrow_i Y^{ZZ}$  then  $X^Y \hookrightarrow_i X^{YZZ}$ .*

**Proof.** For (a) see [11, §2(h)]. Suppose that  $Y^Z$  is saturated. Let us show that

$$(4) \quad X^Y \hookrightarrow_1 X^{YZZ} \hookrightarrow_1 X^{YZZ}.$$

The first inclusion follows from (3). Let us prove the second one. Consider  $\eta \in X^{YZZ}$ . For every  $f \in X$  and  $\xi \in Y^Z$ , we have that  $f\xi \in X^{YZ}$ , since  $f\xi h \in Z$  for all  $h \in X^Y$ . Then  $\eta f\xi \in Z$  and so  $\eta f \in Y^{ZZ}$ . Hence,  $\eta \in X^{YZZ}$ . Moreover,

$$\begin{aligned} \|\eta\|_{X^{YZZ}} &= \sup_{f \in B_X} \|\eta f\|_{Y^{ZZ}} = \sup_{f \in B_X} \sup_{\xi \in B_{Y^Z}} \|\eta f\xi\|_Z \\ &\leq \|\eta\|_{X^{YZZ}} \cdot \sup_{f \in B_X} \sup_{\xi \in B_{Y^Z}} \|f\xi\|_{X^{YZ}} \\ &= \|\eta\|_{X^{YZZ}} \cdot \sup_{f \in B_X} \sup_{\xi \in B_{Y^Z}} \sup_{h \in B_{X^Y}} \|f\xi h\|_Z \\ &\leq \|\eta\|_{X^{YZZ}} \cdot \sup_{f \in B_X} \sup_{h \in B_{X^Y}} \|fh\|_Y \leq \|\eta\|_{X^{YZZ}}. \end{aligned}$$

If  $Y \hookrightarrow_i Y^{ZZ}$ , from Lemma 3.1(a) we have that  $X^Y \hookrightarrow_i X^{YZZ}$ . This fact together with (4) gives  $X^Y \hookrightarrow_i X^{YZZ}$ . So (b) holds.  $\square$

Let us see that being a generalized perfect space is a transitive property. In Section 5, this fact will allow us to recognize B.f.s.'  $X, Y$  such that  $X$  is  $Y$ -perfect passing through an  $L^p$ -space. As can be expected,  $p$ -convexity and  $p$ -concavity properties must be required for the spaces  $X$  and  $Y$ .

**Proposition 3.5.** *Let  $X, Y, Z$  be B.f.s.' with  $X$  and  $X^Z$  saturated.*

- (a) *If  $X$  is  $Y$ -perfect and  $Y$  is  $Z$ -perfect, then  $X$  is  $Z$ -perfect.*
- (b) *If  $X \hookrightarrow_i X^{YY}$  and  $Y \hookrightarrow_i Y^{ZZ}$ , then  $X \hookrightarrow_i X^{ZZ}$ .*

**Proof.** Suppose that  $X^Y$  and  $Y^Z$  are saturated. Then,

$$(5) \quad X \hookrightarrow_1 X^{ZZ} \hookrightarrow_1 X^{YYZZ}$$

This can be directly checked in the same line of the proof of (4).

If  $X$  is  $Y$ -perfect and  $Y$  is  $Z$ -perfect, then  $X^{YY} \equiv X$  and  $Y^{ZZ} \equiv Y$ , and so  $X^{YYZZ} \equiv X$ . This, together with (5), gives  $X \equiv X^{ZZ}$ , that is  $X$  is  $Z$ -perfect. So (a) holds.

Suppose that  $X \hookrightarrow_i X^{YY}$  and  $Y \hookrightarrow_i Y^{ZZ}$ . Then, from Lemma 3.1(a),

$$X \hookrightarrow_i X^{YY} \hookrightarrow_i X^{YYZZ}.$$

This fact, together with (5), gives  $X \hookrightarrow_i X^{ZZ}$ . So, (b) holds.  $\square$

**Remark 3.6.** Taking  $Z = L^1(\mu)$  in Proposition 3.5, it follows:

- (i) In the case when  $Y$  has the Fatou property,  $X$  being  $Y$ -perfect implies that  $X$  has the Fatou property.
- (ii) In the case when  $Y$  is order semi-continuous,  $X \hookrightarrow_i X^{YY}$  implies that  $X$  is order semi-continuous.

The converse of (i) and (ii) does not hold as Example 2.5 shows.

The following useful result, mentioned in [11, §2(b)], can be directly proved.

**Lemma 3.7.** *Let  $X, Y, Z$  be B.f.s.' with  $X$  and  $Y$  saturated. Then,*

$$X^{YZ} \equiv Y^{XZ}.$$

As a consequence of Lemma 3.7, for B.f.s.'  $X, Z$  such that  $X$  and  $X^Z$  are saturated, we have that  $X^{XZZ} \equiv (X^Z)^{XZ} \equiv L^\infty(\mu)$ . Then, for every B.f.s.  $Y$  satisfying  $X \hookrightarrow_1 Y \hookrightarrow_1 X^{ZZ}$ , we have that

$$L^\infty(\mu) \equiv X^X \hookrightarrow_1 X^Y \hookrightarrow_1 X^{XZZ} \equiv L^\infty(\mu),$$

and so

$$(6) \quad X^Y \equiv L^\infty(\mu).$$

In particular,  $X$  is  $Y$ -perfect in the only case of  $X = Y$ , since  $X^{YY} \equiv L^\infty(\mu)^Y \equiv Y$ . For instance, taking  $Z = L^1(\mu)$  and  $Y = X''$ , if  $X$  does not have the Fatou property then  $X$  is not  $X''$ -perfect.

Note that if any of the inclusions in  $X \subseteq Y \subseteq X^{ZZ}$  is continuous with other constant different of 1, then  $X^Y = L^\infty(\mu)$  with equivalent norms.

