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Banach function subspaces of L^1 of a vector measure and related Orlicz spaces

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ABSTRACT

Given a vector measure ν with values in a Banach space X, we consider the space $L^1(\nu)$ of real functions which are integrable with respect to ν. We prove that every order continuous Banach function space Y continuously contained in $L^1(\nu)$ is generated via a certain positive map ρ related to ν and defined on $X^* \times \mathcal{M}$, where X^* is the dual space of X and $\mathcal{M}$ the space of measurable functions. This procedure provides a way of defining Orlicz spaces with respect to the vector measure ν.

1. INTRODUCTION

Let ν be a countably additive vector measure with values in a Banach space X and $L^1(\nu)$ the space of classes of real valued functions which are integrable with respect to ν. Our aim is the study of the following problem: *given a Banach function subspace Y of $L^1(\nu)$, is it possible to describe Y in terms of the vector measure ν?* Consider for example, the simple case of Lebesgue measure m on the interval $[0, 1]$. The space $L^p[0, 1]$ can be described in terms of m, as the space of functions f such that f^p is integrable with respect to m.

The tool for solving this problem is a map $\rho : X^* \times \mathcal{M} \to [0, +\infty]$, where X^* is the dual space of X and $\mathcal{M}$ is a space of measurable functions which has natural properties related to ν. We say that ρ is a ν-norm function. From this map ρ

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we generate a Banach function space $E(\rho_\nu)$, the closure of simple functions with respect to the norm given by

$$\rho_\nu(f) = \sup_{x^* \in B_{X^*}} \rho(x^*, f).$$

This method for generating Banach function spaces allows us to define Orlicz spaces with respect to ν. In particular, it includes the spaces $L^p(\nu)$ of functions whose pth powers are integrable with respect to ν, considered in [13].

The aim of this paper is to establish that every order continuous Banach function subspace Y of $L^1(\nu)$ can be represented as a space $E(\rho_\nu)$, generated by a suitable ν-norm function (see Theorem 5.1). Hence, Y can be described in terms of the vector measure ν.

2. PRELIMINARIES

Throughout the paper, (Ω, Σ) will be a fixed measurable space and $\mathcal{M}$ denotes the space of all measurable real valued functions on Ω. Since we will be considering different measures on Σ, we will identify almost everywhere equal functions only after fixing a measure.

Given a finite positive measure μ on Σ, a *Banach function space*, in short B.f.s., (also called Köthe function space) with respect to μ, is a Banach space Y of functions on Ω which are integrable with respect to μ and satisfying:

(1) If $f \in \mathcal{M}$, $g \in Y$ with $|f| \leqslant |g|$ μ-a.e., then $f \in Y$ and $\|f\|_Y \leqslant \|g\|_Y$,
(2) $\chi_A \in Y$ for every $A \in \Sigma$.

Of course, functions which are equal μ-a.e. are identified and $\|\cdot\|_Y$ denotes the norm of Y. Note that a B.f.s. with respect to μ is a Banach lattice for the μ-a.e. order. By a *Banach function subspace* (in short, B.f.ss.) of a B.f.s. Y, we mean a B.f.s. continuously contained in Y, with the same order structure. A B.f.s. Y is *order continuous* if order bounded increasing sequences are norm convergent. The *Köthe dual* of a B.f.s. Y is the space Y' of measurable functions g such that $fg \in L^1(\mu)$, for every $f \in Y$. The space Y' is a subspace of Y^* (the topological dual space of Y) and is equipped with the relative topology from Y^*. The spaces Y' and Y^* coincide with equality of norms if and only if Y is order continuous. The *Köthe bidual* of a B.f.s. Y is the Köthe dual of Y'. The above claims and further properties of B.f.s. can be found in [9].

Let $\nu : \Sigma \to X$ be a countably additive vector measure with values in a Banach space X. For each $x^* \in X^*$, the variation of the scalar measure $x^*\nu$ is denoted by $|x^*\nu|$. The *semivariation* of ν is the set function on Σ defined by $\|\nu\|(A) = \sup\{|x^*\nu|(A)\colon x^* \in B_{X^*}\}$, where B_{X^*} is the unit ball of X^*. A measurable set A is *ν-null* if $\|\nu\|(A) = 0$. A positive measure λ on Σ is a *control measure* for ν if ν and λ have the same null sets. A *Rybakov control measure* for ν is a control measure of the form $\lambda = |x_0^*\nu|$ for certain $x_0^* \in B_{X^*}$; see [5, Theorem IX.2.2]. In this case, the ν-a.e. order is equivalent to the λ-a.e. order.

