Basis-independent partial matchings induced by morphisms between persistence modules

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Abstract

In this paper, we study how basis-independent partial matchings induced by morphisms between persistence modules (also called ladder modules) can be defined. Besides, we extend the notion of basis-independent partial matchings to the situation of a pair of morphisms with same target persistence module. The relation with the state-of-the-art methods is also given. Apart form the basis-independent property, another important property that makes our partial matchings different to the state-of-the-art ones is their linearity with respect to ladder modules.

1 Introduction

Persistent homology [9] is one of the main tools of topological data analysis. The algebraic structure associated to persitent homology is persistence module [7, 12]. There are still many open questions about persistence modules, specially multidimensional ones. Ladder modules [11] (that is, morphisms between 1-dimensional persistence modules) are concrete cases of multidimensional modules and therefore interesting objects of study. In this section we will introduce such concept together with previous works that motivated our study. Unless stated otherwise, "persistence modules" will refer to "1-dimensional persistence modules".

Partial matchings induced by morphisms between persistence modules are needed to answer questions that arise in topological data analysis. For example, consider a finite filtration K_* of simplicial complexes obtained from a given dataset:

$$K_1 \hookrightarrow K_2 \hookrightarrow \dots \hookrightarrow K_n.$$

The original dataset may vary, for example if it depends on time or if it is modified as part of an experiment. Then, a new filtration L_* may arise. In some cases, there exists a simplicial map $\kappa_i : K_i \to L_i$ between each pair of simplicial complexes (K_i, L_i) making the following diagram commutative:

$$\begin{array}{ccccccc} L_1 & \hookrightarrow & L_2 & \hookrightarrow & \dots & \hookrightarrow & L_n \\ \uparrow{\scriptstyle \kappa_1} & & \uparrow{\scriptstyle \kappa_2} & & & & \uparrow{\scriptstyle \kappa_n} \\ K_1 & \hookrightarrow & K_2 & \hookrightarrow & \dots & \hookrightarrow & K_n \end{array} \tag{1}$$

Nevertheless, these simplicial maps cannot always be defined. Partial simplicial maps [10] (that is, partially defined maps which are simplicial isomorphisms in their domains) are used to model such situations. Let us denote a partial simplicial map between a pair of simplicial complexes (K_i, L_i) by $\mu_i : K_i \nrightarrow L_i$. Actually, μ_i can be expressed using simplicial maps. Consider the filtration:

$$K_* \cup_{\mu_*} L_* = K_* \sqcup L_* / \sim_{\mu_*}$$

where $x \sim_{\mu_*} y$ if $y = \mu_*(x)$. Notice that there exists trivial injections between K_*, L_* and $K_* \cup_{\mu_*} L_*$.

We obtain the following commutative diagram:

$$\begin{array}{ccccccc} L_1 & \hookrightarrow & L_2 & \hookrightarrow & \dots & \hookrightarrow & L_n \\ \downarrow & & \downarrow & & & & \downarrow \\ K_1 \cup_{\mu_1} L_1 & \hookrightarrow & K_2 \cup_{\mu_2} L_2 & \hookrightarrow & \dots & \hookrightarrow & K_n \cup_{\mu_n} L_n \\ \uparrow & & \uparrow & & & & \uparrow \\ K_1 & \hookrightarrow & K_2 & \hookrightarrow & \dots & \hookrightarrow & K_n \end{array} \tag{2}$$

When applying the homology functor with coefficients in a field to Diagrams (1) and (2), we obtain, respectively, the commutative diagram:

$$\begin{array}{ccccccc} U_1 & \longrightarrow & U_2 & \longrightarrow & \dots & \longrightarrow & U_n \\ \alpha_1 \uparrow & & \alpha_2 \uparrow & & & & \alpha_n \uparrow \\ V_1 & \longrightarrow & V_2 & \longrightarrow & \dots & \longrightarrow & V_n \end{array} \tag{3}$$

that can be seen as a morphism $\alpha : \mathbb{V} \longrightarrow \mathbb{U}$ between the persistence modules $\mathbb{V}$ and $\mathbb{U}$ of length n, and

$$\begin{array}{ccccccc} U_1 & \longrightarrow & U_2 & \longrightarrow & \dots & \longrightarrow & U_n \\ \downarrow \beta_1 & & \downarrow \beta_2 & & & & \downarrow \beta_n \\ W_1 & \longrightarrow & W_2 & \longrightarrow & \dots & \longrightarrow & W_n \\ \alpha_1 \uparrow & & \alpha_2 \uparrow & & & & \alpha_n \uparrow \\ V_1 & \longrightarrow & V_2 & \longrightarrow & \dots & \longrightarrow & V_n \end{array} \tag{4}$$

that can be seen as a pair of morphisms $\mathbb{V} \xrightarrow{\alpha} \mathbb{W} \xleftarrow{\beta} \mathbb{U}$ between the persistence modules $\mathbb{V}$, $\mathbb{W}$ and $\mathbb{U}$.

Since $\mathbb{V}$ and $\mathbb{U}$ are characterized up to isomorphism by their respective barcodes $\mathcal{B}(\mathbb{V})$ and $\mathcal{B}(\mathbb{U})$ (this is a consequence of Gabriel's Theorem [13], see [12]), the following question arises: How can Diagrams (3) and (4) induce partial matchings between the multisets $\mathcal{B}(\mathbb{V})$ and $\mathcal{B}(\mathbb{U})$?

It is proven in [11] that ladder modules with $n \leq 4$ (which can be seen as morphisms between persistence modules of length n) are finite decomposable. Such decomposition characterizes up to isomorphism the ladder module (Diagram (3)) and, in particular, the morphism involved when $n \leq 4$. Nevertheless, ladder modules are infinite decomposable for $n > 4$.

A technique for computing partial matchings induced by morphisms between persistence modules of any length is introduced in [3] and extended recently in [4]. Despite being very useful for theoretical purposes, such techniques factorizes through the image of the given morphism and sometimes produces a different partial matching than the expected from the ladder module decomposition.

Moreover, computing a partial matching induced by Diagram (4) is not straightforward. Once we have the matching induced by Diagram (3), we could try to create the new one composing the partial matching between $\mathcal{B}(\mathbb{V})$ and $\mathcal{B}(\mathbb{W})$, and the partial matching between $\mathcal{B}(\mathbb{W})$ and $\mathcal{B}(\mathbb{U})$.

Nevertheless, it is proven in [3] that there does not exist any functor from the category of persistence modules to the category of multisets with partial matchings as morphisms. In other words, composing partial matchings force us to make arbitrary basis choices. In [16], continuing previous work [10], a persistence module $\mathbb{K} \subset \mathbb{W}$ is defined and interpreted as the "common persistence" between $\mathbb{V}$ and $\mathbb{U}$. Nevertheless no explicit relation of the persistence module $\mathbb{K}$ to the persistence modules $\mathbb{V}$ and $\mathbb{U}$ is given.

In this paper we will define basis-independent partial matchings such that:

- They are induced by morphisms between persistence modules (Diagram (3)).
- They are linear with respect to the direct sum of ladder modules (then, they do not factorize through the image of the given morphism.)

Besides, we also extend the definition to basis-independent partial matchings induced by Diagram (4) and describe their relation with the persistence module $\mathbb{K}$ defined in [16] thanks to the new concept of *enriched partial matchings* presented here.

2 Background

In this section, we briefly recall the concept of persistence modules and all the other concepts mentioned in the introduction section. Notice that some of the classical definitions presented here have been slightly modified to suit our situation. Please, pay attention to the remarks in this section, they will be used later in the proofs of the main results of the paper. In addition, some minor lemmas that may be needed later have also been added to this section. We have omitted categorical or quiver concepts when possible to keep the paper readable by the widest audience.

2.1 Vector spaces and linear maps

From now on, we will consider vector spaces to be finite dimensional, and scalars in a fixed field $\mathbf{F}$. The following elementary remarks about vector spaces will be used later.

Remark 2.1. Given three vector spaces A, B, C, with $B \subseteq A$, we have:

$$\dim A \,/\, B = \dim A - \dim B \geq \dim A \cap C - \dim B \cap C = \dim A \cap C \,/\, B \cap C$$

where $\dim V$ denotes the dimension of the vector space V.

Remark 2.2. Given a linear map $\alpha : A \to B$ between two vector spaces A and B, we have that:

$$\dim A = \dim \big[\alpha(A)\big] + \dim \big[\alpha^{-1}(0)\big],$$

$$\dim B = \dim \big[\alpha^{-1}(B)\big] - \dim \big[\alpha^{-1}(0)\big],$$

$$B = \alpha\alpha^{-1}(B) \quad \text{and} \quad A \subseteq \alpha^{-1}\alpha(A),$$

where $\alpha^{-1}(0)$ denotes the kernel of α.

2.2 Zigzag, persistence and interval modules

Let us briefly introduce now the concept of zigzag, persistence and interval modules. See [5].

Definition 2.3. A **zigzag module** $\mathbb{V}$ is a sequence of vector spaces V_i together with linear maps $\varphi_i^{\mathbb{V}}$ (called structure morphisms):

$$V_1 \overset{\varphi_1^{\mathbb{V}}}{\longleftrightarrow} V_2 \overset{\varphi_2^{\mathbb{V}}}{\longleftrightarrow} \cdots \overset{\varphi_{n-1}^{\mathbb{V}}}{\longleftrightarrow} V_n.$$

If the linear map $\varphi_i^{\mathbb{V}}$ goes from V_i to V_{i+1} then $\varphi_i^{\mathbb{V}}$ is denoted by $f_i^{\mathbb{V}}$ and, if the linear map $\varphi_i^{\mathbb{V}}$ goes from V_{i+1} to V_i then $\varphi_i^{\mathbb{V}}$ is denoted by $g_i^{\mathbb{V}}$, that is,

$$f_i^{\mathbb{V}} = \varphi_i^{\mathbb{V}} : V_i \longrightarrow V_{i+1} \qquad \text{and} \qquad g_i^{\mathbb{V}} = \varphi_i^{\mathbb{V}} : V_{i+1} \longrightarrow V_i.$$

This way, the sequence of symbols f and g will determine the type of $\mathbb{V}$. For example, the structure of a zigzag module $\mathbb{V}$ of type $\tau_3 = ffg$ will be:

$$V_1 \xrightarrow{f_1^{\mathbb{V}}} V_2 \xrightarrow{f_2^{\mathbb{V}}} V_3 \xleftarrow{g_3^{\mathbb{V}}} V_4.$$

Definition 2.4. A **persistence module** $\mathbb{V}$ **of length** n is a zigzag module of type $\tau_n = f \overset{(n-1)\text{-times}}{\cdots\cdots\cdots} f$, that is,

$$\mathbb{V} = \; V_1 \xrightarrow{f_1^{\mathbb{V}}} V_2 \xrightarrow{f_2^{\mathbb{V}}} \cdots \xrightarrow{f_{n-1}^{\mathbb{V}}} V_n.$$

Remark 2.5. For the sake of clarity, the composition $f_{b-1}^{\mathbb{V}} \circ f_{b-2}^{\mathbb{V}} \circ \ldots \circ f_{a+1}^{\mathbb{V}} \circ f_a^{\mathbb{V}}$ will be denoted by $f_{a,b}^{\mathbb{V}}$. In particular, observe that $f_a^{\mathbb{V}}$ will also be denoted by $f_{a,a+1}^{\mathbb{V}}$ and $f_{a,a}^{\mathbb{V}}$ is the identity map on V_a.

Remark 2.6. We sometimes add the trivial spaces $V_0 = 0$ and $V_{n+1} = 0$ to a given persistence module $\mathbb{V}$ of length n, together with zero maps $f_0^{\mathbb{V}} : 0 \to V_1$ and $f_n^{\mathbb{V}} : V_n \to 0$. Then, given $a, b \in \mathbb{Z}$, with $0 \leq a \leq b \leq n$, we have that $f_{a,b}^{\mathbb{V}}(V_a) = 0$ if $b = n+1$ or $a = 0$.

Zigzag modules can be decomposed in simple parts.

Definition 2.7. Given two zigzag modules $\mathbb{U}, \mathbb{V}$ of type τ_n, it is said $\mathbb{U}$ to be a **submodule** of $\mathbb{V}$, denoted $\mathbb{U} \subseteq \mathbb{V}$, if $U_i \subseteq V_i$ and, $\varphi_i^{\mathbb{V}}(U_i) \subseteq U_{i+1}$ if $\varphi_i^{\mathbb{V}} = f_i^{\mathbb{V}}$ or $\varphi_i^{\mathbb{V}}(U_i) \subseteq U_{i-1}$ if $\varphi_i^{\mathbb{V}} = g_i^{\mathbb{V}}$, for all i. In particular, $\phi_i^{\mathbb{U}}$ is defined as $\phi_i^{\mathbb{V}}|_{U_i}$.

In addition, $\mathbb{U}$ is said to be a **summand** of $\mathbb{V}$ if there exists another submodule $\mathbb{W}$ of $\mathbb{V}$ such that $V_i = U_i \oplus W_i$ for all i. In that case, we say $\mathbb{V}$ is the **direct sum** of $\mathbb{U}, \mathbb{W}$, denoted $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$.

Indecomposable modules are zigzag modules that cannot be expressed as a direct sum (except themselves with the zero module). Indecomposable modules have the structure of interval modules [5] which are defined as follows.

Definition 2.8. Given $a, b, n \in \mathbb{Z}$, $1 \leq a \leq b \leq n$, an **interval module** $\mathbb{I}[a, b]$ of type τ_n is the following zigzag module of type τ_n:

$$I_1 \overset{\varphi_1^{\mathbb{I}[a,b]}}{\longleftrightarrow} I_2 \overset{\varphi_2^{\mathbb{I}[a,b]}}{\longleftrightarrow} \cdots \overset{\varphi_{n-1}^{\mathbb{I}[a,b]}}{\longleftrightarrow} I_n$$

where, for $1 \leq i \leq n$,

$$I_i = \begin{cases} \mathbf{F}, & \text{if } i \in [a, b], \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\varphi_i^{\mathbb{I}[a,b]} = \begin{cases} \text{the identity map if } i \in [a, b-1], \\ \text{the zero map, otherwise.} \end{cases}$$

Notice that, in topological data analysis, intervals are sometimes consider to be semi-open. If the reader is used to that notation, please recall that in this paper $[a, b+1)$ will be written as $[a, b]$.

