Trajectory statistical solutions and Liouville type equations for evolution equations: Abstract results and applications*

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Abstract

In this article, we first prove, from the viewpoint of infinite dynamical system, sufficient conditions ensuring the existence of trajectory statistical solutions for autonomous evolution equations. Then we establish that the constructed trajectory statistical solutions possess invariant property and satisfy a Liouville type equation. Moreover, we reveal that the equation describing the invariant property of the trajectory statistical solutions is a particular situation of the Liouville type equation. Finally, we study the equations of three-dimensional incompressible magneto-micropolar fluids in detail and illustrate how to apply our abstract results to some concrete autonomous evolution equations.

Keywords: Trajectory statistical solution; Trajectory attractor; Liouville type equation; Autonomous evolution equation; Magneto-micropolar fluids

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1 Introduction

In the theory of turbulence, the invariant measures and statistical solutions have been proved to be very useful in the understanding of the Navier-Stokes equations (see Foias *et al.* [12]). The main reason is that the measurements of several aspects of turbulent flows are actually measurements of time-average quantities. Statistical solutions

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have been introduced as a rigorous mathematical notion to formalize the object of ensemble average in the conventional statistical theory of turbulence. Nowadays, invariant measures and statistical solutions are widely used to describe certain characteristics of the fluids in the real world (see [33]).

There are two prevalent notions of statistical solutions. The one is introduced by Foias and Prodi [11] (will be called Foias-Prodi statistical solutions) and the other by Vishik and Fursikov [36] (will be called Vishik-Fursikov statistical solutions). The Foias-Prodi statistical solutions defined in [11] are a family of Borel measures parametrized by the time variable and defined on the phase space of the Navier-Stokes equations, representing the probability distribution of the velocity field of the flow at each time. While the Vishik-Fursikov statistical solutions given in [36] are a single Borel measure on the space of trajectories, representing the probability distribution of the space-time velocity field. We can discover that the Foias-Prodi statistical solutions are associated to some invariant measures defined on the phase space of the addressed system, while the Vishik-Fursikov statistical solutions are associated to some invariant measures defined on the trajectory space of solutions.

The invariant measures for well-posed dissipative systems were studied in a series of references (see [5, 19, 21, 24–26, 38]). For instance, Łukaszewicz, Real and Robinson [25] used the notion of Generalized Banach limit to construct the invariant measures for general continuous dynamical systems on metric spaces. Later, Chekroun and Glatt-Holtz [5] improved the results of [25] to construct invariant measures for a broad class of dissipative autonomous dynamical systems. Recently, Łukaszewicz and Robinson [26] extended the result of [5] to construct invariant measures for dissipative non-autonomous dynamical systems. The constructions of invariant measures in [5] depend essentially on the existence of global attractors of the continuous semigroup generated by the solution operators, and in [26] the constructions depend heavily on the pullback attractors of the continuous process associated to the solution operators. We recognize that both of the constructions of the invariant measures in [5] and [26] require that the addressed system is globally well-posed.

It is an interesting problem to give a general construction of invariant measures and trajectory statistical solutions for general evolution equations including those systems, which possess global weak solutions but without a known result of global uniqueness, say, the three-dimensional (3D) incompressible Navier-Stokes equations. In [2], Bronzi, Mondaini and Rosa provided a general framework for the theory of trajectory statistical solutions for evolution equations with similar properties to those of the 3D Navier-Stokes equations. In [4], Bronzi, Mondaini and Rosa established an abstract framework for the theory of statistical solutions for general evolution equations. The proofs of existence of trajectory statistical solutions for the initial value problem in [2,4] are based on the Krein-Milman approximation of the initial measure by convex combinations of Dirac deltas, as done in [12,15]. This is an approach via functional analysis, topological analysis and measure theory.

Very recently, Zhao and Caraballo in [45] investigated the existence and regularity of the trajectory statistical solutions for the 3D globally modified Navier-Stokes equations, via the approach of infinite dynamical systems. They successfully used the natural translation semigroup and trajectory attractor to construct the trajectory statistical solutions. In the end of the article [45], the authors pointed out that the approach employed in [45] could be raised to an abstract level and the constructed trajectory statistical solutions should satisfy a Liouville type equation in Statistical Mechanics.

In this article, we continue to investigate the abstract theory concerning the constructions of trajectory statistical solutions for general evolution equations, including not only the systems possessing global well-posedness but also those displaying the property of global existence of weak solutions but without a known result of global uniqueness. Precisely speaking, we will first present sufficient conditions ensuring the existence of trajectory statistical solutions for general autonomous evolution equations. We prove that if a general evolution system satisfies the following two conditions:

(1) the trajectory space of the system is a metrizable topological space;

(2) the translation semigroup has a trajectory attractor in the trajectory space;

then the evolution system possesses trajectory statistical solutions. We then establish that the constructed trajectory statistical solutions possess an invariant property under the action of the translation semigroup and satisfy a Liouville type equation. Further, we will reveal that the equation describing the invariant property of the trajectory statistical solutions is exactly a particular situation of the Liouville type equation.

We remark that our abstract framework to construct trajectory statistical solutions is different from that in [2,4]. As it has already been mentioned, the abstract framework of [2, 4] is formulated mainly through the topological analysis and measure theory, and its proofs of existence of trajectory statistical solutions for the initial value problem are based on the Krein-Milman approximation of the initial measure by convex combinations of Dirac deltas, while our sufficient conditions are merely from the point of view of infinite dynamical system. Our proofs rely on the theory of infinite dynamical systems, the Kakutani-Riesz Representation Theorem and an elementary topological observation. We point out that our idea of the topological observation originates from [5, 26], but the approaches of [5, 26] to construct invariant measures can not be applied in our situation because that the solution operators of the addressed systems here might not generate a semigroup or process. The essential difference between [5,26] and this article is that we consider the invariant measures on the trajectory space, while [5,26] consider the invariant measures on the phase space of the addressed system. The advantage of our abstract results is that it is convenient to check for the purpose of applications in concrete evolution equations.

To illustrate the wide applicability of our abstract results, we will study the equations of 3D incompressible magneto-micropolar fluids in detail. The equations of magneto-micropolar fluids describe the motion of electrically conducting micropolar fluids in the presence of magnetic fields, whose equations consist of a particular mutual coupling of the Navier-Stokes equations with equations for the micro-rotational velocity and magnetic field. Also, our abstract results are valid for the general reaction-diffusion system and nonlinear wave equation discussed in [8, 37]. Both of the systems possess trajectory attractors and their trajectory spaces are metrizable topological spaces. Of course, our results are also valid for the 3D Navier-Stokes equations studied by Foias *et al.* in a series of works [12–14, 16, 17].

The article is organized as follows. In Section 2, we first prove sufficient conditions ensuring the existence of trajectory statistical solutions for general autonomous evolution equations. Then we establish that the constructed trajectory statistical solutions are invariant under the action of the translation semigroup and satisfy a Liouville type equation. Further, we reveal that the equation describing the invariant property of the trajectory statistical solutions is exactly a particular situation of the Liouville type equation. In Section 3, we study the 3D incompressible magneto-micropolar fluids in detail, showing how to apply our abstract results to some concrete autonomous evolution equations.

2 Sufficient conditions for the existence of trajectory statistical solutions for autonomous evolution equations

In this section, we will prove sufficient conditions ensuring the existence of trajectory statistical solutions of general autonomous evolution equations. Then we establish that the constructed trajectory statistical solutions possess an invariant property under the action of the translation semigroup and satisfy a Liouville type equation.

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be two Banach spaces with $\mathcal{Y} \hookrightarrow \mathcal{X}$, where the embedding is continuous. We consider the following autonomous evolution equation with initial value

$$\frac{\partial u(x,t)}{\partial t} = F(u), \quad t > 0, \tag{2.1}$$

$$u(x,t)|_{t=0} = u_0 \in \mathcal{X}, \tag{2.2}$$

where $u(x,t)$: $(x,t) \in \Omega \times [0,+\infty) \longmapsto \mathcal{X}$ is the unknown function, $\Omega \subset \mathbb{R}^n$ is a domain satisfying some conditions, $F(u)$ is some nonlinear differential or abstract operator acting on $\mathcal{X}$. Usually, $F(\cdot) : \mathcal{Y} \longmapsto \mathcal{Y}^*$, where $\mathcal{Y}^*$ is the dual space of $\mathcal{Y}$ and such a relation is typical for differential operators. For example, we shall take $\mathcal{X} = \hat{H}$, $\mathcal{Y} = \hat{V}$ and $\mathcal{Y}^* = \hat{V}^*$ for the 3D incompressible magneto-micropolar fluids in Section 3.

We will investigate global in time solutions $u(\cdot)$ of problem (2.1)-(2.2) defined on the time interval $[0, +\infty)$. A certain sufficiently broad family of such solutions will be called the trajectory space (the definition will be specified in Definition 2.1(2)) of equation (2.1). We can see that the choice of the trajectory space plays an important role in our construction of the trajectory statistical solutions.