#### 4. PERFECT SPACES INVOLVING $P$ -POWERS OF A B.F.S.

Let  $X$  be a saturated B.f.s. The *maximal normed extension* of  $X$  is the B.f.s. defined as

$$[X] := \{g \in L^0(\mu) : \sup\{\|f\|_X : f \in X, 0 \leq f \leq |g|\} < \infty\},$$



endowed with the norm

$$\|g\|_{[X]} := \sup\{\|f\|_X : f \in X, 0 \leq f \leq |g|\}, \quad \text{for } g \in [X].$$

We always have that  $X \hookrightarrow_i [X] \hookrightarrow_1 X''$ . In particular,  $[X]$  is saturated. Note that  $[X]$  is the largest B.f.s. having  $X$  as a closed subspace. Then,  $[X] \equiv X''$  if and only if  $X$  is order semi-continuous. These topics appear in [11, §1], where [1] is referred as original source.

From (6), we have that

$$(7) \quad X^X \equiv X^{[X]} \equiv X^{X''} \equiv L^\infty(\mu).$$

Let  $0 < p < \infty$ . The  $p$ -power of  $X$  is the space defined as

$$X^p = \{f \in L^0(\mu) : |f|^p \in X\},$$

endowed with the quasi-norm  $\|f\|_{X^p} = \| |f|^p \|_X^{1/p}$ , for  $f \in X^p$ . For  $1 \leq p$ ,  $\|\cdot\|_{X^p}$  is a norm, [11, Proposition 1]. In the case when  $0 < p < 1$ , if  $X$  is  $(1/p)$ -convex with constant 1 (see Section 5 for this concept), then  $\|\cdot\|_{X^p}$  is also a norm. This follows from [5, Lemma 3] noting that there our space  $X^p$  is denoted by  $X^{1/p}$ . We will only consider these cases, for which  $X^p$  is a B.f.s. Note that  $X^p$  is saturated, since  $X$  is so, and  $[X^p] \equiv [X]^p$ . Moreover,  $X^p$  has the Fatou property if and only if  $X$  has the Fatou property. Similarly,  $X^p$  is order semi-continuous if and only if  $X$  is so.

Let  $1 \leq q < p < \infty$ . Then,

$$(8) \quad [X^p]^{X^q} \equiv X^r \quad \text{and} \quad (X^p)^{X^q} \equiv [X^p]^{[X^q]} \equiv [X^r],$$

where  $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ , see [11, Theorem 5]. In particular, since  $p$  and  $r$  play the same role, we have that

$$[X^p]^{X^q X^q} \equiv (X^r)^{X^q} \equiv [X^p] \quad \text{and} \quad (X^p)^{X^q X^q} \equiv [X^r]^{X^q} \equiv X^p.$$

That is,  $[X^p]$  and  $X^p$  are  $X^q$ -perfect.

**Example 4.1.** For every  $1 \leq q \leq p \leq \infty$ , we have that

$$L^p(\mu)^{L^q(\mu)} \equiv L^r(\mu)$$

with  $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ . In the case  $q < p < \infty$ , this follows from (8) by taking  $X = L^1(\mu)$ . Note that,  $L^1(\mu) \equiv [L^1(\mu)] \equiv L^1(\mu)''$ , since  $L^1(\mu)$  has the Fatou property. The remaining cases follow from (1) and (2). See also [11, Proposition 3]. Then,  $L^p(\mu)$  is  $L^q(\mu)$ -perfect.

There is another possibility left of generalized dual space as combination of  $p$ -powers and maximal normed extension of  $X$ , namely  $(X^p)^{[X^q]}$ . We give a description for this space when  $X$  satisfies certain conditions.

**Proposition 4.2.** Let  $1 \leq q < p < \infty$  and suppose that  $X$  is order semi-continuous. Then, taking  $r$  such that  $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ ,

$$(9) \quad (X^p)^{[X^q]} \equiv [X^r].$$

**Proof.** From (8) and Lemma 3.1(a) we have that  $[X^r] \equiv (X^p)^{X^q} \hookrightarrow_i (X^p)^{[X^q]}$ . Let  $h \in (X^p)^{[X^q]}$  and consider a sequence of simple functions  $(\varphi_n) \subset X^r$  such that  $0 \leq \varphi_n \uparrow |h|$  (e.g.  $\varphi_n = \psi_n \chi_{\bigcup_1^n A_j}$  with  $\psi_n$  simple functions such that  $0 \leq \psi_n \uparrow |h|$  and  $(A_n) \subset \Sigma$  such that  $\Omega = \bigcup_n A_n$  and  $\chi_{A_n} \in X$ ). Note that  $X^r$  is order semi-continuous, since  $X$  is so. Thus,  $(\varphi_n) \subset X^r \hookrightarrow_i [X^r] \equiv (X^r)''$ . Moreover,

$$\|\varphi_n\|_{(X^r)''} = \|\varphi_n\|_{[X^r]} = \|\varphi_n\|_{(X^p)^{X^q}} = \|\varphi_n\|_{(X^p)^{[X^q]}} \leq \|h\|_{(X^p)^{[X^q]}}.$$

Then, since  $(X^r)''$  has the Fatou property, it follows that  $h \in (X^r)'' \equiv [X^r]$ .  $\square$

Note that under conditions of Proposition 4.2, since

$$(X^p)^{[X^q][X^q]} \equiv [X^r]^{[X^q]} \equiv [X^p],$$

$X^p$  is  $[X^q]$ -perfect if and only if  $X \equiv [X]$ .

