# Factorizing operators on Banach function spaces through spaces of multiplication operators

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A B S T R A C T

In order to extend the theory of optimal domains for continuous operators on a Banach function space $X(\mu)$ over a finite measure $\mu$, we consider operators $T$ satisfying other type of inequalities than the one given by the continuity which occur in several well-known factorization theorems (for instance, Pisier Factorization Theorem through Lorentz spaces, $p$th-power factorable operators...). We prove that such a $T$ factorizes through a space of multiplication operators which can be understood in a certain sense as the optimal domain for $T$. Our extended optimal domain technique does not need necessarily the equivalence between $\mu$ and the measure defined by the operator $T$ and, by using $\delta$-rings, $\mu$ is allowed to be infinite. Classical and new examples and applications of our results are also given, including some new results on the Hardy operator and a factorization theorem through Hilbert spaces.

## 1. Introduction

Let $(\Omega, \Sigma)$ be a measurable space, $X(\mu)$ a Banach function space related to a positive measure $\mu$ on $\Sigma$ and $T : X(\mu) \to E$ a linear operator with values in a Banach space $E$. Supposing that $T$ satisfies a certain property, natural questions arise: Can $T$ be extended to a larger domain in a way that the extension (still with values in $E$) preserves the same property? And, for a positive answer, which is the largest domain in this sense?

These questions have been solved for the continuity of $T$ in the case when $\mu$ is finite and satisfies a compatibility property with $T$ and $X(\mu)$ is order continuous and contains the simple functions. Namely, in this case, we can consider the vector measure $m_T : \Sigma \to E$ defined by $T$ as $m_T(A) = T(\chi_A)$ and then the space $L^1(m_T)$ of integrable functions with respect to $m_T$ is the largest within all the order continuous Banach function spaces related to $\mu$ to which $T$ can be extended as a continuous operator still with values in $E$, see [4, Corollary 3.3]. Note that the continuity property of $T$, i.e. there exists $K > 0$ such that $\|T(f)\|_E \leqslant K\|f\|_{X(\mu)}$ for all $f \in X(\mu)$, can be rewritten as

$$\big\|T(f\varphi)\big\|_E \leqslant K\|f\|_{X(\mu)}\|\varphi\|_{L^\infty(\mu)} \tag{1}$$

for all $f \in X(\mu)$ and $\varphi$ simple function. Then, $L^1(m_T)$ is the largest domain for $T$ satisfying the inequality (1).

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In this paper we consider weaker conditions on $X(\mu)$ (e.g. it does not need to be order continuous or contain the simple functions and $\mu$ does not need to be finite) and study the questions above when $T$ is what we call $Y(\eta)$-extensible, i.e. satisfies an inequality of the type

$$\left\|T(f\varphi)\right\|_E \leqslant K\|f\|_{X(\mu)}\|\varphi\|_{Y(\eta)} \tag{2}$$

for all $f \in X(\mu)$ and $\varphi$ simple function belonging to a Banach function space $Y(\eta)$ related to another positive measure $\eta$ on $\Sigma$. Of course, this type of domination for $T$ includes the continuity given by (1). Other relevant examples of this kind of property are given by the Pisier's factorization theorem through weighted Lorentz spaces [10,19], the $L^p$-product extensible operators [3] and the $p$th power factorable operators [18, §5], all of these are explained in Section 6.

Under minimal conditions on $X(\mu)$ and $T$ which guarantee that $m_T$ is a vector measure defined on certain $\delta$-ring (see Section 3), we prove in Section 5 that the largest domain for $T$ (within all the Banach function spaces related to some measure for which every $\mu$-null set is null) satisfying the inequality (2) can be represented as a subspace of $L^1(m_T)$ which is given by a space of multiplication operators from $Y_b(\eta)$ to $L^1(m_T)$, where $Y_b(\eta)$ is the closure of the intersection of the simple functions with $Y(\eta)$. This subspace is studied in Section 4. Actually, from a technical point of view, what we obtain is not a real optimal extension of $T$ (this is only the case when $m_T$ and $\mu$ have the same null sets) but an optimal factorization of $T$. This extra complication allows to consider special cases as the Pisier's factorization theorem.

Therefore, the main contributions of the present paper are the following. The vector measure technique for determining the optimal domain of an operator has been used recently in a lot of papers (see for instance [4,5,8,9,18]); these applications usually consider the case of B.f.s. (with weak unit) defined over finite measure spaces. Also, the considered extension that is considered is the one that preserves the continuity of the operator. In this paper we generalize this method in two different directions.

(1) We analyze the optimal domain for operators defined on Banach function spaces over measure spaces defined on $\delta$-rings instead of $\sigma$-algebras. This allows us to consider also the cases of $\sigma$-finite measures and even non-$\sigma$-finite ones. We illustrate by means of examples and applications how our extension is meaningful (Section 3).

(2) We find the optimal domain for an operator when a different property stronger than continuity is considered. We show that dominations of operators given by norm inequalities involving norms of other spaces can also be extended to a different domain preserving the same domination, and this extension is optimal. For doing this, we consider spaces of multiplication operators (Sections 4–6).

All along in the paper, we will consider two examples to illustrate the results obtained. The first one is the Hardy operator $S: L^1 \cap L^\infty \to \Lambda_\psi$ given by $S(f)(x) = \frac{1}{x}\int_0^x f(y)\,dy$ for all $x > 0$, where $\Lambda_\psi$ is a classical Lorentz space. We refer the reader to [9] for information about this operator not explained here. The second one, related to the case $\mu$ non-$\sigma$-finite, is a certain kind of kernel operator from $\ell^1(I)$ into $\ell^p(I)$ with $I$ being a non-countable set.

## 2. Preliminaries

### *2.1. Banach function spaces*

Let $(\Omega, \Sigma)$ be a fixed measurable space. For a measure $\mu : \Sigma \to [0, \infty]$, we denote by $L^0(\mu)$ the space of all $\Sigma$-measurable real valued functions on $\Omega$, where functions which are equal $\mu$-a.e. are identified. By a *Banach function space* (briefly, B.f.s.) $X(\mu)$ we mean a Banach space contained in $L^0(\mu)$ with norm $\|\cdot\|_{X(\mu)}$ satisfying that if $f \in X(\mu)$ and $g \in L^0(\mu)$ with $|g| \leqslant |f|$ $\mu$-a.e. then $g \in X(\mu)$ and $\|g\|_{X(\mu)} \leqslant \|f\|_{X(\mu)}$. Note that $X(\mu)$ is a Banach lattice for the $\mu$-a.e. pointwise order, in which the convergence in norm of a sequence implies the convergence $\mu$-a.e. for some subsequence. A function $g \in X(\mu)$ is a *weak unit* if $g > 0$ $\mu$-a.e. If $X(\mu)$ has a weak unit then $X(\mu)$ is *saturated*, i.e. there exists no $A \in \Sigma$ with $\mu(A) > 0$ such that $f\chi_A = 0$ $\mu$-a.e. for all $f \in X(\mu)$, or equivalently, for every $A \in \Sigma$ with $\mu(A) > 0$ there exists $B \in \Sigma \cap 2^A$ with $\mu(B) > 0$ such that $\chi_B \in X(\mu)$. In the case when $\mu$ is $\sigma$-finite, $X(\mu)$ is saturated if and only if it has a weak unit. A B.f.s. is *order continuous* if order bounded increasing sequences are convergent in norm. Denote by $\mathcal{S}(\Sigma)$ the space of the *simple functions*, i.e. $\varphi = \sum_1^n \alpha_j \chi_{A_j}$ with $\alpha_j \in \mathbb{R}$ and $A_j \in \Sigma$. In the following result, parts (a) and (c) are known for the case $\mu$ $\sigma$-finite (see [1, Theorems I.3.11 and I.3.13]). For a general $\mu$, the arguments are a bit different, so we write the proof for the aim of completeness.

**Lemma 1.** *Let $X(\mu)$ be a non-trivial B.f.s. and $X_b(\mu)$ denote the closure of $\mathcal{S}(\Sigma) \cap X(\mu)$ in $X(\mu)$. The following claims hold*:

(a) *$X_b(\mu)$ is a non-trivial B.f.s.*
(b) *If $X(\mu)$ has a weak unit then $X_b(\mu)$ also has a weak unit.*
(c) *If $X(\mu)$ is order continuous then $X_b(\mu) = X(\mu)$.*

**Proof.** (a) Since $X(\mu)$ is a non-trivial B.f.s., there exists $A \in \Sigma$ with $\mu(A) > 0$ such that $\chi_A \in X(\mu)$. So, $X_b(\mu)$ is non-trivial. Obviously, $X_b(\mu)$ is a Banach space with norm $\|\cdot\|_{X(\mu)}$ contained in $L^0(\mu)$. To establish the lattice property, let us prove first that $X_b(\mu)$ is the closure in the space $X(\mu)$ of the ideal $\{f \in X(\mu) \cap L^\infty(\mu):\ \chi_{\mathrm{Supp}(f)} \in X(\mu)\}$. Since $\mathcal{S}(\Sigma) \cap X(\mu)$

is contained in this ideal, we only have to see that given $f \in X(\mu) \cap L^\infty(\mu)$ with $\chi_{\mathrm{Supp}(f)} \in X(\mu)$, there exists $(\varphi_n) \subset \mathcal{S}(\Sigma) \cap X(\mu)$ such that $\varphi_n \to f$ in $X(\mu)$. Take $(\psi_n) \subset \mathcal{S}(\Sigma)$ such that $\psi_n \to f$ in $L^\infty(\mu)$. Then

$$\|f - \psi_n \chi_{\mathrm{Supp}\, f}\|_{X(\mu)} \leqslant \|f - \psi_n\|_\infty \|\chi_{\mathrm{Supp}\, f}\|_{X(\mu)} \to 0$$

as $n \to \infty$ and $\psi_n \chi_{\mathrm{Supp}\, f} \in \mathcal{S}(\Sigma) \cap X(\mu)$ as $|\psi_n| \chi_{\mathrm{Supp}\, f} \leqslant \|\psi_n\|_\infty \chi_{\mathrm{Supp}\, f} \in X(\mu)$.

Let $f \in X_b(\mu)$ and $g \in L^0(\mu)$ be such that $|g| \leqslant |f|$ $\mu$-a.e. In particular, $f$ belongs to the B.f.s. $X(\mu)$ and so $g \in X(\mu)$ and $\|g\|_{X(\mu)} \leqslant \|f\|_{X(\mu)}$. Take $(f_n) \in \mathcal{S}(\Sigma) \cap X(\mu)$ such that $f_n \to f$ in $X(\mu)$ and consider the functions defined by $g_n(x) = \mathrm{sgn}(g(x)) \cdot \min\{|f_n(x)|, |g(x)|\}$. It follows that $g_n \in L^\infty(\mu) \cap X(\mu)$ and $g_n \to g$ in $X(\mu)$ (see the proof of [1, Theorem I.3.11]). Moreover, $\chi_{\mathrm{Supp}\, g_n} \in X(\mu)$, since $\mathrm{Supp}\, g_n \subset \mathrm{Supp}\, f_n$ and $\chi_{\mathrm{Supp}\, f_n} \in X(\mu)$ as $f_n \in \mathcal{S}(\Sigma) \cap X(\mu)$. So, $g \in X_b(\mu)$.

(b) Only note that a B.f.s. $X(\mu)$ has a weak unit if and only if there exists a sequence $(A_n) \subset \Sigma$ such that $\chi_{A_n} \in X(\mu)$ and $\Omega = \bigcup_n A_n$.

(c) Suppose that $X(\mu)$ is order continuous. If $0 \leqslant f \in X(\mu)$, taking $(\varphi_n) \subset \mathcal{S}(\Sigma)$ such that $0 \leqslant \varphi_n \uparrow f$ pointwise, we have that $(\varphi_n) \subset X(\mu)$ and $\varphi_n \to f$ in $X(\mu)$, that is, $f \in X_b(\mu)$. For a general $f \in X(\mu)$, considering the positive and negative parts of $f$, we obtain that $f \in X_b(\mu)$. $\square$

Given two set functions $\mu, \lambda : \Sigma \to [0, \infty]$, we will write $\lambda \ll \mu$ if $\lambda(A) = 0$ whenever $A \in \Sigma$ with $\mu(A) = 0$ (i.e. every $\mu$-null set is $\lambda$-null). If $\lambda \ll \mu$ and $\mu \ll \lambda$ we will say that $\mu$ and $\lambda$ are *equivalent* (i.e. they have the same null sets).