We use $\mathcal{X}_{\mathrm{w}}$ to denote the space $\mathcal{X}$ endowed with its weak topology. For instance, in Section 3, the collection of open sets in $\hat{H}_{\mathrm{w}}$ has a characterization by a basis of neighborhoods given by

$$\mathcal{O}_{\rm w}(u,r,v_1,v_2,\cdots,v_N)=\Big\{w\in\hat{H}_{\rm w}\,\Big|\,\sum_{j=1}^{N}|(u-w,v_j)|^2<L^2\Big\},$$

for $u\in\hat{H}_{\rm w}$, $L>0$, $N\in\mathbb{N}$, and $v_1,v_2,\cdots,v_N\in\hat{H}$, where $(\cdot,\cdot)$ is the inner product in $\hat{H}$ (cf. [3]). Further, we use

$$\mathcal{F}^+_{\rm loc}=\mathcal{C}_{\rm loc}([0,+\infty);\mathcal{X}_{\rm w})$$

to denote the space of continuous functions from $\mathbb{R}_+=[0,+\infty)$ to $\mathcal{X}_{\rm w}$. This space can also be seen as the space of weakly continuous functions from $\mathbb{R}_+$ to $\mathcal{X}$. The topology on $\mathcal{F}^+_{\rm loc}$, which is denoted by $\Theta^+_{\rm loc}$, is that of uniform convergence in $\mathcal{X}_{\rm w}$ on compact subinterval of $\mathbb{R}_+$, that is, by definition, $w_n(t)\longrightarrow w(t)$ $(n\to+\infty)$ in the topology $\Theta^+_{\rm loc}$, if, for every $T>0$, $(w_n(t),\Phi)\longrightarrow(w(t),\Phi)$ uniformly on $[0,T]$ $(n\to+\infty)$ for each Φ in the dual space $\mathcal{X}^*$ of $\mathcal{X}$ (or for each Φ in $\mathcal{X}$ if it is a Hilbert space), here $(\cdot,\cdot)$ is the dual pairing between $\mathcal{X}$ and $\mathcal{X}^*$ (or is the inner product of $\mathcal{X}$ if it is a Hilbert space).

Generally speaking, the topology $\Theta^+_{\rm loc}$ in the space $\mathcal{F}^+_{\rm loc}$ is not metrizable. However, we will use the property that the topology $\Theta^+_{\rm loc}$ in the trajectory space $\mathcal{T}^{\rm tr}_{\mathcal{X}}$ is metrizable. To this end, we usually assume that the trajectory space $\mathcal{T}^{\rm tr}_{\mathcal{X}}\subset\mathcal{M}^+_{\rm loc}$ and the topology $\Theta^+_{\rm loc}$ is metrizable in the space

$$\mathcal{M}^+_{\rm loc}=\mathcal{C}_{\rm loc}([0,+\infty);X_{\rm w}),$$

where X is a fixed bounded subset of $\mathcal{X}$, and $X_{\rm w}$ is the space X endowed the topology inherited from $\mathcal{X}_{\rm w}$. This assumption is quite natural in some concrete evolution equations. For instance, when we investigate the 3D incompressible magneto-micropolar fluids in Section 3, $\mathcal{C}_{\rm loc}([0,+\infty);\hat{H}_{\rm w})$ is not metrizable but $\mathcal{C}_{\rm loc}([0,+\infty);\mathcal{B}(\hat{H})_{\rm w})$ is metrizable for a fixed bounded subset $\mathcal{B}(\hat{H})$ of $\hat{H}$ (see [30]).

We next specify the definitions of global weak solution, global regular weak solution, trajectory space $\mathcal{T}^{\rm tr}_{\mathcal{X}}$ and regular trajectory space $\mathcal{T}^{\rm tr}_{\mathcal{Y}}$ for equation (2.1).

Definition 2.1.

(1) *A function* $u(\cdot)\in\mathcal{F}^+_{\rm loc}\cap L^\infty(\mathbb{R}_+;\mathcal{X})\cap L^2_{\rm loc}(\mathbb{R}_+;\mathcal{Y})$ *is called a global weak solution of equation* (2.1) *if it satisfies equation* (2.1) *in the sense of distribution* $\mathcal{D}'(0,+\infty;\mathcal{Y}^*)$. *A global weak solution* u *belonging to* $L^\infty(\mathbb{R}_+;\mathcal{Y})$ *is called a global regular weak solution.*

(2) *The trajectory space* $\mathcal{T}^{\rm tr}_{\mathcal{X}}$ *and regular trajectory space* $\mathcal{T}^{\rm tr}_{\mathcal{Y}}$ *for equation* (2.1) *are defined as*

$$\begin{aligned}\mathcal{T}^{\rm tr}_{\mathcal{X}}=&\{u(\cdot)\,|\,u(\cdot)\ \text{is a global weak solution of equation (2.1) and}\\ &\quad\|u(t)\|_{\mathcal{X}}\leqslant R\ \text{for some fixed}\ R>0,\ t\in\mathbb{R}_+\},\\ \mathcal{T}^{\rm tr}_{\mathcal{Y}}=&\{u(\cdot)\,|\,u(\cdot)\ \text{is a global regular weak solution of equation (2.1) and}\\ &\quad\|u(t)\|_{\mathcal{Y}}\leqslant R_1\ \text{for some fixed}\ R_1>0,\ t\in\mathbb{R}_+\}.\end{aligned}$$

Elements of $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$ are called bounded, complete trajectories of equation (2.1). Consider the family of translation operators $\{T(t)\}_{t\geqslant 0}$ acting in the space $\mathcal{F}_{\mathrm{loc}}^{+}$ via the formula

$$T(t)u(s) = u(t+s), \ \ \forall\, t \geqslant 0.$$

It is not difficult to check that the family of operators $\{T(t)\}_{t\geqslant 0}$ generate a continuous (with respect to the topology $\Theta_{\mathrm{loc}}^{+}$) semigroup on the space $\mathcal{F}_{\mathrm{loc}}^{+}$ (cf. [8, page 222, Proposition 1.3]). In concrete applications, we will have the following property

$$T(t)\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}} \subseteq \mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}, \ \ \forall\, t \geqslant 0, \tag{2.3}$$

and this property is quite natural for solutions of general autonomous evolution equations.

Since the topology $\Theta_{\mathrm{loc}}^{+}$ in the trajectory space $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$ is metrizable, we use $\mathrm{d}_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}}(\cdot,\cdot)$ to denote the metric which is compatible with the topology $\Theta_{\mathrm{loc}}^{+}$ in $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$. For a set $\mathcal{P} \subset \mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$ and some $\epsilon > 0$, we set

$$\mathcal{O}(\mathcal{P},\epsilon) = \{w \in \mathcal{T}_{\mathcal{X}}^{\mathrm{tr}} \big|\ \mathrm{d}_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}}(w,\mathcal{P}) = \inf_{\phi\in\mathcal{P}} \mathrm{d}_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}}(w,\phi) < \epsilon\}.$$

In addition, we use $\mathcal{C}(\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}})$ to denote the space of continuous functions from $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$ to $\mathbb{R}$.

Definition 2.2.

(1) *A set $\mathcal{P} \subseteq \mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$ is said to be attracting for the semigroup $\{T(t)\}_{t\geqslant 0}$ in the topology $\Theta_{\mathrm{loc}}^{+}$, if for any set $\mathcal{B} \subset \mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$, the set $T(t)\mathcal{B}$ is attracted to $\mathcal{P}$ in the topology $\Theta_{\mathrm{loc}}^{+}$ as $t \to +\infty$, that is, for any $\epsilon > 0$, there exists a $\tau = \tau(\mathcal{B},\epsilon) \geqslant 0$ such that $T(t)\mathcal{B} \subseteq \mathcal{O}(\mathcal{P},\epsilon)$ for all $t \geqslant \tau$.*

(2) *A set $\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}} \subseteq \mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$ is called a trajectory attractor for the semigroup $\{T(t)\}_{t\geqslant 0}$ in the topology $\Theta_{\mathrm{loc}}^{+}$, if*

   (a) *$\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}}$ is compact in the topology $\Theta_{\mathrm{loc}}^{+}$;*

   (b) *$\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}}$ is an attracting set in the topology $\Theta_{\mathrm{loc}}^{+}$;*

   (c) *$\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}}$ is strictly invariant under the action of $T(t)$: $T(t)\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}} = \mathcal{A}_{\mathcal{X}}^{\mathrm{tr}}$, $\forall\, t \geqslant 0$.*

Definition 2.3. *We say a Borel probability measure ρ on $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$ is a $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$-trajectory statistical solution over $[0,+\infty)$ (or simply a trajectory statistical solution) for equation* (2.1) *if*

(1) *ρ is tight for any $\mathcal{B} \in \mathcal{B}(\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}})$ (the collection of Borel sets of $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$) in the sense that*

$$\rho(\mathcal{B}) = \sup\left\{\rho(E)\big| E \in \mathcal{B}(\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}) \ \text{ and } \ E \subset \mathcal{B}\right\};$$

(2) *ρ is supported by a Borel subset of $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$.*

The following two lemmas, which were proved in our recent article [45], play the essential role when we prove the existence of the trajectory statistical solutions. For the convenience of the reader, we reproduce the proofs here.