**Example 4.3.** Let  $m$  be a vector measure on  $\Sigma$  with the same null sets as  $\mu$  and consider the B.f.s.'  $L^1(m)$  and  $L^1_w(m)$  of real measurable functions on  $\Omega$  which are integrable and weakly integrable with respect to  $m$ , respectively. For details on these spaces see for instance [3,16] and the references therein. The containments  $L^\infty(m) \subseteq L^1(m) \subseteq L^1_w(m)$  hold, where  $L^\infty(m)$  denotes the space of  $m$ -a.e. bounded functions (of course it coincides with  $L^\infty(\mu)$ ). An important fact is that  $L^1(m)'' \equiv L^1_w(m)$  ([4, Proposition 2.4]). The space  $L^1(m)$  is *order continuous* (i.e. order bounded increasing sequences are convergent in norm), and  $L^1_w(m)$  has the Fatou property. In particular, both spaces are order semi-continuous. Hence,  $[L^1(m)] \equiv L^1(m)'' \equiv L^1_w(m)$ . The spaces  $L^p(m)$  and  $L^p_w(m)$  are defined in [14] as the  $p$ -power of  $L^1(m)$  and  $L^1_w(m)$  respectively (see also [7]). Let us apply the previous results to this setting. Let  $1 \leq q < p < \infty$  and take  $r$  such that  $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ . From (8) and Proposition 4.2, it follows:

- (i)  $L^p_w(m)^{L^q(m)} \equiv L^r(m)$ .
- (ii)  $L^p(m)^{L^q_w(m)} \equiv L^p(m)^{L^q(m)} \equiv L^p_w(m)^{L^q_w(m)} \equiv L^r_w(m)$ .

In particular,  $L^p_w(m)$  and  $L^p(m)$  are always  $L^q(m)$ -perfect and,  $L^p(m)$  is  $L^q_w(m)$ -perfect in the only case of  $L^q_w(m) \equiv L^q(m)$ , that is, in the only case of  $L^1(m)$  having the Fatou property. Moreover, given  $1 \leq p \leq \infty$ , from (7) we have that

$$(iii) \quad L^p(m)^{L^p_w(m)} \equiv L^\infty(\mu) \equiv L^\infty(m).$$

Note that (i), (ii) for  $q = 1$  and (iii) which have been proved in [6, Theorem 4, 5 and 8] have been obtained here from a more general context.

Let  $1 \leq p \leq \infty$ . A B.f.s.  $X$  is  $p$ -convex if there exists  $C > 0$  such that

$$\left\| \left( \sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_X \leq C \cdot \left( \sum_{i=1}^n \|f_i\|_X^p \right)^{1/p}$$

when  $1 \leq p < \infty$ , and

$$\left\| \sup_{i=1, \dots, n} |f_i| \right\|_X \leq C \cdot \sup_{i=1, \dots, n} \|f_i\|_X$$

when  $p = \infty$ , for every  $(f_i)_{i=1}^n \subset X$ . A B.f.s.  $X$  is  $p$ -concave if it satisfies the converse inequalities of a  $p$ -convex B.f.s. Note that a B.f.s.  $X$  is always 1-convex and  $\infty$ -concave with constant  $C = 1$ . The spaces  $X^p$  are always  $p$ -convex with constant 1. For the particular case of  $L^p(\mu)$ , it also is  $p$ -concave with constant 1. A relevant note regarding the next results is that, from [9, Proposition 1.d.8], every  $p$ -convex ( $p$ -concave) B.f.s.  $X$  has an equivalent norm for which  $X$  is  $p$ -convex ( $p$ -concave) with constant 1.

**Lemma 5.1.** *Let  $X, Y$  be B.f.s. with  $X$  saturated and  $1 \leq p \leq \infty$ . If  $Y$  is  $p$ -convex, then  $X^Y$  is  $p$ -convex with the same constant.*

The proof of Lemma 5.1 is a simple check. Note that the analogous statement of Lemma 5.1 for  $Y$   $p$ -concave does not hold. Indeed, for any  $p$ -concave B.f.s.  $Y$  we have that  $Y^Y \equiv L^\infty(\mu)$  which is not  $p$ -concave for  $p < \infty$ .

**Lemma 5.2.** *Let  $X, Y$  be B.f.s.' with  $X$  and  $X^Y$  saturated and  $1 \leq p < \infty$ .*

- (i)  $X$  is  $Y$ -perfect if and only if  $X^p$  is  $Y^p$ -perfect.
- (ii)  $X \hookrightarrow_i X^{YY}$  if and only if  $X^p \hookrightarrow_i (X^p)^{Y^p Y^p}$

*In particular,*

- (iii)  $X$  has the Fatou property if and only if  $X^p$  is  $L^p(\mu)$ -perfect.
- (iv)  $X$  order semi-continuous if and only if  $X^p \hookrightarrow_i (X^p)^{L^p(\mu) L^p(\mu)}$ .

**Proof.** From [11, §2.(g)], we always have that

$$(10) \quad (X^p)^{Y^p} \equiv (X^Y)^p.$$

In particular,  $(X^p)^{Y^p}$  is saturated. Then,

$$(11) \quad (X^p)^{Y^p Y^p} \equiv ((X^Y)^p)^{Y^p} \equiv (X^{YY})^p.$$

Note that  $X^{YY} \equiv X$  if and only if  $(X^{YY})^p \equiv X^p$ , and by (11), this is equivalent to  $X^p$  being  $Y^p$ -perfect. So, (i) holds.

Similarly,  $X \hookrightarrow_t X^{YY}$  if and only if  $X^p \hookrightarrow_t (X^{YY})^p$ . Then, (ii) follows from (11). For the particular case  $Y = L^1(\mu)$ , we obtain (iii) and (iv).  $\square$

A characterization of the B.f.s.' which are  $L^p(\mu)$ -perfect and of those which are isometrically embedded in its  $L^p(\mu)$ -bidual, is possible.

**Proposition 5.3.** *Let  $X$  be a saturated B.f.s. and  $1 \leq p < \infty$ . Then,*

- (i)  *$X$  is  $L^p(\mu)$ -perfect if and only if  $X$  is  $p$ -convex with constant 1 and has the Fatou property.*
- (ii)  *$X \hookrightarrow_t X^{L^p(\mu)L^p(\mu)}$  if and only if  $X$  is  $p$ -convex with constant 1 and order semi-continuous.*

**Proof.** (i) Suppose  $X$  is  $L^p(\mu)$ -perfect. Since  $L^p(\mu)$  is  $p$ -convex with constant 1, has the Fatou property and  $X \equiv X^{L^p(\mu)L^p(\mu)}$ , from Lemma 5.1 and Proposition 3.3, we have that  $X$  is  $p$ -convex with constant 1 and has the Fatou property.