A function $f \in \mathcal{M}$ is *integrable* with respect to ν (in the sense of Lewis [7]) if it satisfies

(1) f is integrable with respect to $x^*\nu$, for every $x^* \in X^*$, and
(2) for each $A \in \Sigma$, there is a vector $\int_A f\,d\nu$ in X such that

$$x^*\left(\int_A f\,d\nu\right) = \int_A f\,d(x^*\nu), \qquad \text{for all } x^* \in X^*.$$

For $A = \Omega$ we simply write $\int f\,d\nu$ for $\int_\Omega f\,d\nu$.

The space $L^1(\nu)$ of integrable functions with respect to ν, equipped with the norm

$$\|f\|_\nu = \sup_{x^* \in B_{X^*}} \int |f|\,d|x^*\nu|,$$

is an order continuous B.f.s. with respect to any Rybakov control measure λ for ν; [2, Theorem 1]. The space $L^1_w(\nu)$ of functions satisfying just condition (1) in the definition of integrability with respect to ν, equipped with the norm $\|\cdot\|_\nu$, is a B.f.s. with respect to λ, [14, Theorem 9]. A function f in $L^1_w(\nu)$ is in $L^1(\nu)$ if and only if $\lim_{\|\nu\|(A)\to 0} \|f\chi_A\|_\nu = 0$; [7, Theorem 2.6]. Note that $L^1_w(\nu) = L^1(\nu)$ whenever X does not contain a copy of c_0, [6, Theorem II.5.1]. It is known, for arbitrary X, that $L^1_w(\nu) = L^1(\nu)$ if and only if $L^1_w(\nu)$ is order continuous, if and only if $L^1(\nu)$ is weakly sequentially complete, if and only if $L^1_w(\nu)$ is weakly sequentially complete; see [14, Theorem 10] and also [2, Theorem 3].

We refer to the work of Kluvánek and Knowles [6], Curbera [2–4], Ricker [12] and Okada [10] for other results concerning the space $L^1(\nu)$.

3. SPACES GENERATED BY A ν-NORM FUNCTION

Let $\nu : \Sigma \to X$ be a vector measure. In this section we construct a B.f.ss. of $L^1(\nu)$ from a given positive map closely related to ν.

Definition 3.1. A *ν-norm* function is a map $\rho : X^* \times \mathcal{M} \to [0, +\infty]$ with the following properties:

(a) For each $x^* \in X^*$, the map $\rho_{x^*} : \mathcal{M} \to [0, +\infty]$ given by $\rho_{x^*}(f) = \rho(x^*, f)$, satisfies:
 (a1) $\rho_{x^*}(f) = 0$ if and only if $f = 0$ $|x^*\nu|$-a.e.,
 (a2) $\rho_{x^*}(af) = |a|\rho_{x^*}(f)$, for all $a \in \mathbb{R}$ and $f \in \mathcal{M}$,
 (a3) $\rho_{x^*}(f + g) \leqslant \rho_{x^*}(f) + \rho_{x^*}(g)$, for all $f, g \in \mathcal{M}$,
 (a4) if $f, g \in \mathcal{M}$ and $|f| \leqslant |g|$, $|x^*\nu|$-a.e., then $\rho_{x^*}(f) \leqslant \rho_{x^*}(g)$,
 (a5) if $f, f_n \in \mathcal{M}$ and $0 \leqslant f_n \uparrow f$ $|x^*\nu|$-a.e., then $\rho_{x^*}(f_n) \uparrow \rho_{x^*}(f)$,
 (a6) $\rho_{x^*}(\chi_\Omega) < +\infty$,

(a7) there exists $C = C(x^*) > 0$ such that for all $f \in \mathcal{M}$

$$\int |f|\, d|x^*\nu| \leqslant C\rho_{x^*}(f).$$

(b) For each $f \in \mathcal{M}$, the map $\rho_f : X^* \to [0, +\infty]$ given by $\rho_f(x^*) = \rho(x^*, f)$, satisfies:

(b1) $|a|\rho_f(x^*) \leqslant \rho_f(ax^*)$, for all $a \in \mathbb{R}$, $|a| \leqslant 1$ and $x^* \in X^*$,

(b2) for $f = \chi_\Omega$ we have $\sup_{x^* \in B_{X^*}} \rho_f(x^*) < +\infty$.