Lemma 2.1. *Let K be some compact subset of $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$. Then for every $\psi \in \mathcal{C}(\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}})$, there exists some $\epsilon > 0$ such that*

$$\sup_{w \in \mathcal{O}(K,\epsilon)} |\psi(w)| < +\infty.$$

Proof. Fix some $\psi \in \mathcal{C}(\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}})$. For every $w \in K$ we can choose $\delta = \delta_w > 0$ such that for every $v \in \mathcal{O}(w;\delta_w) = \{\Phi \in \mathcal{T}_{\mathcal{X}}^{\mathrm{tr}} \mid \mathrm{d}_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}}(w,\Phi) < \delta_w\}$ there holds $|\psi(w) - \psi(v)| < 1$. Picking numbers $\delta_w > 0$ in this way we can construct an open covering $\Lambda = \{\mathcal{O}(w; \frac{\delta_w}{3}) \mid w \in K\}$ for K. Since K is compact in $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$, we may extract from this open covering a finite one

$$\Lambda_m = \Big\{\mathcal{O}(w_1; \frac{\delta_{w_1}}{3}), \mathcal{O}(w_2; \frac{\delta_{w_2}}{3}), \cdots, \mathcal{O}(w_m; \frac{\delta_{w_m}}{3})\Big\}.$$

Set

$$\epsilon = \frac{\min\{\delta_{w_1}, \delta_{w_2}, \cdots, \delta_{w_m}\}}{3}, \quad C = 1 + \max_{1 \leqslant j \leqslant m} |\psi(w_j)|.$$

Given any $w \in \mathcal{O}(K,\epsilon)$ we can choose $v \in K$ so that $\mathrm{d}_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}}(w,v) < 2\epsilon$. Since Λ_m covers K we can choose w_j such that $\mathrm{d}_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}}(v,w_j) < \frac{\delta_{w_j}}{3}$. Hence we obtain

$$\mathrm{d}_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}}(w,w_j) < \mathrm{d}_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}}(w,v) + \mathrm{d}_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}}(v,w_j) < 2\epsilon + \frac{\delta_{w_j}}{3} \leqslant \delta_{w_j},$$

and conclude that $|\psi(w)| \leqslant C$. By the arbitrariness of $w \in \mathcal{O}(K,\epsilon)$, we end the proof. $\square$

Lemma 2.2. *Let K be some compact subset of $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$ and let ψ, $\phi \in \mathcal{C}(\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}})$ satisfying $\psi(w) = \phi(w)$ for every $w \in K$. Then for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $\sup_{w \in \mathcal{O}(K,\delta)} |\psi(w) - \phi(w)| < \epsilon$.*

Proof. Consider given $\epsilon > 0$. For every $w \in K$ we pick $\gamma_w > 0$ so that $|\phi(w) - \phi(v)| + |\psi(w) - \psi(v)| < \epsilon$ whenever $v \in \mathcal{O}(w;\gamma_w)$. Due to the compactness of K in $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$, we can cover K with a finite collection

$$\Lambda_k = \Big\{\mathcal{O}(w_1; \frac{\gamma_{w_1}}{3}), \mathcal{O}(w_2; \frac{\gamma_{w_2}}{3}), \cdots, \mathcal{O}(w_k; \frac{\gamma_{w_k}}{3})\Big\} \,\text{with}\, w_j \in K, j = 1, 2, \cdots, k.$$

Set $3\delta = \min\{\gamma_{w_1}, \gamma_{w_2}, \cdots, \gamma_{w_k}\}$, then for every $v \in K_\delta$ we can choose $w \in K$ so that $\mathrm{d}_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}}(v,w) < 2\delta$. Notice that K is covered by Λ_k, we may take w_j such that $\mathrm{d}_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}}(w,w_j) < \frac{\gamma_{w_j}}{3}$. Thus

$$\mathrm{d}_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}}(v,w_j) \leqslant \mathrm{d}_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}}(v,w) + \mathrm{d}_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}}(w,w_j) \leqslant 2\delta + \frac{\gamma_{w_j}}{3} \leqslant \gamma_{w_j}.$$

Therefore for arbitrary $v \in \mathcal{O}(K,\delta)$, there exists some j such that $v \in \mathcal{O}(w_j;\gamma_{w_j})$. Keeping in mind that $\psi(w_j) = \phi(w_j)$, we have

$$|\psi(v) - \phi(v)| \leqslant |\psi(v) - \psi(w_j)| + |\phi(w_j) - \phi(v)| < \epsilon.$$

The proof of Lemma 2.2 is completed. $\square$

We next recall the definition of generalized Banach limit and a useful property.

Definition 2.4. ([12,26]) *A generalized Banach limit is any linear functional, which we denote by* $\mathrm{LIM}_{t\to+\infty}$, *defined on the space of all bounded real-valued functions on* $[0,+\infty)$ *satisfying*

(1) $\mathrm{LIM}_{t\to+\infty} f(t) \geqslant 0$ *for nonnegative functions* $f(\cdot)$ *on* $[0,+\infty)$;

(2) $\mathrm{LIM}_{t\to+\infty} f(t) = \lim\limits_{t\to+\infty} f(t)$ *if the usual limit* $\lim\limits_{t\to+\infty} f(t)$ *exists.*

Let B_+ be the collection of all bounded real-valued functions on $[0,+\infty)$. For any generalized Banach limit $\mathrm{LIM}_{t\to+\infty}$, the following useful property

$$|\mathrm{LIM}_{t\to+\infty} f(t)| \leqslant \limsup_{t\to+\infty} |f(t)|, \ \ \forall\, f(\cdot) \in B_+, \tag{2.4}$$

is presented in [12, (1.38)] and in [5, (2.3)].

We now begin to state and prove our abstract results. First, a sufficient condition ensuring the existence of trajectory statistical solutions for autonomous evolution equations reads as follows.

Theorem 2.1. *Suppose that the autonomous evolution equation* (2.1) *satisfies the following two conditions:*

(A1) *Equation* (2.1) *possesses a nonempty trajectory space* $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$ *which is metrizable with the topology* $\Theta_{\mathrm{loc}}^{+}$.

(A2) *The translation semigroup* $\{T(t)\}_{t\geqslant 0}$ *possesses a trajectory attractor* $\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}} \subseteq \mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$ *in the topology* $\Theta_{\mathrm{loc}}^{+}$.

Then, equation (2.1) *possesses at least one trajectory statistical solution which is supported by the trajectory attractor* $\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}}$.

Proof. Let $\mathrm{LIM}_{t\to+\infty}$ be a given generalized Banach limit. Since $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$ is nonempty, we can pick some $v \in \mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$. Consider some $\psi \in \mathcal{C}(\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}})$. By assumption (A2), the translation semigroup $\{T(t)\}_{t\geqslant 0}$ possesses a trajectory attractor $\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}} \subseteq \mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$. By the attracting property of the trajectory attractor we see that for every $\epsilon > 0$, there exists a time $t_\epsilon \geqslant 0$ such that

$$T(t)v \in \mathcal{O}(\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}}, \epsilon), \ \text{ for every } \ t \geqslant t_\epsilon. \tag{2.5}$$

By Lemma 2.1 we can choose $\epsilon > 0$ such that

$$C_1 = \sup_{u\in\mathcal{O}(\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}},\epsilon)} |\psi(u)| < +\infty. \tag{2.6}$$

Since $T(t)$ maps $\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}}$ continuously into itself, the function $t \longmapsto |\psi(T(t)v)|$ is continuous on $[0,+\infty)$ and thus $|\psi(T(t)v)|$ is bounded on the compact interval $[0,t_\epsilon]$. Hence, we can take t_ϵ as required in (2.5) for the picked ϵ, and

$$C_2 = \sup_{t\in[0,t_\epsilon]} |\psi(T(t)v)| < +\infty. \tag{2.7}$$

It then follows from (2.6) and (2.7) that

$$\frac{1}{t}\int_0^t \psi(T(s)v)\mathrm{d}s = \frac{1}{t}\int_0^{t_\epsilon} \psi(T(s)v)\mathrm{d}s + \frac{1}{t}\int_{t_\epsilon}^t \psi(T(s)v)\mathrm{d}s \leqslant \frac{C_2 t_\epsilon}{t} + \frac{C_1(t-t_\epsilon)}{t} < +\infty,$$

which implies that the map defined by $t \longmapsto \frac{1}{t}\int_0^t \psi(T(s)v)\mathrm{d}s$ is bounded over $[0,+\infty)$. Therefore, if $\psi \in \mathcal{C}(\mathcal{T}_\mathcal{X}^{\mathrm{tr}})$ is non-negative, then

$$\mathcal{L}_v(\psi) = \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t \psi(T(s)v)\mathrm{d}s \tag{2.8}$$

is well defined as a positive linear functional on $\mathcal{C}(\mathcal{T}_\mathcal{X}^{\mathrm{tr}})$.