Conversely, if  $X$  is  $p$ -convex with constant 1 and has the Fatou property, we can consider the B.f.s.  $X^{1/p}$  which also has the Fatou property. From Lemma 5.2(iii), we have that  $X \equiv (X^{1/p})^p$  is  $L^p(\mu)$ -perfect.

(ii) Suppose  $X \hookrightarrow_t X^{L^p(\mu)L^p(\mu)}$ . Since  $X^{L^p(\mu)L^p(\mu)}$  is  $p$ -convex with constant 1 and the norm of this space coincides with the norm of  $X$ , then  $X$  is also  $p$ -convex with constant 1. Thus, we can consider the B.f.s.  $X^{1/p}$  which satisfies

$$(X^{1/p})^p \equiv X \hookrightarrow_t X^{L^p(\mu)L^p(\mu)} \equiv ((X^{1/p})^p)^{L^p(\mu)L^p(\mu)}.$$

Then, by Lemma 5.2(iv),  $X^{1/p}$  is order semi-continuous and hence  $X$  is so.

Conversely, suppose that  $X$  is  $p$ -convex with constant 1 and order semi-continuous. Then, we can consider the B.f.s.  $X^{1/p}$  which is also order semi-continuous. From Lemma 5.2(iv), we have that

$$X \equiv (X^{1/p})^p \hookrightarrow_t ((X^{1/p})^p)^{L^p(\mu)L^p(\mu)} \equiv X^{L^p(\mu)L^p(\mu)}. \quad \square$$

Note that in Proposition 5.3 it has been implicitly used the fact that  $X^{L^p(\mu)}$  is saturated for every  $p$ -convex (with constant 1) saturated B.f.s.  $X$ . This fact is due to  $X^{L^p(\mu)} \equiv ((X^{1/p})^p)^{L^p(\mu)} \equiv ((X^{1/p})')^p$  (see (10) for  $Y = L^1(\mu)$ ).

We have solved Questions 2.4 for the particular case  $Y = L^p(\mu)$ . Now, we consider the "dual" problem: When  $L^p(\mu)$  is  $Y$ -perfect? We will give conditions guaranteeing  $L^p(\mu)$  is  $Y$ -perfect, using the following result obtained by Reisner [12, Theorem 1] as a generalization of the case  $p = 1, q = \infty$  due to Lozanovskii [10, Theorem 6]. Denote by  $L_F^\infty(\mu)$  the space of all functions in  $L^\infty(\mu)$  with support having finite measure and by  $L_{loc}^1(\mu)$  the space of all functions which are locally integrable (i.e. integrable on measurable sets with finite measure).

**Theorem 5.4.** *Let  $Y$  be a B.f.s. such that  $L_F^\infty(\mu) \subseteq Y \subseteq L_{loc}^1(\mu)$ . Given  $1 \leq p < q \leq \infty$ , consider  $r$  defined as  $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ . Then,  $Y$  is  $p$ -convex and  $q$ -concave if*

and only if there exists  $K > 0$  such that for every  $g \in L^r(\mu)$  and  $\varepsilon > 0$ , there exist  $h_1^\varepsilon \in L^q(\mu)^Y$  and  $h_2^\varepsilon \in Y^{L^p(\mu)}$  satisfying that  $g = h_1^\varepsilon h_2^\varepsilon$  and

$$(12) \quad \|h_1^\varepsilon\|_{L^q(\mu)^Y} \|h_2^\varepsilon\|_{Y^{L^p(\mu)}} \leq (1 + \varepsilon)K \|g\|_{L^r(\mu)}.$$

Moreover, if  $K_1$  is the  $p$ -convexity constant of  $Y$  and  $K_2$  is the  $q$ -concavity constant of  $Y$ , we can take  $K = K_1 K_2$ . Also, if  $K$  satisfies (12), then it follows  $K_1 K_2 \leq K$ .

**Remark 5.5.** For  $1 < q \leq \infty$ , every  $q$ -concave B.f.s.  $Y$  with  $L_F^\infty(\mu) \subseteq Y \subseteq L_{\text{loc}}^1(\mu)$ , satisfies that  $L^q(\mu)^Y$  is saturated. Indeed, from Theorem 5.4 applied for  $p = 1$ , taking  $0 < g \in L^r(\mu)$  and any  $\varepsilon > 0$ , there exist  $h_1^\varepsilon \in L^q(\mu)^Y$  and  $h_2^\varepsilon \in Y'$  satisfying  $g = h_1^\varepsilon h_2^\varepsilon$  and (12). In particular, it must be  $h_1^\varepsilon > 0$  and so it is a weak unit of  $L^q(\mu)^Y$ .

**Proposition 5.6.** Let  $Y$  be a B.f.s. such that  $L_F^\infty(\mu) \subseteq Y \subseteq L_{\text{loc}}^1(\mu)$  and  $1 < q \leq \infty$ . If  $Y$  is  $q$ -concave with constant 1, then  $L^q(\mu)$  is  $Y$ -perfect.