An example of a ν-norm function is the map $\rho : X^* \times \mathcal{M} \to [0, +\infty]$ given by

$$\rho(x^*, f) = \int |f|\, d|x^*\nu|.$$

Remark 3.2. Property (a4) of Definition 3.1 implies that $\rho_{x^*}(f) = \rho_{x^*}(|f|)$ for every $f \in \mathcal{M}$ and $x^* \in X^*$.

Remark 3.3. The definition we have given of Banach function space is the one in [9, Definition 1.b.17]. In [1, Definition I.1.3] a different definition of Banach function space is given. Namely, for $\mathcal{M}^+$ the cone of positive functions in $\mathcal{M}$ and $\xi : \mathcal{M}^+ \to [0, +\infty]$ a map satisfying properties (a1)–(a7) of Definition 3.1 for a finite positive measure μ, a Banach function space is defined as $\{f \in \mathcal{M}\colon \xi(|f|) < +\infty\}$. The map ξ is called a *function norm*. Although coming from different approaches, the two definitions only differ in property (a5), called the Fatou property; see [9, p. 30], [1, Theorem I.1.7]. Hence, in both cases we can speak of Banach function spaces.

From the previous remark, for a fixed $x^* \in X^*$, the space

$$E_{x^*} = \{f \in \mathcal{M}\colon \rho_{x^*}(|f|) < +\infty\},$$

where $|x^*\nu|$-a.e. equal functions are identified, is a B.f.s. with respect to the measure $|x^*\nu|$ with norm ρ_{x^*}.

Definition 3.4. Given a ν-norm function ρ, we define the map $\rho_\nu : \mathcal{M} \to [0, +\infty]$ by

$$\rho_\nu(f) = \sup_{x^* \in B_{X^*}} \rho_{x^*}(f), \qquad f \in \mathcal{M},$$

and the space $E_w(\rho_\nu) = \{f \in \mathcal{M}\colon \rho_\nu(|f|) < +\infty\}$, where ν-a.e. equal functions are identified. By Remark 3.2, $\rho_\nu(f) = \rho_\nu(|f|)$ for all $f \in \mathcal{M}$.

Proposition 3.5. *The space $E_w(\rho_\nu)$ with norm ρ_ν is a B.f.ss. of $L^1_w(\nu)$.*

Proof. Let $\lambda = |x_0^*\nu|$ be a fixed Rybakov control measure for ν. We first show that ρ_ν is a function norm for the measure λ. Properties (a1)–(a5) for ρ_ν follow from the corresponding properties for each ρ_{x^*}. Property (a6) for ρ_ν is property (b2) for ρ. Property (a7) is satisfied for the constant $C = C(x_0^*)$. Then, from Remark 3.3, the space $E_w(\rho_\nu)$ is a B.f.s. with respect to λ.

For each $x^* \in B_{X^*}$, from property (a7) for ρ_{x^*}, we have

$$\int |f|\, d|x^*\nu| \leqslant C(x^*)\rho_{x^*}(f) \leqslant C(x^*)\rho_\nu(f),$$

so the integration operator with respect to $|x^*\nu|$ can be defined in $E_w(\rho_\nu)$ by

$$f \in E_w(\rho_\nu) \mapsto I_{x^*}(f) = \int f\, d|x^*\nu| \in \mathbb{R},$$

and is continuous. Moreover, $E_w(\rho_\nu)$ is contained in $L^1_w(\nu)$, and the embedding is continuous. This follows from the fact that positive linear maps between Banach lattices are continuous (see [9, p. 2]) and, as noted before, B.f.s. are Banach lattices. □

For each $x^* \in X^*$, the measure $|x^*\nu|$ is absolutely continuous with respect to the Rybakov control measure λ. So, from the definition of ρ_ν and from property (b1), the natural inclusion from $E_w(\rho_\nu)$ into E_{x^*} is well defined and continuous. Moreover, it is one to one if and only if $|x^*\nu|$ is also a control measure for ν.