Zigzag modules can be expressed as direct sums of interval modules.

Theorem 2.9 (Interval Decomposition Theorem [5])**.** *Let $\mathbb{V}$ be a zigzag module of type τ_n then:*

$$\mathbb{V} \simeq \bigoplus_{1 \leq a \leq b \leq n} \mathbb{I}[a, b]^{m_{a,b}^{\mathbb{V}}}$$

where $m_{a,b}^{\mathbb{V}}$ is the multiplicity of the interval module $\mathbb{I}[a, b]$.

2.3 Persistence diagrams and barcodes

Zigzag modules are unambiguously described (up to isomorphism) by interval modules $\mathbb{I}[a, b]$ and their multiplicities. This information is usually represented in two equivalent ways: persistence barcodes and persistence diagrams. Please, notice that the word "diagram" in "persistence diagram" does not have any categorical meaning.

Definition 2.10. Let $\mathbb{V} \simeq \bigoplus_{1 \leq a \leq b \leq n} \mathbb{I}[a, b]^{m_{a,b}^{\mathbb{V}}}$ be a zigzag modules decomposed in interval modules. The **persistence diagram** of $\mathbb{V}$ is the function

$$\begin{aligned} \mathcal{D}^{\mathbb{V}} : &\Delta_+ \longrightarrow \mathbb{Z}_{\geq 0} \\ &(a, b) \mapsto m_{a,b}^{\mathbb{V}} \end{aligned}$$

where $\Delta_+ = \{(a, b) \in \mathbb{Z}^2_{\geq 0} \text{ such that } a \leq b\}$.

Definition 2.11. Let $\mathbb{V} \simeq \bigoplus_{1 \leq a \leq b \leq n} \mathbb{I}[a, b]^{m_{a,b}^{\mathbb{V}}}$ be a zigzag modules decomposed in interval modules. The **persistence barcode** of $\mathbb{V}$ is a multiset[1] formed by the intervals whose associated interval modules have no null multiplicity, mathematically:

$$\mathcal{B}^{\mathbb{V}} = \{([a, b], m_{a,b}^{\mathbb{V}}) : m_{a,b}^{\mathbb{V}} > 0\}.$$

The collection of all persistence barcodes will be denoted by **Б**. Although we will use the concept of persistence diagrams along the paper, the concept of persistence barcodes will be useful when defining enriched partial matchings in Section 4.

Before finishing this subsection, let us recall the formula for computing multilplicities of the interval modules in the decomposition of a given persistence module. Let $(a, b) \in \Delta_+$, then:

$$\mathcal{D}^{\mathbb{V}}(a, b) = \dim \left[f_{a,b}^{\mathbb{V}}(V_a) \cap \ker f_b^{\mathbb{V}} \right] - \dim \left[f_{a-1,b}^{\mathbb{V}}(V_{a-1}) \cap \ker f_b^{\mathbb{V}} \right]. \tag{5}$$

This expression is well-defined taking into account Remark 2.6. This formula for persistence modules and analogous formulas for zigzag modules can be found in [5].

[1] A multiset is a set where each element has associated a multiplicity.

2.4 Ladder modules and morphisms between zigzag modules

In this subsection, morphisms between zigzag modules (also called 2-dimensional modules and ladder modules [11]) are recalled.

Definition 2.12. A **morphism** $\alpha : \mathbb{V} \to \mathbb{U}$ **between zigzag modules** of type τ_n is a set of linear maps

$$\alpha_i : V_i \to U_i$$

such that the following diagram is commutative:

$$\begin{array}{ccccccc}
U_1 & \overset{\varphi_1^{\mathbb{U}}}{\longleftrightarrow} & U_2 & \overset{\varphi_2^{\mathbb{U}}}{\longleftrightarrow} & \cdots & \overset{\varphi_{n-1}^{\mathbb{U}}}{\longleftrightarrow} & U_n \\
{\scriptstyle \alpha_1}\uparrow & & {\scriptstyle \alpha_2}\uparrow & & & & {\scriptstyle \alpha_n}\uparrow \\
V_1 & \overset{\varphi_1^{\mathbb{V}}}{\longleftrightarrow} & V_2 & \overset{\varphi_2^{\mathbb{V}}}{\longleftrightarrow} & \cdots & \overset{\varphi_{n-1}^{\mathbb{V}}}{\longleftrightarrow} & V_n
\end{array}$$

When all α_i are injective (resp., surjective), we say α is injective (resp., surjective). Such diagrams are called **ladder modules** of type τ_n.

Observe that ladder modules and morphisms between zigzag modules are different point of views of the same concept as shown in [1].

Remark 2.13. Consider a morphism between two interval modules

$$\mathbb{I}[a,b] \xrightarrow{\alpha} \mathbb{I}[a',b'].$$

Then, all linear maps α_i must be zero unless $a' \leq a \leq b' \leq b$.

Definition 2.14. The **image of a morphism** $\alpha : \mathbb{V} \to \mathbb{U}$ is the submodule of $\mathbb{U}$ whose vector spaces are $\alpha_i(V_i)$ for $i \in \mathbb{Z}_{\geq 0}$.

There is a decomposition theorem for ladder modules of type τ_n in indecomposable ladder module when $n \leq 4$. Nevertheless, when $n > 4$, ladder modules are, in general, representation-infinite, that is, they can not be expressed as a finite direct sum of indecomposable ladder modules.

Theorem 2.15. *[11] Let $\mathbb{A}$ be a ladder module of type τ_n with $n \leq 4$. Then,*

$$\mathbb{A} \simeq \bigoplus_{\mathbb{I} \in \Gamma_0} \mathbb{I}^{m_{\mathbb{I}}^{\mathbb{A}}}$$

where $m_{\mathbb{I}}^{\mathbb{A}}$ is the multiplicity of the indecomposable ladder module $\mathbb{I}$. The set of indecomposable ladder modules of type τ_n with $n \leq 4$ corresponds to the finite set of vertices Γ_0 of the Auslander-Reiten quiver [2] (Γ_0, Γ_1) of ladder modules of type τ_n with $n \leq 4$.

In this paper, we will not define the Auslander-Reiten quiver (Γ_0, Γ_1) since it is not needed for the understanding of our main results. Besides, the only decomposable ladder modules appearing in this paper are the ones of type $\tau_1 = f$, $\tau_2 = ff$ and $\tau_3 = fff$ since we only deal with morphisms between persistence modules. To illustrate Theorem 2.15, let us describe the indecomposable ladder modules of type $\tau_2 = ff$ (that will be represented by integer 2×3 matrices) and give an example.

If an indecomposable ladder module $\mathbb{I}$ of type $\tau_2 = ff$ is represented by an integer 2×3 matrix with all entries being 0 or 1, then the linear maps α_i of $\mathbb{I}$ are the identity map when possible and the zero map otherwise. For example, $\begin{smallmatrix}0&1&1\\0&0&1\end{smallmatrix}$ represents the following indecomposable ladder module of type $\tau_2 = ff$:

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathbf{F} & \xrightarrow{Id} & \mathbf{F} \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle Id} \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbf{F}.
\end{array}$$

There are another 28 indecomposable ladder modules of type $\tau_2 = ff$ (see [11]), but only two of them, represented by the matrices $\begin{smallmatrix}1&2&1\\0&1&1\end{smallmatrix}$ and $\begin{smallmatrix}1&1&0\\1&2&1\end{smallmatrix}$, are not made up of only 0 and 1. Concretely, such two matrices respectively represent the following indecomposable ladder modules:

$$\begin{array}{ccccc}
\mathbf{F} & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & \oplus_2\mathbf{F} & \xrightarrow{[0\,1]} & \mathbf{F} \\
\uparrow & & {\scriptstyle \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}\uparrow & & {\scriptstyle Id}\uparrow \\
0 & \longrightarrow & \mathbf{F} & \xrightarrow{Id} & \mathbf{F}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccccc}
\mathbf{F} & \xrightarrow{Id} & \mathbf{F} & \longrightarrow & 0 \\
{\scriptstyle Id}\uparrow & & \uparrow{\scriptstyle [1\,1]} & & \uparrow \\
\mathbf{F} & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & \oplus_2\mathbf{F} & \xrightarrow{[0\,1]} & \mathbf{F}
\end{array}$$

Example 2.16. The ladder module

$$\mathbb{A} = \begin{array}{ccccc} \oplus_2\mathbf{F} & \xrightarrow{\left[\begin{smallmatrix}1&0\\0&0\\0&1\end{smallmatrix}\right]} & \oplus_3\mathbf{F} & \xrightarrow{[0\ 1\ 0]} & \mathbf{F} \\ \uparrow & & \uparrow \left[\begin{smallmatrix}1&0&0\\1&0&0\\0&0&1\end{smallmatrix}\right] & & \uparrow [1\ 0] \\ 0 & \longrightarrow & \oplus_3\mathbf{F} & \xrightarrow{\left[\begin{smallmatrix}1&0&0\\0&0&1\end{smallmatrix}\right]} & \oplus_2\mathbf{F} \end{array}$$

of type $\tau_2 = ff$, has the following decomposition in indecomposable ladder modules:

$$\mathbb{A} \simeq \begin{smallmatrix}1&2&1\\0&1&1\end{smallmatrix} \oplus \begin{smallmatrix}1&1&0\\0&1&1\end{smallmatrix} \oplus \begin{smallmatrix}0&0&0\\0&1&0\end{smallmatrix}$$

Recall that the ladder module $\mathbb{A}$ can also be interpreted as a morphism $\alpha : \mathbb{V} \to \mathbb{U}$ between two persistence modules $\mathbb{U}$ and $\mathbb{V}$ where

$$\alpha_1 : 0 \to \oplus_2\mathbf{F}, \qquad \alpha_2 = \left[\begin{smallmatrix}1&0&0\\1&0&0\\0&0&1\end{smallmatrix}\right] : \oplus_3\mathbf{F} \to \oplus_3\mathbf{F} \quad \text{and} \quad \alpha_3 = [\,1\ 0\,] : \oplus_2\mathbf{F} \to \mathbf{F},$$

and $\mathbb{V}, \mathbb{U}$ have the following decomposition in interval modules:

$$\mathbb{V} \simeq \mathbb{I}[2,2] \oplus \mathbb{I}[2,3]^2 \qquad \text{and} \qquad \mathbb{U} \simeq \mathbb{I}[1,2]^2 \oplus \mathbb{I}[2,3].$$

2.5 Partial matchings

In this subsection, we recall how persistence barcodes are related to each other via "partial matchings".

Definition 2.17. A **partial matching between sets** A and C is a partial bijection $\sigma : A \to C$ or, in other words, a bijection σ between the domain of σ, $\operatorname{dom}\sigma \subseteq A$, and the image of σ, $\operatorname{im}\sigma \subseteq C$.

Given a map $\sigma : A \to C$ between two sets of points, the codomain and the coimage of σ are defined, respectively, as follows:

$$\operatorname{cod}\sigma = A \setminus \operatorname{dom}\sigma \text{ and } \operatorname{coim}\sigma = C \setminus \operatorname{im}\sigma.$$

Remark 2.18. A representation set of the multiset A, denoted by $s(A)$, is a set obtained by enumerating the elements of A.

Let us see an example.

Example 2.19. Consider the following two persistence barcodes:

$$\mathcal{B}^{\mathbb{V}} = \{([2,3],2), ([3,3],2)\} \text{ and } \mathcal{B}^{\mathbb{U}} = \{([1,2],1), ([2,3],2), ([3,3],2)\}.$$

Then,

$$s(\mathcal{B}^{\mathbb{V}}) = \{[2,3]_1, [2,3]_2, [3,3]_1, [3,3]_2\}$$

and

$$s(\mathcal{B}^{\mathbb{U}}) = \{[1,2]_1, [2,3]_1, [2,3]_2, [3,3]_1, [3,3]_2\}.$$

A partial matching between $s(\mathcal{B}^{\mathbb{V}})$ and $s(\mathcal{B}^{\mathbb{U}})$ can be, for example, $\sigma([2,3]_2) = [3,3]_1$.

By abuse of language, we will say "partial matchings between persistence modules $\mathbb{V}, \mathbb{U}$" when we mean "partial matchings between sets $s(\mathcal{B}^{\mathbb{V}})$ and $s(\mathcal{B}^{\mathbb{U}})$".

2.5.1 Bauer-Lesnick induced partial matching

A method for computing a partial matching induced by a morphism between persistence modules is given in [3]. Such method is introduced in this subsection to show similarities and differences with our approach. First, we have to give more notations.

Notation 2.20. Given $a, b \in \mathbb{N}$, consider the sets

$$\{[\,\cdot\,, b]_*\} = \{[x, b]_n \,:\, x, n \in \mathbb{N}\}$$

and

$$\{[a,\,\cdot\,]_*\} = \{[a, y]_n \,:\, y, n \in \mathbb{N}\}.$$

For $x, x', y, y', n, m \in \mathbb{N}$, we write:

$$(x, b)_n < (x', b)_m \text{ if } \begin{cases} x < x', \text{ or} \\ x = x' \text{ and } n < m. \end{cases}$$

and, respectively,

$$(a, y)_n < (a, y')_m \text{ if } \begin{cases} y > y', \text{ or} \\ y = y' \text{ and } n < m. \end{cases}$$

To compute a partial matching induced by an injective morphism between persistence modules $\alpha : \mathbb{V} \to \mathbb{U}$, fix $b \in \mathbb{N}$ and consider the ordered sets $A = \{[\,\cdot\,, b]_*\} \subset s(\mathcal{B}^{\mathbb{V}})$ and $B = \{[\,\cdot\,, b]_*\} \subset s(\mathcal{B}^{\mathbb{U}})$. It is proven in [3] that the number of elements of A is less or equal than the number of elements of B, that is, $\#A \leq \#B$, so a partial matching between the two sets can be defined by matching the i-th element of A with the i-th element of B, for $1 \leq i \leq \#A$. Putting together the partial matchings obtained for all $b \in \mathbb{N}$, we obtain a new partial matching between $s(\mathcal{B}^{\mathbb{V}})$ and $s(\mathcal{B}^{\mathbb{U}})$ denote by ι_α.