**Proof.** The space  $L^q(\mu)^Y$  is saturated by Remark 5.5, so  $L^q(\mu) \hookrightarrow_1 L^q(\mu)^{YY}$ . Let  $z \in L^q(\mu)^{YY}$ . Applying Theorem 5.4 for  $p = 1$ ,  $q$  and  $r = \hat{q}$  with  $\frac{1}{q} + \frac{1}{\hat{q}} = 1$ , we have that, for every  $\varepsilon > 0$  and  $g \in L^{\hat{q}}(\mu)$ , there exist  $h_1^\varepsilon \in L^q(\mu)^Y$  and  $h_2^\varepsilon \in Y'$  satisfying  $g = h_1^\varepsilon h_2^\varepsilon$  and (12). Then,

$$\begin{aligned} \int |zg| d\mu &= \int |zh_1^\varepsilon h_2^\varepsilon| d\mu \leq \|zh_1^\varepsilon\|_Y \|h_2^\varepsilon\|_{Y'} \\ &\leq \|z\|_{L^q(\mu)^{YY}} \|h_1^\varepsilon\|_{L^q(\mu)^Y} \|h_2^\varepsilon\|_{Y'} \\ &\leq \|z\|_{L^q(\mu)^{YY}} (1 + \varepsilon) \|g\|_{L^{\hat{q}}(\mu)}. \end{aligned}$$

So,  $z \in L^{\hat{q}}(\mu)' \equiv L^q(\mu)$  and  $\|z\|_{L^q(\mu)} \leq \|z\|_{L^q(\mu)^{YY}}$ . That is,  $L^q(\mu)^{YY} \hookrightarrow_1 L^q(\mu)$ . Hence,  $L^q(\mu)$  is  $Y$ -perfect.  $\square$

Finally, using the transitivity of the fact of being a generalized perfect space with  $L^p(\mu)$  as intermediary space, we give conditions on two B.f.s.'  $X$  and  $Y$  guaranteeing that  $X$  is  $Y$ -perfect.

**Theorem 5.7.** Let  $X, Y$  be B.f.s.' with  $X, X^Y$  saturated and  $1 < p < \infty$ . Suppose that  $X$  is  $p$ -convex,  $Y$  is  $p$ -concave (both with constant 1) and  $L_F^\infty(\mu) \subseteq Y \subseteq L_{\text{loc}}^1(\mu)$ .

- (i) If  $X$  has the Fatou property, then  $X$  is  $Y$ -perfect.
- (ii) If  $X$  is order semi-continuous, then  $X \hookrightarrow_1 X^{YY}$ .

**Proof.** If  $X$  has the Fatou property, from Proposition 5.3(i) we have that  $X$  is  $L^p(\mu)$ -perfect. On other hand, from Proposition 5.6,  $L^p(\mu)$  is  $Y$ -perfect. Then, from Proposition 3.5(i), it follows that  $X$  is  $Y$ -perfect. So (i) holds.

If  $X$  is order semi-continuous, from Proposition 5.3(ii) we have that  $X \hookrightarrow_t X^{L^p(\mu)L^p(\mu)}$ . Since  $L^p(\mu)$  is  $Y$ -perfect then, from Proposition 3.5(ii), it follows that  $X \hookrightarrow_t X^{YY}$ . So (ii) holds.  $\square$

## 6. GENERALIZED DUALITY FOR REARRANGEMENT INVARIANT B.F.S.'

Let  $I = [0, a)$  with  $0 < a \leq \infty$  and consider the measure space  $(I, \mathcal{B}(I), \lambda)$ , where  $\mathcal{B}(I)$  is the  $\sigma$ -algebra of the Borel sets of  $I$  and  $\lambda$  is the Lebesgue measure on  $I$ . A B.f.s.  $X$  is said to be *rearrangement invariant* (r.i.) whenever  $f \in X$  if and only if  $f^* \in X$  and in this case  $\|f^*\|_X = \|f\|_X$ . Here,  $f^*$  denotes the *decreasing rearrangement* of  $f$ , that is,

$$f^*(s) := \inf\{r > 0: \lambda(\{x \in I: |f(x)| > r\}) \leq s\}.$$

For issues related to r.i. B.f.s.' see [2,8,9]. Remark that  $(I, \mathcal{B}(I), \lambda)$  is considered here in order to simplify. Actually, the analogous of this section for a non-atomic  $\sigma$ -finite measure space  $(\Omega, \Sigma, \mu)$  holds since, in this case, every r.i. B.f.s. related to  $\mu$  is order isometric to an r.i. B.f.s. related to the Lebesgue measure on  $[0, \mu(\Omega))$ , via composition with a measure homomorphism from  $\Omega$  to  $[0, \mu(\Omega))$ , see [9, Ch. 2a].

A non-trivial r.i. B.f.s.  $X$  always satisfies that

$$(13) \quad L^1(\lambda) \cap L^\infty(\lambda) \subseteq X \subseteq L^1(\lambda) + L^\infty(\lambda).$$

In particular,  $X$  is saturated. As noted in [11, §3] (see also [13]), for every non-trivial r.i. B.f.s.'  $X$  and  $Y$  we have that  $X^Y$  is r.i. The following result gives an equivalent condition to  $X^Y$  being non-trivial, and so saturated. Let  $X_F$  denote the space of functions in  $X$  with support having finite measure.

**Lemma 6.1.** *Let  $X, Y$  be non-trivial r.i. B.f.s.'. Then,*

$$X^Y \neq \{0\} \iff X_F \subseteq Y.$$

**Proof.** Only note that  $X^Y$  is an r.i. B.f.s.,  $L_F^\infty(\lambda) \subseteq L^\infty(\lambda) \cap L^1(\lambda)$  and, as it can be directly checked,  $X_F \subseteq Y$  if and only if  $L_F^\infty(\lambda) \subseteq X^Y$ .  $\square$

Note that, from Lemma 6.1,  $X^{L^\infty(\lambda)} \neq \{0\}$  implies that  $X \subset L^\infty(\lambda)$ .

In the case of finite measure (i.e.  $a < \infty$ ), in which  $L^1(\lambda) \cap L^\infty(\lambda) = L^\infty(\lambda)$ , the  $L^\infty(\mu)$ -dual space of every non-trivial r.i. B.f.s.  $X$  is the trivial space, except if  $X$  is  $L^\infty(\lambda)$  itself.