Consider two measures $\mu, \lambda : \Sigma \to [0, \infty]$ such $\lambda \ll \mu$. The map $[i] : L^0(\mu) \to L^0(\lambda)$ which takes a $\mu$-a.e. class in $L^0(\mu)$ represented by $f$ into the $\lambda$-a.e. class represented by the same $f$, is a well-defined linear map. In order to simplify notation $[i](f)$ will be denoted again as $f$. Of course, in the case when $\lambda$ and $\mu$ are equivalent we have that $L^0(\mu) = L^0(\lambda)$ and $[i]$ is the identity map. Note that if $X(\mu)$ and $Y(\lambda)$ are B.f.s.' such that $[i] : X(\mu) \to Y(\lambda)$ is well defined then it is automatically continuous, since it is a positive map between Banach lattices (see [14, p. 2]). Given $h \in L^0(\lambda)$, we can consider the linear map $\mathcal{P}_h : L^0(\mu) \to L^0(\lambda)$ defined as $\mathcal{P}_h(f) = fh$ (formally $[i](f) \cdot h$). If $X(\mu)$ and $Y(\lambda)$ are B.f.s.' such that $\mathcal{P}_h : X(\mu) \to Y(\lambda)$ is well defined then it is automatically continuous, since we can write $\mathcal{P}_h = \mathcal{P}_{h^+} - \mathcal{P}_{h^-}$, where $h^+$ and $h^-$ are the positive and negative parts of $h$ respectively, and $\mathcal{P}_{h^+}$, $\mathcal{P}_{h^-}$ are continuous as they are positive operators between Banach lattices. Denote by $M(X(\mu), Y(\lambda))$ the space of all $h \in L^0(\lambda)$ such that $\mathcal{P}_h : X(\mu) \to Y(\lambda)$ is well defined, i.e.

$$M\big(X(\mu), Y(\lambda)\big) = \big\{h \in L^0(\lambda):\ fh \in Y(\lambda) \text{ for all } f \in X(\mu)\big\},$$

which can be endowed with the natural seminorm

$$\|h\|_{M(X(\mu), Y(\lambda))} = \sup_{f \in B_{X(\mu)}} \|fh\|_{Y(\lambda)}$$

where $B_{X(\mu)}$ is the open unit ball of $X(\mu)$ (i.e. the usual operator norm of $\mathcal{P}_h$). It can be checked that this seminorm is a norm if and only if $X(\mu)$ satisfies a variation of the saturation condition, namely, there exists no $A \in \Sigma$ with $\lambda(A) > 0$ such that $f\chi_A = 0$ $\lambda$-a.e. for all $f \in X(\mu)$, or equivalently, for every $A \in \Sigma$ with $\lambda(A) > 0$ there exists $B \in \Sigma \cap 2^A$ with $\lambda(B) > 0$ such that $\chi_B \in X(\mu)$. If $X(\mu)$ has a weak unit, this condition holds. Moreover, in this case, $M(X(\mu), Y(\lambda))$ is complete and so it is a B.f.s. This fact is proved in [15, Proposition 2] in the case when $\mu$ and $\lambda$ are equivalent and $\sigma$-finite. For the general case, the proof is similar.

### *2.2. Integration with respect to vector measures defined on a $\delta$-ring*

The classical theory of integration with respect to vector measures defined on a $\sigma$-algebra (see for instance [18, Chapter 3]) was extended to the case of vector measures defined on a $\delta$-ring by Lewis [13] and Masani and Niemi [16,17].

Let $\mathcal{R}$ be a *$\delta$-ring* of subsets of $\Omega$ (i.e. a ring closed under countable intersections) and $\mathcal{R}^{\mathrm{loc}}$ the $\sigma$-algebra of all subsets $A$ of $\Omega$ such that $A \cap B \in \mathcal{R}$ for all $B \in \mathcal{R}$. Note that if $\mathcal{R}$ is a $\sigma$-algebra then $\mathcal{R}^{\mathrm{loc}} = \mathcal{R}$. Denote by $\mathcal{S}(\mathcal{R})$ the space of all *$\mathcal{R}$-simple functions* (i.e. simple functions supported in $\mathcal{R}$).

Given a real measure $\lambda : \mathcal{R} \to \mathbb{R}$, that is $\sum \lambda(A_n)$ converges to $\lambda(\bigcup A_n)$ for every pairwise disjoint sequence $(A_n) \subset \mathcal{R}$ with $\bigcup A_n \in \mathcal{R}$, the *variation* of $\lambda$ is the measure $|\lambda| : \mathcal{R}^{\mathrm{loc}} \to [0, \infty]$ given by

$$|\lambda|(A) = \sup\Big\{\sum \big|\lambda(A_i)\big|:\ (A_i) \text{ finite disjoint sequence in } \mathcal{R} \cap 2^A\Big\}.$$

The space $L^1(\lambda)$ of *integrable functions with respect to* $\lambda$ is defined as the space $L^1(|\lambda|)$ with the usual norm $|f|_\lambda = \int_\Omega |f|\, d|\lambda|$. Note that $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\lambda)$. The integral of an $\mathcal{R}$-simple function $\varphi = \sum_{i=1}^n a_i \chi_{A_i}$ over $A \in \mathcal{R}^{\mathrm{loc}}$ is defined in the natural way by $\int_A \varphi\, d\lambda = \sum_{i=1}^n a_i \lambda(A_i \cap A)$. For $f \in L^1(\lambda)$ the integral over $A \in \mathcal{R}^{\mathrm{loc}}$ is defined by $\int_A f\, d\lambda = \lim \int_A \varphi_n\, d\lambda$, where $(\varphi_n)$ is a sequence in $\mathcal{S}(\mathcal{R})$ converging to $f$ in $L^1(\lambda)$.

Let $E$ be a real Banach space and $m : \mathcal{R} \to E$ a *vector measure*, that is $\sum m(A_n)$ converges to $m(\bigcup A_n)$ in $E$ for every pairwise disjoint sequence $(A_n) \subset \mathcal{R}$ with $\bigcup A_n \in \mathcal{R}$. Denote by $E^*$ the topological dual of $E$ and by $B_{E^*}$ its open unit ball. The *semivariation* of $m$ is the set function $\|m\| : \mathcal{R}^{\mathrm{loc}} \to [0, \infty]$ defined by

$$\|m\|(A) = \sup_{x^* \in B_{E^*}} \big|x^* m\big|(A),$$

where $|x^*m|$ is the variation of the real measure $x^*m:\mathcal{R}\to\mathbb{R}$ given by the composition of $m$ with $x^*$. Note that $\|m\|$ is finite on $\mathcal{R}$. A set $A\in\mathcal{R}^{\mathrm{loc}}$ is *$m$-null* if $\|m\|(A)=0$, or equivalently, $m(B)=0$ whenever $B\in\mathcal{R}\cap 2^A$. A property holds *$m$-almost everywhere* (briefly, $m$-a.e.) if it holds except on an $m$-null set. From [2, Theorem 3.2], there always exists a measure $\lambda:\mathcal{R}\to[0,\infty]$ with the same null sets as $m$ (i.e. $\|m\|$ and $|\lambda|$ are equivalent). An $\mathcal{R}^{\mathrm{loc}}$-measurable real function $f$ is *integrable with respect to $m$* if

(i) $f\in L^1(|x^*m|)$ for every $x^*\in E^*$, and
(ii) for each $A\in\mathcal{R}^{\mathrm{loc}}$ there exists $x_A\in E$ such that

$$x^*(x_A)=\int_A f\,dx^*m,\quad \text{for every } x^*\in E^*.$$

The vector $x_A$ is unique and will be written as $\int_A f\,dm$. We denote by $L^1(m)$ the space of all integrable functions with respect to $m$ and by $L^1_w(m)$ the space of functions satisfying only condition (i). In both spaces, functions which are equal $m$-a.e. are identified. Taking a measure $\lambda$ equivalent to $m$, we have that $L^1(m)$ and $L^1_w(m)$ are B.f.s.' related to $(\Omega,\mathcal{R}^{\mathrm{loc}},|\lambda|)$ with the norm

$$\|f\|_m=\sup_{x^*\in B_{E^*}}\int_\Omega |f|\,d\left|x^*m\right|,\quad \text{for all } f\in L^1_w(m).$$

Of course, $L^1(m)$ is a closed subspace of $L^1_w(m)$. In the case when $E$ does not contain an isomorphic copy of $c_0$, we have that $L^1(m)=L^1_w(m)$. Moreover, $L^1(m)$ is order continuous and contains $\mathcal{S}(\mathcal{R})$ as a dense set. Note that $\|\cdot\|_m$ is defined for any $\mathcal{R}^{\mathrm{loc}}$-measurable function and has the *Fatou property*, that is, if $0\leqslant f_j\uparrow f$ $m$-a.e. then $\|f_j\|_m\uparrow\|f\|_m$. Actually, $L^1_w(m)$ can be described as the space of all $\mathcal{R}^{\mathrm{loc}}$-measurable functions $f$ with $\|f\|_m<\infty$. The norm of $f\in L^1(m)$ can also be computed by means of the formula

$$\|f\|_m=\sup\left\{\left\|\int_\Omega f\varphi\,dm\right\|_E:\ \varphi\in\mathcal{S}\left(\mathcal{R}^{\mathrm{loc}}\right)\cap B_{L^\infty(\lambda)}\right\}. \tag{3}$$

The *integration operator* $I_m:L^1(m)\to E$ defined by $I_m(f)=\int_\Omega f\,dm$ for all $f\in L^1(m)$, is a continuous linear operator. Given $f\in L^1(m)$, the *indefinite integral of $f$ with respect to $m$* is the vector measure $m_f:\mathcal{R}^{\mathrm{loc}}\to E$ defined by

$$m_f(A)=\int_A f\,dm,\quad \text{for all } A\in\mathcal{R}^{\mathrm{loc}}.$$

Note that $\|g\|_{m_f}=\|gf\|_m$ for every $\mathcal{R}^{\mathrm{loc}}$-measurable function $g$. Then, it follows that $g\in L^1_w(m_f)$ if and only if $gf\in L^1_w(m)$ and $g\in L^1(m_f)$ if and only if $gf\in L^1(m)$. Moreover, $\|m_f\|(A)=\|\chi_A\|_{m_f}=\|f\chi_A\|_m$ for every $A\in\mathcal{R}^{\mathrm{loc}}$. For further issues related to integration with respect to vector measures defined on a $\delta$-ring see [7].

Let us end this section by showing two results which will be used in this paper.

**Lemma 2.** *Let $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ be two $\delta$-rings of subsets of $\Omega$ such that $\widetilde{\mathcal{R}}\subset\mathcal{R}\subset\widetilde{\mathcal{R}}^{\mathrm{loc}}=\mathcal{R}^{\mathrm{loc}}$. If $m:\mathcal{R}\to E$ is a vector measure and $\widetilde{m}$ denotes the restriction of $m$ to $\widetilde{\mathcal{R}}$, then the following assertions hold*:

(a) *For every $\mathcal{R}^{\mathrm{loc}}$-measurable function $f$ we have that $\|f\|_{\widetilde{m}}\leqslant\|f\|_m$ and so $[i]:L^1_w(m)\to L^1_w(\widetilde{m})$ is well defined.*
(b) *If $m$ satisfies that*

$$A\in\mathcal{R},\ e^*\in E^*\quad\text{and}\quad\sup_{B\in\widetilde{\mathcal{R}}\cap 2^A}\left|e^*m(B)\right|=0\quad\Rightarrow\quad e^*m(A)=0 \tag{4}$$

*then $[i]:L^1(m)\to L^1(\widetilde{m})$ is well defined and $\int_\Omega f\,d\widetilde{m}=\int_\Omega f\,dm$ for every $f\in L^1(m)$.*

**Proof.** (a) Note that for every $e^*\in E^*$ and $A\in\widetilde{\mathcal{R}}^{\mathrm{loc}}=\mathcal{R}^{\mathrm{loc}}$ we have that $|e^*\widetilde{m}|(A)\leqslant|e^*m|(A)$. So, for every $\mathcal{R}^{\mathrm{loc}}$-measurable function $f$ and $e^*\in E^*$, it follows that $\int_\Omega|f|\,d|e^*\widetilde{m}|\leqslant\int_\Omega|f|\,d|e^*m|$ and thus $\|f\|_{\widetilde{m}}\leqslant\|f\|_m$. In particular, $\|\widetilde{m}\|(A)\leqslant\|m\|(A)$ for all $A\in\mathcal{R}^{\mathrm{loc}}$ (take $f=\chi_A$). Then, $\|\widetilde{m}\|\ll\|m\|$ and $[i]:L^1_w(m)\to L^1_w(\widetilde{m})$ is well defined.