A similar procedure is followed in [3] when α is surjective. In this case, fix $a \in \mathbb{N}$ and consider the sets $A = \{[a,\,\cdot\,]_*\} \subseteq s(\mathcal{B}^{\mathbb{V}})$ and $B = \{[a,\,\cdot\,]_*\} \subseteq s(\mathcal{B}^{\mathbb{U}})$. We have that $\#B \leq \#A$. Again, a partial matching between the two sets can be defined by matching the i-th element of B with the i-th element of A, for $1 \leq i \leq \#B$. Putting together the partial matchings obtained for all $a \in \mathbb{N}$, we obtain a new partial matching between $s(\mathcal{B}^{\mathbb{V}})$ and $s(\mathcal{B}^{\mathbb{U}})$ denoted by λ_α.

Finally, since any morphism α between persistence modules can be decomposed in a surjective and an injective morphism between persistence modules, a partial matching induced by α can always be computed.

Definition 2.21. Given a morphism between persistence modules $\alpha : \mathbb{V} \to \mathbb{U}$, and its deocomposition $\alpha = \gamma \circ \beta$ where β is surjective and γ is injective:

$$\mathbb{V} \xrightarrow{\beta} \operatorname{im}\alpha \xrightarrow{\gamma} \mathbb{U},$$

the **Bauer-Lesnick partial matching (or BL-matching) induced by** α is the partial matching $\sigma = \lambda_\gamma \circ \iota_\beta$ obtained by the composition of partial matchings ι_β and λ_γ:

$$s(\mathcal{B}^{\mathbb{V}}) \xrightarrow{\lambda_\beta} s(\mathcal{B}^{\operatorname{im}\alpha}) \xrightarrow{\iota_\gamma} s(\mathcal{B}^{\mathbb{U}}).$$

Let us see a very simple example of a BL-matching.

Example 2.22. Let us consider the ladder module $\alpha : \mathbb{V} \to \mathbb{U}$ where $\mathbb{V} = \mathbb{I}[a, b]$ and $\mathbb{U} = \mathbb{I}[a', b']$, being $a, b, a', b' \in \mathbb{N}$, $a \leq a' \leq b \leq b'$ and $\alpha_i = Id$ when $i \in [a', b]$. Then,

$$\mathbb{I}[a, b] \xrightarrow{\beta} \operatorname{im}\alpha = \mathbb{I}[a', b] \xrightarrow{\gamma} \mathbb{I}[a', b']$$

and

$$s(\mathcal{B}^{\mathbb{V}}) = \{[a, b]_1\} \xrightarrow{\lambda_\beta} s(\mathcal{B}^{\operatorname{im}\alpha}) = \{[a', b]_1\} \xrightarrow{\iota_\gamma} s(\mathcal{B}^{\mathbb{V}}) = \{[a', b']_1\}$$

where

$$\lambda_\beta\big([a, b]_1\big) = [a', b]_1 \qquad \text{and} \qquad \iota_\gamma\big([a', b]_1\big) = [a', b']_1.$$

Therefore, the BL-matching $\sigma = \iota_\gamma \circ \lambda_\beta$ satisfies that:

$$\sigma\big([a, b]_1\big) = [a', b']_1.$$

and it is zero otherwise.

Notice that fixed two persistence modules, the BL-matchings induced by morphisms between them coincide if the images of such morphisms coincide. In other words, consider two different morphisms α_1 and α_2 between the same persistence modules. If the persistence modules $\operatorname{im}\alpha_1$ and $\operatorname{im}\alpha_2$ coincide then the BL-matching induced by α_1 and the BL-matching induced by α_2 coincide. As we will see later, this is not always the case when we compute basis-independent partial matchings following our approach.

Example 2.23. Let us consider the following ladder module $\alpha : \mathbb{V} \to \mathbb{U}$ of type $\tau_2 = ff$:

$$\mathbb{A} = \begin{array}{ccccc} \oplus_2 F & \xrightarrow{\left[\begin{smallmatrix}1&0\\0&0\\0&1\end{smallmatrix}\right]} & \oplus_3 F & \xrightarrow{[0\ 1\ 0]} & F \\ \uparrow & & \uparrow{\scriptstyle \left[\begin{smallmatrix}1&0&0\\1&0&0\\0&0&1\end{smallmatrix}\right]} & & \uparrow{\scriptstyle [1\ 0]} \\ 0 & \longrightarrow & \oplus_3 F & \xrightarrow{f_2^{\mathbb{V}}=\left[\begin{smallmatrix}1&0&0\\0&0&1\end{smallmatrix}\right]} & \oplus_2 F \end{array}$$

Then, $\alpha = \gamma \circ \beta$ where

$$\mathbb{V} \xrightarrow{\beta} \operatorname{im}\alpha \xrightarrow{\gamma} \mathbb{U},$$

β is surjective and γ is injective. The persistence module $\operatorname{im}\alpha$ is:

$$0 \longrightarrow \oplus_2 F \xrightarrow{[1\ 0]} F$$

and its decomposition in interval modules is:

$$\operatorname{im}\alpha \simeq \mathbb{I}[2,3] \oplus \mathbb{I}[2,2].$$

To construct the BL-matching between $\mathbb{V}$ and $\mathbb{U}$, recall that $\mathbb{V} \simeq \mathbb{I}[2,2] \oplus \mathbb{I}[2,3]^2$ and $\mathbb{U} \simeq \mathbb{I}[1,2]^2 \oplus \mathbb{I}[2,3]$ (see Example 2.16 in page 6). Then:

$$\begin{aligned} s(\mathcal{B}^{\mathbb{V}}) &= \{[2,3]_1, [2,3]_2, [2,2]_1\}, \\ s(\mathcal{B}^{\operatorname{im}\alpha}) &= \{[2,3]_1, [2,2]_1\} \quad \text{and} \\ s(\mathcal{B}^{\mathbb{U}}) &= \{[1,2]_1, [1,2]_2, [2,3]_1\}. \end{aligned}$$

Therefore,

$$\lambda_\beta([2,3]_1) = [2,3]_1,\ \lambda_\beta([2,3]_2) = [2,2]_1,\ \lambda_\beta([2,2]x_1) = \emptyset,$$
$$\iota_\gamma([2,2]_1) = [1,2]_1, \ \text{ and } \ \iota_\gamma([2,3]_1) = [2,3]_1.$$

Finally, the composition σ of λ_β and ι_γ produces the BL-matching σ induced by α:

$$\sigma([2,3]_1) = [2,3]_1,\ \sigma([2,3]_2) = [1,2]_1,\ \sigma([2,2]_1) = \emptyset.$$

Observe that replacing $\alpha_2 = \left[\begin{smallmatrix}1&0&0\\1&0&0\\0&0&1\end{smallmatrix}\right]$ by, for example, $\alpha_2 = \left[\begin{smallmatrix}1&0&0\\1&0&0\\0&1&0\end{smallmatrix}\right]$, will not change σ since both $\operatorname{im}\alpha$ and the decomposition of $\mathbb{V}$ do not change.

3 Basis-independent partial matchings

Observe that when we transform a multiset into a set using the map s, we are making arbitrary basis choices. If we want to define a basis-independent partial matching we should not perform such operation. As an alternative, we can define the partial matching using multiplicities instead of pairings. Let us define what basis-independent partial matchings mean in this paper.

Definition 3.1. **A basis-independent partial matching between two persistence modules** $\mathbb{V}, \mathbb{U}$ of length n is a function

$$\mathcal{M}_{\mathbb{V}}^{\mathbb{U}} : \Delta_+ \times \Delta_+ \to \mathbb{Z}_{\geq 0}$$

such that:

$$\sum_{1\leq b'\leq n} \sum_{1\leq a'\leq b'} \mathcal{M}_{\mathbb{V}}^{\mathbb{U}}(a,b,a',b') \leq \mathcal{D}^{\mathbb{V}}(a,b) \quad \text{and}$$
$$\sum_{1\leq b\leq n} \sum_{1\leq a\leq b} \mathcal{M}_{\mathbb{V}}^{\mathbb{U}}(a,b,a',b') \leq \mathcal{D}^{\mathbb{U}}(a',b').$$

Notice that a partial matching σ between $s(\mathcal{B}^{\mathbb{V}})$ and $s(\mathcal{B}^{\mathbb{U}})$ always induces a basis-independent partial matching between the persistence modules $\mathbb{V}$ and $\mathbb{U}$ as follows. Assume that a partial matching σ between $s(\mathcal{B}^{\mathbb{V}})$ and $s(\mathcal{B}^{\mathbb{U}})$ exists. Define

$$\mathcal{M}_{\mathbb{V}}^{\mathbb{U}}(a,b,a',b') = \#\big\{[a,b]_x \in s(\mathcal{B}^{\mathbb{V}}) \,\big|\, \exists\, y \in \mathbb{N} \text{ such that } \sigma([a,b]_x) = [a',b']_y\big\}.$$

Then,

$$\sum_{1\leq b'\leq n}\sum_{1\leq a'\leq b'} \mathcal{M}_{\mathbb{V}}^{\mathbb{U}}(a,b,a',b') = \#\big\{[a,b]_x \,\big|\, [a,b]_x \in \operatorname{dom}\sigma\big\} \leq \mathcal{D}^{\mathbb{V}}(a,b)$$

and

$$\sum_{1\leq b\leq n}\sum_{1\leq a\leq b} \mathcal{M}_{\mathbb{V}}^{\mathbb{U}}(a,b,a',b') = \#\big\{[a',b']_x \,\big|\, [a',b']_y \in \operatorname{im}\sigma\big\} \leq \mathcal{D}^{\mathbb{U}}(a',b').$$

In particular, given a a BL-matching, we can always compute its corresponding basis-independent partial matching that will be denoted by $\mathcal{M}_{BL}^{\alpha}$.

Example 3.2. Consider the ladder module $\alpha : \mathbb{V} \to \mathbb{U}$ of type $\tau_2 = ff$ described in Example 2.16 and the BL-matching σ between $s(\mathcal{B}^{\mathbb{V}}) = \{[2,3]_1, [2,3]_2, [2,2]_1\}$ and $s(\mathcal{B}^{\mathbb{U}}) = \{[1,2]_1, [1,2]_2, [2,3]_1\}$ given in Example 2.23 and detailed below:

$$\sigma([2,3]_1) = [2,3]_1,\ \sigma([2,3])_2) = [1,2]_1,\ \sigma([2,2]_1) = \emptyset.$$

Then

$$\mathcal{M}_{BL}^{\alpha}(2,3,2,3) = 1, \quad \mathcal{M}_{BL}^{\alpha}(2,3,1,2) = 1 \text{ and } \mathcal{M}_{BL}^{\alpha}(a,b,a',b') = 0 \text{ otherwise.}$$

Besides, due to the properties of the function $\mathcal{M}_{\mathbb{V}}^{\mathbb{U}}$ given in Definition 3.1, we always can obtain a (non-unique) partial matching between persistence modules starting from a concrete basis-independent partial matching. Nevertheless, contrary to what happens with composition of partial matchings, the composition of basis-independent partial matchings is not well-defined. Specifically, it has been proven in [3] that no functor from the category of peristence modules to the category of multisets with partial matchings as morphisms can exist. Nevertheless, using basis-independent partial matchings, we do not need any arbitrary choice of the elements of the multisets $\mathcal{B}^{\mathbb{V}}$ and $\mathcal{B}^{\mathbb{U}}$, that is, any choice of the basis of $\mathbb{V}$ and $\mathbb{U}$, gaining in versatility.

3.1 Basis-independent partial matchings induced by ladder modules

In this subsection, we will define a basis-independent partial matching with the following characteristics (already mentioned in the introduction):

- It is induced by a given morphism between persistence modules.
- It is a linear map with respect to the direct sum of ladder modules and then it does not factorize through the image of the given morphism.

The following subspaces will play a very important role in this subsection.

Notation 3.3. Let $\mathbb{V}$ be a persistence module of length n. Let $a, b \in \mathbb{Z}_{\geq 0}$. Then, $S_{a,b}^{\mathbb{V}}$ denotes the following subspace:

$$S_{a,b}^{\mathbb{V}} = \begin{cases} f_{a,b}^{\mathbb{V}}(V_a) \cap \ker f_b^{\mathbb{V}} & \text{if } 1 \leq a \leq b \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $S_{a,b}^{\mathbb{V}}$ can be seen as the subspace of V_a which "persists" in V_b through $f_{a,b}^{\mathbb{V}}$. Now, observe that Expression (5) in page 4 can be written as

$$\mathcal{D}^{\mathbb{V}}(a,b) = \dim S_{a,b}^{\mathbb{V}} - \dim S_{a-1,b}^{\mathbb{V}}. \tag{6}$$

Therefore, the multiplicity $\mathcal{D}^{\mathbb{V}}(a,b)$ can be seen as the dimension of the subspace of V_a which is not a subspace of V_{a-1} and "persists" in V_b through $f_{a,b}^{\mathbb{V}}$. This interpretation is usually called as "the elder rule" in the literature [9].

The following diagram can help us to figure out how we can define a basis-independent partial matching between two persistent modules $\mathbb{U}$ and $\mathbb{V}$ induced by a morphism $\alpha : \mathbb{V} \to \mathbb{U}$:

$$\begin{array}{ccccccc} \mathbf{U_{a'}} & \xrightarrow{f^{\mathbb{U}}_{a',b'}} & \mathbf{U_{b'}} & \xrightarrow{f^{\mathbb{U}}_{b'}} & U_{b'+1} & & \\ & & \Big\uparrow{\scriptstyle \alpha_{b'}} & & & & \\ \mathbf{V_a} & \xrightarrow{f^{\mathbb{V}}_{a,b'}} & V_{b'} & \xrightarrow{f^{\mathbb{V}}_{b',b}} & \mathbf{V_b} & \xrightarrow{f^{\mathbb{V}}_{b}} & V_{b+1}. \end{array} \tag{7}$$

But before giving such definition, we need to introduce a new notation.

Definition 3.4. Let $\alpha : \mathbb{V} \to \mathbb{U}$ be a morphism between persistence modules of length n. Then,

$$\mathcal{X}^\alpha : \mathbb{Z}^4_{\geq 0} \to \mathbb{Z}_{\geq 0}$$

is defined as follows:

$$\mathcal{X}^\alpha(a,b,a',b') = \begin{cases} \dim\left[S^{\mathbb{V}}_{a,b} \cap f^{\mathbb{V}}_{b',b}\, \alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'})\right] \\ \qquad\qquad - \dim\left[S^{\mathbb{V}}_{a,b} \cap f^{\mathbb{V}}_{b',b}\alpha^{-1}_{b'}(0)\right] & \text{if } b' \leq b \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

For the sake of simplicity, we sometimes denote $\mathcal{X}^\alpha(a,b,a',b')$ by ${}^{b}_{a}\mathcal{X}^{b'}_{a'}$ if there is no need to explicitly mention the morphism α.