We always have

$$(1) \qquad E_w(\rho_\nu) \subset \{f \in \mathcal{M}\colon\ \rho_{x^*}(|f|) < +\infty \text{ for all } x^* \in X^*\} = \bigcap_{x^* \in X^*} E_{x^*}.$$

Observe that the two vector spaces will coincide, after identifying functions which are equal ν-a.e., if $\rho_f(B_{X^*})$ is a bounded set whenever f satisfies $\rho_f(X^*) \subset [0, +\infty)$.

Proposition 3.6. *Let ρ be a ν-norm function. If, for every simple function φ, the map $\rho_\varphi : X^* \to [0, +\infty)$ is subadditive and continuous, then*

$$E_w(\rho_\nu) = \{f \in \mathcal{M}\colon\ \rho_{x^*}(|f|) < +\infty \textit{ for all } x^* \in X^*\}.$$

Proof. Let $f \in \mathcal{M}$ be such that $\rho_{x^*}(|f|) < +\infty$ for all $x^* \in X^*$. Given a sequence (φ_n) of simple functions with $0 \leqslant \varphi_n \uparrow |f|$, for each n we consider the map $T_n : X^* \to [0, +\infty)$ defined by $T_n(x^*) = \rho_{\varphi_n}(x^*)$, for $x^* \in X^*$. By hypothesis, T_n is subadditive and continuous, and from property (b1) of ρ, T_n satisfies $|a|T_n(x^*) \leqslant T_n(ax^*)$ for $|a| \leqslant 1$ and $x^* \in X^*$. Also, (T_n) is a pointwise bounded family of functions, since $T_n(x^*) = \rho_{x^*}(\varphi_n) \leqslant \rho_{x^*}(|f|) < +\infty$ for all n. These properties of (T_n) allow us, as in the proof of the classical Banach–Steinhaus theorem, to get a constant $M > 0$ such that $\sup_{x^* \in B_{X^*}} T_n(x^*) \leqslant M$ for all n. From property (a5) of ρ, for each $x^* \in X^*$ we can take n_{x^*} such that $\rho_{x^*}(|f|) - \rho_{x^*}(\varphi_{n_{x^*}}) \leqslant 1$, then

$$\rho_\nu(|f|) = \sup_{x^* \in B_{X^*}} \rho_{x^*}(|f|) \leqslant 1 + \sup_{x^* \in B_{X^*}} \rho_{x^*}(\varphi_{n_{x^*}}) \leqslant 1 + M.$$

Hence, $f \in E_w(\rho_\nu)$. □

For each ν-norm function ρ we have constructed a B.f.ss. $E_w(\rho_\nu)$ of $L^1_w(\nu)$. We are now interested in similar subspaces of the smaller space $L^1(\nu)$. In order to obtain such spaces we introduce the following definition.

Definition 3.7. The space $E(\rho_\nu)$ is defined to be the closure of the simple functions in the Banach space $E_w(\rho_\nu)$.

Proposition 3.8. *The space $E(\rho_\nu)$ with norm ρ_ν is a B.f.ss. of $L^1(\nu)$.*

Proof. The space $E(\rho_\nu)$ is clearly a Banach subspace of $E_w(\rho_\nu)$. Also, $E(\rho_\nu)$ satisfies the lattice property, that is, given $f \in \mathcal{M}$ and $g \in E(\rho_\nu)$ with $|f| \leqslant |g|$ ν-a.e., then $f \in E(\rho_\nu)$ and $\rho_\nu(f) \leqslant \rho_\nu(g)$; see [1, Theorem I.3.11]. So, $E(\rho_\nu)$ is a B.f.ss. of $E_w(\rho_\nu)$.