We next prove that the positive linear functional $\mathcal{L}_v(\psi)$ depends only on the values of ψ on the trajectory attractor $\mathcal{A}_\mathcal{X}^{\mathrm{tr}}$. To this end, we shall prove that if $\psi(w) = \phi(w)$ for every $w \in \mathcal{A}_\mathcal{X}^{\mathrm{tr}}$ then $\mathcal{L}_v(\psi) = \mathcal{L}_v(\phi)$. Indeed, for any given $\epsilon > 0$, we can choose, by Lemma 2.2, a corresponding $\delta > 0$ such that

$$\sup_{v\in\mathcal{O}(\mathcal{A}_\mathcal{X}^{\mathrm{tr}},\delta)} |\psi(v) - \phi(v)| < \epsilon. \tag{2.9}$$

We now pick $t_\delta > 0$ such that $T(t)v \in \mathcal{O}(\mathcal{A}_\mathcal{X}^{\mathrm{tr}}, \delta)$ for every $t \geqslant t_\delta$. Set

$$C_\delta = \sup_{t\in[0,t_\delta]} (|\psi(T(t)v)| + |\phi(T(t)v)|).$$

Analogous to (2.7), we see that $C_\delta < +\infty$. Combining (2.4), (2.7) and (2.9) yields

$$\begin{aligned}
|\mathcal{L}_v(\psi-\phi)| =& \Big|\mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t \big(\psi(T(s)v) - \phi(T(s)v)\big)\mathrm{d}s\Big| \\
\leqslant& \limsup_{t\to+\infty}\frac{1}{t}\Big|\int_0^t \big(\psi(T(s)v) - \phi(T(s)v)\big)\mathrm{d}s\Big| \\
\leqslant& \limsup_{t\to+\infty}\frac{1}{t}\int_0^{t_\delta} \big|\psi(T(s)v) - \phi(T(s)v)\big|\mathrm{d}s \\
&+ \limsup_{t\to+\infty}\frac{1}{t}\int_{t_\delta}^t \big|\psi(T(s)v) - \phi(T(s)v)\big|\mathrm{d}s \\
\leqslant& \limsup_{t\to+\infty}\frac{t_\delta C_\delta}{t} + \limsup_{t\to+\infty}\frac{(t-t_\delta)\epsilon}{t} \leqslant \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the positive linear functional $\mathcal{L}_v(\psi)$ depends only on the values of ψ on the trajectory attractor $\mathcal{A}_\mathcal{X}^{\mathrm{tr}}$.

Now we define $G(\psi) = \mathcal{L}_v(\widetilde{\psi})$, where $\widetilde{\psi}$ is a zero extension of ψ from $\mathcal{C}(\mathcal{A}_\mathcal{X}^{\mathrm{tr}})$ to $\mathcal{C}(\mathcal{T}_\mathcal{X}^{\mathrm{tr}})$ given by the Tietze theorem (see [12, Theorem A.7]). By Definition 2.4(1) and (2.8), we can find that $G(\cdot)$ is a positive linear functional on $\mathcal{C}(\mathcal{A}_\mathcal{X}^{\mathrm{tr}})$. Notice that $\mathcal{A}_\mathcal{X}^{\mathrm{tr}} \subseteq \mathcal{T}_\mathcal{X}^{\mathrm{tr}}$ is compact with respect to the topology Θ_{loc}^+. Then, $\mathcal{A}_\mathcal{X}^{\mathrm{tr}}$ is obviously a locally compact topological space with respect to the topology Θ_{loc}^+. By the Kakutani-Riesz

Representation Theorem (see [12, Theorem A.1]), we assert that there exists a unique positive, finite, Borel measure ρ_v on $\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}}$ such that

$$G(\psi) = \int_{\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}}} \psi(u)\mathrm{d}\rho_v(u). \tag{2.10}$$

Taking $\rho_v(E) = \rho_v(\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}} \cap E)$ for $E \in \mathcal{B}(\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}})$, we extend ρ_v by zero to a Borel measure on $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$. Thus for every $\psi \in \mathcal{C}(\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}})$, we have

$$G(\psi) = \mathcal{L}_v(\psi) = \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t \psi(T(s)v)\mathrm{d}s = \int_{\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}}} \psi(u)\mathrm{d}\rho_v(u) = \int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \psi(u)\mathrm{d}\rho_v(u), \tag{2.11}$$

and obviously $\rho_v(\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}\backslash\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}}) = 0$. Now, by assumption (A1), $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$ is a metrizable topological space, every finite Borel measure is tight in the sense of Definition 2.3(1) (see [1, Theorem 12.5]. To see that ρ_v is a Borel probability measure we pick $\psi \equiv 1$ in (2.11). Therefore, by Definition 2.3, ρ_v is a trajectory statistical solution for equation (2.1). The proof of Theorem 2.1 is completed. □

Remark 2.1. *From Definition 2.1 we see that the elements in $\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}}$ are bounded, complete trajectories of equation* (2.1)*. Thus $\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}}$ is called weak trajectory attractor. Notice that $\mathcal{Y} \hookrightarrow \mathcal{X}$ and in concrete evolution partial differential equations the space $\mathcal{Y}$ is usually more regular than $\mathcal{X}$. A trajectory attractor (if exists) $\mathcal{A}_{\mathcal{Y}}^{\mathrm{tr}}$ consisting of all bounded global regular weak solutions of equation* (2.1) *is called a regular trajectory attractor. If further we have*

$$\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}} = \mathcal{A}_{\mathcal{Y}}^{\mathrm{tr}}, \tag{2.12}$$

then we say that equation (2.1) *possesses the property of "trajectory asymptotic smoothing effect". Recall that the regularity of the trajectory statistical solutions means that it is supported by a set in the trajectory space in which all weak solutions are in fact regular weak solutions. Therefore, from our constructions of the trajectory statistical solutions we see that another interesting point of our abstract results is that the regularity problem of the trajectory statistical solutions come down to the regularity problem of the trajectory attractor, that is, the regularity of the trajectory statistical solutions for equation* (2.1) *will be a direct consequence of the relation* (2.12)*. The regularity relation* (2.12) *of the trajectory attractors for the 2D incompressible non-Newtonian fluids and the 3D globally modified Navier-Stokes equations was established respectively in [43] and [45]. As a byproduct of our abstract results, we have the assertion that the 2D incompressible non-Newtonian fluids and the 3D globally modified Navier-Stokes equations possess regular trajectory statistical solutions.*

Remark 2.2. *If problem* (2.1)*-*(2.2) *is globally well-posed in $\mathcal{X}$, then its solution operator $S(t)$, defined by*

$$S(t) : u_0 \in \mathcal{X} \longmapsto u(t) = u(t; u_0) = S(t)u_0 \in \mathcal{X},$$

generates a continuous semigroup $\{S(t)\}_{t\geqslant 0}$ *in* $\mathcal{X}$*, where* $u(t)$ *is the solution of problem* (2.1)-(2.2) *corresponding to the initial value* u_0*. In this case, by the uniqueness of the weak solutions, we have*

$$S(t)u(s) = T(t)u(s), \ \ \forall\, u(s) \in \mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}.$$

Therefore, if the conditions of Theorem 2.1 hold, then we can also use the semigroup $\{S(t)\}_{t\geqslant 0}$ *of the solution operators and its trajectory attractor to construct the same trajectory statistical solutions as that as in Theorem 2.1 for equation* (2.1)*. This implies that we present a unified approach to construct the trajectory statistical solutions for those general evolution equations which possess nonempty metrizable trajectory space and trajectory attractor.*

We next establish that the trajectory statistical solution ρ_v is invariant under the action of the translation semigroup $\{T(t)\}_{t\geqslant 0}$. Moreover, if we set $\mu_t = T(t)\rho_v$ by define

$$\mu_t(E) = T(t)\rho_v(E) = \rho_v(T(t)^{-1}(E)) \tag{2.13}$$

for every set $E \subseteq \mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$ that is ρ_v-measurable, then we prove that μ_t satisfies a Liouville type equation in Statistical Mechanics. To this end, we first introduce the definition of the class $\mathcal{T}$ of test functions. We expect that the function $\Phi(\cdot) \in \mathcal{T}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u(t)) = \langle \Phi'(u(t)), F(u(t))\rangle, \ \ t > 0, \tag{2.14}$$

for every global weak solution $u(t)$ of equation (2.1), where $\langle\cdot,\cdot\rangle$ is the dual pairing between $\mathcal{Y}$ and $\mathcal{Y}^*$.

Definition 2.5. (cf. [12, pages 178-179, Definition 1.2]) We define the class $\mathcal{T}$ of test functions to be the set of real-valued functionals $\Phi = \Phi(u)$ on $\mathcal{X}$ that are bounded on bounded subset of $\mathcal{X}$ and satisfy

(a) for any $u \in \mathcal{Y}$, the Fréchet derivative $\Phi'(u)$ exists: for each $u \in \mathcal{Y}$ there exists an element $\Phi'(u)$ such that

$$\frac{|\Phi(u+v) - \Phi(u) - \langle \Phi'(u), v\rangle|}{\|v\|_{\mathcal{Y}}} \longrightarrow 0 \ \ as \ \ \|v\|_{\mathcal{Y}} \to 0, \ \ v \in \mathcal{Y};$$

(b) $\Phi'(u) \in \mathcal{Y}$ for all $u \in \mathcal{Y}$, and the mapping $\psi \longmapsto \Phi'(u)$ is continuous and bounded as a functional from $\mathcal{Y}$ to $\mathcal{Y}$;

(c) for every global weak solution $u(t)$ of equation (2.1), (2.14) holds true.