(b) Take $A\in\mathcal{R}$. Since $\chi_A\in L^1(m)\subset L^1_w(m)$ and (a) holds, we have that $\chi_A\in L^1_w(\widetilde{m})$ and so condition (i) of the definition of integrable function with respect to $\widetilde{m}$ holds for $f=\chi_A$. Let us prove that (b) also holds. Let $B\in\widetilde{\mathcal{R}}^{\mathrm{loc}}=\mathcal{R}^{\mathrm{loc}}$ and $e^*\in E^*$. Since $|e^*\widetilde{m}|(A\cap B)=\sup\{|e^*\widetilde{m}|(H):\ H\in\widetilde{\mathcal{R}}\cap 2^{A\cap B}\}<\infty$, there exists an increasing sequence $(H_n)$ of sets in $\widetilde{\mathcal{R}}\cap 2^{A\cap B}$ satisfying that $|e^*\widetilde{m}|(A\cap B\backslash\cup H_n)=0$. Condition (4) implies that $|e^*m|(A\cap B\backslash\cup H_n)=0$. Thus, $\chi_{H_n}\uparrow\chi_{A\cap B}$ $e^*\widetilde{m}$-a.e. and $e^*m$-a.e., and so

$$\int_B \chi_A \, de^*\widetilde{m} = \lim_{n\to\infty} \int \chi_{H_n} \, de^*\widetilde{m} = \lim_{n\to\infty} e^*\widetilde{m}(H_n) = \lim_{n\to\infty} e^*m(H_n)$$

$$= \lim_{n\to\infty} \int \chi_{H_n} \, de^*m = \int_B \chi_A \, de^*m = e^*\left( \int_B \chi_A \, dm \right).$$

Hence, for every $A \in \mathcal{R}$ we have that $\chi_A \in L^1(\widetilde{m})$ and $\int_B \chi_A \, d\widetilde{m} = \int_B \chi_A \, dm$ for all $B \in \mathcal{R}^{\mathrm{loc}}$. Now, consider $f \in L^1(m) \subset L^1_w(m)$. By (a), $f \in L^1_w(\widetilde{m})$. Taking a sequence $(\varphi_n)$ of $\mathcal{R}$-simple functions converging to $f$ in $L^1(m)$, we have that $\|f - \varphi_n\|_{\widetilde{m}} \leqslant \|f - \varphi_n\|_m \to 0$. Then, since $\varphi_n \in L^1(\widetilde{m})$ and $L^1(\widetilde{m})$ is a closed subspace of $L^1_w(\widetilde{m})$, we have that $f \in L^1(\widetilde{m})$. Moreover, as the integration operator is continuous,

$$\int_\Omega f \, d\widetilde{m} = \lim_n \int_\Omega \varphi_n \, d\widetilde{m} = \lim_n \int_\Omega \varphi_n \, dm = \int_\Omega f \, dm. \qquad \square$$

If the vector measure $m$ considered in Lemma 2 satisfies certain $\sigma$-finiteness condition, the maps $[i]$ given in (a) and (b) are isometries.

**Lemma 3.** *Let $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ be two $\delta$-rings of subsets of $\Omega$ such that $\widetilde{\mathcal{R}} \subset \mathcal{R} \subset \widetilde{\mathcal{R}}^{\mathrm{loc}}$ and $\Omega = \bigcup_n A_n$ with $A_n \in \widetilde{\mathcal{R}}$. If $m : \mathcal{R} \to E$ is a vector measure and $\widetilde{m}$ denotes the restriction of $m$ to $\widetilde{\mathcal{R}}$, then $L^1_w(m) = L^1_w(\widetilde{m})$ with equal norms, $L^1(m) = L^1(\widetilde{m})$ and $\int_\Omega f \, dm = \int_\Omega f \, d\widetilde{m}$ for every $f \in L^1(m)$.*

**Proof.** The containments $\widetilde{\mathcal{R}} \subset \mathcal{R} \subset \widetilde{\mathcal{R}}^{\mathrm{loc}}$ imply that $\mathcal{R}^{\mathrm{loc}} \subset \widetilde{\mathcal{R}}^{\mathrm{loc}}$, see [16, Lemma A.3]. For every $A \in \widetilde{\mathcal{R}}^{\mathrm{loc}}$ we have that $A = \bigcup_n A \cap A_n \in \mathcal{R}^{\mathrm{loc}}$ since $A \cap A_n \in \widetilde{\mathcal{R}} \subset \mathcal{R}$. So, $\mathcal{R}^{\mathrm{loc}} = \widetilde{\mathcal{R}}^{\mathrm{loc}}$. Note that if $A \in \widetilde{\mathcal{R}}$ then $\mathcal{R} \cap 2^A = \widetilde{\mathcal{R}} \cap 2^A$. Indeed, if $B \in \mathcal{R}$ with $B \subset A$ then $B = B \cap A \in \widetilde{\mathcal{R}}$. The converse containment is clear. Then, it follows that $|x^*m|(A) = |x^*\widetilde{m}|(A)$ for all $A \in \widetilde{\mathcal{R}}$. Given $A \in \mathcal{R}^{\mathrm{loc}}$, since $A = \bigcup_n A \cap A_n$ where $A \cap A_n \in \widetilde{\mathcal{R}}$, noting that $(A_n)$ can be taken to be pairwise disjoint, we have that

$$\left|x^*m\right|(A) = \sum \left|x^*m\right|(A \cap A_n) = \sum \left|x^*\widetilde{m}\right|(A \cap A_n) = \left|x^*\widetilde{m}\right|(A)$$

for all $x^* \in E^*$. Hence, $\|f\|_{\widetilde{m}} = \|f\|_m$ for all $\mathcal{R}^{\mathrm{loc}}$-measurable function $f$ and thus $L^1_w(m) = L^1_w(\widetilde{m})$. Condition (4) in Lemma 2(b) holds. Indeed, if $A \in \mathcal{R}$, $e^* \in E^*$ and $\sup\{|e^*m(B)|: \ B \in \widetilde{\mathcal{R}} \cap 2^A\} = 0$, it follows that $e^*m(A) = \sum e^*m(A \cap A_n) = 0$. Then, $L^1(m) \subset L^1(\widetilde{m})$ and $\int_\Omega f \, d\widetilde{m} = \int_\Omega f \, dm$. Moreover, $L^1(\widetilde{m}) \subset L^1(m)$ as $\mathcal{S}(\widetilde{\mathcal{R}}) \subset \mathcal{S}(\mathcal{R})$ and $\| \cdot \|_{\widetilde{m}} = \| \cdot \|_m$. $\square$

## 3. Optimal domain for order-w continuous operators

Fix $(\Omega, \Sigma)$ a measurable space. Let $X(\mu)$ be a B.f.s. and consider the $\delta$-ring of subsets of $\Omega$ given by

$$\Sigma_{X(\mu)} = \left\{ A \in \Sigma : \ \chi_A \in X(\mu) \right\},$$

which satisfies $\Sigma \subset \Sigma_{X(\mu)}^{\mathrm{loc}}$. Throughout the paper we will assume $\Sigma = \Sigma_{X(\mu)}^{\mathrm{loc}}$. This condition holds for instance if $X(\mu)$ has a weak unit, which is equivalent to the existence of a sequence $(A_n) \subset \Sigma_{X(\mu)}$ such that $\Omega = \bigcup_n A_n$.

Given a linear operator $T : X(\mu) \to E$ with values in a Banach space $E$, we can consider the finitely additive set function $m_T : \Sigma_{X(\mu)} \to E$ given by $m_T(A) = T(\chi_A)$. Let us require $T$ to satisfy that there exists $A \in \Sigma_{X(\mu)}$ such that $T(\chi_A) \neq 0$ (in particular $\mu(A) > 0$), as in other case $m_T$ is null. Note that $m_T \ll \mu$, that is, $\|m_T\| \ll \mu$ or equivalently $|\lambda_T| \ll \mu$ for any measure $\lambda_T : \Sigma_{X(\mu)} \to [0, \infty]$ with the same null sets as $m_T$. So, the map $[i] : L^0(\mu) \to L^0(|\lambda_T|)$ is well defined.

We will say that $T$ is *order-w continuous* if $Tf_n \to Tf$ weakly in $E$ whenever $f_n, f \in X(\mu)$ with $0 \leqslant f_n \uparrow f$ $\mu$-a.e. This property holds for instance if $X(\mu)$ is order continuous and $T$ is continuous. Note that if $T$ is order-w continuous, then the condition $T(\chi_A) \neq 0$ for some $A \in \Sigma_{X(\mu)}$ is equivalent to $T$ being non-null.

If $T$ is order-w continuous, all the conditions required in [8, Proposition 2.3] are satisfied, so $m_T$ is a vector measure and for every $f \in X(\mu)$ we have that $f \in L^1(m_T)$ with $\int_\Omega f \, dm_T = Tf$. Therefore, $T$ factorizes as follows

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "{[i]}"'] & & E \\ & L^1(m_T) \arrow[ur, dotted, "I_{m_T}"'] & \end{tikzcd} \tag{5}$$

where $I_{m_T}$ is the integration operator with respect to $m_T$. As a consequence we have that every order-w continuous linear operator $T : X(\mu) \to E$ is continuous, since $T = I_{m_T} \circ [i]$ with $[i]$ and $I_{m_T}$ being continuous. Note that $I_{m_T} : L^1(m_T) \to E$ is order-w continuous, since $L^1(m_T)$ is order continuous and $I_{m_T}$ is continuous. Also note that if $A \in \Sigma$ with $\chi_A \in L^1(m_T)$ and $e^* \in E^*$ satisfy that $\sup\{|e^*I_{m_T}(\chi_B)|: \ B \in \Sigma_{X(\mu)} \cap 2^A\} = 0$, then $e^*I_{m_T}(\chi_A) = 0$. The following proposition shows that the factorization (5) is optimal. This outcome is an extended version of the optimal domain result given in [4, Corollary 3.3]. See also [8, Corollary 2.6].

**Proposition 4.** *Let $T : X(\mu) \to E$ be a non-null order-w continuous linear operator. Suppose that $Z(\zeta)$ is a B.f.s. such that $\zeta \ll \mu$ and $T$ factorizes as*

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "{[i]}"'] & & E \\ & Z(\zeta) \arrow[ur, dotted, "S"'] & \end{tikzcd} \tag{6}$$

*with $S$ being an order-w continuous linear operator satisfying that, for $A \in \Sigma$ with $\chi_A \in Z(\zeta)$ and $e^* \in E^*$,*

$$\sup_{B \in \Sigma_{X(\mu)} \cap 2^A} \left| e^* S(\chi_B) \right| = 0 \quad \Rightarrow \quad e^* S(\chi_A) = 0. \tag{7}$$

*Then, $[i] : Z(\zeta) \to L^1(m_T)$ is well defined and $S(f) = I_{m_T}(f)$ for all $f \in Z(\zeta)$.*

**Proof.** Consider the $\delta$-ring $\Sigma_{Z(\zeta)} = \{A \in \Sigma\colon\ \chi_A \in Z(\zeta)\}$. Note that $\Sigma = \Sigma^{\mathrm{loc}}_{Z(\zeta)}$. In other case, there exists $A \in \Sigma^{\mathrm{loc}}_{Z(\zeta)}$ such that $A \notin \Sigma = \Sigma^{\mathrm{loc}}_{X(\mu)}$. Then, there exists $B \in \Sigma_{X(\mu)}$ such that $A \cap B \notin \Sigma_{X(\mu)}$. Since $\chi_{A\cap B} \in X(\mu)$ (as $\chi_{A \cap B} \leqslant \chi_B \in X(\mu)$), it follows that $A \cap B \notin \Sigma$. In other hand, $B \in \Sigma_{Z(\zeta)}$ (as $\Sigma_{X(\mu)} \subset \Sigma_{Z(\zeta)}$), so $A \cap B \in \Sigma_{Z(\zeta)}$ and, in particular, $A \cap B \in \Sigma$ which is a contradiction. Since $S : Z(\zeta) \to E$ is a non-null order-w continuous linear operator, it factorizes as (5), i.e.

$$\begin{tikzcd} Z(\zeta) \arrow[rr, "S"] \arrow[dr, dotted, "{[i]}"'] & & E \\ & L^1(m_S) \arrow[ur, dotted, "I_{m_S}"'] & \end{tikzcd} \tag{8}$$

where $m_S : \Sigma_{Z(\zeta)} \to E$ is the vector measure given by $m_S(A) = S(\chi_A)$ and $I_{m_S}$ is the integration operator with respect to $m_S$. By (6) we have that $m_S(A) = S(\chi_A) = T(\chi_A) = m_T(A)$ for all $A \in \Sigma_{X(\mu)}$, that is, $m_T$ is the restriction of $m_S$ to $\Sigma_{X(\mu)}$. Condition (7) is just condition (4) in Lemma 2 for $m_S$, then $m_T \ll m_S$, $[i] : L^1(m_S) \to L^1(m_T)$ is well defined and $I_{m_S}(f) = I_{m_T}(f)$ for all $f \in L^1(m_S)$. So, by (8) we have that $[i] : Z(\zeta) \to L^1(m_T)$ is well defined and $S(f) = I_{m_T}(f)$ for all $f \in Z(\zeta)$. $\square$

Note that in the case when $X(\mu)$ has a weak unit, or equivalently, $\Omega = \bigcup_n A_n$ for some $(A_n) \subset \Sigma_{X(\mu)}$, condition (7) in Proposition 4 always holds for every order-w continuous linear operator $S$ satisfying (6).