From the fact that the simple functions are dense in both spaces $E(\rho_\nu)$ and $L^1(\nu)$, and Proposition 3.5, we see that $E(\rho_\nu)$ is continuously included in $L^1(\nu)$. □

The order continuity of an abstract Banach lattice Y with weak unit (see [9, p. 9]), is a strong property which allows one to obtain important results, e.g., such as representing Y as a space $L^1(\tilde{\nu})$ for a suitable vector measure $\tilde{\nu}$; see [2, Theorem 8]. In the case of a B.f.s. Y with respect to a finite positive measure μ, Y is order continuous if and only if all functions $f \in Y$ have *absolutely continuous norm*, that is, $\|f\chi_A\|_Y \to 0$ whenever $\mu(A) \to 0$. This follows from [1, Proposition I.3.5] (the proof holds for Banach function spaces without the Fatou property). Moreover, if the simple functions are dense in Y, then Y is order continuous if and only if χ_Ω has absolutely continuous norm; see [1, Theorem I.3.13] (the proof holds for Banach function spaces without the Fatou property). Moreover, if Y is an order continuous B.f.s., then

$$Y = \Big\{ f \in \mathcal{M}\colon \lim_{\mu(A)\to 0} \|f\chi_A\|_Y = 0 \Big\}.$$

Indeed, suppose that $f \in \mathcal{M}$ satisfies $\|f\chi_A\|_Y \to 0$ whenever $\mu(A) \to 0$ and each function $f_n = |f|\chi_{[|f|\leqslant n]} \in Y$. Then $f_n \uparrow |f|$ and $\|f_m - f_n\|_Y = \|f\chi_{[n<|f|\leqslant m]}\|_Y \to 0$ whenever $m > n \to +\infty$, since $\mu([n < |f| \leqslant m]) \to 0$. So, $f \in Y$.

Remark 3.9. Let ρ be a ν-norm function. Since $\|\nu\|(A) \to 0$ if and only if $\lambda(A) \to 0$, with λ a Rybakov control measure for ν, the above comments imply that:

(a) $E(\rho_\nu)$ is order continuous if and only if $\rho_\nu(\chi_A) \to 0$ when $\|\nu\|(A) \to 0$.
(b) If $E(\rho_\nu)$ is order continuous, then

$$E(\rho_\nu) = \Big\{ f \in \mathcal{M}\colon \lim_{\|\nu\|(A)\to 0} \rho_\nu(f\chi_A) = 0 \Big\}.$$

Example 3.10. Given $p \in [1, +\infty)$, the map $\rho : X^* \times \mathcal{M} \to [0, +\infty]$ defined by

$$\rho(x^*, f) = \left(\int |f|^p \, d|x^* \nu| \right)^{1/p},$$

is a ν-norm function. For $x^* \in X^*$, the space E_{x^*} is precisely the space $L^p(|x^*\nu|)$. Moreover,

$$\rho_\nu(f) = \sup_{x^* \in B_{X^*}} \rho_{x^*}(f) = \sup_{x^* \in B_{X^*}} \left(\int |f|^p \, d|x^* \nu| \right)^{1/p} = \|f^p\|_\nu^{1/p},$$

shows that $E_w(\rho_\nu) = \{f \in \mathcal{M}\colon \ f^p \in L^1_w(\nu)\}$, and so equality holds in (1). Accordingly, the space $E(\rho_\nu)$ is just the space $L^p(\nu) = \{f \in \mathcal{M}\colon \ f^p \in L^1(\nu)\}$, since $L^p(\nu)$ endowed with the norm ρ_ν is a Banach space in which the simple functions are dense; see [13, Proposition 4]. From Remark 3.9(a), the space $L^p(\nu)$ is order continuous, since

$$\lim_{\|\nu\|(A) \to 0} \rho_\nu(\chi_A) = \lim_{\|\nu\|(A) \to 0} \|\chi_A\|_\nu^{1/p} = \lim_{\|\nu\|(A) \to 0} \|\nu\|(A)^{1/p} = 0;$$

see also [13, Proposition 6].

4. ORLICZ SPACES WITH RESPECT TO VECTOR MEASURES

The previous construction of B.f.s. through a ν-norm function also gives a procedure for defining Orlicz spaces with respect to a vector measure.

Let $\nu : \Sigma \to X$ be a vector measure and $\Phi : [0, +\infty) \to [0, +\infty)$ be a convex continuous increasing function such that $\Phi(t) = 0$ if and only if $t = 0$. For results concerning Orlicz spaces; see [11]. The map $\rho : X^* \times \mathcal{M} \to [0, +\infty]$ defined by

$$\rho(x^*, f) = \inf\left\{ k > 0\colon \ \int \Phi\left(\frac{|f|}{k} \right) d|x^* \nu| \leqslant 1 \right\}, \tag{2}$$

is a ν-norm function. Actually, the infimum in (2) is a minimum, whenever it is positive, [1, p. 268]. Observe, for a fixed $x^* \in X^*$, that the space $E_{x^*} = \{f \in \mathcal{M}\colon \rho_{x^*}(|f|) < +\infty\}$ is a classical Orlicz space.