For example, we can consider the cylindrical test function depending only on a finite number m of component of u. In fact, $\mathcal{Y} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{Y}^*$ and the embeddings are continuous. We consider the cylindrical test function defined on $\mathcal{Y}$. Let m be some positive integer, $\varphi_1, \ \varphi_2, \ \cdots \ \varphi_m$ belong to $\mathcal{Y}$ and γ be a continuously differentiable real-valved function on $\mathbb{R}^m$ with compact support. For each $u \in \mathcal{Y}$, define $\Phi(u)$ via

$$\Phi(u) = \gamma(\langle \varphi_1, u\rangle, \langle \varphi_2, u\rangle, \cdots, \langle \varphi_m, u\rangle),$$

where $\langle \varphi_j, u\rangle$ is the dual pairing between $\varphi_j \in \mathcal{Y}$ and $u \in \mathcal{Y} \subset \mathcal{Y}^*$. Then the function $\Phi(\cdot)$ is obviously continuous from $\mathcal{Y}$ to $\mathbb{R}$ and in fact is differentiable on $\mathcal{Y}$, with differential $\Phi'(\cdot)$ at $u \in \mathcal{Y}$ given by

$$\Phi'(u) = \sum_{j=1}^{m} \partial_j \gamma(\langle \varphi_1, u\rangle, \langle \varphi_2, u\rangle, \cdots, \langle \varphi_m, u\rangle)\varphi_j, \tag{2.15}$$

where $\partial_j \gamma$ denotes the derivative of γ with respect to its j-th coordinate. (2.15) shows that $\Phi'(\cdot) \in \mathcal{Y}$. Above analyses show that the cylindrical test functions of above form satisfy Definition 2.5.

Theorem 2.2.

(1) *Let ρ_v be the trajectory statistical solution proved in Theorem 2.1, then ρ_v satisfies the following invariant property*

$$\int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \psi(T(t)u)\mathrm{d}\rho_v(u) = \int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \psi(u)\mathrm{d}\rho_v(u) = \int_{\mathcal{A}_{\mathcal{X}}^{\mathrm{tr}}} \psi(u)\mathrm{d}\rho_v(u), \ \ \forall\, t \geqslant 0, \tag{2.16}$$

for every $\psi \in \mathcal{C}(\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}})$.

(2) *$\mu_t = T(t)\rho_v$ satisfies the following Liouville type equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \Phi(u)\mathrm{d}\mu_t(u) = \int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \langle \Phi'(u), F(u)\rangle \mathrm{d}\mu_t(u), \tag{2.17}$$

for all test functions $\Phi \in \mathcal{T}$.

Proof. We first prove that ρ_v is invariant under the action of the translation semigroup $\{T(t)\}_{t\geqslant 0}$, that is ρ_v satisfies (2.16). To this end, we fix any $t^* \geqslant 0$ and any $\psi \in \mathcal{C}(\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}})$. Since the interval $[0, t^*]$ is compact in $\mathbb{R}$ and $t \mapsto |\psi(T(t)u)|$ is continuous, we have, using the property of generalized Banach limit (2.4),

$$\left|\mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^{t^*} \psi(T(s)u)\mathrm{d}s\right| \leqslant \limsup_{t\to+\infty}\left|\frac{1}{t}\int_0^{t^*} \psi(T(s)u)\mathrm{d}s\right| = 0,$$

and thus

$$\mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^{t^*} \psi(T(s)u)\mathrm{d}s = 0. \tag{2.18}$$

At the same time, we use (2.6) and also the property of generalized Banach limit (2.4) to deduce

$$\left|\mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_t^{t+t^*} \psi(T(s)u)\mathrm{d}s\right| \leqslant \limsup_{t\to+\infty}\frac{t^*}{t}C_1 = 0,$$

and hence

$$\mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_t^{t+t^*} \psi(T(s)u)\mathrm{d}s = 0. \tag{2.19}$$

Therefore, (2.11) and (2.18)-(2.19) imply

$$
\begin{aligned}
\int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \psi(T(t^*)u)\mathrm{d}\rho_v(u) =& \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t \psi(T(t^*)T(s)u)\mathrm{d}s \\
=& \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^t \psi(T(t^*+s)u)\mathrm{d}s \\
=& \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_{t^*}^{t+t*} \psi(T(s)u)\mathrm{d}s \\
=& \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_{t}^{t+t^*} \psi(T(s)u)\mathrm{d}s - \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^{t^*} \psi(T(s)u)\mathrm{d}s \\
&+ \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^{t} \psi(T(s)u)\mathrm{d}s \\
=& \mathrm{LIM}_{t\to+\infty}\frac{1}{t}\int_0^{t} \psi(T(s)u)\mathrm{d}s = \int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \psi(u)\mathrm{d}\rho_v(u).
\end{aligned}
$$

The invariant property of ρ_v under the action of $\{T(t)\}_{t\geqslant 0}$ is proved.

We next establish that $\mu_t = T(t)\rho_v$, which is defined by (2.13), satisfies the Liouville type equation in Statistical Mechanics. In fact, since the mapping $T(t)$ from $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$ into itself is continuous and ρ_v is a probability measure on $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$, it is not difficult to check that $\mu_t = T(t)\rho_v$ is a probability measure on $\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}$. Now, for any $\Phi \in \mathcal{T}$, we have

$$
\int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \Phi(u)\mathrm{d}\mu_t(u) = \int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \Phi(u)\mathrm{d}(T(t)\rho_v)(u) = \int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \Phi(T(t)u)\mathrm{d}\rho_v(u). \tag{2.20}
$$

Notice that the function $t \longmapsto \Phi(T(t)u)$ is differentiable and (2.14) holds true. We use the generalized chain differentiation rule to differentiate (2.20) and obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \Phi(u)\mathrm{d}\mu_t(u) =& \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \Phi(u)\mathrm{d}(T(t)\rho_v)(u) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \Phi(T(t)u)\mathrm{d}\rho_v(u) \\
=& \int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \Big\langle \Phi'(T(t)u), \frac{\mathrm{d}}{\mathrm{d}t}T(t)u\Big\rangle \mathrm{d}\rho_v(u) \\
=& \int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \langle \Phi'(T(t)u), F(T(t)u)\rangle \mathrm{d}\rho_v(u) \\
=& \int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \langle \Phi'(u), F(u)\rangle \mathrm{d}\mu_t(u).
\end{aligned}
$$

This proves (2.17) and the proof of Theorem 2.2 is therefore finished. $\square$

We point out that, if a statistical equilibrium has been reached by the system, then the statistical informations do not change with time, that is $\Phi'(u(t)) = 0$. In this situation, (2.17) implies

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \Phi(u)\mathrm{d}\mu_t(u) = \int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \langle \Phi'(u), F(u)\rangle \mathrm{d}\mu_t(u) = 0. \tag{2.21}
$$

(2.21) shows that μ_t is independent of time t. Thus (2.20) and (2.21) imply

$$\int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \Phi(T(t)u)\mathrm{d}\rho_v(u) = \int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \Phi(u)\mathrm{d}\mu_t(u) = \int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \Phi(u)\mathrm{d}\mu_0(u) = \int_{\mathcal{T}_{\mathcal{X}}^{\mathrm{tr}}} \Phi(u)\mathrm{d}\rho_v(u). \quad (2.22)$$

(2.22) describes exactly the invariant property of the trajectory statistical solutions under the action of the translation semigroup $\{T(t)\}_{t\geqslant 0}$, which has been proved in Theorem 2.2(1). The previous analysis reveals that the equation describing the invariant property of the trajectory statistical solutions is a particular situation of the Liouville type equation.

3 Trajectory attractor and trajectory statistical solution for the 3D incompressible magneto-micropolar fluids

In this section, we study the 3D incompressible magneto-micropolar fluids in details, illustrating how to apply our abstract results to some concrete autonomous evolution equations.

3.1 The 3D incompressible magneto-micropolar fluids

We consider the following system of incompressible magneto-micropolar fluids equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} - (\nu+\nu_r)\Delta u + (u\cdot\nabla)u + \nabla(p+\dfrac{1}{2}h\cdot h) = 2\nu_r\nabla\times\omega + r(h\cdot\nabla)h + f, \\ j\dfrac{\partial \omega}{\partial t} - \alpha\Delta\omega - \beta\nabla\mathrm{div}\omega + 4\nu_r\omega + j(u\cdot\nabla)\omega = 2\nu_r\nabla\times u + g, \\ \dfrac{\partial h}{\partial t} - \mu\Delta h + (u\cdot\nabla)h - (h\cdot\nabla)u = 0, \\ \mathrm{div}\, u = 0, \mathrm{div}\, h = 0, \end{cases} \quad (3.1)$$

with the initial and boundary conditions

$$u(x,0) = u_0(x),\ \omega(x,0) = \omega_0(x),\ h(x,0) = h_0(x),\ x\in\Omega, \quad (3.2)$$

$$u(x,t) = \omega(x,t) = h(x,t) = 0,\ (x,t)\in\partial\Omega\times[0,+\infty), \quad (3.3)$$

where

$$\begin{cases} u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \\ \omega = \omega(x,t) = (\omega_1(x,t), \omega_2(x,t), \omega_3(x,t)) \\ h = h(x,t) = (h_1(x,t), h_2(x,t), h_3(x,t)) \\ p = p(x,t), \end{cases}$$

are unknown functions, here $u(x,t)$ denotes the velocity of the fluid at a physical point $x=(x_1,x_2,x_3)$ and at the moment of time t; $\omega(x,t)$, $h(x,t)$ and $p(x,t)$ denote respectively the micro-rotational velocity, magnetic field and hydrostatic pressure of the fluid;

$f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t))$ and $g = g(x,t) = (g_1(x,t), g_2(x,t), g_3(x,t))$ are the external force and the angular momentum, respectively; ν, ν_r, r, j, α, β and μ are positive constants associated to properties of the material. In this article, we consider above equations (3.1) in a 3D bounded domain $\Omega \subset \mathbb{R}^3$ with suitable smooth boundary $\partial\Omega$, and take $r = j = \mu = 1$ for simplicity.