**Example 5.** Denote by $\mathbb{R}^+$ the interval $[0, \infty)$, by $\mathcal{B}$ the $\sigma$-algebra of all Borel subsets of $\mathbb{R}^+$ and by $\lambda$ the Lebesgue measure on $\mathcal{B}$. Consider the *Hardy operator* $S$ defined on $L^1 \cap L^\infty(\lambda)$ as $S(f)(x) = \frac{1}{x}\int_0^x f(y)\,dy$. Note that $L^1 \cap L^\infty(\lambda)$ is a B.f.s. endowed with the usual norm $\|f\|_{L^1 \cap L^\infty(\lambda)} = \max\{\|f\|_{L^1(\lambda)}, \|f\|_{L^\infty(\lambda)}\}$ and has a weak unit. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing concave map with $\psi(0) = 0$, $\psi(0^+) = 0$, $\psi(\infty) = \infty$ and $\Lambda_\psi(\lambda) = \{f \in L^0(\lambda)\colon\ \|f\|_{\Lambda_\psi(\lambda)} = \int_0^\infty f^*(s)\,d\psi(s) < \infty\}$ the related Lorentz space ($f^*$ being the decreasing rearrangement of $f$), which is an order continuous B.f.s. endowed with the norm $\|f\|_{\Lambda_\psi(\lambda)}$. For issues related to Lorentz spaces see for instance [11, §II.5]. Assume that $\theta_\psi(t) = \int_t^\infty \frac{\psi'(s)}{s}\,ds < \infty$ for all $t > 0$, where $\psi'$ denotes the derivative of $\psi$. Then, it can be checked that $S : L^1 \cap L^\infty(\lambda) \to \Lambda_\psi(\lambda)$ is well defined and continuous with $\|S(f)\|_{\Lambda_\psi(\lambda)} \leqslant (\psi(a) + \theta_\psi(a))\|f\|_{L^1 \cap L^\infty(\lambda)}$ for any $a > 0$. Moreover, $S$ is order-w continuous, since if $0 \leqslant f_n \uparrow f \in L^1 \cap L^\infty(\lambda)$ $\lambda$-a.e., we have that $0 \leqslant S(f_n) \uparrow S(f) \in \Lambda_\psi(\lambda)$ pointwise (by the monotone convergence theorem) and so $S(f_n) \to S(f)$ in $\Lambda_\psi(\lambda)$ (by order continuity of the Lorentz space). Therefore, for the $\delta$-ring $\mathcal{B}_{L^1 \cap L^\infty(\lambda)} = \{A \in \mathcal{B}\colon\ \lambda(A) < \infty\}$, we have that $m_S : \mathcal{B}_{L^1 \cap L^\infty(\lambda)} \to \Lambda_\psi(\lambda)$, given by $m_S(A) = S(\chi_A)$, is a vector measure with $m_S \ll \lambda$ and $S$ optimally factorizes as

$$\begin{tikzcd} L^1 \cap L^\infty(\lambda) \arrow[rr, "S"] \arrow[dr, dotted, "{[i]}"'] & & \Lambda_\psi(\lambda) \\ & L^1(m_S) \arrow[ur, dotted, "I_{m_S}"'] & \end{tikzcd} \tag{9}$$

Note that in this case, $[i]$ is actually the inclusion map $i$, since $m_S$ and $\lambda$ are equivalent, and $I_{m_S}$ is just $S$, in fact $L^1(m_S) = \{f \in L^0(\lambda)\colon\ S(|f|) \in \Lambda_\psi(\lambda)\}$, see [9, Proposition 3.4] and the previous comments. Moreover, if there exists $C > 0$ such that $\frac{\psi(t)}{t} \leqslant C\theta_\psi(t)$ for all $t > 0$, from [9, Theorem 4.4], we can give a precise description of the optimal domain of $S$ in the sense of Proposition 4. Namely, $L^1(m_S) = L^1(\theta_\psi(t)\,dt)$, that is the space of integrable functions with respect to the Lebesgue measure with density $\theta_\psi$. An example of function satisfying all the above conditions is $\psi(t) = t^{1/p}$ with $1 < p < \infty$, for which $\Lambda_\psi(\lambda) = L^{p,1}(\lambda)$, see for instance [1, §4.4].

**Example 6.** Let $I$ be a non-countable set and $K : I \times I \to [0, \infty)$ a non-null map such that $\beta = (\|K_i\|_\infty)_{i\in I} \in \ell^p(I)$ $(1 \leqslant p < \infty)$, where $K_i : I \to [0,\infty)$ is defined as $K_i(j) = K(i, j)$. Note that $\ell^p(I)$ is an order continuous saturated B.f.s. related to $(I, 2^I, \mu)$, where $2^I$ is the $\sigma$-algebra of all parts on $I$ and $\mu$ is the counting measure. Also note that $(2^I)_{\ell^1(I)} = \{A \subset I$: $A$ is finite$\}$ and $(2^I)^{\mathrm{loc}}_{\ell^1(I)} = 2^I$. Consider the map $T : \ell^1(I) \to \ell^p(I)$ defined as

$$Tx = \left(\sum_{j\in I} x_j K(i, j)\right)_{i\in I}$$

for every $x = (x_j)_{j\in I} \in \ell^1(I)$. Since

$$\left(\sum_{i\in I}\left|\sum_{j\in I} x_j K(i, j)\right|^p\right)^{1/p} \leqslant \left(\sum_{i\in I} \|K_i\|_\infty^p \left(\sum_{j\in I} |x_j|\right)^p\right)^{1/p} = \|x\|_{\ell^1(I)} \|\beta\|_{\ell^p(I)},$$

the map $T$ is a well-defined continuous linear operator. Then, since $\ell^1(I)$ is order continuous, $T$ is order-w continuous. Thus, $m_T : (2^I)_{\ell^1(I)} \to \ell^p(I)$, given by $m_T(A) = T(\chi_A)$, is a vector measure with $m_T \ll \mu$ and $T$ optimally factorizes as

$$\begin{tikzcd}
\ell^1(I) \arrow[rr, "T"] \arrow[dr, dotted, "{[i]}"'] & & \ell^p(I) \\
& L^1(m_T) \arrow[ur, dotted, "I_{m_T}"'] &
\end{tikzcd} \tag{10}$$

Since $m_T$ is positive (i.e. $m_T(A) \geqslant 0$ for all $A \in (2^I)_{\ell^1(I)}$), we have that $I_{m_T}(f) \geqslant 0$ for all $0 \leqslant f \in L^1(m_T)$. Then, by using formula (3), it follows that $\|f\|_{m_T} = \|I_{m_T}(|f|)\|_{\ell^p(I)}$ for all $f \in L^1(m_T)$. Note that $A \in 2^I$ is $m_T$-null if and only if $K(i, j) = 0$ for all $(i, j) \in I \times A$. Hence, $m_T$ and $\mu$ are equivalent (and so $[i]$ is an inclusion map) if and only if for every $j \in I$ there exists $i \in I$ such that $K(i, j) > 0$. Since $\ell^p(I)$ does not contain any isomorphic copy of $c_0$, we have that $L^1(m_T) = L^1_w(m_T)$. For every ($2^I$-measurable) function $f$ and $e^* \in \ell^{p'}(I) = \ell^p(I)^*$ (where $\frac{1}{p} + \frac{1}{p'} = 1$), it is routine to prove that

$$\int_\Omega |f|\, d\left|e^* m_T\right| = \sup_{\substack{B\subset I\\ \text{finite}}} \sum_{j\in B} \left|f(j)\langle e^*, K_j\rangle\right|$$

where $K_j = (K(i, j))_{i\in I} \in \ell^p(I)$ (as $K_j \leqslant \beta$). Then,

$$L^1(m_T) = \left\{f = \big(f(j)\big)_{j\in I} \subset \mathbb{R}\colon\ \big(f(j)\langle e^*, K_j\rangle\big)_{j\in I} \in \ell^1(I) \text{ for all } e^* \in \ell^{p'}(I)\right\}$$

and for every $f \in L^1(m_T)$ we have that

$$\|f\|_{m_T} = \sup_{e^*\in B_{\ell^{p'}(I)}} \sum_{j\in I} \left|f(j)\langle e^*, K_j\rangle\right|.$$

Moreover, if $f \in L^1(m_T)$, for each $i \in I$ we have that $(f(j)K(i, j))_{j\in I} \in \ell^1(I)$ (take $e_i^* = (\delta_{i,s})_{s\in I} \in \ell^{p'}(I)$ with $\delta_{i,s} = 1$ if $s = i$ and $\delta_{i,s} = 0$ in other case). So, we can consider the element $x_f = (\sum_{j\in I} f(j)K(i, j))_{i\in I} \subset \mathbb{R}$. Since $\beta \in \ell^p(I)$, there exists a countable set $J \subset I$ such that $K(i, j) = 0$ for all $(i, j) \in (I\backslash J) \times I$. Set $J = \{i_n\}_{n\geqslant 1}$. Then, for each $n$ there exists a countable set $\Gamma_n \subset I$ such that $f(j)K(i_n, j) = 0$ for all $j \in I\backslash\Gamma_n$. Consider $\Gamma = \bigcup_{n\geqslant 1} \Gamma_n = \{j_m\}_{m\geqslant 1}$ and the $(2^I)_{\ell^1(I)}$-simple functions $\varphi_m = f\chi_{\{j_1,\dots,j_m\}}$. Since $0 \leqslant |\varphi_m| \uparrow |f|\chi_\Gamma \in L^1(m_T)$, by the order continuity of $L^1(m_T)$, we have that $\varphi_m \to f\chi_\Gamma$ in $L^1(m_T)$ and so $I_{m_T}(\varphi_m) \to I_{m_T}(f\chi_\Gamma)$ in $\ell^p(I)$. Note that $I\backslash\Gamma \cap \operatorname{Supp} f$ is $m_T$-null, as $K(i, j) = 0$ for all $(i, j) \in I \times (I\backslash\Gamma \cap \operatorname{Supp} f)$. So, $I_{m_T}(f) = I_{m_T}(f\chi_\Gamma)$. Then,

$$\begin{aligned}
I_{m_T}(f)(i) &= \lim_{m\to\infty} I_{m_T}(\varphi_m)(i) = \lim_{m\to\infty} T(\varphi_m)(i) \\
&= \lim_{m\to\infty} \sum_{j\in I} \varphi_m(j)K(i, j) = \sum_{j\in I} f(j)\chi_\Gamma(j)K(i, j) = x_f(i)
\end{aligned}$$

for all $i \in I$ and thus $x_f = I_{m_T}(f) \in \ell^p(I)$. Therefore, for every $f \in L^1(m_T)$ we have that $(f(j)K(i, j))_{j\in I} \in \ell^1(I)$ for all $i \in I$, $(\sum_{j\in I} f(j)K(i, j))_{i\in I} \in \ell^p(I)$ and $I_{m_T}(f) = (\sum_{j\in I} f(j)K(i, j))_{i\in I}$. Moreover,

$$\|f\|_{m_T} = \left\|I_{m_T}\big(|f|\big)\right\|_{\ell^p(I)} = \left(\sum_{i\in I}\left(\sum_{j\in I}\left|f(j)\right|K(i, j)\right)^p\right)^{1/p}.$$

Let us show a kernel satisfying all the above conditions. Take $I=[0,\infty)$ and $\phi \in \ell^p(I)$ (e.g. $\phi(i)=\frac{1}{i^r}\chi_{\mathbb{N}}(i)$ with $r>\frac{1}{p}$). Then $K: I\times I\to[0,\infty)$ defined by $K(i,j)=\phi(i)\chi_{[0,i]}(j)$, satisfies that $\beta=(\|K_i\|_\infty)_{i\in I}\in \ell^p(I)$ (as $\beta=|\phi|$). For this kernel, the operator $T:\ell^1(I)\to\ell^p(I)$ is given by

$$T(x)=\biggl(\phi(i)\biggl(\sum_{j\leqslant i}x_j\biggr)\biggr)_{i\in I}$$

for every $x=(x_j)_{j\in I}\in\ell^1(I)$. For this "version of Hardy operator", the optimal factorization (10) holds.