**Proposition 6.2.** *Let  $X$  be a non-trivial r.i. B.f.s. and suppose that  $\lambda$  is finite. If  $X \neq L^\infty(\lambda)$  then  $X^{L^\infty(\lambda)} = \{0\}$ .*

**Proof.** Note that  $L^\infty(\lambda) \subseteq X = X_F$ , since  $X$  is r.i. and  $\lambda$  is finite. Suppose  $X \neq L^\infty(\lambda)$ . Then  $X \not\subseteq L^\infty(\lambda)$  and so from Lemma 6.1,  $X^{L^\infty(\lambda)} = \{0\}$ .  $\square$

Note that if  $X = L^\infty(\lambda)$  as sets (with  $\lambda$  finite), even if the norms are only equivalent, we always have  $X^{L^\infty(\lambda)} \equiv L^\infty(\lambda)$ . This is due to the following:  $L^\infty(\lambda)$  is  $\infty$ -convex with convexity constant equal to 1, so by Lemma 5.1 we have that  $X^{L^\infty(\lambda)}$  is  $\infty$ -convex with constant 1. Then, since every B.f.s. is  $\infty$ -concave with constant 1, from Remark 2 after Proposition 2.b.3 in [9] we have that  $X^{L^\infty(\lambda)} \equiv L^\infty(\lambda)$ . Consequently, a non-trivial r.i. B.f.s.  $X$  satisfies that  $X^{L^\infty(\lambda)L^\infty(\lambda)} = X$  if and only if  $X = L^\infty(\lambda)$  isomorphically.

Also, in the context of the r.i. B.f.s.' related to a finite measure, Section 4 concerning  $p$ -powers is complemented by the following result. Note that the  $p$ -power of an r.i. B.f.s. is always r.i.

**Proposition 6.3.** *Let  $X$  be a non-trivial r.i. B.f.s. and suppose that  $\lambda$  is finite. If  $X \neq L^\infty(\lambda)$  then  $(X^p)^{X^q} = \{0\}$  whenever  $1 \leq p < q < \infty$ .*

**Proof.** Given  $1 \leq p < q < \infty$  we can consider  $r \in (1, \infty)$  such that  $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ . Since  $\chi_I \in L^\infty(\lambda) \subset X^r$ , by [11, Lemma 1] it follows that  $f = f\chi_I \in X^p$  for every  $f \in X^q$ . That is,  $X^q \subset X^p$ .

Suppose that  $(X^p)^{X^q} \neq \{0\}$ . From Lemma 6.1 and the above comment we have that  $X^q = X^p$  and so  $X^{q/p} = X$ . Let  $n \in \mathbb{N}$  be such that  $2 < (\frac{q}{p})^n$ . Then,  $X^{(q/p)^n} \subset X^2$  and  $X^{(q/p)^n} = (X^{q/p})^{(q/p)^{n-1}} = X^{(q/p)^{n-1}} = \dots = X$ . Since  $X^2 \subset X$ , it follows that  $X = X^2$ . Then, using (8) we obtain that

$$L^\infty(\lambda) \subset X \subset [X] = [X^2] = (X^2)^X = X^X = L^\infty(\lambda),$$

which contradicts  $X \neq L^\infty(\lambda)$ .  $\square$

Let us show now some particular cases of generalized perfect spaces involving classical r.i. B.f.s.'.

### 6.1. Lorentz and Marcinkiewicz spaces

Let  $\varphi: I \rightarrow [0, \infty)$  be an increasing concave map vanishing only at zero. The Lorentz space related to  $\varphi$  is the r.i. B.f.s. defined as

$$\Lambda_\varphi := \left\{ f \in L^0(\lambda): \int f^*(t) d\varphi(t) < \infty \right\},$$

endowed with the norm  $\|f\|_{\Lambda_\varphi} := \int f^*(t) d\varphi(t)$ . The Marcinkiewicz space related to  $\varphi$  is the r.i. B.f.s. defined as

$$M_\varphi := \left\{ f \in L^0(\lambda): \sup_{0 < t \in I} \frac{1}{\varphi(t)} \int_0^t f^*(s) ds < \infty \right\},$$

endowed with the norm  $\|f\|_{M_\varphi} := \sup_{0 < t \in I} \frac{1}{\varphi(t)} \int_0^t f^*(s) ds$ .

Every r.i. B.f.s.  $X$  is an intermediate space between a Lorentz and a Marcinkiewicz space. Namely,

$$\Lambda_{\varphi_X} \hookrightarrow_1 X \hookrightarrow_1 M_{t/\varphi_X},$$

where  $\varphi_X$  is the *fundamental function* of  $X$  (i.e.  $\varphi_X(t) := \|\chi_{[0,t]}\|_X$  for  $t \in I$ , which can be assumed to be concave). The spaces  $\Lambda_\varphi$  and  $M_\varphi$  have fundamental function  $\varphi$  and  $t/\varphi$  respectively.

An r.i. B.f.s.  $X$  has the *majorant property* if whenever  $g \in L^0(\lambda)$ ,  $f \in X$  and  $\int_0^t g^*(s) ds \leq \int_0^t f^*(s) ds$  for all  $0 < t \in I$ , it follows that  $g \in X$  and  $\|g\|_X \leq \|f\|_X$ . For instance, this property is satisfied if  $X$  is separable (which is equivalent to being order continuous in the setting of r.i. B.f.s.) or has the Fatou property. The Fatou property holds for  $\Lambda_\varphi$  and  $M_\varphi$  and so does the majorant property.

**Proposition 6.4.** *Let  $X$  be an r.i. B.f.s. such that  $(M_\varphi)_F \subseteq X$ . Then,  $M_\varphi^{XX} = M_\varphi$  with equivalent norms if and only if there exists  $k > 0$  satisfying*

$$(14) \quad \frac{t}{\varphi(t)} \leq k \varphi_{M_\varphi^{XX}}(t), \quad \text{for all } 0 < t \in I.$$

Moreover,  $M_\varphi$  is  $X$ -perfect if and only if (14) holds for  $k = 1$ .

**Proof.** Suppose that  $M_\varphi^{XX} = M_\varphi$ . Then, the fundamental functions of both spaces are equivalent and in particular, (14) holds. Note that if  $M_\varphi^{XX} \equiv M_\varphi$ , then (14) holds for  $k = 1$ .