Above equations (3.1) describe the motion of electrically conducting micropolar fluids (see [23]) in the presence of magnetic fields. The theory of magneto-micropolar fluids was first introduced by Galdi and Rionero in [18]. If the magnetic field $h = (0,0,0)$, equations (3.1) reduce to the 3D micropolar fluids system, which was first proposed by Eringen [10] and the pullback dynamical behavior was investigated by Zhao *et al.* [44]. If we ignore the micro-rotation of particles, equations (3.1) reduce to the 3D incompressible magneto-hydrodynamic (MHD) equations, which can model the magnetic properties of electrically conducting fluids (see e. g. [40, 42, 47]). When the magnetic field is absent ($h = (0,0,0)$) and there is no micro-structure ($\nu_r = 0$), then equations (3.1) turn to the classical 3D incompressible Navier-Stokes equations.

The incompressible magneto-micropolar fluids equations have been extensively studied due to its abundant physical background. In the 2D space, the global existence of solutions for the equations with mixed partial viscosity was studied in [28]; the global regularity of solutions was investigated in [6, 29, 34, 39]; the existence of uniform attractor was obtained in [22]. In the 3D space, Li and Shang studied the large time decay of solutions in [20]; Yuan studied the regularity of weak solutions and the blow-up criteria for smooth solutions in [41]; Rojas-Medar proved the local existence and uniqueness of strong solutions in [31]. The global existence of weak solutions for the 3D case and the uniqueness of weak solution for the 2D case were obtained in [32]. However, as far as we know, there are few references investigating the invariant measures and trajectory statistical solutions for the 3D magneto-micropolar fluids equations.

The purpose of this section is to investigate the existence of trajectory attractors and trajectory statistical solutions for the 3D incompressible magneto-micropolar fluids equations. Similar to the 3D incompressible Navier-Stokes equations, the global uniqueness of weak solution for the 3D incompressible magneto-micropolar fluids equations is still unproved. Thus the semigroup of solution operators is not well defined in the phase space.

3.2 Global existence and estimates of weak solutions

We first introduce some notations. Let $\mathbb{L}^q(\Omega) = (L^q(\Omega))^3$ to denote the 3D Lebesgue space, with the norm denoted by $\|\cdot\|_{\mathbb{L}^q}$. The norm in $\mathbb{L}^2(\Omega)$ will be denoted just by $\|\cdot\|$. Let $\mathbb{W}_0^{m,p} = (W_0^{m,p})^3$ to denote the 3D Sobolev spaces, with the norm denoted by $\|\cdot\|_{m,p}$. Especially, we write $\mathbb{W}_0^{m,p}(\Omega) = \mathbb{H}_0^m$ for $p = 2$. We also use the following function spaces:

$\mathcal{V} = \Big\{\varphi = (\varphi_1, \varphi_2, \varphi_3) \in (\mathcal{C}_0^\infty(\Omega))^3 | \nabla\cdot\varphi = 0\Big\}$,

$H =$ the closure of $\mathcal{V}$ in $\mathbb{L}^2$ with inner product $(\cdot,\cdot)$ and norm as in $\mathbb{L}^2$, $\|\cdot\|_H = \|\cdot\|$,

$V =$ the closure of $\mathcal{V}$ in $\mathbb{H}^1$ with norm $\|\cdot\|_V = \|\cdot\|_{1,2}$,

$\hat{H} = H \times \mathbb{L}^2 \times H$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|_{\hat{H}} = \|\cdot\|$ defined as

$$(\Phi, \Psi) = (\phi, \xi) + (\varphi, \eta) + (\psi, \zeta),\ \ \Phi = (\phi, \varphi, \psi), \Psi = (\xi, \eta, \zeta) \in \hat{H},$$
$$\|\Phi\| = (\|\phi\|^2 + \|\varphi\|^2 + \|\psi\|^2)^{1/2},\ \ \Phi = (\phi, \varphi, \psi) \in \hat{H},$$

$\hat{V} = V \times \mathbb{H}_0^1 \times V$ with norm $\|\cdot\|_{\hat{V}}$ defined as

$$\|\Phi\|_{\hat{V}} = (\|\phi\|_V^2 + \|\varphi\|_{1,2}^2 + \|\psi\|_V^2)^{1/2},\ \Phi = (\phi, \varphi, \psi) \in \hat{V},$$

here, we use the same notation $(\cdot,\cdot)$ to denote the inner product in $\mathbb{L}^2(\Omega)$, H and $\hat{H}$ when no confusion is possible. At the same time, we use $\hat{V}^*$ to denote the dual space of $\hat{V}$ and use $\langle\cdot,\cdot\rangle$ to denote the dual pairing between $\hat{V}$ and $\hat{V}^*$.

Now we use $\hat{H}_{\mathrm{w}}$ to denote the space $\hat{H}$ endowed with its weak topology. In fact, the collection of open sets in $\hat{H}_{\mathrm{w}}$ has a characterization by a basis of neighborhoods given by

$$\mathcal{O}_{\mathrm{w}}(u, r, v_1, v_2, \cdots, v_N) = \Big\{ w \in \hat{H}_{\mathrm{w}} \Big| \sum_{j=1}^{N} |(u - w, v_j)|^2 < L^2 \Big\},$$

for $u \in \hat{H}_{\mathrm{w}}$, $L > 0$, $N \in \mathbb{N}$, and $v_1, v_2, \cdots, v_N \in \hat{H}$. Further, let $\mathcal{F}_{\mathrm{loc}}^+ = \mathcal{C}_{\mathrm{loc}}([0, +\infty); \hat{H}_{\mathrm{w}})$ be the space of continuous functions from $[0, +\infty)$ to $\hat{H}_{\mathrm{w}}$. This space can also be seen as the space of weakly continuous function from $\mathbb{R}_+$ to $\hat{H}$. The topology on $\mathcal{F}_{\mathrm{loc}}^+$, which is still denoted by Θ_{loc}^+, is that of uniform convergence in $\hat{H}_{\mathrm{w}}$ on compact subinterval of $\mathbb{R}_+$, that is, by definition, $w_n(t) \longrightarrow w(t)$ $(n \to +\infty)$ in the topology Θ_{loc}^+, if, for every $T > 0$, $(w_n(t), \Phi) \longrightarrow (w(t), \Phi)$ uniformly on $[0, T]$ $(n \to +\infty)$ for each $\Phi \in \hat{H}$.

We next introduce some operators. Define a trilinear form $b(\cdot,\cdot,\cdot)$ as

$$b(u, \omega, h) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial \omega_j}{\partial x_i} h_j \mathrm{d}x,\ \ \forall u \in V,\ \forall \omega,\ h \in \mathbb{H}_0^1.$$

It is not difficult to check that $b(u, \omega, h)$ is continuous on $V \times \mathbb{H}_0^1 \times \mathbb{H}_0^1$ and

$$b(u, \omega, h) = -b(u, h, \omega),\ \ b(u, \omega, \omega) = 0,\ \ \forall u \in V,\ \omega,\ h \in \mathbb{H}_0^1. \tag{3.4}$$

We also define the operator B: $\hat{V} \times \hat{V} \longmapsto \hat{V}^*$ by

$$\langle B(w, \Phi), \Psi \rangle = b(u, \phi, \xi) + b(u, \varphi, \eta) + b(u, \psi, \zeta) - b(h, \phi, \zeta) - b(h, \psi, \xi),$$

for any $w = (u, \omega, h)$, $\Phi = (\phi, \varphi, \psi)$, $\Psi = (\xi, \eta, \zeta) \in \hat{V}$. As a consequence of (3.4), we have

$$\langle B(w, \Phi), \Phi \rangle = 0,\ \ \forall\, w, \Phi \in \hat{V}. \tag{3.5}$$

Further, for each $w = (u, \omega, h) \in \hat{V}$ we define two operators A and N: $\hat{V} \longmapsto \hat{V}^*$ as

$$\langle Aw, \Phi \rangle = (\nu + \nu_r)(\nabla u, \nabla \phi) + \alpha(\nabla \omega, \nabla \varphi) + (\nabla h, \nabla \psi) + \beta(\mathrm{div}\omega, \mathrm{div}\varphi),\ \forall \Phi = (\phi, \varphi, \psi) \in \hat{V},$$

$$\langle N(w), \Phi \rangle = -2\nu_r(\nabla \times \omega, \phi) - 2\nu_r(\nabla \times u, \varphi) + 4\nu_r(\omega, \varphi),\ \forall \Phi = (\phi, \varphi, \psi) \in \hat{V}.$$

In the sequel, we let $F=(f,g,0)\in\hat{H}$ be independent of time t. Using the above notations and operators, we can write the initial boundary values problem (3.1)-(3.3) as following:

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t}+Aw(t)+B(w(t),w(t))+N(w(t))=F,\ \ t>0, \tag{3.6}$$

$$w|_{t=0}=w_0=(u_0,\omega_0,h_0)\in\hat{H}, \tag{3.7}$$

where (3.6) is understood in the sense of distributions $\mathcal{D}'(0,+\infty;\hat{V}^*)$.