## 4. Subspaces of $L^1(m)$ generated by generalized semivariations

Fix $(\Omega,\Sigma)$ a measurable space, $E$ a Banach space and $Y(\eta)$ a B.f.s. For a vector measure $n:\Sigma\to E$ with $n\ll\eta$, we define the *$Y(\eta)$-semivariation of $n$* by

$$\|n\|_{Y(\eta)}=\sup\biggl\{\biggl\|\int_\Omega\varphi\,dn\biggr\|_E:\ \varphi\in\mathcal{S}(\Sigma)\cap B_{Y(\eta)}\biggr\}.$$

This concept generalizes the total semivariation of $n$ which is obtained as $\|n\|(\Omega)=\|n\|_{L^\infty(\lambda)}$ for any $\lambda$ equivalent to $n$, see (3) for $f=\chi_\Omega\in L^1(n)$.

Let $\mathcal{R}$ be a $\delta$-ring of subsets of $\Omega$ such that $\mathcal{R}^{\mathrm{loc}}=\Sigma$ and $m:\mathcal{R}\to E$ a vector measure with $m\ll\eta$. For every $f\in L^1(m)$, the set function $m_f:\Sigma\to E$, given by $m_f(A)=\int_\Omega f\chi_A\,dm$, is a vector measure with $m_f\ll\eta$ and

$$\|m_f\|_{Y(\eta)}=\sup\biggl\{\biggl\|\int_\Omega\varphi f\,dm\biggr\|_E:\ \varphi\in\mathcal{S}(\Sigma)\cap B_{Y(\eta)}\biggr\}.$$

Let us consider the space

$$L^1_{Y(\eta)}(m)=\bigl\{f\in L^1(m):\ \|m_f\|_{Y(\eta)}<\infty\bigr\}$$

equipped with the norm $\|f\|_{L^1_{Y(\eta)}(m)}=\max\{\|f\|_m,\|m_f\|_{Y(\eta)}\}$. Of course, $L^1_{Y(\eta)}(m)$ is a subspace of $L^1(m)$. The norm in $L^1(m)$ and the $Y(\eta)$-semivariation are related as the following result shows.

**Lemma 7.** *For every $f\in L^1(m)$ and $\varphi\in\mathcal{S}(\Sigma)\cap Y(\eta)$ we have that*

$$\|f\varphi\|_m\leqslant\|\varphi\|_{Y(\eta)}\|m_f\|_{Y(\eta)}.$$

**Proof.** For $f\in L^1(m)$ and $\varphi\in\mathcal{S}(\Sigma)\cap Y(\eta)$ we have that $f\varphi\in L^1(m)$ and so, by (3),

$$\|f\varphi\|_m=\sup\biggl\{\biggl\|\int_\Omega f\varphi\psi\,dm\biggr\|_E:\ \psi\in\mathcal{S}(\Sigma)\cap B_{L^\infty(\lambda)}\biggr\},$$

where $\lambda:\mathcal{R}\to[0,\infty]$ is a measure equivalent to $m$. Note that for every $\psi\in\mathcal{S}(\Sigma)$ such that $|\psi|\leqslant1$ $\lambda$-a.e., there exists $A\in\Sigma$ such that $\Omega\backslash A$ is $\lambda$-null (and so $m$-null) and $|\psi\varphi|\chi_A\leqslant|\varphi|$ pointwise. Then, $\int_\Omega f\varphi\psi\,dm=\int_\Omega f\varphi\psi\chi_A\,dm$ where $\varphi\psi\chi_A\in\mathcal{S}(\Sigma)\cap Y(\eta)$ and $\|\varphi\psi\chi_A\|_{Y(\eta)}\leqslant\|\varphi\|_{Y(\eta)}$. So, it follows that $\|f\varphi\|_m\leqslant\|\varphi\|_{Y(\eta)}\|m_f\|_{Y(\eta)}$. $\square$

**Proposition 8.** *The space $L^1_{Y(\eta)}(m)$ is a B.f.s.*

**Proof.** Let us see that $L^1_{Y(\eta)}(m)$ satisfies the Riesz–Fischer property an so we will have that $L^1_{Y(\eta)}(m)$ is a Banach space. Let $(f_n)\subset L^1_{Y(\eta)}(m)$ be such that $\sum\|f_n\|_{L^1_{Y(\eta)}(m)}<\infty$. Then, $\sum\|f_n\|_m<\infty$ and since $L^1(m)$ is complete, we have that $f=\sum f_n\in L^1(m)$ and

$$\biggl\|\int_\Omega\varphi f\,dm\biggr\|_E\leqslant\sum\biggl\|\int_\Omega\varphi f_n\,dm\biggr\|_E\leqslant\sum\|m_{f_n}\|_{Y(\eta)}$$

for all $\varphi\in\mathcal{S}(\Sigma)\cap B_{Y(\eta)}$. So, $\|m_f\|_{Y(\eta)}\leqslant\sum\|m_{f_n}\|_{Y(\eta)}\leqslant\sum\|f_n\|_{L^1_{Y(\eta)}(m)}<\infty$ and thus $f\in L^1_{Y(\eta)}(m)$.

In other hand, if $f$ is a measurable function such that $|f|\leqslant|g|$ $m$-a.e. for some $g\in L^1_{Y(\eta)}(m)$, then $f\in L^1(m)$ and $\|f\|_m\leqslant\|g\|_m$. Moreover, for every $\varphi\in\mathcal{S}(\Sigma)\cap B_{Y(\eta)}$, using Lemma 7, it follows

$$\biggl\|\int_\Omega\varphi f\,dm\biggr\|_E\leqslant\|\varphi f\|_m\leqslant\|\varphi g\|_m\leqslant\|\varphi\|_{Y(\eta)}\|m_g\|_{Y(\eta)}\leqslant\|m_g\|_{Y(\eta)}$$

and so $\|m_f\|_{Y(\eta)}\leqslant\|m_g\|_{Y(\eta)}<\infty$. Hence, $f\in L^1_{Y(\eta)}(m)$ and $\|f\|_{L^1_{Y(\eta)}(m)}\leqslant\|g\|_{L^1_{Y(\eta)}(m)}$. Therefore, $L^1_{Y(\eta)}(m)$ is a B.f.s. $\square$

The subspace of $L^1(m)$ generated by $Y(\eta)=L^\infty(\eta)$ is $L^1(m)$ itself. Indeed, for each $f\in L^1(m)$, by (3) and since $m\ll\eta$, we have that $\|m_f\|_{L^\infty(\eta)}\leqslant\|f\|_m$ and by Lemma 7 with $\varphi=\chi_\Omega$, we have that $\|f\|_m\leqslant\|m_f\|_{L^\infty(\eta)}$. So, $L^1_{L^\infty(\eta)}(m)=L^1(m)$ with equal norms.

**Remark 9.** In the case when $\chi_\Omega\in Y(\eta)$, by Lemma 7 we have that $\|f\|_m\leqslant\|\chi_\Omega\|_{Y(\eta)}\|m_f\|_{Y(\eta)}$ for every $f\in L^1(m)$. Then, it follows that

$$\|m_f\|_{Y(\eta)}\leqslant\|f\|_{L^1_{Y(\eta)}(m)}\leqslant\max\bigl\{\|\chi_\Omega\|_{Y(\eta)},1\bigr\}\|m_f\|_{Y(\eta)}$$

for every $f\in L^1_{Y(\eta)}(m)$, that is, $\|m_f\|_{Y(\eta)}$ is an equivalent norm to $\|f\|_{L^1_{Y(\eta)}(m)}$.

Recall that $Y_b(\eta)$ denotes the closure of $\mathcal{S}(\Sigma)\cap Y(\eta)$ in $Y(\eta)$, which is a B.f.s. by Lemma 1(a). Since $m\ll\eta$ we can consider the space $M(Y_b(\eta),L^1(m))$ endowed with the seminorm $\|f\|_{M(Y_b(\eta),L^1(m))}=\sup\{\|hf\|_m:\ h\in B_{Y_b(\eta)}\}$. Note that $M(Y_b(\eta),L^1(m))$ may be not a B.f.s. (see Section 2.1). The following result shows that the intersection of this space with $L^1(m)$ with the natural norm (the maximum of the seminorm and the norm) coincides isometrically with our space $L^1_{Y(\eta)}(m)$.

**Proposition 10.** *The equality*

$$L^1_{Y(\eta)}(m)=L^1(m)\cap M\bigl(Y_b(\eta),L^1(m)\bigr)$$

*holds and* $\|m_f\|_{Y(\eta)}=\|f\|_{M(Y_b(\eta),L^1(m))}$ *for all* $f\in L^1_{Y(\eta)}(m)$.

**Proof.** Let $f\in L^1_{Y(\eta)}(m)$. In particular $f\in L^1(m)$ and so, by Lemma 7, we have that $\|f\varphi\|_m\leqslant\|\varphi\|_{Y(\eta)}\|m_f\|_{Y(\eta)}$ for every $\varphi\in\mathcal{S}(\Sigma)\cap Y(\eta)$. Given $h\in Y_b(\eta)$, we can take $(\varphi_n)\subset\mathcal{S}(\Sigma)\cap Y(\eta)$ such that $\varphi_n\to h$ in $Y(\eta)$ and $\eta$-a.e. pointwise (and so $m$-a.e.). Since

$$\|f\varphi_n-f\varphi_k\|_m\leqslant\|\varphi_n-\varphi_k\|_{Y(\eta)}\|m_f\|_{Y(\eta)}\to 0$$

as $n,k\to\infty$, there exists $g\in L^1(m)$ such that $f\varphi_n\to g$ in $L^1(m)$. Then, for some subsequence $(\varphi_{n_k})$ we have that $f\varphi_{n_k}\to g$ $m$-a.e. and so $fh=g$ $m$-a.e. Thus, $fh\in L^1(m)$ and

$$\|fh\|_m=\lim_n\|f\varphi_n\|_m\leqslant\|m_f\|_{Y(\eta)}\lim_n\|\varphi_n\|_{Y(\eta)}=\|m_f\|_{Y(\eta)}\|h\|_{Y(\eta)}.$$

Therefore, $f\in M(Y_b(\eta),L^1(m))$ and $\|f\|_{M(Y_b(\eta),L^1(m))}\leqslant\|m_f\|_{Y(\eta)}$.

Conversely, let $f\in L^1(m)\cap M(Y_b(\eta),L^1(m))$. For every $\varphi\in\mathcal{S}(\Sigma)\cap B_{Y(\eta)}$ we have that $\|\int_\Omega\varphi f\,dm\|_E\leqslant\|\varphi f\|_m\leqslant\|f\|_{M(Y_b(\eta),L^1(m))}$. Then, $\|m_f\|_{Y(\eta)}\leqslant\|f\|_{M(Y_b(\eta),L^1(m))}<\infty$ and so $f\in L^1_{Y(\eta)}(m)$. $\square$

In the case when $\chi_\Omega\in Y(\eta)$, we have that $M(Y_b(\eta),L^1(m))\subset L^1(m)$. Then, from Proposition 10 and Remark 9, it follows that $L^1_{Y(\eta)}(m)=M(Y_b(\eta),L^1(m))$ isomorphically (isometrically if $\|\chi_\Omega\|_{Y(\eta)}\leqslant 1$).

Let us show some examples of subspaces of $L^1(m)$ generated by particular B.f.s.' $Y(\eta)$.

**Example 11.** Consider $Y(\eta)=L^p(\eta)$ with $1\leqslant p<\infty$. Since $L^p(\eta)$ is an order continuous B.f.s., from Proposition 10 and Lemma 1(c), we have that $L^1_{L^p(\eta)}(m)=L^1(m)\cap M(L^p(\eta),L^1(m))$ isometrically.