Conversely, suppose that (14) holds. Then,

$$M_\varphi^{XX} \hookrightarrow_1 M_{t/\varphi_{M_\varphi^{XX}}} \hookrightarrow_k M_\varphi.$$

Since we always have  $M_\varphi \hookrightarrow_1 M_\varphi^{XX}$ , it follows that  $M_\varphi^{XX} = M_\varphi$  with equivalent norms. If  $k = 1$  then,  $M_\varphi^{XX} \equiv M_\varphi$ .  $\square$

Note that, under conditions of Proposition 6.4, if  $X$  has the majorant property,  $\varphi(0^+) = 0$ ,  $\varphi(\infty) = \infty$  (in the case of  $a = \infty$ ) and  $k > 0$  satisfies

$$(15) \quad \|\varphi' \chi_{[0,t]}\|_X \leq k \varphi_X(t) \frac{\varphi(t)}{t}, \quad \text{for all } 0 < t \in I,$$

then (14) holds for  $k$ . Indeed, since  $\chi_{[0,t]} \in M_\varphi^X$  and  $\|\chi_{[0,t]}\|_{M_\varphi^X} = \|\varphi' \chi_{[0,t]}\|_X$ , where  $\varphi'$  denotes the derivative function of  $\varphi$  (see [11, Theorem 3]), then

$$\varphi_{M_\varphi^{XX}}(t) = \sup_{h \in B_{M_\varphi^X}} \|h \chi_{[0,t]}\|_X \geq \frac{\|\chi_{[0,t]}\|_X}{\|\varphi' \chi_{[0,t]}\|_X}.$$

**Example 6.5.** Let  $\varphi, \psi : I \rightarrow [0, \infty)$  be increasing concave maps vanishing only at zero with  $\varphi(0^+) = 0$ ,  $\varphi(\infty) = \infty$  (if  $a = \infty$ ) and satisfying that  $\frac{\varphi}{\psi}$  is increasing.



Then, every  $f \in M_\varphi$  with  $\alpha := \lambda(\text{Supp } f) < \infty$  belongs to  $M_\psi$  with  $\|f\|_{M_\psi} \leq \frac{\varphi(\alpha)}{\psi(\alpha)} \|f\|_{M_\varphi}$  for  $\alpha > 0$ . That is,  $(M_\varphi)_F \subseteq M_\psi$ . Moreover,

$$\begin{aligned} \|\varphi' \chi_{[0,t]}\|_{M_\psi} &= \sup_{0 < u \in I} \frac{1}{\psi(u)} \int_0^u \varphi'(s) \chi_{[0,t]}(s) ds \\ &= \frac{\varphi(t)}{\psi(t)} = \varphi_{M_\psi}(t) \frac{\varphi(t)}{t}, \end{aligned}$$

for every  $0 < t \in I$ . That is, (15) holds for  $X = M_\psi$  and  $k = 1$ . Since  $M_\psi$  has the majorant property, then (14) holds for  $X = M_\psi$  and  $k = 1$ . Therefore, from Proposition 6.4, it follows that  $M_\varphi$  is  $M_\psi$ -perfect.

Functions satisfying the above required conditions are for instance  $\varphi(t) = t^{1/p}$  and  $\psi(t) = t^{1/q}$  with  $1 \leq p \leq q \leq \infty$ . Note that in this case,  $M_\varphi$  and  $M_\psi$  are just the Lorentz spaces  $L^{\hat{p},\infty}(\lambda)$  and  $L^{\hat{q},\infty}(\lambda)$  respectively, with  $\frac{1}{p} + \frac{1}{\hat{p}} = \frac{1}{q} + \frac{1}{\hat{q}} = 1$  (see [2, Theorem IV.4.6]).

**Example 6.6.** Let  $\varphi, \psi : I \rightarrow [0, \infty)$  be increasing concave maps vanishing only at zero with  $\varphi(0^+) = 0$ ,  $\varphi(\infty) = \infty$  (if  $a = \infty$ ) and satisfying:

- (i)  $\int_0^b \frac{\varphi(s)}{s} d\psi(s) < \infty$  for all (some)  $0 < b < a$ .
- (ii) There exists  $k > 0$  such that  $\int_0^t \varphi'(s) d\psi(s) \leq k \frac{\psi(t)\varphi(t)}{t}$  for all  $0 < t \in I$ .

Condition (i) implies that  $(M_\varphi)_F \subseteq \Lambda_\psi$ . Indeed, given  $f \in M_\varphi$  with  $\alpha := \lambda(\text{Supp } f) < \infty$  we have that

$$\begin{aligned} \int f^*(s) d\psi(s) &= \int_0^\alpha f^*(s) d\psi(s) \leq \int_0^\alpha \frac{1}{s} \int_0^s f^*(u) du d\psi(s) \\ &\leq \|f\|_{M_\varphi} \int_0^\alpha \frac{\varphi(s)}{s} d\psi(s). \end{aligned}$$

Note that, since  $\frac{\varphi}{s}$  is decreasing, if (i) holds for some  $b < \infty$  then it holds for all  $b < \infty$ . Condition (ii) is just (15) for  $X = \Lambda_\psi$  which has the majorant property, so (14) holds for  $X = \Lambda_\psi$ . Then, from Proposition 6.4 we have that  $M_\varphi^{\Lambda_\psi, \Lambda_\psi} = M_\varphi$  with equivalent norms. It can be proved that (ii) holds for  $k = 1$  only in the case  $\varphi(t) = t$ , for which  $M_\varphi \equiv L^\infty(\lambda)$  and so  $M_\varphi$  is obviously  $\Lambda_\psi$ -perfect.

The functions  $\varphi(t) = t^{1/p}$  and  $\psi(t) = t^{1/q}$  with  $1 < p < \infty$  and  $1 \leq q < \frac{p}{p-1}$ , satisfy the conditions (i) and (ii) with constant  $k = (q + p - qp)^{-1}$ . In this case,  $\Lambda_\psi$  is just the Lorentz space  $L^{q,1}(\lambda)$  (see [2, Theorem IV.4.3]).