Looking at Diagram (7), we can interpret $\mathcal{X}^\alpha$ as the dimension of the subspace of $S^{\mathbb{U}}_{a',b'}$ that "persists" in $S^{\mathbb{V}}_{a,b}$ "through" α^{-1}. In the context of zigzag homology, observe that $\mathcal{X}^\alpha$ is the value $D[2,6]$ of the following zigzag module:

$$\begin{array}{ccccccccc} U_{b'+1} & \xleftarrow{f^{\mathbb{U}}_{b'}} & f^{\mathbb{U}}_{a',b'}(U_{a'}) & \hookrightarrow & \mathbf{U_{b'}} & & & & \\ & & & & \Big\uparrow{\scriptstyle \alpha_{b'}} & & & & \\ & & & & \mathbf{V_{b'}} & \hookleftarrow & f^{\mathbb{V}}_{a,b'}(V_a) & \xrightarrow{f^{\mathbb{V}}_{b',b}} V_b & \xrightarrow{f^{\mathbb{V}}_{b}} V_{b+1} \end{array}$$

Remark 3.5. Since $U_0 = V_0 = V_{n+1} = U_{n+1} = 0$, then ${}^{b}_{a}\mathcal{X}^{b'}_{a'} = 0$ if any of the variables a, b, a', b' is 0 or $n+1$. Besides, by definition of S we have that ${}^{b}_{a}\mathcal{X}^{b'}_{a'} = 0$ if $b < a$, $b' < a'$.

To define the basis-independent partial matching induced by α, the function $\mathcal{X}^\alpha$ will play a similar role to $\dim S$ in the definition of $\mathcal{D}$ in Expression (6) of page 9. This way, we will also make use of an "elder rule operator" defined as follows.

Definition 3.6. Given a function $\mathcal{F} : \mathbb{Z}^n \to \mathbb{Z}$, let us define $E_i(\mathcal{F})$ as follows:

$$\begin{aligned} E_i(\mathcal{F})(x_1, \ldots, x_n) =& \mathcal{F}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \\ & - \mathcal{F}(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_n) \end{aligned}$$

where E denotes the **elder rule operator**. We will write $E_j(E_i(\mathcal{F}))$ as $E_{i,j}(\mathcal{F})$.

For example, let $\mathbb{V}$ be a persistence module. Consider the function $\dim S^{\mathbb{V}} : (a,b) \mapsto \dim S^{\mathbb{V}}_{a,b}$. Then, Expression (6) in page 9 can be written as:

$$\mathcal{D}^{\mathbb{V}}(a,b) = E_1(\dim S^{\mathbb{V}})(a,b).$$

Definition 3.7. Let $\alpha : \mathbb{V} \to \mathbb{U}$ be a morphism between persistence modules. Define $\mathcal{M}^\alpha : \Delta_+ \times \Delta_+ \to \mathbb{Z}_{\geq 0}$ as:

$$\mathcal{M}^\alpha = E_{1,3}(\mathcal{X}^\alpha)$$

or, equivalently,

$$\mathcal{M}^\alpha(a,b,a',b') = {}^{b}_{a}\mathcal{X}^{b'}_{a'} - {}^{\;\;\;b}_{a-1}\mathcal{X}^{b'}_{a'} - {}^{b}_{a}\mathcal{X}^{b'}_{a'-1} + {}^{\;\;\;b}_{a-1}\mathcal{X}^{b'}_{a'-1}.$$

The non-negative integer $\mathcal{M}^\alpha(a,b,a',b')$ can be interpreted as the amount of interval modules $\mathbb{I}[a,b]$ in the decomposition of $\mathbb{V}$ that are "sent" by α to interval modules $\mathbb{I}[a',b']$ in the decomposition of $\mathbb{U}$. Before proving that $\mathcal{M}^\alpha$ is, in fact, a basis-independent partial matching, let us introduce some technical results.

Lemma 3.8. *If $a \leq a'$ then ${}^{b}_{a}\mathcal{X}^{b'}_{a'} = {}^{b}_{a}\mathcal{X}^{b'}_{c}$ for any $c \in \mathbb{Z}_{\geq 0}$ such that $a \leq c \leq a'$.*

Proof. Note that, by definition,

$$ {}^{b}_{a}\mathcal{X}^{b'}_{a'} - {}^{b}_{a}\mathcal{X}^{b'}_{c} = \dim \left[S^{\mathbb{V}}_{a,b} \cap f^{\mathbb{V}}_{b',b}\, \alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'})\right] - \dim \left[S^{\mathbb{V}}_{a,b} \cap f^{\mathbb{V}}_{b',b}\, \alpha^{-1}_{b'}(S^{\mathbb{U}}_{c,b'})\right]. $$

Since $c \leq a'$ then $f^{\mathbb{U}}_{c,b'}(U_c) \subseteq f^{\mathbb{U}}_{a',b'}(U_a)$. Therefore, $S^{\mathbb{U}}_{c,b'} \subseteq S^{\mathbb{U}}_{a',b'}$ then

$$ C = S^{\mathbb{V}}_{a,b} \cap f^{\mathbb{V}}_{b',b}\, \alpha^{-1}_{b'}(S^{\mathbb{U}}_{c,b'}) \subseteq A' = S^{\mathbb{V}}_{a,b} \cap f^{\mathbb{V}}_{b',b}\, \alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'}). $$

Let us see that $A' \subseteq C$. Consider $x \in A'$. Then, in particular, there exists $y \in V_a$ such that $f^{\mathbb{V}}_{a,b}(y) = x$ and there exists $z \in \alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'})$ such that $f^{\mathbb{V}}_{b',b}(z) = x$. Then, necessarily, $f^{\mathbb{V}}_{a,b'}(y) = z$ and $\alpha_{b'}\, f^{\mathbb{V}}_{a,b'}(y) \in S^{\mathbb{U}}_{a',b'} = f^{\mathbb{U}}_{a',b'}(U_{a'}) \cap \ker f^{\mathbb{U}}_{b',b'+1}$. Since $a \leq c \leq a' \leq b'$ then

$$ \alpha_{b'}\, f^{\mathbb{V}}_{a,b'}(y) = \alpha_{b'}\, f^{\mathbb{V}}_{c,b'}\, f^{\mathbb{V}}_{a,c}(y) = f^{\mathbb{U}}_{c,b'}\, \alpha_c\, f^{\mathbb{V}}_{a,c}(y) \in f^{\mathbb{U}}_{c,b'}(U_{a'}) \cap \ker f^{\mathbb{U}}_{b',b'+1} = S^{\mathbb{U}}_{c,b'}. $$

Therefore, $x \in S^{\mathbb{V}}_{a,b} \cap f^{\mathbb{V}}_{b',b}\, \alpha^{-1}_{b'}(S^{\mathbb{U}}_{c,b'}) = C$, concluding the proof. □ □

Lemma 3.9. *If $b' \leq a$ then ${}^{b}_{a}\mathcal{X}^{b'}_{a'} = {}^{b}_{c}\mathcal{X}^{b'}_{a'}$ for any $c \in \mathbb{Z}_{\geq 0}$ such that $b' \leq c \leq a$.*

Proof. Similar to the proof of Lemma 3.8, since $b' \leq c \leq a$, then $S^{\mathbb{V}}_{c,b} \subseteq S^{\mathbb{V}}_{a,b}$ and therefore

$$ C = S^{\mathbb{V}}_{c,b} \cap f^{\mathbb{V}}_{b',b}\ \alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'}) \subseteq A = S^{\mathbb{V}}_{a,b} \cap f^{\mathbb{V}}_{b',b}\ \alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'}). $$

Let us prove that A$\subseteq C$. Note that $f^{\mathbb{V}}_{b',b}\ \alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'}) = f^{\mathbb{V}}_{c,b}\, f^{\mathbb{V}}_{b',c}\ \alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'})$, then

$$ f^{\mathbb{V}}_{b',b}\ \alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'}) \subseteq f^{\mathbb{V}}_{c,b}(V_c). $$

Then,

$$ S^{\mathbb{V}}_{a,b} \cap f^{\mathbb{V}}_{b',b}\, \alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'}) \subseteq \ker f^{\mathbb{V}}_{b,b+1} \cap f^{\mathbb{V}}_{b',b}\, \alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'}) \subseteq f^{\mathbb{V}}_{b,c}(V_c) $$

concluding that $A \subseteq C$. □ □

The following lemma is deduced directly from the definition of the elder rule operator.

Lemma 3.10. *If $i \neq j$ then*

$$ \sum_{a \leq x_j \leq b} E_i(\mathcal{F}) = E_i \left(\sum_{a \leq x_j \leq b} \mathcal{F} \right). $$

Another useful lemmas are the following.

Lemma 3.11. *If $1 \leq i \leq n$ then:*

$$ \begin{aligned} \sum_{a+1 \leq x_i \leq b} E_i(\mathcal{F})(x_1, \ldots, x_{i-1}, x_i, x_{i+1} \ldots, x_n) =& \mathcal{F}(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n) \\ & - \mathcal{F}(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n). \end{aligned} $$

Proof. Writing down the sum we have

$$ \begin{aligned} \sum_{a+1 \leq x_i \leq b} E_i(\mathcal{F})(\ldots, x_i, \ldots) = & \mathcal{F}(\ldots, b, \ldots) - \mathcal{F}(\ldots, b-1, \ldots) \\ & + \mathcal{F}(\ldots, b-1, \ldots) - \mathcal{F}(\ldots, b-2, \ldots) \\ & \vdots \\ & + \mathcal{F}(\ldots, a+1, \ldots) - \mathcal{F}(\ldots, a, \ldots). \end{aligned} $$

Cancelling addends with opposite signs, we obtain the desired result. □

Lemma 3.12. *Let $\mathbb{V}$ be a persistence module of length n. Then, for any $b \in \mathbb{Z}$ with $1 \leq b \leq n$ and any subspace $A_b \subseteq V_b$, we have that:*

$$\dim A_b = \sum_{b \leq i \leq n} \dim \left[f^{\mathbb{V}}_{b,i}(A_b) \cap \ker f^{\mathbb{V}}_{i,i+1}\right].$$

Proof. First, by Remark 2.2, we have that:

$$\dim A_b = \dim \left[f^{\mathbb{V}}_{b,b+1}(A_b)\right] + \dim \left[\,\ker f^{\mathbb{V}}_{b,b+1}\right]. \tag{8}$$

Second, for any $i \in \mathbb{Z}$ with $b \leq i \leq n$ and for any subspace $A_i \subseteq V_i$ we have:

$$\dim A_i = \dim \left[f^{\mathbb{V}}_{i,i+1}(A_i)\right] + \dim \left[A_i \cap \ker f^{\mathbb{V}}_{i,i+1}\right] \tag{9}$$

Then, applying recursively Property (9) to Expression (8), we have:

$$\begin{aligned}
\dim A_b &= \dim \left[f^{\mathbb{V}}_{b,b+1}(A_b)\right] + \dim \left[\,\ker f^{\mathbb{V}}_{b,b+1}\right] \\
&= \dim \left[f^{\mathbb{V}}_{b,b+2}(A_b)\right] + \dim \left[f_{b,b+1}(A_b) \cap \ker f^{\mathbb{V}}_{b+1,b+2}\right] + \dim \left[\,\ker f^{\mathbb{V}}_{b,b+1}\right] \\
&= \cdots \\
&= \dim \left[f^{\mathbb{V}}_{n,n+1} f^{\mathbb{V}}_{b,n}(A_b)\right] + \dim \left[f^{\mathbb{V}}_{b,n}(A_b) \cap \ker f^{\mathbb{V}}_{n-1,n}\right] + \cdots \\
&\quad + \dim \left[f^{\mathbb{V}}_{b,b+1}(A_b) \cap \ker f^{\mathbb{V}}_{b+1,b+2}\right] + \dim \left[\,\ker f^{\mathbb{V}}_{b,b+1}\right].
\end{aligned}$$

Now, since $A_{n+1} = 0$, $f^{\mathbb{V}}_{n,n+1} f^{\mathbb{V}}_{b,n}(A_b) = 0$ and $f_{b,b}$ is the identity, we have that $\ker f^{\mathbb{V}}_{b,b+1} = f^{\mathbb{V}}_{b,b}(A_b) \cap \ker f^{\mathbb{V}}_{b,b+1}$. Then,

$$\dim A_b = \dim \left[f^{\mathbb{V}}_{b,n}(A_b) \cap \ker f^{\mathbb{V}}_{n-1,n}\right] + \cdots + \dim \left[f^{\mathbb{V}}_{b,b}(A_b) \cap \ker f^{\mathbb{V}}_{b,b+1}\right]$$

concluding the proof. □ □

Lemma 3.13. *Let $\alpha : \mathbb{V} \to \mathbb{U}$ be a morphism between persistence modules. Let $a, b \in \mathbb{Z}_{\geq 0}$ with $a \leq b$ and consider a subspace $A_a \subseteq U_a$. Then:*

$$f^{\mathbb{V}}_{a,b}\, \alpha_a^{-1}(A_a) \subseteq \alpha_b^{-1}\, f^{\mathbb{U}}_{a,b}(A_a).$$

Proof. By commutativity and Remark 2.2:

$$f^{\mathbb{V}}_{a,b}\, \alpha_a^{-1}(A_a) \subseteq\; \alpha_b^{-1}\, \alpha_b\, f^{\mathbb{V}}_{a,b}\, \alpha_a^{-1}(A_a) =\; \alpha_b f^{\mathbb{U}}_{a,b}\, \alpha_a\, \alpha_a^{-1}(A_a) =\; \alpha_b f^{\mathbb{U}}_{a,b}(A_A).$$

□ □

Lemma 3.14. *Let α be a morphism between persistence modules. Then:*

$$\sum_{1 \leq a' \leq b'} \mathcal{M}^{\alpha}(a,b,a',b') = {}^{b}_{a}\mathcal{X}^{b'}_{b'} - {}^{\;\;b}_{a-1}\mathcal{X}^{b'}_{b'}$$

and

$$\sum_{1 \leq a \leq b} \mathcal{M}^{\alpha}(a,b,a',b') = {}^{b}_{b}\mathcal{X}^{b'}_{a'} - {}^{b}_{b}\mathcal{X}^{b'}_{a'-1}.$$

Proof. Both relations can be proven in an analogous way. Let us prove the first one. Using Lemmas 3.10 and 3.11 we have:

$$\begin{aligned}
&\sum_{1 \leq a' \leq b'} \mathcal{M}^{\alpha}(a,b,a',b') = \sum_{1 \leq a' \leq b'} E_{1,3}(\mathcal{X}^{\alpha})(a,b,a',b') \\
&= E_1 \sum_{1 \leq a' \leq b'} E_3(\mathcal{X}^{\alpha})(a,b,a',b') = E_1 \left(\mathcal{X}^{\alpha}(a,b,b',b') - \mathcal{X}^{\alpha}(a,b,0,b')\right).
\end{aligned}$$

Using $\mathcal{X}^{\alpha}(a,b,0,b') = 0$ by Remark 3.5, we have that:

$$\sum_{1 \leq a' \leq b'} \mathcal{M}^{\alpha}(a,b,a',b') = E_1\left(\mathcal{X}^{\alpha}\right)(a,b,b',b') = {}^{b}_{a}\mathcal{X}^{b'}_{b'} - {}^{\;\;b}_{a-1}\mathcal{X}^{b'}_{b'}.$$

□ □

Finally, let us prove that $\mathcal{M}^\alpha$ is, in fact, a basis-independent partial matching induced by a morphism α between persistence modules.