In this setting, we denote the B.f.s. $E_w(\rho_\nu)$ and $E(\rho_\nu)$ by $L^\Phi_w(\nu)$ and $L^\Phi(\nu)$, respectively, and call the last one the Orlicz space with respect to the vector measure ν and the function Φ. The spaces $L^\Phi(\nu)$ generalize the spaces $L^p(\nu)$, that are obtained by taking the function $\Phi(t) = t^p$.

Proposition 4.1. *For the space $L^\Phi_w(\nu)$, the containment* (1) *is an equality. Moreover, the space $L^\Phi(\nu)$ is order continuous.*

Proof. Let f be a function bounded ν-a.e. by a constant $C > 0$. Given $\varepsilon > 0$, we have

$$\int \Phi\left(\frac{|f|}{\varepsilon} \right) d|x^* \nu| \leqslant \Phi\left(\frac{C}{\varepsilon} \right) |x^* \nu|(\Omega) \leqslant \Phi\left(\frac{C}{\varepsilon} \right) \|\nu\|(\Omega) \|x^*\|_{X^*} \leqslant 1$$

whenever $\|x^*\|_{X^*} \leqslant \delta_\varepsilon = (\|\nu\|(\Omega)\Phi(C/\varepsilon))^{-1}$. Hence, from (2) we conclude that $\rho_f : X^* \to [0, +\infty)$ satisfies $\rho_f(x^*) \leqslant \varepsilon$ for all x^* with $\|x^*\|_{X^*} \leqslant \delta_\varepsilon$ and thus, ρ_f is continuous at $x^* = 0$. Since ρ_f is subadditive, we deduce that ρ_f is continuous at every $x^* \in X^*$. Therefore, from Proposition 3.6, the containment (1) is an equality.

To obtain the second claim, by Remark 3.9(a) it suffices to show that $\rho_\nu(\chi_A) \to 0$ whenever $\|\nu\|(A) \to 0$. Given $\varepsilon > 0$, for every $x^* \in B_{X^*}$ we have

$$\int \Phi\left(\frac{\chi_A}{\varepsilon}\right) d|x^*\nu| = \Phi\left(\frac{1}{\varepsilon}\right)|x^*\nu|(A) \leqslant \Phi\left(\frac{1}{\varepsilon}\right)\|\nu\|(A) \leqslant 1$$

whenever $\|\nu\|(A) \leqslant \delta_\varepsilon = \Phi(1/\varepsilon)^{-1}$. From (2) we conclude that $\rho_{x^*}(\chi_A) \leqslant \varepsilon$. Hence, $\rho_\nu(\chi_A) \leqslant \varepsilon$ whenever $\|\nu\|(A) \leqslant \delta_\varepsilon$. □

An important property of Orlicz functions is the Δ_2-property, that is, there exists a constant $b > 0$ such that $\Phi(2t) \leqslant b\Phi(t)$ for all $t \geqslant 0$. In our case, this property allows us to give a simple description, in terms of ν, of the spaces $L^\Phi_w(\nu)$ and $L^\Phi(\nu)$. If Φ has the Δ_2-property then, for each $x^* \in X^*$ and $f \in \mathcal{M}$, we have that $\rho_{x^*}(f) < +\infty$ if and only if $\int \Phi(|f|)\, d|x^*\nu| < +\infty$. So, by the first part of Proposition 4.1, we have

$$L^\Phi_w(\nu) = \{f \in \mathcal{M} \colon \Phi(|f|) \in L^1_w(\nu)\}.$$

Proposition 4.2. *Let the function Φ possess the Δ_2-property. A sequence (f_n) converges to zero in the norm of $L^\Phi_w(\nu)$ if and only if the sequence $(\Phi(|f_n|))$ converges to zero in the norm of $L^1_w(\nu)$.*