Definition 3.1. *Let $w_0=(u_0,\omega_0,h_0)\in\hat{H}$. A function $w(t)=(u(t),\omega(t),h(t))$ is said to be a weak solution of problem* (3.6)-(3.7) *satisfying the initial condition w_0, if*

(1) $\dfrac{\mathrm{d}w(t)}{\mathrm{d}t}\in L^{4/3}([0,T];\hat{V}^*)$, $w(t)\in\mathcal{C}([0,T];\hat{H}_{\mathrm{w}})\cap L^2([0,T];\hat{V})$ *for all $T>0$;*

(2) $w(t)$ *satisfies the equation*

$$(\frac{\mathrm{d}w(t)}{\mathrm{d}t},\Phi)+\langle Aw(t),\Phi\rangle+\langle B(w(t),w(t)),\Phi\rangle+\langle N(w(t)),\Phi\rangle=\langle F,\Phi\rangle,\ \ \forall\Phi\in\hat{V},$$

in the distribution sense $\mathcal{D}'(0,+\infty)$, and $w(t)$ satisfies the following energy inequality

$$\begin{aligned}&\|w(t)\|^2+2\int_0^t\langle Aw(s),w(s)\rangle\mathrm{d}s+2\int_0^t\langle N(w(s)),w(s)\rangle\mathrm{d}s\\ \leqslant&\|w_0\|^2+2\int_0^t(F,w(s))\mathrm{d}s\end{aligned} \tag{3.8}$$

in the sense that

$$\begin{aligned}&-\frac{1}{2}\int_0^t\|w(t)\|^2\phi'(t)\mathrm{d}t+\int_0^t\langle Aw(t),w(t)\rangle\phi(t)\mathrm{d}t+\int_0^t\langle N(w(t)),w(t)\rangle\phi(t)\mathrm{d}t\\ \leqslant&\int_0^t(F(t),w(t))\phi(t)\mathrm{d}t,\ \forall\phi(\cdot)\in\mathcal{C}_0^\infty([0,t])\ \textit{with}\ \phi(t)\geqslant 0,\ \forall t\geqslant 0.\end{aligned} \tag{3.9}$$

In the sequel of the article, we will use for brevity the notation $q_1\lesssim q_2$ to mean that $q_1\leqslant cq_2$ for a universal constant $c>0$ that only depends on the parameters coming from the problem.

For the existence and some estimates of weak solutions to problem (3.6)-(3.7), we have the following result.

Lemma 3.1. *Assume $F\in\hat{H}$. Then, for any $w_0=(u_0,\omega_0,h_0)\in\hat{H}$, there exists at least one corresponding weak solution $w(t)$ to problem* (3.6)-(3.7)*. Moreover, for the positive constant $R=1+\|f\|^2+\|g\|^2$ there exists a positive time $t_*\geqslant 0$ depending only on Ω, f, g, α, ν and w_0 such that*

$$w(t)\in X\triangleq\{v\in\hat{H}\big|\ \|v\|\leqslant R\},\ \forall t\geqslant t_*. \tag{3.10}$$

Proof. The proof of the existence of weak solutions to problem (3.6)-(3.7) can be found in [32]. We next prove (3.10). Taking the dual pairing of $w(t)$ with equation (3.6) yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2 + (\nu+\nu_r)\|\nabla u(t)\|^2 + \alpha\|\nabla\omega(t)\|^2 + \|\nabla h(t)\|^2 \\
&- 2\nu_r\int_\Omega \nabla\times\omega(t)\cdot u(t)\mathrm{d}x - 2\nu_r\int_\Omega \nabla\times u(t)\cdot\omega(t)\mathrm{d}x \\
&+ 4\nu_r\|\omega(t)\|^2 + \beta\|\mathrm{div}\omega(t)\|^2 = \int_\Omega fu(t)\mathrm{d}x + \int_\Omega g\omega(t)\mathrm{d}x. \qquad (3.11)
\end{aligned}$$

Since $\nabla\cdot v = 0$, direct computations imply

$$\nabla\times(\nabla\times v) = -\Delta v,\ \forall v\in V. \qquad (3.12)$$

Taking inner product of uv with (3.12) yields

$$\|\nabla\times v\|^2 = \|\nabla v\|^2,\ \forall v\in V.$$

Then using Cauchy inequality and the following Poincaré inequality

$$\|\phi\|^2 \leqslant \lambda\|\nabla\phi\|^2,\ \forall\phi\in\mathbb{H}_0^1(\Omega),\ \lambda \text{ is a constant depending only on } \Omega, \qquad (3.13)$$

we have

$$\begin{aligned}
2\nu_r\int_\Omega \nabla\times\omega(t)\cdot u(t)\mathrm{d}x &= 2\nu_r\int_\Omega \nabla\times u(t)\cdot\omega(t)\mathrm{d}x \\
&\leqslant 2\nu_r\|\omega(t)\|^2 + \frac{\nu_r}{2}\|\nabla u(t)\|^2, \qquad (3.14)\\
\int_\Omega fu(t)\mathrm{d}x + \int_\Omega g\omega(t)\mathrm{d}x &\lesssim \frac{\alpha}{2\lambda}\|\omega(t)\|^2 + \|g\|^2 + \frac{\nu}{2\lambda}\|u(t)\|^2 + \|f\|^2 \\
&\lesssim \frac{\alpha}{2}\|\nabla\omega(t)\|^2 + \|g\|^2 + \frac{\nu}{2}\|\nabla u(t)\|^2 + \|f\|^2. \qquad (3.15)
\end{aligned}$$

Combining (3.11) and (3.14)-(3.15), we see that there exists a constant δ depending only on α and ν such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2 + \delta\|\nabla w(t)\|^2 \lesssim \|g\|^2 + \|f\|^2. \qquad (3.16)$$

It then follows from (3.13), (3.16) and the Gronwall inequality that

$$\|w(t)\|^2 \lesssim \|w_0\|^2 e^{-\frac{\delta}{\lambda}t} + \|g\|^2 + \|f\|^2,\ \forall t>0,$$

from which we readily obtain (3.10). The proof of Lemma 3.1 is completed. □

3.3 Existence of the trajectory attractor

We will prove the existence of trajectory attractor for equation (3.6). Hereinafter, we let X be the fixed set defined by (3.10). We begin with the definition of the trajectory space.

Definition 3.2. *The trajectory space $\mathcal{T}_{\hat{H}}^{+}$ of equation* (3.6) *consists of functions $w(t) \in L^{\infty}([0,+\infty);\hat{H}) \cap L^2_{\mathrm{loc}}([0,+\infty);\hat{V})$ such that $w(t)$ is a weak solution of* (3.6) *on* $[0,+\infty)$ *and $w(t) \in X$ for all $t \in [0,+\infty)$.*

Let $\mathcal{M}_{\mathrm{loc}}^{+} = \mathcal{C}_{\mathrm{loc}}([0,+\infty);X_{\mathrm{w}})$ be the space of continuous functions from $[0,+\infty)$ to X_{w}, where X_{w} denotes the space X endowed with the weak topology inherited from $\hat{H}_{\mathrm{w}}$. We endow $\mathcal{M}_{\mathrm{loc}}^{+}$ also with the topology $\Theta_{\mathrm{loc}}^{+}$. Since X is a fixed bounded subset of the Banach space $\hat{H}$, the topology $\Theta_{\mathrm{loc}}^{+}$ in $\mathcal{M}_{\mathrm{loc}}^{+}$ is metrizable. Note that $\mathcal{M}_{\mathrm{loc}}^{+}$ is a Hausdorff topological space.

The natural translation semigroup $\{T(t)\}_{t\geqslant 0}$ on $\mathcal{F}_{\mathrm{loc}}^{+}$ is defined as

$$(T(t)w)(s) = w(t+s),\ \forall\, w \in \mathcal{F}_{\mathrm{loc}}^{+}.$$

We obviously have $T(t)\mathcal{T}_{\hat{H}}^{+} \subseteq \mathcal{T}_{\hat{H}}^{+}$, $\forall t \geqslant 0$.

Now we have given the definitions of spaces $\mathcal{F}_{\mathrm{loc}}^{+}$, $\mathcal{M}_{\mathrm{loc}}^{+}$, the trajectory space $\mathcal{T}_{\hat{H}}^{+}$ and its corresponding topology $\Theta_{\mathrm{loc}}^{+}$, as well as the translation semigroup $\{T(t)\}_{t\geqslant 0}$ acting on $\mathcal{F}_{\mathrm{loc}}^{+}$. Then the definitions of trajectory attracting set and trajectory attractor for the equations of 3D incompressible magneto-micropolar fluids are similar to those in Definition 2.2. To prove the existence of trajectory attractor for equation (3.6), we next establish some lemmas.