## 6.2. Orlicz spaces

Let  $\Phi$  be a *Young function*, i.e.  $\Phi: [0, \infty) \rightarrow [0, \infty)$  is continuous, convex, increasing,  $\Phi(0) = 0$  and  $\lim_{t \rightarrow \infty} \Phi(t) = \infty$ . The *Orlicz space* related to  $\Phi$  is the r.i. B.f.s. defined as

$$L^\Phi := \left\{ f \in L^0(\lambda): \int \Phi\left(\frac{|f|}{c}\right) d\lambda < \infty \text{ for some } c > 0 \right\},$$

endowed with the *Luxemburg norm*

$$\|f\|_{L^\Phi} := \inf \left\{ c > 0: \int \Phi\left(\frac{|f|}{c}\right) d\lambda \leq 1 \right\}.$$

The following result about generalized duality of Orlicz spaces is a little variation of [11, Theorem 4] (the constant  $\frac{1}{2}$  in (i)) and the proof is almost the same, so we omit it. Note that the reason of this variation is for the representation obtained to be isometric.

**Theorem 6.7.** *Let  $\Phi, \Phi_0, \Phi_1$  be Young functions satisfying:*

- (i)  $\Phi(st) \leq \frac{1}{2}(\Phi_0(s) + \Phi_1(t))$ , for all  $s, t \geq 0$ ,
- (ii)  $\Phi^{-1}(t) \leq \Phi_0^{-1}(t)\Phi_1^{-1}(t)$ , for all  $t \geq 0$ .

*Then,  $(L^{\Phi_0})^{L^\Phi} \equiv L^{\Phi_1}$ .*

The hypothesis of Theorem 6.7 are satisfied for instance if  $\Phi, \Phi_0$  are Young functions such that  $\frac{\Phi_0}{\Phi}$  is increasing and, for any  $s > 0$ ,

$$(16) \quad \limsup_{t \rightarrow \infty} \frac{\Phi(st)}{\Phi_0(t)} = \limsup_{t \rightarrow 0} \frac{\Phi_0(t)}{\Phi(st)} = 0,$$

by taking  $\Phi_1(t) := \sup_{s>0} (2\Phi(st) - \Phi_0(s))$  for all  $t \geq 0$ . Again, the reason of this fact can be found in [11, Example 2], noting that in the notation there, our  $\Phi_1$  is just  $(2\Phi) \ominus \Phi_0$ .

**Remark 6.8.** Let  $\Phi$  be a Young function such that  $\frac{\Phi(t)}{t}$  is increasing and

$$(17) \quad \limsup_{t \rightarrow \infty} \frac{t}{\Phi(t)} = \limsup_{t \rightarrow 0} \frac{\Phi(t)}{t} = 0.$$

Then, taking  $\hat{\Phi}(t) = \sup_{s>0} (2st - \Phi(s))$  for all  $t \geq 0$ , we obtain that  $(L^\Phi)' \equiv L^{\hat{\Phi}}$ . Note that this improves the representation obtained for  $(L^\Phi)'$  via the complementary Young function  $\Psi$  of  $\Phi$ .

The Young function  $t^p$  ( $1 < p < \infty$ ) whose Orlicz space is just the classical  $L^p$  (order isometrically), and the Young functions  $t \log^+(t)$  and  $\exp(t) - 1$  whose Orlicz

spaces, in the case of finite measure, are (order isomorphically) the Zygmund spaces  $L \log L$  and  $L_{\exp}$  respectively (see [2, Definition IV.6.1]), satisfy the conditions of Remark 6.8.

As a corollary of Theorem 6.7 we obtain that  $L^{\Phi_0}$  is  $L^{\Phi}$ -perfect.

**Corollary 6.9.** *Let  $\Phi, \Phi_0$  be Young functions satisfying the conditions (i) and (ii) of Theorem 6.7 for some other Young function  $\Phi_1$ . Then,*

$$(L^{\Phi_0})^{L^{\Phi}L^{\Phi}} \equiv L^{\Phi_0}.$$

**Proof.** Note that the Young functions  $\Phi_0, \Phi_1$  play the same role in the hypothesis of Theorem 6.7. Hence, it follows that  $(L^{\Phi_0})^{L^{\Phi}} \equiv L^{\Phi_1}$  and  $(L^{\Phi_1})^{L^{\Phi}} \equiv L^{\Phi_0}$ . So,  $(L^{\Phi_0})^{L^{\Phi}L^{\Phi}} \equiv (L^{\Phi_1})^{L^{\Phi}} \equiv L^{\Phi_0}$ .  $\square$

Note that under the assumptions of Corollary 6.9,  $L^{\Phi_1}$  is also  $L^{\Phi}$ -perfect, since it can be interchanged with  $L^{\Phi_0}$ .

Finally let us show an example of functions satisfying the hypothesis of Corollary 6.9. Consider  $\phi(t) := t^p \log^{-\beta}(2+t)$  and  $\phi_0(t) := \exp(t^\alpha) - 1$  which are Young functions for  $2 < p < \infty$ ,  $0 < \beta \leq p - 2$  and  $1 \leq \alpha < \infty$ . In the case when  $p < \alpha$ , the function  $\phi_0/\phi$  is increasing and (16) is satisfied. So,  $L^{\phi_0}$  is  $L^{\phi}$ -perfect. Note that if the measure is finite, for these Young functions,  $L^{\phi}$  and  $L^{\phi_0}$  are order isomorphic to the Zygmund spaces  $L^p(\log L)^{-\beta/p}$  and  $L_{\exp}^{1/\alpha}$  respectively (see [2, Definition IV.6.11]).

**Note.** Just before submission for publication of this paper we became aware of the preprint by A.R. Schep [15] in which some of our results concerning  $p$ -convexity and  $p$ -concavity, have been independently obtained in the setting of products of Banach function spaces.

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