Proof. Suppose that (f_n) converges to zero in $L^\Phi_w(\nu)$. Then, for large enough n, we have $\rho_{x^*}(f_n) \leqslant \rho_\nu(f_n) \leqslant 1$ for $x^* \in B_{X^*}$. If $\rho_{x^*}(f_n) > 0$, from the convexity of Φ and by using the fact that (2) is a minimum, it follows that

$$\int \Phi(|f_n|)\, d|x^*\nu| \leqslant \rho_{x^*}(f_n) \int \Phi\left(\frac{|f_n|}{\rho_{x^*}(f_n)}\right) d|x^*\nu| \leqslant \rho_{x^*}(f_n) \leqslant \rho_\nu(f_n).$$

For the case $\rho_{x^*}(f_n) = 0$, we have that $f_n = 0$ $|x^*\nu|$-a.e. Then $\Phi(|f_n|) = 0$ $|x^*\nu|$-a.e. and so $\int \Phi(|f_n|)\, d|x^*\nu| = 0 \leqslant \rho_\nu(f_n)$. Hence, $\|\Phi(|f_n|)\|_\nu \leqslant \rho_\nu(f_n)$ and so $(\Phi(|f_n|))$ converges to zero in $L^1_w(\nu)$. Conversely, suppose that $(\Phi(|f_n|))$ converges to zero in the norm $\|\cdot\|_\nu$. Given $\varepsilon > 0$ we take $k_\varepsilon \in \mathbb{N}$ such that $1/2^{k_\varepsilon} < \varepsilon$. By the Δ_2-property of Φ, we have

$$\int \Phi(2^{k_\varepsilon}|f_n|)\, d|x^*\nu| \leqslant b^{k_\varepsilon} \int \Phi(|f_n|)\, d|x^*\nu| \leqslant b^{k_\varepsilon} \|\Phi(|f_n|)\|_\nu \leqslant 1$$

for all $x^* \in B_{X^*}$ and for large enough n (only depending on ε). Hence, $\rho_\nu(f_n) \leqslant 1/2^{k_\varepsilon} < \varepsilon$ for large enough n. □

Remark 4.3. As seen from the proof, necessity in Proposition 4.2 holds for any Φ. If the Δ_2-property holds for $t \geqslant t_0 > 0$, then sufficiency is only obtained for a subsequence (f_{n_k}), but the following result still holds.

Proposition 4.4. *If Φ has the Δ_2-property, then*

$$L^{\Phi}(\nu)=\{f\in\mathcal{M}:\ \Phi(|f|)\in L^1(\nu)\}.$$

Proof. Since $L^{\Phi}(\nu)$ is order continuous, from Remark 3.9(b) we have

$$L^{\Phi}(\nu)=\Big\{f\in\mathcal{M}:\ \lim_{\|\nu\|(A)\to 0}\rho_\nu(f\chi_A)=0\Big\}.$$

If Φ has the Δ_2-property, then from Proposition 4.2 we have

$$\begin{aligned}L^{\Phi}(\nu)&=\Big\{f\in\mathcal{M}:\ \lim_{\|\nu\|(A)\to 0}\big\|\Phi(|f|)\chi_A\big\|_\nu=0\Big\}\\&=\{f\in\mathcal{M}:\ \Phi(|f|)\in L^1(\nu)\},\end{aligned}$$

where the last equality is obtained from Remark 3.9(b) applied to the space $L^1(\nu)$. □

5. REPRESENTATION OF B.F.SS. OF $L^1(\nu)$ AS SPACES $E(\rho_\nu)$

The aim of this final section is to establish our main result, namely,

Theorem 5.1. *Let ν be a vector measure and Y be an order continuous Banach function subspace of $L^1(\nu)$. Then there exists a ν-norm function ρ such that $Y=E(\rho_\nu)$ and $\|f\|_Y=\rho_\nu(f)$, for every $f\in Y$.*

Let $\nu:\Sigma\to X$ be a vector measure and $\lambda=|x_0^*\nu|$ be a fixed Rybakov control measure for ν.