Theorem 3.15. *Let α be a morphism between two persistence modules $\mathbb{U}$ and $\mathbb{V}$ of length n. Then,*

$$\sum_{1\le b'\le n}\sum_{1\le a'\le b'}\mathcal{M}^\alpha(a,b,a',b')\le \dim S^{\mathbb{V}}_{a,b}-\dim S^{\mathbb{V}}_{a-1,b}=\mathcal{D}^{\mathbb{V}}(a,b)$$

and

$$\sum_{1\le b'\le n}\sum_{1\le a\le b}\mathcal{M}^\alpha(a,b,a',b')\le \dim S^{\mathbb{U}}_{a',b'}-\dim S^{\mathbb{U}}_{a'-1,b'}=\mathcal{D}^{\mathbb{U}}(a',b').$$

Proof. Let us start with the first inequality. By Lemma 3.14, we have:

$$\sum_{1\le b'\le n}\sum_{1\le a'\le b'}\mathcal{M}^\alpha(a,b,a',b')=\sum_{1\le b'\le n}{}^{\,b}_{a}\mathcal{X}^{b'}_{b'}-{}^{\,b}_{a-1}\mathcal{X}^{b'}_{b'}.$$

By definition,

$$\begin{aligned}{}^{\,b}_{a}\mathcal{X}^{b'}_{b'}-{}^{\,b}_{a-1}\mathcal{X}^{b'}_{b'}=&\dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{b',b'})\right]-\dim\left[S^{\mathbb{V}}_{a-1,b}\cap f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{b',b'})\right]\\ &-\dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b',b}\alpha^{-1}_{b'}(0)\right]+\dim\left[S^{\mathbb{V}}_{a-1,b}\cap f^{\mathbb{V}}_{b',b}\alpha^{-1}_{b'}(0)\right].\end{aligned}\tag{10}$$

Now, denote

$$A=S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b',b}\alpha^{-1}_{b'}(0),\ B=S^{\mathbb{V}}_{a-1,b}\cap f^{\mathbb{V}}_{b',b}\alpha^{-1}_{b'}(0)\text{ and }C=f^{\mathbb{V}}_{b'-1,b}\alpha^{-1}_{b'-1}(S^{\mathbb{U}}_{b'-1,b'-1}).$$

Since $S^{\mathbb{V}}_{a-1,b}\subseteq S^{\mathbb{V}}_{a,b}$ then $B\subseteq A$. Besides, using Lemma 3.13 we have:

$$C=f^{\mathbb{V}}_{b',b}f^{\mathbb{V}}_{b'-1,b'}\,\alpha^{-1}_{b'-1}(S^{\mathbb{U}}_{b'-1,b'-1})\subseteq f_{b',b}\,\alpha^{-1}_{b'}f^{\mathbb{U}}_{b'-1,b'}(S^{\mathbb{U}}_{b'-1,b'-1})=f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(0).$$

Therefore,

$$A\cap C=S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b'-1,b}\alpha^{-1}_{b'-1}(S^{\mathbb{U}}_{b'-1,b'-1})$$

and

$$B\cap C=S^{\mathbb{V}}_{a-1,b}\cap f^{\mathbb{V}}_{b'-1,b}\alpha^{-1}_{b'-1}(S^{\mathbb{U}}_{b'-1,b'-1})$$

and from Remark 2.1 we have:

$$\begin{aligned}&-\dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(0)\right]+\dim\left[S^{\mathbb{V}}_{a-1,b}\cap f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(0)\right]\\ &\le-\dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b'-1,b}\,\alpha^{-1}_{b'-1}(S^{\mathbb{U}}_{b'-1,b'-1})\right]+\dim\left[S^{\mathbb{V}}_{a-1,b}\cap f^{\mathbb{V}}_{b'-1,b}\alpha^{-1}_{b'-1}(S^{\mathbb{U}}_{b'-1,b'-1})\right].\end{aligned}$$

Then, applying this last result to Expression (10):

$$\begin{aligned}&\sum_{1\le b'\le b}{}^{\,b}_{a}\mathcal{X}^{b'}_{b'}-{}^{\,b}_{a-1}\mathcal{X}^{b'}_{b'}\\ &\le\sum_{1\le b'\le b}\dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{b',b'})\right]-\dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b'-1,b}\,\alpha^{-1}_{b'-1}(S^{\mathbb{U}}_{b'-1,b'-1})\right]\\ &\qquad-\dim\left[S^{\mathbb{V}}_{a-1,b}\cap f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{b',b'})\right]+\dim\left[S^{\mathbb{V}}_{a-1,b}\cap f^{\mathbb{V}}_{b'-1,b}\,\alpha^{-1}_{b'-1}(S^{\mathbb{U}}_{b'-1,b'-1})\right].\end{aligned}$$

Cancelling addends with opposite signs, we get:

$$\begin{aligned}\sum_{1\le b'\le b}{}^{\,b}_{a}\mathcal{X}^{b'}_{b'}-{}^{\,b}_{a-1}\mathcal{X}^{b'}_{b'}\le&\dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b,b}\,\alpha^{-1}_{b}(S^{\mathbb{U}}_{b,b})\right]-\dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{0,b}\,\alpha^{-1}_{0}(S^{\mathbb{U}}_{0,0})\right]\\ &-\dim\left[S^{\mathbb{V}}_{a-1,b}\cap f^{\mathbb{V}}_{b,b}\,\alpha^{-1}_{b}(S^{\mathbb{U}}_{b,b})\right]+\dim\left[S^{\mathbb{V}}_{a-1,b}\cap f^{\mathbb{V}}_{0,b}\,\alpha^{-1}_{0}(S^{\mathbb{U}}_{0,0})\right].\end{aligned}$$

Since $f^{\mathbb{V}}_{0,b}$ and $f^{\mathbb{V}}_{b,b}$ are, respectively, the zero and the identity map, we have:

$$\sum_{1\le b'\le b}\sum_{1\le a'\le b'}\mathcal{M}^\alpha(a,b,a',b')\le\dim\left[S^{\mathbb{V}}_{a,b}\cap\alpha^{-1}_{b}(S^{\mathbb{U}}_{b,b})\right]-\dim\left[S^{\mathbb{V}}_{a-1,b}\cap\alpha^{-1}_{b}(S^{\mathbb{U}}_{b,b})\right].$$

Applying again Remark 2.1 with $A = S^{\mathbb{V}}_{a,b}$, $B = S^{\mathbb{V}}_{a-1,b}$ and $C = S^{\mathbb{U}}_{b,b}$, we obtain:

$$\sum_{1\le b'\le b}\sum_{1\le a'\le b'} \mathcal{M}^{\alpha}(a,b,a',b') \le \dim S^{\mathbb{V}}_{a,b} - \dim S^{\mathbb{V}}_{a-1,b} = \mathcal{D}^{\mathbb{V}}(a,b).$$

Besides, since ${}^{b}_{a}\mathcal{X}^{b'}_{a'} = 0$ if $b < b'$ by Remark 3.5, then:

$$\sum_{1\le b'\le n}\sum_{1\le a'\le b'} \mathcal{M}^{\alpha}(a,b,a',b') = \sum_{1\le b'\le b}\sum_{1\le a'\le b'} \mathcal{M}^{\alpha}(a,b,a',b')$$

obtaining the desired result. For the second inequality, we will proceed in a similar way. First, by definition,

$$\begin{aligned}{}^{b}_{a}\mathcal{X}^{b'}_{a'} - {}^{b}_{a}\mathcal{X}^{b'}_{a'-1} =& \dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'})\right] - \dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b',b}(\ker\alpha_{b'})\right]\\ &- \dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a'-1,b'})\right] + \dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b',b}(\ker\alpha_{b'})\right]\\ =& \dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'})\right] - \dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a'-1,b'})\right].\end{aligned}$$

Now, applying Remark 2.1 to

$$A = f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'})\cap\ker f^{\mathbb{V}}_{b},\ B = f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a'-1,b'})\cap\ker f^{\mathbb{V}}_{b} \text{ and } C = S^{\mathbb{V}}_{a,b},$$

we obtain that:

$$\begin{aligned}&\dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'})\right] - \dim\left[S^{\mathbb{V}}_{a,b}\cap f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a'-1,b'})\right]\\ &\le \dim\left[f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'})\cap\ker f^{\mathbb{V}}_{b}\right] - \dim\left[f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a'-1,b'})\cap\ker f^{\mathbb{V}}_{b}\right].\end{aligned} \tag{11}$$

Besides, since ${}^{b}_{a}\mathcal{X}^{b'}_{a'} = 0$ if $b < b'$, again by Remark 3.5, then:

$$\sum_{1\le b\le n}\sum_{1\le a\le b} \mathcal{M}^{\alpha}(a,b,a',b') = \sum_{b'\le b\le n}\sum_{1\le a\le b} \mathcal{M}^{\alpha}(a,b,a',b').$$

By Lemma 3.14 and Expression (11), we have:

$$\begin{aligned}&\sum_{b'\le b\le n}\sum_{1\le a\le b} \mathcal{M}^{\alpha}(a,b,a',b') = \sum_{b'\le b\le n} {}^{b}_{a}\mathcal{X}^{b'}_{a'} - {}^{b}_{a}\mathcal{X}^{b'}_{a'-1}\\ &\le \sum_{b'\le b\le n} \dim\left[f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'})\cap\ker f^{\mathbb{V}}_{b}\right] - \dim\left[f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a'-1,b'})\cap\ker f^{\mathbb{V}}_{b}\right]\end{aligned}$$

and by Lemma 3.12,

$$\begin{aligned}&\sum_{b'\le b\le n} \dim\left[f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'})\cap\ker f^{\mathbb{V}}_{b}\right] - \dim\left[f^{\mathbb{V}}_{b',b}\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a'-1,b'})\cap\ker f^{\mathbb{V}}_{b}\right]\\ &= \dim\left[\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'})\right] - \dim\left[\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a'-1,b'})\right].\end{aligned}$$

Finally, by Remark 2.2:

$$\begin{aligned}&\dim\left[\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'})\right] - \dim\left[\,\alpha^{-1}_{b'}(S^{\mathbb{U}}_{a'-1,b'})\right]\\ &= \dim\left[S^{\mathbb{U}}_{a',b'}\right] + \dim\left[\,\ker\alpha_{b'}\right] - \dim\left[S^{\mathbb{U}}_{a'-1,b'}\right] - \dim\left[\,\ker\alpha_{b'}\right]\\ &= \dim\left[S^{\mathbb{U}}_{a',b'}\right] - \dim\left[S^{\mathbb{U}}_{a'-1,b'}\right] = \mathcal{D}^{\mathbb{U}}(a',b'),\end{aligned}$$

concluding the proof. □ □

Proposition 3.16. $\mathcal{M}^{\alpha}(a,b,a',b') = 0$ *unless* $a' \le a \le b' \le b$.

Proof. By Remark 3.5, if $b < b'$ then ${}^{b}_{a}\mathcal{X}^{b'}_{a'} = 0$ and so $\mathcal{M}^{\alpha}(a,b,a',b') = 0$. If $a < a'$, then, by Lemma 3.8,

$${}^{b}_{a}\mathcal{X}^{b'}_{a'} = {}^{b}_{a}\mathcal{X}^{b'}_{a'-1} \text{ and } {}^{\ \ \ b}_{a-1}\mathcal{X}^{b'}_{a'} = {}^{\ \ \ b}_{a-1}\mathcal{X}^{b'}_{a'-1}.$$

Similarly, when $b' < a$, by Lemma 3.9,

$${}^{b}_{a}\mathcal{X}^{b'}_{a'} = {}^{\ \ \ b}_{a-1}\mathcal{X}^{b'}_{a'} \text{ and } {}^{b}_{a}\mathcal{X}^{b'}_{a'-1} = {}^{\ \ \ b}_{a-1}\mathcal{X}^{b'}_{a'-1}.$$

Then $\mathcal{M}^{\alpha}(a,b,a',b') = 0$ in both cases. □ □

Recall that ladder modules of type $\tau = ff\dots f$ and morphism between persistence modules are different points of view of the same concept. We will write $\mathcal{M}^{\mathbb{L}}$ instead of $\mathcal{M}^{\alpha}$ when we want to focus on the point of view of ladder modules.

Remark 3.17. Denote the collection of ladder modules of type $\tau = ff\dots f$ by PLM and define $\Delta^4_+ = \{(a,b,a',b') \in \mathbb{Z}^4_{\geq 0}$ such that $a' \leq a \leq b' \leq b\}$. By the previous proposition, we can think in $\mathcal{M}$ as a function

$$\mathcal{M} : PLM \times \Delta^4_+ \to \mathbb{Z}_{\geq 0}$$

such that $\mathcal{M}(\mathbb{L}, *) = \mathcal{M}^{\mathbb{L}}(*)$. Let PM denote the collection of persistence modules, then let $\mathcal{D}$ denote the function:

$$\mathcal{D} : PM \times \Delta_+ \to \mathbb{Z}_{\geq 0}$$

such that $\mathcal{D}(\mathbb{U}, *) = \mathcal{D}^{\mathbb{U}}(*)$. This way, $\mathcal{M}$ could be seen as a generalization of persistence barcodes for ladder modules of type $\tau = ff\dots f$.