If $\eta$ is finite (and so $\chi_\Omega\in L^p(\eta)$), we have that $L^1_{L^p(\eta)}(m)=M(L^p(\eta),L^1(m))$. Under certain requirements, this space coincides isomorphically with the space $L^1_{p',\eta}(m)$ ($\frac{1}{p}+\frac{1}{p'}=1$) studied in [3], given by all functions $f\in L^1(m)$ such that the so called *$p'$-semivariation* of $m_f$ with respect to $\eta$ is finite, that is,

$$\|m_f\|_{p',\eta}=\sup_{\pi\in\mathcal{P}(\Omega)}\ \sup_{x^*\in B_{E^*}}\left(\sum_{A\in\pi}\frac{|\int_A f\,dx^*m|^{p'}}{\eta(A)^{p'-1}}\right)^{1/p'}<\infty,$$

where $\mathcal{P}(\Omega)$ is the set of the finite disjoint partitions $\pi$ of $\Omega$. Namely, if $p>1$, $\mathcal{R}=\Sigma$, $m$ is equivalent to $\eta$ and the $p'$-semivariation of $m$ with respect to $\eta$ is finite, then $L^1_{L^p(\eta)}(m)=L^1_{p',\eta}(m)$ isomorphically, see [3, Theorem 3]. In particular, the $L^p(\eta)$-semivariation and the $p'$-semivariation of $m$ are related as follows

$$\|m\|_{p',\eta}\leqslant\|m\|_{L^p(\eta)}\leqslant 2^{1/p}\max\bigl\{1,\eta(\Omega)^{1/p}\bigr\}\|m\|_{p',\eta}.$$

**Example 12.** Given $1\leqslant p<\infty$, we consider the *$p$-power space* of $L^1(m)$ defined by $L^p(m)=\{f\in L^0(|\lambda|):\ |f|^p\in L^1(m)\}$, where $\lambda:\mathcal{R}\to[0,\infty]$ is a measure with the same null sets as $m$. Following the same proof of [15, Proposition 1], we obtain that $L^p(m)$ is a B.f.s. with the norm $\|f\|_{L^p(m)} = \||f|^p\|_m^{1/p}$ (see also [14, §1.d]). Take $\eta = |\lambda|$ and $Y(\eta) = L^p(m)$. Since $L^p(m)$ is order continuous (as $L^1(m)$ is) then, from Proposition 10 and Lemma 1(c), we have that

$$L^1_{L^p(m)}(m) = L^1(m) \cap M\big(L^p(m), L^1(m)\big)$$

with equal norms.

Suppose now that $m$ is *$\sigma$-finite*, that is, $\Omega = (\bigcup A_n) \cup N$ with $A_n \in \mathcal{R}$ and $N$ $m$-null. From [7, Theorem 3.3], this is equivalent to the fact that $L^1(m)$ has a weak unit (so $L^p(m)$ also does) and also to the fact that there exists $\lambda : \Sigma \to [0, \infty)$ with the same null sets as $m$. In this case and for $p > 1$, from [15, Theorem 5 and §1(2)], it follows that $M(L^p(m), L^1(m)) = [L^1(m)]^{p'}$ isometrically, where $1/p + 1/p' = 1$ and $[L^1(m)]$ denotes the *maximal normed extension* of $L^1(m)$, that is the space

$$\big\{g \in L^0(\lambda)\colon\ \|g\|_{[L^1(m)]} = \sup\big\{\|f\|_m\colon\ f \in L^1(m),\ 0 \leqslant f \leqslant |g|\big\} < \infty\big\}$$

which is a B.f.s. with norm $\|\cdot\|_{[L^1(m)]}$. Let us see that $[L^1(m)] = L^1_w(m)$ isometrically. If $g \in L^1_w(m)$, for every $f \in L^1(m)$ such that $0 \leqslant f \leqslant |g|$ we have that $\|f\|_m \leqslant \|g\|_m$. So, $g \in [L^1(m)]$ and $\|g\|_{[L^1(m)]} \leqslant \|g\|_m$. Conversely, let $g \in [L^1(m)]$ and consider $(\varphi_n) \subset \mathcal{S}(\mathcal{R}) \subset L^1(m)$ such that $0 \leqslant \varphi_n \uparrow |g|$ (e.g. $\varphi_n := \psi_n \chi_{\bigcup_1^n A_j}$ with $(\psi_n) \subset \mathcal{S}(\Sigma)$ such that $0 \leqslant \psi_n \uparrow |g|$ and $(A_n)$ given by the $\sigma$-finiteness of $m$). Then, $\|g\|_m = \lim_n \|\varphi_n\|_m \leqslant \|g\|_{[L^1(m)]}$ and so $g \in L^1_w(m)$.

Therefore, $L^1_{L^p(m)}(m) = L^1(m) \cap L^{p'}_w(m)$ isometrically whenever $p > 1$ and $1/p + 1/p' = 1$. For $p = 1$, since $M(L^1(m), L^1(m)) = L^\infty(\lambda)$ (see [15, Theorem 1]), we have that $L^1_{L^1(m)}(m) = L^1(m) \cap L^\infty(\lambda)$ isometrically.

The measure $m$ is said to be *strongly additive* if $m(A_n) \to 0$ whenever $(A_n)$ is a disjoint sequence in $\mathcal{R}$. For instance, this is the case of the classical vector measures defined on a $\sigma$-algebra. From [7, Corollary 3.2(b)], $m$ is strongly additive if and only if $\chi_\Omega \in L^1(m)$ or equivalently $\chi_\Omega \in L^p(m)$. In this case, $L^1_{L^p(m)}(m) = L^{p'}_w(m)$ for $1 < p < \infty$ and $L^1_{L^1(m)}(m) = L^\infty(\lambda)$ isomorphically (isometrically if $\|m\|(\Omega) \leqslant 1$).

An important consequence of the fact $[L^1(m)] = L^1_w(m)$ must be noted. Since $[L^1(m)]$ coincides with $L^1(m)''$ (i.e. the Köthe bidual of $L^1(m)$) as $L^1(m)$ is order continuous (see [15, §1]), then $L^1_w(m) = L^1(m)''$ isometrically. This extends the result in [5, Proposition 2.4] for classical vector measures defined on $\sigma$-algebras to the case of vector measures defined on $\delta$-rings which are $\sigma$-finite.

**Example 13.** Consider the vector measure $m_S : \mathcal{B}_{L^1 \cap L^\infty(\lambda)} \to \Lambda_\psi(\lambda)$ given by the Hardy operator $S$ as in Example 5. We can take $\eta = \lambda$ and $Y(\eta) = \Lambda_\phi(\lambda)$ for any function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with the same properties as $\psi$. Then,

$$L^1_{\Lambda_\phi(\lambda)}(m_S) = L^1(m_S) \cap M\big(\Lambda_\phi(\lambda), L^1(m_S)\big).$$

Let us describe more precisely this space. For every $f \in L^0(\lambda)$ we define

$$\rho(f) = \sup_{0<t<\infty} \sup_{\substack{A \in \mathcal{B} \\ \lambda(A)=t}} \frac{1}{\phi(t)} \|f\chi_A\|_{m_S}.$$

Similarly to the proof of [15, Theorem 3(7)], it follows that

$$\rho(f) = \sup_{h \in B_{\Lambda_\phi(\lambda)}} \|fh\|_{m_S}.$$

So, noting that, since $\Lambda_\psi(\lambda)$ is order continuous and has the Fatou property then $L^1(m_S) = L^1_w(m_S)$ (see [9, Proposition 3.4(e), (f)]), we have that

$$M\big(\Lambda_\phi(\lambda), L^1(m_S)\big) = \big\{f \in L^0(\lambda)\colon\ \rho(f) < \infty\big\}. \tag{11}$$

Suppose that there exists $C > 0$ such that $\frac{\psi(t)}{t} \leqslant C\theta_\psi(t)$ for all $t > 0$. In this case, $L^1(m_S) = L^1(\theta_\psi(t)\,dt)$ and every $f \in L^1(m_S)$ satisfies

$$\int_0^\infty \big|f(t)\big|\theta_\psi(t)\,dt \leqslant \|f\|_{m_S} \leqslant (1 + C)\int_0^\infty \big|f(t)\big|\theta_\psi(t)\,dt, \tag{12}$$

see [9, §4]. Consider the Marcinkiewicz space

$$M_\phi(\lambda) = \left\{f \in L^0(\lambda)\colon\ \|f\|_{M_\phi(\lambda)} = \sup_{0<t<\infty} \frac{1}{\phi(t)} \int_0^t f^*(s)\,ds < \infty\right\},$$

which is a B.f.s. with the norm $\|f\|_{M_\phi(\lambda)}$ (see [11, §II.5]). From [1, Proposition II.3.3(a)], for every $f \in L^0(\lambda)$ we have that

$$\|f\|_{M_\phi(\lambda)} = \sup_{0<t<\infty} \sup_{\substack{A\in\mathcal{B}\\ \lambda(A)=t}} \frac{1}{\phi(t)} \int_A \big|f(s)\big|\,ds. \tag{13}$$

By using (11)–(13), it is direct to check that

$$M\big(\Lambda_\phi(\lambda), L^1(m_S)\big) = \big\{f \in L^0(\lambda)\colon\ f\theta_\psi \in M_\phi(\lambda)\big\}$$

and $\|f\theta_\psi\|_{M_\phi(\lambda)} \leqslant \rho(f) \leqslant (1+C)\|f\theta_\psi\|_{M_\phi(\lambda)}$ for all $f \in M(\Lambda_\phi(\lambda), L^1(m_S))$. Therefore,

$$L^1_{\Lambda_\phi(\lambda)}(m_S) = \big\{f \in L^0(\lambda)\colon\ f\theta_\psi \in M_\phi(\lambda) \cap L^1(\lambda)\big\}$$

and

$$\|f\theta_\psi\|_{M_\phi\cap L^1(\lambda)} \leqslant \|f\|_{L^1_{\Lambda_\phi(\lambda)}(m_S)} \leqslant (1+C)\|f\theta_\psi\|_{M_\phi\cap L^1(\lambda)}$$

for all $f \in L^1_{\Lambda_\phi(\lambda)}(m_S)$.

Note that for a general $Y(\eta)$, instead of $\Lambda_\phi(\lambda)$, we always have

$$L^1_{Y(\eta)}(m_S) = \big\{f \in L^0(\lambda)\colon\ S\big(|f|\big), S\big(|fh|\big) \in \Lambda_\psi(\lambda) \text{ for all } h \in Y_b(\eta)\big\}.$$

This follows from Proposition 10 and the description of $L^1(m_S)$ given in Example 5.

**Example 14.** Let $I$ be a non-countable set and consider a map $\theta : I \to (0,\infty)$. The space defined as

$$\ell^1_\theta(I) = \big\{x = (x_j)_{j\in I} \subset \mathbb{R}\colon\ \big(x_i\theta(i)\big)_{i\in I} \in \ell^1(I)\big\}$$

and endowed with the norm $\|x\|_{\ell^1_\theta(I)} = \sum_{i\in I}|x_i\theta(i)|$, is an order continuous B.f.s. related to $(I, 2^I, \mu)$, with $\mu$ being the counting measure on $I$.

Let $K : I \times I \to [0,\infty)$ be a kernel as in Example 6 and $m_T : \mathcal{R} \to \ell^p(I)$ the vector measure generated by $K$, where $\mathcal{R} = \{A \subset I\colon\ A \text{ is finite}\}$. Then,

$$L^1_{\ell^1_\theta(I)}(m_T) = L^1(m_T) \cap M\big(\ell^1_\theta(I), L^1(m_T)\big).$$

From the description of $L^1(m_T)$ given in Example 6, it is routine to check that $L^1_{\ell^1_\theta(I)}(m_T)$ can be described as the space of functions $f : I \to \mathbb{R}$ such that

$$\big(f(j)\langle e^*, K_j\rangle\big)_{j\in I} \in \ell^1(I) \quad \text{and} \quad \left(\frac{f(j)\langle e^*, K_j\rangle}{\theta(j)}\right)_{j\in I} \in \ell^\infty(I)$$

for every $e^* \in \ell^{p'}(I)$, and

$$\|f\|_{L^1_{\ell^1_\theta(I)}(m_T)} = \max\left\{\sup_{e^*\in B_{\ell^{p'}(I)}} \sum_{j\in I}\big|f(j)\langle e^*, K_j\rangle\big|,\ \sup_{e^*\in B_{\ell^{p'}(I)}} \sup_{j\in I}\left|\frac{f(j)\langle e^*, K_j\rangle}{\theta(j)}\right|\right\}.$$

If $p = 1$ a simpler description can be given. In this case, $\beta = (\|K_i\|_\infty)_{i\in I} \in \ell^1(I)$ and so $K_j = (K(i,j))_{i\in I} \in \ell^1(I)$ (as $K_j \leqslant \beta$). Then, we can consider $\psi = (\|K_j\|_{\ell^1(I)})_{j\in I}$. Let us require $m_T$ and $\mu$ to be equivalent, or equivalently, for every $j \in I$ there exists $i \in I$ such that $K(i,j) > 0$. Thus, $\psi : I \to (0,\infty)$. Note that if this condition fails on a set $J \subset I$, that is, $K_j = 0$ for all $j \in J$, then $L^1_{\ell^1_\theta(I)}(m_T)$ does not change if we consider $m_T$ defined on $I\backslash J$. From the description above, it follows that $L^1_{\ell^1_\theta(I)}(m_T) = \ell^1_\psi(I) \cap \ell^\infty_{\frac{\psi}{\theta}}(I)$ isometrically. Note that for $\psi : I \to (0,\infty)$, we denote

$$\ell^\infty_\psi(I) = \big\{x = (x_j)_{j\in I} \subset \mathbb{R}\colon\ \big(x_i\psi(i)\big)_{i\in I} \in \ell^\infty(I)\big\},$$

which endowed with the norm $\|x\|_{\ell^\infty_\psi(I)} = \sup_{i\in I}|x_i\psi(i)|$, is a B.f.s. related to $(I, 2^I, \mu)$.