Definition 5.2. For each $x^*\in X^*$, define the space

$$Y'_{x^*}=\{g\in Y':\ g\chi_{[h_{x^*}=0]}=0\ \lambda\text{-a.e.}\}\subset Y',$$

where Y' is the Köthe dual of Y (with respect to λ) and h_{x^*} is the Radon–Nikodym derivative of the measure $|x^*\nu|$ with respect to λ. Of course, Y'_{x^*} is equipped with the norm from Y'. Note that $Y\hookrightarrow L^1(\nu)\hookrightarrow L^1(x^*\nu)$ implies that $h_{x^*}\in Y'_{x^*}$.

The space Y'_{x^*} is a closed ideal of Y', that is, a closed subspace for which $f\in Y'_{x^*}$ whenever $|f|\leqslant|g|$ ν-a.e. for some $g\in Y'_{x^*}$. In general, the simple functions may not be included in Y'_{x^*}. In fact, this inclusion holds if and only if λ is absolutely continuous with respect to $|x^*\nu|$, or equivalently, $Y'_{x^*}=Y'$.

The spaces Y'_{x^*} allow us to define a ν-norm function ρ for which the space $E_w(\rho_\nu)$ is just Y'', the Köthe bidual of Y.

Proposition 5.3. *The map $\rho:X^*\times\mathcal{M}\to[0,+\infty]$ defined by*

$$\rho(x^*,f)=\sup_{g\in B_{Y'_{x^*}}}\int|gf|\,d\lambda,$$

is a ν-norm function.

Proof. Fix $x^* \in X^*$. We show that the map ρ_{x^*} satisfies part (a) of Definition 3.1. If $f = 0$ $|x^*\nu|$-a.e., then $\lambda([|f| \neq 0] \cap [h_{x^*} \neq 0]) = 0$ and so $f\chi_{[h_{x^*} \neq 0]} = 0$ λ-a.e. Hence, $fg = 0$ λ-a.e. for every $g \in Y'_{x^*}$ and thus $\rho_{x^*}(f) = 0$. Conversely, suppose that $\rho_{x^*}(f) = 0$. Since $h_{x^*} \in Y'_{x^*}$, we have $0 = \int |f| h_{x^*}\, d\lambda = \int |f|\, d|x^*\nu|$ and so $f = 0$ $|x^*\nu|$-a.e. Therefore ρ_{x^*} satisfies property (a1). Properties (a2)–(a5) are also satisfied by ρ_{x^*} and can be checked directly. For all $A \in \Sigma$ we have $\rho_{x^*}(\chi_A) \leqslant \|\chi_A\|_Y$; this establishes property (a6). Property (a7) holds, since

$$\int |f|\, d|x^*\nu| = \int |f| h_{x^*}\, d\lambda \leqslant \|h_{x^*}\|_{Y'} \rho_{x^*}(f).$$

Fix $f \in \mathcal{M}$. Since $h_{ax^*} = |a| h_{x^*}$ λ-a.e. for all $a \in \mathbb{R}$, we have that $Y'_{ax^*} = Y'_{x^*}$ whenever $a \neq 0$ and so $\rho_f(ax^*) = \rho_f(x^*)$. Hence, property (b1) for the map ρ_f holds for $a \neq 0$ with $|a| \leqslant 1$. The case $a = 0$ is obvious. Property (b2) holds, since $\sup_{x^* \in B_{X^*}} \rho_{\chi_\Omega}(x^*) \leqslant \|\chi_\Omega\|_Y$. □

For each $x^* \in X^*$, we have

$$\rho_{x^*}(f) = \sup_{g \in B_{Y'_{x^*}}} \int |fg|\, d\lambda \leqslant \sup_{g \in B_{Y'}} \int |fg|\, d\lambda = \rho_{x_0^*}(f),$$

since $Y'_{x_0^*} = Y'$. Then $\rho_\nu(f) = \rho_{x_0^*}(f)$ and so $E_w(\rho_\nu) = Y''$. Hence, $E(\rho_\nu)$ is just the closure of the simple functions in Y''.

Finally we are in a position to prove Theorem 5.1. In the case when Y is order continuous, we noted earlier that Y' and Y^* coincide and hence,

$$\|f\|_Y = \sup_{g \in B_{Y'}} \left| \int gf\, d\lambda \right| = \|f\|_{Y''}, \quad f \in Y.$$

Also, in this case, the simple functions are dense in Y. So, Y is the closure of the simple functions in Y''. These observations, together with Proposition 5.3, complete the proof of Theorem 5.1.

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