Before giving an example, let us prove that $\mathcal{M}$ is linear with respect to the direct sum of ladder modules.

Proposition 3.18 (Linearity of the function $\mathcal{M}$)**.** *Let $\mathbb{L}, \mathbb{L}_1, \mathbb{L}_2$ be ladders modules of type $\tau = ff\dots f$ such that $\mathbb{L} \simeq \mathbb{L}_1 \oplus \mathbb{L}_2$. Then,*

$$\mathcal{M}^{\mathbb{L}} = \mathcal{M}^{\mathbb{L}_1} + \mathcal{M}^{\mathbb{L}_2}.$$

Proof. It is enough to prove that ${}^{a}_{b}\mathcal{X}^{b'}_{a'}(\mathbb{L}) = {}^{a}_{b}\mathcal{X}^{b'}_{a'}(\mathbb{L}_1) + {}^{a}_{b}\mathcal{X}^{b'}_{a'}(\mathbb{L}_2)$. Notice that since $\mathbb{L} \simeq \mathbb{L}_1 \oplus \mathbb{L}_2$ we can decompose $V_a = (V_a)_1 \oplus (V_a)_2$, $U_{a'} = (U_{a'})_1 \oplus (U_{a'})_2$ and $\alpha_a = (\alpha_a)_1 \oplus (\alpha_a)_2$. Then,

$$\begin{aligned} S^{\mathbb{V}}_{a,b} &= f^{\mathbb{V}}_{a,b}((V_a)_1 \oplus (V_a)_2) \cap \ker f^{\mathbb{V}}_b \\ &= \left(f^{\mathbb{V}}_{a,b}(V_a)_1 \cap \ker f^{\mathbb{V}}_b\right) \oplus \left(f^{\mathbb{V}}_{a,b}(V_a)_1 \cap \ker f^{\mathbb{V}}_b\right) \\ &= (S^{\mathbb{V}}_{a,b})_1 \oplus (S^{\mathbb{V}}_{a,b})_2 \end{aligned}$$

Analogously, $S^{\mathbb{U}}_{a',b'} = (S^{\mathbb{U}}_{a',b'})_1 \oplus (S^{\mathbb{U}}_{a',b'})_2$. We also have

$$f^{\mathbb{V}}_{b',b}\, \alpha^{-1}_{b'} \left((S^{\mathbb{U}}_{a',b'})_1 \oplus (S^{\mathbb{U}}_{a',b'})_2\right) = f^{\mathbb{V}}_{b',b}(\alpha_{b'})^{-1}_1 (S^{\mathbb{U}}_{a',b'})_1 \oplus f^{\mathbb{V}}_{b',b}(\alpha_{b'})^{-1}_2 (S^{\mathbb{U}}_{a',b'})_2$$

Since $(S^{\mathbb{U}}_{a',b'})_1 \cap (S^{\mathbb{U}}_{a',b'})_2 = 0$, we get:

$$\begin{aligned} {}^{a}_{b}\mathcal{X}^{b'}_{a'}(\mathbb{L}) &= \dim\left[S^{\mathbb{V}}_{a,b} \cap f^{\mathbb{V}}_{b',b}\, \alpha^{-1}_{b'}(S^{\mathbb{U}}_{a',b'})\right] - \dim\left[S^{\mathbb{V}}_{a,b} \cap f^{\mathbb{V}}_{b',b}\, \alpha^{-1}_{b'}(0)\right] \\ &= \dim\left[\left((S^{\mathbb{V}}_{a,b})_1 \oplus (S^{\mathbb{V}}_{a,b})_2\right) \cap \left(f^{\mathbb{V}}_{b',b}(\alpha_{b'})^{-1}_1 (S^{\mathbb{U}}_{a',b'})_1 \oplus f^{\mathbb{V}}_{b',b}(\alpha_{b'})^{-1}_2 (S^{\mathbb{U}}_{a',b'})_2\right)\right] \\ &\quad - \dim\left[\left((S^{\mathbb{V}}_{a,b})_1 \oplus (S^{\mathbb{V}}_{a,b})_2\right) \cap \alpha^{-1}_{b'}(0)\right] \\ &= \dim\left[\left((S^{\mathbb{V}}_{a,b})_1 \cap f^{\mathbb{V}}_{b',b}(\alpha_{b'})^{-1}_1 (S^{\mathbb{U}}_{a',b'})_1\right) \oplus \left((S^{\mathbb{V}}_{a,b})_2 \cap f^{\mathbb{V}}_{b',b}(\alpha_{b'})^{-1}_2 (S^{\mathbb{U}}_{a',b'})_2\right)\right] \\ &\quad - \dim\left[\left((S^{\mathbb{V}}_{a,b})_1 \cap (\alpha_{b'})^{-1}_1(0)\right) \oplus \left((S^{\mathbb{V}}_{a,b})_2 \cap (\alpha_{b'})^{-1}_2(0)\right)\right] \\ &= \dim\left[(S^{\mathbb{V}}_{a,b})_1 \cap f^{\mathbb{V}}_{b',b}(\alpha_{b'})^{-1}_1 (S^{\mathbb{U}}_{a',b'})_1\right] + \dim\left[(S^{\mathbb{V}}_{a,b})_2 \cap f^{\mathbb{V}}_{b',b}(\alpha_{b'})^{-1}_2 (S^{\mathbb{U}}_{a',b'})_2\right] \\ &\quad - \dim\left[(S^{\mathbb{V}}_{a,b})_1 \cap (\alpha_{b'})^{-1}_1(0)\right] - \dim\left[(S^{\mathbb{V}}_{a,b})_2 \cap (\alpha_{b'})^{-1}_2(0)\right] \\ &= {}^{a}_{b}\mathcal{X}^{b'}_{a'}(\mathbb{L}_1) + {}^{a}_{b}\mathcal{X}^{b'}_{a'}(\mathbb{L}_2). \end{aligned}$$

□ □

Example 3.19. Recall that the ladder module $\mathbb{A}$ from Example 2.16 can be decomposed as follows:

$$\mathbb{A} \simeq \begin{smallmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix} = \mathbb{L}_1 \oplus \mathbb{L}_2 \oplus \mathbb{L}_3.$$

By Proposition 3.18, we have:

$$\mathcal{M}^{\mathbb{A}} = \mathcal{M}^{\mathbb{L}_1} + \mathcal{M}^{\mathbb{L}_2} + \mathcal{M}^{\mathbb{L}_3}.$$

In the case of $\mathbb{L}_1$, observe that $\mathbb{V} \simeq \mathbb{I}[2,3]$ and $\mathbb{U} \simeq \mathbb{I}[1,2] \oplus \mathbb{I}[2,3]$. Then,

$$\begin{aligned} &S^{\mathbb{V}}_{2,3} = \mathbf{F}, \qquad S^{\mathbb{U}}_{1,2} = \langle\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\rangle, \qquad S^{\mathbb{U}}_{2,3} = \mathbf{F}, \\ &f^{\mathbb{V}}_{2,3}\, \alpha^{-1}_2(S^{\mathbb{U}}_{1,2}) = 0 \quad \text{and} \quad f^{\mathbb{V}}_{3,3}\, \alpha^{-1}_3(S^{\mathbb{U}}_{2,3}) = \mathbf{F}. \end{aligned}$$

In addition, $\ker\alpha_2 = \ker\alpha_3 = 0$. The only non-trivial calculations for $\mathcal{X}$ is:

$$ {}^3_2\mathcal{X}^3_2 = \dim\left[S^{\mathbb{V}}_{2,3} \cap f^{\mathbb{V}}_{3,3}\,\alpha_3^{-1}(S^{\mathbb{U}}_{2,3})\right] - \dim\left[S^{\mathbb{V}}_{2,3} \cap f^{\mathbb{V}}_{3,3}(\ker\alpha_3)\right] = 1 - 0 = 1. $$

Then, the resulting matching is:

$$ \mathcal{M}^{\mathbb{L}_1}(2,3,2,3) = {}^3_2\mathcal{X}^3_2 - {}^3_1\mathcal{X}^3_2 - {}^3_2\mathcal{X}^3_1 + {}^3_1\mathcal{X}^3_1 = 1 - 0 - 0 + 0 = 1 $$

and $\mathcal{M}^{\mathbb{L}_1}(a,b,a',b') = 0$ otherwise. Similar calculation gives $M^{\mathbb{L}_2}(2,3,1,2) = 1$ and $M^{\mathbb{L}_2}(a,b,a',b') = 0$ otherwise. $M^{\mathbb{L}_3}(*)$ is always zero. Finally,

$$ \mathcal{M}^{\mathbb{A}}(2,3,2,3) = 1, \qquad \mathcal{M}^{\mathbb{A}}(2,3,1,2) = 1 \quad \text{and} \quad \mathcal{M}^{\mathbb{A}}(a,b,a',b') = 0 \text{ otherwise.} $$

Let us notice that, in this case, $\mathcal{M}^{\mathbb{A}} = \mathcal{M}^{\alpha}_{BL}$, where $\mathcal{M}^{\alpha}_{BL}$ is the basis-independent partial matching obtained in Example 3.2 in page 9. Nevertheless, $\mathcal{M}^{\mathbb{A}}$ and $\mathcal{M}^{\alpha}_{BL}$ do not always coincide, as the following example shows.

Example 3.20. Consider again the ladder module $\mathbb{A}$ of Example 2.23 but replacing α_2 by $\left[\begin{smallmatrix}1&0&0\\1&0&0\\0&1&0\end{smallmatrix}\right]$. As mentioned in that example, $\mathcal{M}^{\mathbb{A}}_{BL}$ will not change since $\mathbb{V}$, $\mathbb{U}$ and $\operatorname{im}\alpha$ remain the same up to isomorphism. Nevertheless, now

$$ \mathbb{A} \simeq \begin{smallmatrix}1&2&1\\0&1&1\end{smallmatrix} \oplus \begin{smallmatrix}1&1&0\\0&1&0\end{smallmatrix} \oplus \begin{smallmatrix}0&0&0\\0&1&1\end{smallmatrix} $$

and applying Proposition 3.18, we have:

$$ \mathcal{M}^{\mathbb{A}} = \mathcal{M}^{\mathbb{L}_1} + \mathcal{M}^{\mathbb{L}_2} + \mathcal{M}^{\mathbb{L}_3} $$

with $M^{\mathbb{L}_i}(*)$ being zero for $i = 1,2,3$, except for

$$ \mathcal{M}^{\mathbb{L}_1}(2,3,2,3) = 1 \quad \text{and} \quad \mathcal{M}^{\mathbb{L}_2}(2,2,1,2) = 1. $$

Then

$$ \mathcal{M}^{\mathbb{A}}(2,3,2,3) = 1, \qquad \mathcal{M}^{\mathbb{A}}(2,2,1,2) = 1 \quad \text{and} \quad \mathcal{M}^{\mathbb{A}}(a,b,a',b') = 0 \text{ otherwise.} $$

Observe that, in this case, $\mathcal{M}^{\mathbb{A}} \neq \mathcal{M}^{\alpha}_{BL}$ where $\mathcal{M}^{\alpha}_{BL}$ is the basis-independent partial matching obtained in Example 3.2 in page 9. In particular, $\mathcal{M}^{\mathbb{A}}$ is the basis-independent partial matching that we might expect by looking at the decomposition of the ladder module $\mathbb{A}$.

Remark 3.21. Taking into account Remark 3.17 we might ask: does $\mathcal{M}$ characterize up to isomorphism ladder modules as $\mathcal{D}$ characterizes up to isomorphism persistence modules? Unfortunately, the answer is no. For example, ladder modules $\mathbb{A} = \begin{smallmatrix}1&2&1\\0&1&1\end{smallmatrix}$ and $\mathbb{B} = \begin{smallmatrix}1&1&0\\0&0&0\end{smallmatrix} \oplus \begin{smallmatrix}0&1&1\\0&1&1\end{smallmatrix}$ are not isomorphic but $\mathcal{M}^{\mathbb{A}} = \mathcal{M}^{\mathbb{B}}$.

4 Enriched partial matchings

The aim of this section is to compute a basis-independent partial matching between persistence modules $\mathbb{V}$ and $\mathbb{U}$ from a pair of morphism $\mathbb{V} \xrightarrow{\alpha} \mathbb{W} \xleftarrow{\beta} \mathbb{U}$ between persistence modules. As said in the introduction, this problem is related to the persistence module $\mathbb{K} \subset \mathbb{W}$ described in [16] which can be seen as the persistence submodules of $\mathbb{U}$ and $\mathbb{V}$ that "matches" and "persists" in $\mathbb{W}$ through the morphisms α and β. Before explaining our approach, let us introduce a motivating example.

Example 4.1. Consider the following diagrams:

$$ \begin{array}{ccccccccc}
\mathbf{F} & \xrightarrow{1} & \mathbf{F} & \xrightarrow{1} & \dots & \xrightarrow{1} & \mathbf{F} & \xrightarrow{1} & \mathbf{F} \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 0} \\
\mathbf{F} & \xrightarrow{1} & \mathbf{F} & \xrightarrow{1} & \dots & \xrightarrow{1} & \mathbf{F} & \xrightarrow{0} & 0 \\
\uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 1} & & & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 0} \\
\mathbf{F} & \xrightarrow{1} & \mathbf{F} & \xrightarrow{1} & \dots & \xrightarrow{1} & \mathbf{F} & \xrightarrow{1} & \mathbf{F}
\end{array} \tag{12} $$

$$
\begin{array}{ccccccccc}
\mathbf{F} & \xrightarrow{1} & \mathbf{F} & \xrightarrow{1} & \dots & \xrightarrow{1} & \mathbf{F} & \xrightarrow{1} & \mathbf{F} \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
\mathbf{F} & \xrightarrow{0} & 0 & \xrightarrow{0} & \dots & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 \\
\uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle 0} & & & & {\scriptstyle 0}\uparrow & & \uparrow{\scriptstyle 0} \\
\mathbf{F} & \xrightarrow{1} & \mathbf{F} & \xrightarrow{1} & \dots & \xrightarrow{1} & \mathbf{F} & \xrightarrow{1} & \mathbf{F}
\end{array}
\tag{13}
$$

In both of them, the upper and lower persistence modules consist of the interval module $\mathbb{I}[1, n]$. The top and bottom interval modules "should be matched" if we compute partial matchings induced by any of the two diagrams. Nevertheless, we should make a difference between both situations: in Diagram (13), a "minor change" of the persistence module in the middle "may remove" the matching while Diagram (12) is much more robust. A solution could be giving "persistence" to our matching: in the first case, the persistence of the matching should be $[1, n-1]$ and, in the second case, $[1, 2]$.