## 5. Factorization of B.f.s.-extensible operators

Fix $(\Omega, \Sigma)$ a measurable space, $X(\mu)$ a B.f.s. satisfying that $\Sigma^{\text{loc}}_{X(\mu)} = \Sigma$ and $T : X(\mu) \to E$ a non-null order-w continuous linear operator with values in a Banach space $E$. Then, we have that $T$ factorizes as in (5).

Let $Y(\eta)$ be a B.f.s. with $m_T \ll \eta$. We will say that $T$ is $Y(\eta)$-*extensible* if there exists a constant $K > 0$ such that

$$\big\|T(f\varphi)\big\|_E \leqslant K\|f\|_{X(\mu)}\|\varphi\|_{Y(\eta)}$$

for all $f \in X(\mu)$ and $\varphi \in \mathcal{S}(\Sigma) \cap Y(\eta)$. The $Y(\eta)$-extensibility is well defined since, if $\varphi = \widetilde{\varphi}$ $\eta$-a.e. (and so $m_T$-a.e.), there exists $A \in \Sigma$ such that $\Omega \backslash A$ is $m_T$-null and $\varphi \chi_A = \widetilde{\varphi} \chi_A$ pointwise and so, noting that $f\varphi \in X(\mu)$, by (5) we have that

$$T(f\varphi) = \int_\Omega f\varphi \, dm_T = \int_\Omega f\varphi\chi_A \, dm_T = \int_\Omega f\widetilde{\varphi}\chi_A \, dm_T = \int_\Omega f\widetilde{\varphi} \, dm_T = T(f\widetilde{\varphi}).$$

In this section we will see that if $T$ is $Y(\eta)$-extensible, then it factorizes through the B.f.s. $L^1_{Y(\eta)}(m_T)$ via the maps $[i]$ and $I_{m_T}$, in a way that $I_{m_T}$ on $L^1_{Y(\eta)}(m_T)$ preserves the $Y(\eta)$-extensibility, and the factorization is optimal.

**Proposition 15.** *The following assertions are equivalent*:

(a) *$T$ is $Y(\eta)$-extensible.*
(b) *$[i] : X(\mu) \to L^1_{Y(\eta)}(m_T)$ is well defined.*
(c) *$\mathcal{P} : X(\mu) \times Y_b(\eta) \to L^1(m_T)$ given by $\mathcal{P}(f, h) = fh$, is well defined.*

**Proof.** Suppose (a) holds. If $f \in X(\mu)$, by (5) we have that $f \in L^1(m_T)$ and

$$\left\| \int_\Omega f\varphi \, dm_T \right\|_E = \left\| T(f\varphi) \right\|_E \leqslant K \|f\|_{X(\mu)} \|\varphi\|_{Y(\eta)} \leqslant K \|f\|_{X(\mu)}$$

for all $\varphi \in \mathcal{S}(\Sigma) \cap B_{Y(\eta)}$. Then, $\|(m_T)_f\|_{Y(\eta)} \leqslant K\|f\|_{X(\mu)} < \infty$ and so $f \in L^1_{Y(\eta)}(m_T)$. Therefore, (b) holds.

Note that since $m_T \ll \mu$ and $m_T \ll \eta$, the map $\mathcal{P}$ which takes a $\mu$-a.e. class in $L^0(\mu)$ represented by $f$ and an $\eta$-a.e. class in $L^0(\eta)$ represented by $h$ into the $m_T$-a.e. class represented by $fh$, is a well-defined bilinear map.

Suppose (b) holds. Then, given $f \in X(\mu)$ and $h \in Y_b(\eta)$, since $f \in L^1_{Y(\eta)}(m_T) = L^1(m_T) \cap M(Y_b(\eta), L^1(m_T))$ (see Proposition 10), we have that $fh \in L^1(m_T)$. So, (c) holds.

Finally, suppose (c) holds. Then, $\mathcal{P} : X(\mu) \times Y_b(\eta) \to L^1(m_T)$ is continuous, since every bilinear map $\mathcal{B} : E \times F \to G$ between Banach lattices such that $\mathcal{B}(x, y) \geqslant \mathcal{B}(\tilde{x}, \tilde{y})$ whenever $x \geqslant \tilde{x} \geqslant 0$ and $y \geqslant \tilde{y} \geqslant 0$, is automatically continuous. This fact can be proved similarly to the case of positive maps between Banach lattices, see [14, p. 2]. So, there exists a constant $K > 0$ such that $\|fh\|_{m_T} \leqslant K\|f\|_{X(\mu)}\|h\|_{Y(\eta)}$ for all $f \in X(\mu)$ and $h \in Y_b(\eta)$. Hence, by (5), we have that

$$\left\| T(f\varphi) \right\|_E = \left\| \int_\Omega f\varphi \, dm_T \right\|_E \leqslant \|f\varphi\|_{m_T} \leqslant K\|f\|_{X(\mu)}\|\varphi\|_{Y(\eta)}$$

for all $f \in X(\mu)$ and $\varphi \in \mathcal{S}(\Sigma) \cap Y(\eta)$. That is, (a) holds. $\square$

Of course, if $T$ is $Y(\eta)$-extensible, by (5) and Proposition 15(b), it factorizes as

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "{[i]}"'] & & E \\ & L^1_{Y(\eta)}(m_T) \arrow[ur, dotted, "I_{m_T}"'] & \end{tikzcd} \tag{14}$$

Note that $I_{m_T} : L^1_{Y(\eta)}(m_T) \to E$ is order-w continuous and satisfies condition (7) of Proposition 4 (as $I_{m_T} : L^1(m_T) \to E$ does), $m_{I_{m_T}} \ll \eta$ (as $m_{I_{m_T}}(A) = \int_\Omega \chi_A \, dm_T$ for all $A \in \Sigma_{L^1_{Y(\eta)}(m_T)}$ and $m_T \ll \eta$) and

$$\left\| I_{m_T}(f\varphi) \right\|_E = \left\| \int_\Omega f\varphi \, dm_T \right\|_E \leqslant \|\varphi\|_{Y(\eta)} \left\| (m_T)_f \right\|_{Y(\eta)} \leqslant \|\varphi\|_{Y(\eta)} \|f\|_{L^1_{Y(\eta)}(m_T)}$$

for all $f \in L^1_{Y(\eta)}(m_T)$ and $\varphi \in \mathcal{S}(\Sigma) \cap Y(\eta)$. That is, $I_{m_T} : L^1_{Y(\eta)}(m_T) \to E$ is $Y(\eta)$-extensible. Moreover, the factorization (14) is optimal in the sense of the following result.

**Theorem 16.** *Suppose that $T$ is $Y(\eta)$-extensible. If $Z(\zeta)$ is a B.f.s. such that $\zeta \ll \mu$ and $T$ factorizes as*

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "{[i]}"'] & & E \\ & Z(\zeta) \arrow[ur, dotted, "S"'] & \end{tikzcd} \tag{15}$$

*with $S$ being order-w continuous, $Y(\eta)$-extensible and satisfying condition* (7) *of Proposition* 4*, then $[i]: Z(\zeta) \to L^1_{Y(\eta)}(m_T)$ is well defined and $S(f) = I_{m_T}(f)$ for all $f \in Z(\zeta)$.*

**Proof.** Let $Z(\zeta)$ be a B.f.s. such that $\zeta \ll \mu$ and $T$ factorizes as in (15). From Proposition 4, $[i]: Z(\zeta) \to L^1(m_T)$ is well defined and $S(f) = I_{m_T}(f)$ for all $f \in Z(\zeta)$. In particular, we have that $m_S \ll \eta$ as $m_S(A) = S(\chi_A) = I_{m_T}(\chi_A) = \int_\Omega \chi_A \, dm_T$ for all $A \in \Sigma_{Z(\zeta)}$ and $m_T \ll \eta$. Since $S$ is supposed to be $Y(\eta)$-extensible, then there exists $K > 0$ such that

$$\left\| \int_\Omega f\varphi \, dm_T \right\|_E = \left\| S(f\varphi) \right\|_E \leqslant K \|f\|_{Z(\zeta)} \|\varphi\|_{Y(\eta)} \leqslant K \|f\|_{Z(\zeta)}$$

for all $f \in Z(\zeta)$ and $\varphi \in \mathcal{S}(\Sigma) \cap B_{Y(\eta)}$. Hence, for every $f \in Z(\zeta)$ it follows that $\|(m_T)_f\|_{Y(\eta)} \leqslant K \|f\|_{Z(\zeta)} < \infty$ and so $f \in L^1_{Y(\eta)}(m_T)$. $\square$

**Example 17.** Let $S: L^1 \cap L^\infty(\lambda) \to \Lambda_\psi(\lambda)$ be the Hardy operator given in Example 5. Consider the Lorentz space $\Lambda_\phi(\lambda)$ with $\phi(t) = \int_0^t \theta_\psi(s)\, ds$ for all $t > 0$. Given $f \in L^1 \cap L^\infty(\lambda)$ and $\varphi \in \mathcal{S}(\mathcal{B}) \cap \Lambda_\phi(\lambda)$, we have that

$$\begin{aligned} \left\| S(f\varphi) \right\|_{\Lambda_\psi(\lambda)} &= \int_0^\infty \left( S(f\varphi) \right)^*(s) \psi'(s)\, ds \leqslant \int_0^\infty \frac{\psi'(s)}{s} \int_0^s f^*(t) \varphi^*(t)\, dt\, ds \\ &\leqslant \|f\|_{L^\infty(\lambda)} \int_0^\infty \varphi^*(t) \int_t^\infty \frac{\psi'(s)}{s}\, ds\, dt \leqslant \|f\|_{L^1 \cap L^\infty(\lambda)} \|\varphi\|_{\Lambda_\phi(\lambda)}, \end{aligned}$$

where the first inequality follows from the definition of $S$ and the properties of the decreasing rearrangement of functions, see for instance [1, §II.1, 2]. This shows that $S$ is $\Lambda_\phi(\lambda)$-extensible. Therefore, $S$ factorizes through $L^1_{\Lambda_\phi(\lambda)}(m_S)$ (see Example 13) as in (14) and the factorization is optimal in the sense of Theorem 16.

**Example 18.** Let $T: \ell^1(I) \to \ell^p(I)$ be the operator defined by the kernel $K$ given in Example 6. Assume that for every $j \in I$ there exists $i \in I$ such that $K(i, j) > 0$. Then, since $\beta = (\|K_i\|_\infty)_{i \in I} \in \ell^p(I)$ and $K_j \leqslant \beta$, we can consider $\psi = (\|K_j\|_{\ell^p(I)})_{j \in I} \subset (0, \infty)$ and the space $\ell^1_\psi(I)$ (see Example 14). Given $f \in \ell^1(I)$ and $\varphi \in \mathcal{S}(2^I) \cap \ell^1_\psi(I)$, we have that

$$\begin{aligned} \left\| T(f\varphi) \right\|_{\ell^p(I)} &= \left( \sum_{i \in I} \left| \sum_{j \in I} f(j)\varphi(j)K(i,j) \right|^p \right)^{1/p} \leqslant \left( \sum_{j \in I} \left| f(j) \right|^{p'} \right)^{1/p'} \left( \sum_{i \in I} \sum_{j \in I} \left| \varphi(j) \right|^p K(i,j)^p \right)^{1/p} \\ &\leqslant \|f\|_{\ell^1(I)} \left( \sum_{j \in I} \left| \varphi(j) \right|^p \sum_{i \in I} K(i,j)^p \right)^{1/p} \leqslant \|f\|_{\ell^1(I)} \left( \sum_{j \in I} \left| \varphi(j) \right|^p \|K_j\|^p_{\ell^p(I)} \right)^{1/p} \\ &\leqslant \|f\|_{\ell^1(I)} \sum_{j \in I} \left| \varphi(j) \right| \|K_j\|_{\ell^p(I)} = \|f\|_{\ell^1(I)} \|\varphi\|_{\ell^1_\psi(I)}. \end{aligned}$$

Hence, $T$ is $\ell^1_\psi(I)$-extensible and so factorizes through $L^1_{\ell^1_\psi(I)}(m_T)$ (see Example 14) as in (14) and the factorization is optimal in the sense of Theorem 16. In the case $p = 1$, note that $L^1_{\ell^1_\psi(I)}(m_T) = \ell^1_\psi(I) \cap \ell^\infty(I)$.

Finally, we show a result which will be used in the following section.

**Lemma 19.** *Suppose that $T$ is $Y(\eta)$-extensible. Then, $T$ is $Z(\zeta)$-extensible for all B.f.s. $Z(\zeta)$ such that $\eta \ll \zeta$ and $[i]: Z(\zeta) \to Y(\eta)$ is well defined.*

**Proof.** Let $Z(\zeta)$ be a B.f.s. as above. By Proposition 15 and Proposition 10, we have that $[i]: X(\mu) \to L^1_{Y(\eta)}(m_T) = L^1(m_T) \cap M(Y_b(\eta), L^1(m_T))$. Since $[i]: Z(\zeta) \to Y(\eta)$ is well defined, $[i]: Z_b(\zeta) \to Y_b(\eta)$ is also well defined and so $M(Y_b(\eta), L^1(m_T)) \subset M(Z_b(\zeta), L^1(m_T))$. Then, $[i]: X(\mu) \to L^1_{Y(\eta)}(m_T) \subset L^1_{Z(\zeta)}(m_T)$ and thus $T$ is $Z(\zeta)$-extensible by Proposition 15. $\square$

## 6. Special cases of B.f.s.-extensible operators

### 6.1. $L^p$-product extensible operators

Let $(\Omega, \Sigma, \mu)$ be a finite measure space, $X(\mu)$ a B.f.s. such that $\mathcal{S}(\Sigma) \subset X(\mu) \subset L^1(\mu)$ (i.e. $X(\mu)$ is a B.f.s. in the sense of Lindenstrauss and Tzafriri [14, Definition 1.b.17]) and $T: X(\mu) \to E$ a non-null order-w continuous linear operator with values in a Banach space $E$. Note that in this case $\chi_\Omega \in X(\mu)$ is a weak unit and so $\Sigma^{\text{loc}}_{X(\mu)} = \Sigma$. In fact, $\Sigma_{X(\mu)} = \Sigma$. Given $1 < p < \infty$, the operator $T$ is said to be *$L^p$-product extensible* if there exists a constant $K > 0$ satisfying that

$$\sup\left\{\left\|T(f\varphi)\right\|_E\colon \varphi \in \mathcal{S}(\Sigma) \cap B_{L^{p'}(\mu)}\right\} \leqslant K\|f\|_{X(\mu)} \tag{16}$$

for all $f \in X(\mu)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. This class of operators has been introduced in [3, Definition 8] as a tool for studying unconditional convergence of series in Banach function spaces. It is clear that $T$ is $L^p$-product extensible if and only if it is $L^{p'}(\mu)$-extensible. Then, in this case, $T$ can be optimally "extended" preserving the inequality (16) to the space $L^1_{L^{p'}(\mu)}(m_T) = L^1_{p,\mu}(m_T)$ (see Example 11) by $I_{m_T}$. So, [3, Theorem 6] can be obtained as a particular case of Proposition 15 and Theorem 16.

### 6.2. Pisier's factorization theorem

Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space, $E$ a Banach space and $T: L^s(\mu) \to E$ a non-null continuous linear operator, where $1 < s < \infty$. Note that $T$ is order-w continuous (as $L^s(\mu)$ is order continuous), $L^s(\mu)$ has a weak unit (as $\mu$ is a $\sigma$-finite measure) and $\Sigma_{L^s(\mu)} = \{A \in \Sigma\colon\ \mu(A) < \infty\}$. Then $\Sigma^{\text{loc}}_{L^s(\mu)} = \Sigma$. For $1 \leqslant p < s$, Pisier's factorization theorem establishes that $T$ factorizes through a weighted Lorentz space $L^{p,1}(\omega d\mu)$ if and only if it satisfies a *lower $p$-estimate*, that is, if there exists $C > 0$ such that

$$\left(\sum_{i=1}^n \left\|T(f_i)\right\|_E^p\right)^{1/p} \leqslant C\left\|\sum_{i=1}^n |f_i|\right\|_{L^s(\mu)} \tag{17}$$

for all disjoint $f_1, \ldots, f_n \in L^s(\mu)$, see [19]. On other hand, from [10, Theorem 4.2], condition (17) is equivalent to the existence of a probability measure $\lambda$ on $\Sigma$ and a constant $K > 0$ such that

$$\left\|T(fg)\right\|_E \leqslant K\|f\|_{L^s(\mu)}\|g\|_{L^{t,1}(\lambda)} \tag{18}$$

for all $f \in L^s(\mu)$ and $g$ a $\Sigma$-measurable function with $|g| \leqslant 1$ pointwise, where $\frac{1}{t} = \frac{1}{p} - \frac{1}{s}$. Then, it follows that $T$ satisfies a lower $p$-estimate if and only if $T$ is $L^{t,1}(\lambda)$-extensible. In this case, Theorem 16 provides the largest "extension" of $T$ preserving the inequality (18), namely, $I_{m_T}: L^1_{L^{t,1}(\lambda)}(m_T) \to E$.

### 6.3. $p$th power factorable operators

Let $(\Omega, \Sigma)$ be a measurable space, $X(\mu)$ a B.f.s. with a weak unit (so, $\Sigma^{\text{loc}}_{X(\mu)} = \Sigma$) and $T: X(\mu) \to E$ a non-null order-w continuous linear operator with values in a Banach space $E$. Note that $m_T$ is $\sigma$-finite, as $X(\mu)$ has a weak unit. Given $0 < r < \infty$, consider the quasi-B.f.s. (B.f.s. if $r \geqslant 1$)

$$X^r(\mu) = \left\{f \in L^0(\mu)\colon\ |f|^r \in X(\mu)\right\},$$

equipped with the quasi-norm (norm if $r \geqslant 1$) $\|f\|_{X^r(\mu)} = \||f|^r\|^{1/r}_{X(\mu)}$. Let us see that the factorization of $T$ through $L^p(m_T)$ is related with certain $X^r(\mu)$-extensibility. Note that although throughout all this paper we have considered B.f.s.' for the aim of simplicity, actually the main arguments can be adapted for quasi-B.f.s.'.

**Proposition 20.** *Assume that $\mathcal{S}(\Sigma) \cap X(\mu)$ is dense in $X(\mu)$ and let $1 < p < \infty$. The following assertions are equivalent*:

(a) $[i]: X(\mu) \to L^1(m_T) \cap L^p(m_T)$ *is well defined.*
(b) $T$ *is* $X^{\frac{1}{p-1}}(\mu)$*-extensible.*

*Moreover, if* (a)–(b) *hold, $T$ factorizes as*

$$\begin{tikzcd}
X(\mu) \arrow[r, "T"] \arrow[d, dotted, "{[i]}"'] & E \\
L^1(m_T) \cap L^p(m_T) \arrow[r, dotted, "i"] & L^1_{X^{\frac{1}{p-1}}(\mu)}(m_T) \arrow[u, dotted, "I_{m_T}"']
\end{tikzcd} \tag{19}$$

**Proof.** Suppose (a) holds. Then, $[i]: X(\mu) \to L^1(m_T) \cap L^p_w(m_T)$ is also well defined. Since $L^1(m_T) \cap L^p_w(m_T) = L^1_{L^{p'}(m_T)}(m_T)$ for $\frac{1}{p} + \frac{1}{p'} = 1$ (Example 12), by Proposition 15 we have that $T$ is $L^{p'}(m_T)$-extensible. Moreover, since $[i]: X(\mu) \to L^p(m_T)$ is well defined, $[i]: X^{\frac{p'}{p}}(\mu) \to L^{p'}(m_T)$ is also well defined. Noting that $\frac{p'}{p} = \frac{1}{p-1}$, (b) follows from Lemma 19.

Conversely, suppose (b) holds. Noting that $X_b^{\frac{1}{p-1}}(\mu)=X^{\frac{1}{p-1}}(\mu)$ as $\mathcal{S}(\Sigma)\cap X(\mu)$ is dense in $X(\mu)$, we have that

$$[i]:X(\mu)\to L^1_{X^{\frac{1}{p-1}}(\mu)}(m_T)=L^1(m_T)\cap M\big(X^{\frac{1}{p-1}}(\mu),L^1(m_T)\big)$$

is well defined. So, given $f\in X(\mu)$, we have that $f\in L^1(m_T)$ and

$$|f|^p=|f|\cdot|f|^{p-1}\in L^1(m_T)\quad\big(\text{i.e. } f\in L^p(m_T)\big),$$

since $|f|\in M(X^{\frac{1}{p-1}}(\mu),L^1(m_T))$ and $|f|^{p-1}\in X^{\frac{1}{p-1}}(\mu)$. Therefore, (a) holds.

Suppose (a)–(b) hold. Since $[i]:X(\mu)\to L^p(m_T)$ is well defined, it follows that $[i]:X^{\frac{1}{p-1}}(\mu)\to L_w^{p'}(m_T)$ is so and then

$$M\big(L_w^{p'}(m_T),L^1(m_T)\big)\subset M\big(X^{\frac{1}{p-1}}(\mu),L^1(m_T)\big).$$

From [15, Theorem 5] and the comments in Example 12, it is deduced that $L^p(m_T)=M(L_w^{p'}(m_T),L^1(m_T))$. Hence, $L^1(m_T)\cap L^p(m_T)\subset L^1_{X^{\frac{1}{p-1}}(\mu)}(m_T)$ and thus $T$ factorizes as above (see for instance [12, Theorem II.5.1]). $\square$

**Remark 21.** The condition $\mathcal{S}(\Sigma)\cap X(\mu)$ dense in $X(\mu)$ in Proposition 20 is only needed for (b) implies (a). If (a)–(b) hold, then $T$ is $L_w^{p'}(m_T)$-extensible. The converse holds if $\mathcal{S}(\Sigma)\cap L_w^{p'}(m_T)$ is dense in $L_w^{p'}(m_T)$ (for instance, this is the case when $L_w^1(m_T)=L^1(m_T)$).

Under the more restrictive setting of $\mu$ finite and $X(\mu)$ order continuous such that $\mathcal{S}(\Sigma)\subset X(\mu)\subset L^1(\mu)$, since $X(\mu)\subset X^{\frac{1}{p}}(\mu)$ for $1\leqslant p<\infty$, it can be considered the property for $T$ of being *pth power factorable*, i.e. $T$ can be extended to $X^{\frac{1}{p}}(\mu)$ by a continuous linear operator $T_p$. In this case, $T$ factorizes as

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "i"'] & & E \\ & X^{\frac{1}{p}}(\mu) \arrow[ur, dotted, "T_p"'] & \end{tikzcd} \tag{20}$$

This class of operators has been recently introduced and thoroughly studied in [18, §5]. Note that in this context, $m_T$ is defined in the $\sigma$-algebra $\Sigma$ and so $L^p(m_T)\subset L^1(m_T)$. Then, from Proposition 20 and [18, Theorem 5.7], we have that $T$ is $p$th power factorable if and only if $T$ is $X^{\frac{1}{p-1}}(\mu)$-extensible, provided $m_T$ and $\mu$ are equivalent. Finally, let us show (in the last setting) an interesting consequence of Proposition 20 for $p=2$.

**Corollary 22.** *Suppose that $E$ is a 2-concave Banach lattice and $T$ is $X(\mu)$-extensible with $m_T$ being equivalent to $\mu$. Then, $T$ factorizes through $L^2(\mu)$ as*

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "M_g"'] & & E \\ & L^2(\mu) \arrow[ur, dotted, "S"'] & \end{tikzcd}$$

*where $M_g$ is a multiplication operator and $S$ a continuous linear operator.*

**Proof.** Since $T$ is $X(\mu)$-extensible, by Proposition 20, it factorizes as

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, dotted, "i"'] & & E \\ & L^2(m_T) \arrow[ur, dotted, "I_{m_T}"'] & \end{tikzcd}$$

Noting that $L^2(m_T)$ is a B.f.s. over $\mu$ and 2-convex and $I_{m_T}:L^2(m_T)\to E$ is 2-concave (by [14, Theorem 1.f.14] and since $E$ is 2-concave), from the Maurey–Rosenthal factorization theorem (see for instance [6, Corollary 5] or [18, Corollary 6.17]) we have that $I_{m_T}$ factorizes as

$$\begin{tikzcd} L^2(m_T) \arrow[rr, "I_{m_T}"] \arrow[dr, dotted, "M_g"'] & & E \\ & L^2(\mu) \arrow[ur, dotted, "S"'] & \end{tikzcd}$$

with $M_g$ being a multiplication operator and $S$ a continuous linear operator. The composition of the two diagrams above gives the result. $\square$

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