Let us now give a new definition of partial matching that takes into account the idea of "persistent matchings". Recall that we denoted the collection of persistence barcodes as **Б**.

Definition 4.2. An **enriched partial matching** between two persistence modules $\mathbb{U}, \mathbb{V}$ of length n is a map

$$\mathcal{G}^{\mathbb{U}}_{\mathbb{V}} : \Delta_+ \times \Delta_+ \to \mathbf{Б}$$

such that:

$$\mathcal{G}^{\mathbb{U}}_{\mathbb{V}}(a, b, a', b') = \emptyset \;\text{ if }\; n < a, a', b, b',$$

$$\sum_{1 \le b' \le n} \sum_{1 \le a' \le b'} \#\mathcal{G}^{\mathbb{U}}_{\mathbb{V}}(a, b, a', b') \le \mathcal{D}^{\mathbb{V}}(a, b) \;\text{ and}$$

$$\sum_{1 \le b \le n} \sum_{1 \le a \le b} \#\mathcal{G}^{\mathbb{U}}_{\mathbb{V}}(a, b, a', b') \le \mathcal{D}^{\mathbb{U}}(a', b').$$

Its associated basis-independent partial matching is:

$$\mathcal{M}^{\mathbb{U}}_{\mathbb{V}}(a, b, a', b') = \#\mathcal{G}^{\mathbb{U}}_{\mathbb{V}}(a, b, a', b').$$

Example 4.3. An enriched partial matching of Diagram (12) in Example 4.1 is:

$$\mathcal{G}^{\mathbb{U}}_{\mathbb{V}}(1, n, 1, n) = \{([1, n-1], 1)\} \quad \text{and} \quad \mathcal{G}^{\mathbb{U}}_{\mathbb{V}}(a, b, a', b') = \emptyset \text{ otherwise.}$$

4.1 Enriched partial matchings induced by morphisms between persistence modules

Our motivation in this subsection is to obtain an enriched partial matching induced by the following diagram:

$$
\begin{array}{ccccccc}
U_1 & \xrightarrow{f^{\mathbb{U}}_1} & U_2 & \xrightarrow{f^{\mathbb{U}}_2} & \dots & \xrightarrow{f^{\mathbb{U}}_{n-1}} & U_n \\
{\scriptstyle \beta_1}\downarrow & & {\scriptstyle \beta_2}\downarrow & & & & {\scriptstyle \beta_n}\downarrow \\
W_1 & \xrightarrow{f^{\mathbb{W}}_1} & W_2 & \xrightarrow{f^{\mathbb{W}}_2} & \dots & \xrightarrow{f^{\mathbb{W}}_{n-1}} & W_n \\
{\scriptstyle \alpha_1}\uparrow & & {\scriptstyle \alpha_2}\uparrow & & & & {\scriptstyle \alpha_n}\uparrow \\
V_1 & \xrightarrow{f^{\mathbb{V}}_1} & V_2 & \xrightarrow{f^{\mathbb{V}}_2} & \dots & \xrightarrow{f^{\mathbb{V}}_{n-1}} & V_n
\end{array}
\tag{14}
$$

Following a similar procedure than before, we will study the relation between the subspaces $S^{\mathbb{V}}_{a,b}$ and $S^{\mathbb{U}}_{a',b'}$ through the morphisms α and β. Consider the following subspaces:

$$R(a, b, c, d) = \begin{cases} \alpha_d \left(f^{\mathbb{V}}_{a,d}(V_a) \cap \ker f^{\mathbb{V}}_{d,b+1} \right) \cap S^{\mathbb{W}}_{c,d} & \text{if } a, c \le d \le b, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$L(a', b', c, d) = \begin{cases} \beta_d \left(f^{\mathbb{U}}_{a',d}(U_{a'}) \cap \ker f^{\mathbb{U}}_{d,b'+1} \right) \cap S^{\mathbb{W}}_{c,d} & \text{if } a', c \le d \le b', \\ 0 & \text{otherwise.} \end{cases}$$

Taking into account that:

$$f^{\mathbb{V}}_{d,b}\left(f^{\mathbb{V}}_{a,d}(V_a)\cap\ker f^{\mathbb{V}}_{d,b+1}\right)=S^{\mathbb{V}}_{a,b}\quad\text{and}$$
$$f^{\mathbb{V}}_{a,d}(V_a)\cap\ker f^{\mathbb{V}}_{d,b+1}=f^{\mathbb{V}}_{a,d}(V_a)\cap(f^{\mathbb{V}}_{d,b})^{-1}\big(S^{\mathbb{V}}_{a,b}\big),$$

the intersection $R(a,b,c,d)\cap L(a',b',c,d)$ may be interpreted as the common subspace of $S^{\mathbb{V}}_{a,b}$ and $S^{\mathbb{U}}_{a',b'}$ that "persists" in the subspace $S^{\mathbb{W}}_{c,d}$.

Lemma 4.4. *The following relations hold:*

$$R(a,b,c,d)=f^{\mathbb{W}}_{a,d}\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b+1}\right)\cap S^{\mathbb{W}}_{c,d}\quad\textit{and}$$
$$L(a',b',c,d)=f^{\mathbb{W}}_{a',d}\,\beta_{a'}\left(\ker f^{\mathbb{U}}_{a',b'+1}\right)\cap S^{\mathbb{W}}_{c,d}.$$

Proof. First, observe that:

$$V_a\cap\ker f^{\mathbb{V}}_{a,b+1}=\ker f^{\mathbb{V}}_{a,b+1}\quad\text{and}\quad f^{\mathbb{V}}_{a,d}(V_a)\cap\ker f^{\mathbb{V}}_{d,b+1}=f^{\mathbb{V}}_{a,d}(\ker f^{\mathbb{V}}_{a,b+1}).$$

Then,

$$R(a,b,c,d)=\alpha_d\,f^{\mathbb{V}}_{a,d}\left(\ker f^{\mathbb{V}}_{a,b+1}\right)\cap S^{\mathbb{W}}_{c,d}\quad\text{and}$$
$$L(a',b',c,d)=\beta_d\,f^{\mathbb{U}}_{a',d}\left(\ker f^{\mathbb{U}}_{a',b'+1}\right)\cap S^{\mathbb{W}}_{c,d}.$$

By commutativity of Diagram (14), we obtain the desired result. □ □

As in the previous section, we need to "count" the dimensions before applying the elder rule operator E. Define the function $\mathcal{Y}:\mathbb{Z}^6_{\geq 0}\to\mathbb{Z}_{\geq 0}$ as follows:

$$\mathcal{Y}(a,b,a',b',c,d)=\dim(R(a,b,c,d)\cap L(a',b',c,d)).$$

Notice that if $a>b'$ or $a'>b$ then

$$\mathcal{Y}(a,b,a',b',c,d)=0.$$

Now, apply the elder rule operator E to $\mathcal{Y}$ five times to define the enriched partial matching.

Definition 4.5. The **enriched partial matching induced by morphisms** $\mathbb{V}\xrightarrow{\alpha}\mathbb{W}\xleftarrow{\beta}\mathbb{U}$ between persistence modules of length n, is the map

$$\mathcal{G}^{\alpha,\beta}:\Delta_+\times\Delta_+\to\mathbf{Б}$$

defined as:

$$\mathcal{G}^{\alpha,\beta}(a,b,a',b')=\{([c,d],e_{c,d}):e_{c,d}>0\}$$

where $(c,d)\in\Delta_+$ and $e_{c,d}=E_{1,2,3,4,5}(\mathcal{Y})(a,b,a',b',c,d)$.

Theorem 4.6. *Fix a pair of morphisms* $\mathbb{V}\xrightarrow{\alpha}\mathbb{W}\xleftarrow{\beta}\mathbb{U}$ *between persistence modules of length n. The operator $\mathcal{G}^{\alpha,\beta}$ is, in fact, an enriched partial matching. In other words, $\mathcal{G}^{\alpha,\beta}(a,b,a',b')=\emptyset$ if $n<a,a',b,b'$,*

$$\sum_{1\leq b'\leq n}\sum_{1\leq a'\leq b'}\#\mathcal{G}^{\alpha,\beta}(a,b,a',b')\leq\mathcal{D}^{\mathbb{V}}(a,b)\quad\textit{and}$$
$$\sum_{1\leq b\leq n}\sum_{1\leq a\leq b}\#\mathcal{G}^{\alpha,\beta}(a,b,a',b')\leq\mathcal{D}^{\mathbb{U}}(a',b').$$

Proof. The property $\mathcal{G}^{\alpha,\beta}(a,b,a',b')=\emptyset$ if $n<a,a',b,b'$ can be followed directly from the definition of $\mathcal{G}^{\alpha,\beta}$. In order to prove the above inequalities, note that

$$\sum_{1\leq b\leq n}\sum_{1\leq a\leq b}\#\mathcal{G}^{\alpha,\beta}(a,b,a',b')=\sum_{1\leq b\leq n}\sum_{1\leq a\leq b}\#\mathcal{G}^{\beta,\alpha}(a',b',a,b).$$

Therefore, we only have to proof one of the two equalities. Let us prove the first one. Let $m = \max\{a, a'\}$ and $M = \min\{b, b'\}$. Observe that:

$$\begin{aligned}\#\mathcal{G}^{\alpha,\beta}(a,b,a',b') &= \#\left\{([c,d],e) \text{ such that } m \le d \le M \text{ and } c \le d\right\} \\ &= \sum_{m\le d\le M}\sum_{1\le c\le d} e_{c,d} = \sum_{m\le d\le M}\sum_{1\le c\le d} E_{1,2,3,4,5}(\mathcal{Y})(a,b,a',b',c,d).\end{aligned}$$

Notice that $\mathcal{Y}(a,b,a',b',0,d) = 0$. By Lemma 3.11, we have:

$$\#\mathcal{G}^{\alpha,\beta}(a,b,a',b') = \sum_{m\le d\le M} E_{1,2,3,4}(\mathcal{Y})(a,b,a',b',d,d).$$

In particular, since $a \le m$, $M \le b$ and all addends are positive, then:

$$\#\mathcal{G}^{\alpha,\beta}(a,b,a',b') \le \sum_{a\le d\le b} E_{1,2,3,4}(\mathcal{Y})(a,b,a',b',d,d).$$

By Remark 3.10, we have

$$\begin{aligned}&\sum_{d\le b'\le n}\sum_{1\le a'\le d}\sum_{a\le d\le b} E_{1,2,3,4}(\mathcal{Y})(a,b,a',b',d,d) \\ &\qquad = \sum_{d\le b'\le n}\sum_{1\le a'\le d} E_{3,4}\left(\sum_{a\le d\le b} E_{1,2}(\mathcal{Y})\right)(a,b,a',b',d,d).\end{aligned}$$

Again, by Lemma 3.11 and since

$$\mathcal{Y}(a,b,d,d-1,d,d) = \mathcal{Y}(a,b,b',0,d,d) = 0,$$

we obtain:

$$\#\mathcal{G}^{\alpha,\beta}(a,b,a',b') \le \sum_{a\le d\le b} E_{1,2}\mathcal{Y}(a,b,d,n,d,d).$$

By Lemma 4.4, we have:

$$\begin{aligned}\mathcal{Y}(a,b,d,n,d,d) &= \dim(R(a,b,d,d)\cap L(d,n,d,d)) \\ &= \dim\left(f^{\mathbb{W}}_{a,d}\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b+1}\right)\cap f^{\mathbb{W}}_{d,d}\beta_d\left(\ker f^{\mathbb{U}}_{d,n+1}\right)\cap S^{\mathbb{W}}_{d,d}\right) \\ &= \dim\left(f^{\mathbb{W}}_{a,d}\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b+1}\right)\cap \beta_d\left(U_d\right)\cap S^{\mathbb{W}}_{d,d}\right) \\ &= \dim\left(f^{\mathbb{W}}_{a,d}\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b+1}\right)\cap \beta_d\left(U_d\right)\cap \ker f^{\mathbb{W}}_{d}\right).\end{aligned}$$

Notice that

$$\begin{aligned}&f^{\mathbb{W}}_{a-1,d}\,\alpha_{a-1}\left(\ker f^{\mathbb{V}}_{a-1,b+1}\right) \subset f^{\mathbb{W}}_{a,d}\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b+1}\right), \\ &f^{\mathbb{W}}_{a,d}\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b}\right) \subset f^{\mathbb{W}}_{a,d}\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b+1}\right).\end{aligned}$$

Let

$$\begin{aligned}A &= f^{\mathbb{W}}_{a,d}\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b+1}\right)\cap\ker f^{\mathbb{W}}_{d}, \\ B &= f^{\mathbb{W}}_{a-1,d}\,\alpha_{a-1}\left(\ker f^{\mathbb{V}}_{a-1,b+1}\right)\cap\ker f^{\mathbb{W}}_{d}, \\ C &= \beta_d\left(U_d\right).\end{aligned}$$

Then $\dim(A\cap C) - \dim(B\cap C) \le \dim(A) - \dim(B)$ by Remark 2.1. Again, let

$$B' = f^{\mathbb{W}}_{a,d}\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b}\right)\cap\ker f^{\mathbb{W}}_{d}.$$

Then, $\dim(A\cap C) - \dim(B'\cap C) \le \dim(A) - \dim(B')$ by Remark 2.1. Putting together both inequalities, we obtain:

$$2\dim(A\cap C) - \dim(B\cap C) - \dim(B'\cap C) \le 2\dim(A) - \dim(B) - \dim(B'),$$

and, in particular,

$$\dim(A\cap C)-\dim(B\cap C)-\dim(B'\cap C)\leq \dim(A)-\dim(B)-\dim(B'). \tag{15}$$

Therefore,

$$\begin{aligned}E_{1,2}\mathcal{Y}(a,b,d,n,d,d) &= \mathcal{Y}(a,b,d,n,d,d)-\mathcal{Y}(a-1,b,d,n,d,d)\\ &\quad -\mathcal{Y}(a,b-1,d,n,d,d)+\mathcal{Y}(a-1,b-1,d,n,d,d)\\ &= \dim(A\cap C)-\dim(B\cap C)-\dim(B'\cap C)\\ &\quad +\mathcal{Y}(a-1,b-1,d,n,d,d).\end{aligned}$$

Using Expression (15) and that

$$\mathcal{Y}(a-1,b-1,d,n,d,d)\leq \dim\left(f^{\mathbb{W}}_{a-1,d}\,\alpha_{a-1}\left(\ker f^{\mathbb{V}}_{a-1,b}\right)\cap \ker f^{\mathbb{W}}_{d}\right),$$

we obtain:

$$\begin{aligned}E_{1,2}\mathcal{Y}(a,b,d,n,d,d) &\leq \dim\left(f^{\mathbb{W}}_{a,d}\,\alpha_{a-1}\left(\ker f^{\mathbb{V}}_{a,b+1}\right)\cap \ker f^{\mathbb{W}}_{d}\right)\\ &\quad -\dim\left(f^{\mathbb{W}}_{a-1,d}\,\alpha_{a-1}\left(\ker f^{\mathbb{V}}_{a-1,b+1}\right)\cap \ker f^{\mathbb{W}}_{d}\right)\\ &\quad -\dim\left(f^{\mathbb{W}}_{a,d}\,\alpha_{a}\left(\ker f^{\mathbb{V}}_{a,b}\right)\cap \ker f^{\mathbb{W}}_{d}\right)\\ &\quad +\dim\left(f^{\mathbb{W}}_{a-1,d}\,\alpha_{a-1}\left(\ker f^{\mathbb{V}}_{a-1,b}\right)\cap \ker f^{\mathbb{W}}_{d}\right).\end{aligned} \tag{16}$$

Besides, by Lemma 3.12, we have:

$$\dim\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b+1}\right)=\sum_{a\leq i\leq n}\dim\left[f^{\mathbb{W}}_{a,i}\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b+1}\right)\cap\ker f^{\mathbb{W}}_{i}\right]$$

that is equal to

$$\sum_{a\leq i\leq b}\dim\left[f^{\mathbb{W}}_{a,i}\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b+1}\right)\cap\ker f^{\mathbb{W}}_{i}\right]$$

since $f^{\mathbb{W}}_{a,i}\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b+1}\right)=0$ for $i\geq b+1$. Extending this reasoning to the other cases, we have:

$$\begin{aligned}\dim f^{\mathbb{W}}_{a-1,a}\,\alpha_{a-1}\left(\ker f^{\mathbb{V}}_{a,b+1}\right) &= \sum_{a\leq i\leq b}\dim\left[f^{\mathbb{W}}_{a-1,i}\,\alpha_{a-1}\left(\ker f^{\mathbb{V}}_{a-1,b+1}\right)\cap\ker f^{\mathbb{W}}_{i}\right],\\ \dim\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b}\right) &= \sum_{a\leq i\leq b-1}\dim\left[f^{\mathbb{W}}_{a,i}\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b}\right)\cap\ker f^{\mathbb{W}}_{i}\right]\\ &= \sum_{a\leq i\leq b}\dim\left[f^{\mathbb{W}}_{a,i}\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b}\right)\cap\ker f^{\mathbb{W}}_{i}\right],\\ \dim f^{\mathbb{W}}_{a-1,a}\,\alpha_{a-1}\left(\ker f^{\mathbb{V}}_{a-1,b}\right) &= \sum_{a\leq i\leq b-1}\dim\left[f^{\mathbb{W}}_{a-1,i}\,\alpha_{a-1}\left(\ker f^{\mathbb{V}}_{a-1,b}\right)\cap\ker f^{\mathbb{W}}_{i}\right]\\ &= \sum_{a\leq i\leq b}\dim\left[f^{\mathbb{W}}_{a-1,i}\,\alpha_{a-1}\left(\ker f^{\mathbb{V}}_{a-1,b+1}\right)\cap\ker f^{\mathbb{W}}_{i}\right].\end{aligned}$$

Using Expression (16) and the relations above, we have:

$$\begin{aligned}\sum_{a\leq d\leq b}E_{1,2}\mathcal{Y}(a,b,d,n,d,d) &\leq \dim\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b+1}\right)-\dim f^{\mathbb{W}}_{a-1,a}\,\alpha_{a-1}\left(\ker f^{\mathbb{V}}_{a-1,b+1}\right)\\ &\quad -\dim\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b}\right)+\dim f^{\mathbb{W}}_{a-1,a}\,\alpha_{a-1}\left(\ker f^{\mathbb{V}}_{a-1,b}\right).\end{aligned}$$

Using the second expression of Remark 2.2 we have:

$$\dim\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b+1}\right)-\dim\,\alpha_a\left(\ker f^{\mathbb{V}}_{a,b}\right)=\dim\ker f^{\mathbb{V}}_{a,b+1}-\dim\ker f^{\mathbb{V}}_{a,b}$$

which is equal to $\dim S^{\mathbb{V}}_{a,b}$ by definition. Analogously,

$$\begin{aligned}&-\dim f^{\mathbb{W}}_{a-1,a}\,\alpha_{a-1}\left(\ker f^{\mathbb{V}}_{a-1,b+1}\right)+\dim f^{\mathbb{W}}_{a-1,a}\,\alpha_{a-1}\left(\ker f^{\mathbb{V}}_{a-1,b}\right)\\ &\qquad =-\dim\ker f^{\mathbb{V}}_{a-1,b+1}+\dim\ker f^{\mathbb{V}}_{a-1,b}=-\dim S^{\mathbb{V}}_{a-1,b}\end{aligned}$$

Finally, putting all together

$$\sum_{1\leq b'\leq n}\sum_{1\leq a'\leq b'} \#\mathcal{G}^{\alpha,\beta}(a,b,a',b') \leq \sum_{a\leq d\leq b} E_{1,2}\mathcal{Y}(a,b,d,n,d,d)$$
$$\leq \dim S^{\mathbb{V}}_{a,b} - \dim S^{\mathbb{V}}_{a-1,b} = \mathcal{D}^{\mathbb{V}}(a,b).$$

concluding the proof. □ □

4.2 Relation between $\mathcal{G}^{\alpha,\beta}$ and the persistence module $\mathbb{K}$

As mentioned in the introduction, our work is related to the work developed in [16]. Although the persistence module $\mathbb{K}$ is constructed in that paper for a diagram different than Diagram (14), we can adapt such construction to our case. Notice that all the columns $V_i \longrightarrow W_i \longleftarrow U_i$ in Diagram (14) are zigzag modules. For those familiar with zigzag modules, observe that $\mathcal{D}[1,3] = \dim(\alpha_i(V_i)\cap\beta_i(U_i))$. Therefore, the vector spaces

$$K_i = \alpha_i(V_i)\cap\beta_i(U_i)$$

encode the relation between $\mathbb{V}$ and $\mathbb{U}$ through $\mathbb{W}$, but this is true only if we consider each column separately in the diagram. How can we extend this construction to the whole diagram? The answer is the following result.

Proposition 4.7. *The persistence module $\mathbb{K}$, formed by vector spaces K_i and structure linear maps $f^{\mathbb{K}}_{a,b} = f^{\mathbb{W}}_{a,b}|_{K_a}$, is well-defined and is a submodule of $\mathbb{W}$.*

Proof. We have to prove that $\operatorname{im} f^{\mathbb{W}}_{a,b}|_{K_a} \subset K_b$ or equivalently,

$$f^{\mathbb{W}}_{a,b}\left(\alpha_a(V_a)\cap\beta_a(U_a)\right) \subset \alpha_b(V_b)\cap\beta_b(U_b).$$

Notice that if $x\in\alpha_a(V_a)\cap\beta_a(U_a)$ then, in particular, $x\in\alpha_a(V_a)$ and there exists $y\in V_a$ such that $f^{\mathbb{W}}_{a,b}\,\alpha_a(y) = f^{\mathbb{W}}_{a,b}(x)$. Then, $\alpha_b\, f^{\mathbb{V}}_{a,b}(y) = f^{\mathbb{W}}_{a,b}(x)$ by commutativity of Diagram (14). Therefore, $f^{\mathbb{W}}_{a,b}(x)\in\alpha_b(V_b)$. Following a similar procedure for $\mathbb{U}$, we have that $f^{\mathbb{W}}_{a,b}(x)\in\beta_b(U_b)$, concluding the proof. □ □

This way, $\mathbb{K}$ is a persistence module obtained without fixing any basis on the given persistence modules. Nevertheless, $\mathbb{K}$ gives no explicit relation between the decomposition of $\mathbb{V}$ and the decomposition of $\mathbb{U}$ as the operator $\mathcal{G}^{\alpha,\beta}$ does. The relation between $\mathcal{G}^{\alpha,\beta}$ and $\mathbb{K}$ is given by the following theorem that uses induced enriched partial matchings.

Theorem 4.8. *Let us consider a pair of morphisms $\mathbb{V}\xrightarrow{\alpha}\mathbb{W}\xleftarrow{\beta}\mathbb{U}$ between persistence modules of length n. Let $(a,b,a',b',c,d)\in\mathbb{Z}^6_{\geq 0}$ such that $1\leq a,a',c\leq d\leq b,b'\leq n$. Then,*

$$\sum_{1\leq c\leq d} E_1\mathcal{D}^{\mathbb{K}}(c,d)$$
$$= \sum_{d\leq b\leq n}\sum_{1\leq a\leq d}\sum_{d\leq b'\leq n}\sum_{1\leq a'\leq d}\sum_{1\leq c\leq d} \#\left\{([c,d],e)\in\mathcal{G}^{\alpha,\beta}(a,b,a',b')\right\}.$$

In other words, the number of intervals in $\mathcal{B}^{\mathbb{K}}$ with endpoint d equals the number of intervals in $\mathcal{G}^{\alpha,\beta}$ with endpoint d.

Proof. By the definition of $\mathcal{G}^{\alpha,\beta}$ and $\mathcal{Y}$. and by Lemma 3.11 we have:

$$\sum_{d\leq b\leq n}\sum_{1\leq a\leq d}\sum_{d\leq b'\leq n}\sum_{1\leq a'\leq d} \#\left\{([c,d],e) : ([c,d],e)\in\mathcal{G}^{\alpha,\beta}(a,b,a',b')\right\}$$
$$= \sum_{d\leq b\leq n}\sum_{1\leq a\leq d}\sum_{d\leq b'\leq n}\sum_{1\leq a'\leq d}\sum_{1\leq c\leq d} E_{1,2,3,4,5}(\mathcal{Y})(a,b,a',b',c,d)$$
$$= \mathcal{Y}(d,n,d,n,d,d).$$

By Lemma 3.11, we have:

$$\sum_{1\leq c\leq d} E_1\mathcal{D}^{\mathbb{K}}(c,d) = S^{\mathbb{K}}_{d,d}.$$

Besides, by Lemma 4.4, we get:

$$\begin{aligned}\mathcal{Y}(d,n,d,n,d,d) &= f^{\mathbb{W}}_{d,d}\,\alpha_d\left(\ker f^{\mathbb{V}}_{d,n+1}\right) \cap f^{\mathbb{W}}_{d,d}\beta_d\left(\ker f^{\mathbb{U}}_{d,n+1}\right) \cap S^{\mathbb{W}}_{d,d} \\ &= \alpha_d\left(V_d\right) \cap \beta_d\left(U_d\right) \cap S^{\mathbb{W}}_{c,d} = K_d \cap W_d \cap \ker f^{\mathbb{W}}_{d,d+1} = S^{\mathbb{K}}_{d,d},\end{aligned}$$

concluding the proof. □ □

We conclude the section with the following final result.

Corollary 4.9. *Let us consider a pair of morphisms* $\mathbb{V} \xrightarrow{\alpha} \mathbb{W} \xleftarrow{\beta} \mathbb{U}$ *between persistence modules of length* n*. Let* $(a,b,a',b') \in \Delta_+ \times \Delta_+$*. Then,*

$$\sum_{1\le b\le n}\sum_{1\le b'\le n}\sum_{1\le a\le b}\sum_{1\le a'\le b'} \#\mathcal{G}^{\alpha,\beta}(a,b,a',b') = \#\mathcal{B}^{\mathbb{K}}$$

In other words, the sum of the cardinals of all multisets appearing in $\mathcal{G}^{\alpha,\beta}$ *is equal to the cardinal of* $\mathcal{B}^{\mathbb{K}}$*.*

5 Concluding remarks and future work

In this paper, we have studied how a morphism between persistence modules (also called ladder modules) can induce a basis-independent partial matching between their corresponding persistence barcodes or diagrams. We have also proved the linearity of our method with respect to direct sum of ladder modules. In addition, the concept of enriched partial matching have been introduced. It has been used to study the relation between persistence modules $\mathbb{V}, \mathbb{U}$ through a pair of morphisms $\mathbb{V} \xrightarrow{\alpha} \mathbb{W} \xleftarrow{\beta} \mathbb{U}$. Explicit relations with other state-of-the-art tools (Bauer-Lesnick matching [3], ladder modules [11] and $\mathbb{K}$ [16]) have been given. Future work could follow these directions:

- **Implementation:** although our definitions and proofs are constructive, the implicit algorithm can only be applied to vector spaces. It would be interesting to implement algorithms acting directly in more common spaces like simplicial complexes. The algorithms of [8] and [6] may be good starting points.
- **Stability:** is the induced basis-independent partial matching stable with respect to modifications of the morphism α? and is the enriched partial matching stable with respect to morphisms α and β? Stability theorems from [10] and [16] offer a great background to proceed.
- **Generalization:** persistence modules can be defined in any poset and not only for finite sequences. Can we extend our partial matching to persistence modules over a real parameter or to zigzag modules? What is the exact relation of induced (enriched) basis-independent partial matchings with multidimensional persistence modules?
- **Applications:** we think induced (enriched) basis-independent partial matching can be used to model real world application. One example could be given by dynamical metric spaces (see [14, 15]) where the object of study is point clouds evolving in time.

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