Lemma 3.2. *Let $w(t) = (u(t),\omega(t),h(t)) \in L^{\infty}([0,T];\hat{H}) \cap L^2([0,T];\hat{V})$ for all $T > 0$, then*

$$t \longmapsto Aw(t) \in L^2([0,T];\hat{V}^*), \tag{3.17}$$

$$t \longmapsto B(w(t),w(t)) \in L^{4/3}([0,T];\hat{V}^*), \tag{3.18}$$

$$t \longmapsto N(w(t)) \in L^2([0,T];\hat{V}^*). \tag{3.19}$$

Proof. Consider a fixed $T > 0$ and $w(t) = (u(t),\omega(t),h(t)) \in L^{\infty}([0,T];\hat{H}) \cap L^2([0,T];\hat{V})$. Then for a.e. $t \in [0,T]$, we see from the definitions of the operators A, $B(\cdot,\cdot)$ and $N(\cdot)$ that $Aw(t)$, $B(w(t),w(t))$ and $N(w(t))$ belong to $\hat{V}^*$. Also, the measurability of the functions $t \longmapsto Aw(t)$, $t \longmapsto B(w(t),w(t))$, $t \longmapsto N(w(t))$ are not difficult to check.

We first prove that $t \longmapsto Aw(t) \in L^2([0,T];\hat{V}^*)$. In fact, using the Cauchy inequality we obtain

$$\begin{aligned}
|\langle Aw(t),\Phi\rangle| &\lesssim |(\nabla u(t),\nabla\phi)| + |(\nabla\omega(t),\nabla\varphi)| + |(\nabla h(t),\nabla\psi)| + |(\mathrm{div}\omega,\mathrm{div}\varphi)| \\
&\lesssim \|\nabla u(t)\|\|\nabla\phi\| + \|\nabla\omega(t)\|\|\nabla\varphi\| + \|\nabla h(t)\|\|\nabla\psi\| \\
&\lesssim \|u(t)\|_V\|\phi\|_V + \|\omega(t)\|_{1,2}\|\varphi\|_{1,2} + \|h(t)\|_V\|\psi\|_V \\
&\lesssim \|w(t)\|_{\hat{V}}\|\Phi\|_{\hat{V}},\ \ \forall\,\Phi = (\phi,\varphi,\psi) \in \hat{V}.
\end{aligned} \tag{3.20}$$

Hence, $\|Aw(t)\|_{\hat{V}^*} \lesssim \|w(t)\|_{\hat{V}}$ and

$$\int_0^T \|Aw(t)\|_{\hat{V}^*}^2 \mathrm{d}t \lesssim \int_0^T \|w(t)\|_{\hat{V}}^2 \mathrm{d}t < \infty.$$

We next prove that $t \longmapsto B(w(t), w(t)) \in L^{4/3}([0,T]; \hat{V}^*)$. By using the Hölder inequality, the Gagliardo-Nirenberg inequality and the embedding $\hat{V} \hookrightarrow \mathbb{L}^6 \times \mathbb{L}^6 \times \mathbb{L}^6$, we have

$$
\begin{aligned}
|b(u,u,\phi)| &\lesssim \|(u \cdot \nabla)u(t)\|_{\mathbb{L}^{6/5}} \|\phi\|_{\mathbb{L}^6} \lesssim \|u(t)\|_{\mathbb{L}^3} \|\nabla u(t)\| \|\phi\|_{\mathbb{L}^6} \\
&\lesssim \|u(t)\|^{1/2} \|\nabla u(t)\|^{1/2} \|u(t)\|_V \|\phi\|_{\mathbb{L}^6} \lesssim \|u(t)\|^{1/2} \|u(t)\|_V^{3/2} \|\phi\|_{\mathbb{L}^6} \\
&\lesssim \|w(t)\|^{1/2} \|w(t)\|_{\hat{V}}^{3/2} \|\Phi\|_{\mathbb{L}^6 \times \mathbb{L}^6 \times \mathbb{L}^6} \\
&\lesssim \|w(t)\|^{1/2} \|w(t)\|_{\hat{V}}^{3/2} \|\Phi\|_{\hat{V}}, \ \ \forall\, \Phi = (\phi, \varphi, \psi) \in \hat{V}.
\end{aligned}
$$

Thus, it is obvious that

$$
|\langle B(w(t), w(t)), \Phi \rangle| \lesssim \|w(t)\|^{1/2} \|w(t)\|_{\hat{V}}^{3/2} \|\Phi\|_{\hat{V}}, \, \forall \Phi \in \hat{V}. \tag{3.21}
$$

Hence, $\|B(w(t), w(t))\|_{\hat{V}^*} \lesssim \|w(t)\|_{\hat{V}}^{3/2} \|w(t)\|^{1/2}$ for $w(t) \in L^\infty([0,T]; \hat{H}) \cap L^2([0,T]; \hat{V})$, and we have

$$
\int_0^T \|B(w(t), w(t))\|_{\hat{V}^*}^{4/3} \mathrm{d}t \lesssim \int_0^T \|w(t)\|^{2/3} \|w(t)\|_{\hat{V}}^2 \mathrm{d}t < \infty.
$$

Also, using the Cauchy inequality and (3.13) yields

$$
\begin{aligned}
|\langle N(w(t)), \Phi \rangle| &\lesssim |(\nabla \times \omega(t), \phi)| + |(\nabla \times u(t), \varphi)| + |(\omega(t), \varphi)| \\
&\lesssim \|\nabla \omega(t)\| \|\phi\| + \|\nabla u(t)\| \|\varphi\| + \|\omega(t)\| \|\varphi\| \\
&\lesssim \|\omega(t)\|_{1,2} \|\phi\|_V + \|u(t)\|_V \|\varphi\|_V + \|\omega(t)\|_{1,2} \|\varphi\|_V \\
&\lesssim \|w(t)\|_{\hat{V}} \|\Phi\|_{\hat{V}}, \, \forall \Phi = (\phi, \varphi, \psi) \in \hat{V}.
\end{aligned} \tag{3.22}
$$

Whence, $\|N(w(t))\|_{\hat{V}^*} \lesssim \|w(t)\|_{\hat{V}}$ and

$$
\int_0^T \|N(w(t))\|_{\hat{V}^*}^2 \mathrm{d}t \lesssim \int_0^T \|w(t)\|_{\hat{V}}^2 \mathrm{d}t < \infty.
$$

The proof of Lemma 3.2 is completed. □

Lemma 3.3. *For any $\Phi = (\phi, \varphi, \psi) \in \hat{V}$, we have*

$$
\|\Phi\|_{\hat{V}}^2 \lesssim \langle A\Phi, \Phi \rangle + \langle N(\Phi), \Phi \rangle \lesssim \|\Phi\|_{\hat{V}}^2.
$$

Proof. According to (3.20) and (3.22),

$$
\langle A\Phi, \Phi \rangle + \langle N(\Phi), \Phi \rangle \lesssim \|\Phi\|_{\hat{V}}^2.
$$

From the definition of operator A, we see that

$$
\begin{aligned}
\langle A\Phi, \Phi \rangle &= (\nu + \nu_r)(\nabla \phi, \nabla \phi) + \alpha(\nabla \varphi, \nabla \varphi) + (\nabla \psi, \nabla \psi) + \beta(\mathrm{div}\varphi, \mathrm{div}\varphi) \\
&\geqslant (\nu + \nu_r)\|\nabla \phi\|^2 + \alpha \|\nabla \varphi\|^2 + \|\nabla \psi\|^2.
\end{aligned} \tag{3.23}
$$

At the same time, using (3.14), we have

$$\begin{aligned}\langle N(\Phi),\Phi\rangle &\geqslant -2\nu_r\|\varphi\|^2-\frac{\nu_r}{2}\|\nabla\phi\|^2-2\nu_r\|\varphi\|^2-\frac{\nu_r}{2}\|\nabla\phi\|^2+4\nu_r\|\varphi\|^2\\ &\geqslant -\nu_r\|\nabla\phi\|^2.\end{aligned}\tag{3.24}$$

Combining (3.23) and (3.24) yields

$$\langle A\Phi,\Phi\rangle+\langle N(\Phi),\Phi\rangle\geqslant\nu\|\nabla\phi\|^2+\alpha\|\nabla\varphi\|^2+\|\nabla\psi\|^2\gtrsim\|\Phi\|^2_{\hat{V}}.$$

The proof of Lemma 3.3 is thus completed. □

Lemma 3.4. *Let $\{w_n(t)\}_{n\geqslant1}$ be a sequence of weak solutions to equation (3.6) such that $w_n(t)\in X$ for all $t\geqslant0$. Then*

(1) *for all $T>0$,*

$$\{w_n(t)\}_{n\geqslant1}\ \textit{is bounded in}\ L^2([0,T];\hat{V})\cap L^\infty([0,T];\hat{H}),\tag{3.25}$$

$$\Big\{\frac{\mathrm{d}}{\mathrm{d}t}w_n(t)\Big\}_{n\geqslant1}\ \textit{is bounded in}\ L^{4/3}([0,T];\hat{V}^*),\tag{3.26}$$

(2) *there is a subsequence $\{w_{n_j}(t)\}_{n_j\geqslant1}$ of $\{w_n(t)\}_{n\geqslant1}$ that converges in $\mathcal{C}([0,T];\hat{H}_{\mathrm{w}})$ to some weak solution $w(t)$ of equation (3.6) and $w(t)\in X$, i.e.,*

$$(w_{n_j}(t),\Phi)\to(w(t),\Phi)\ \textit{uniformly on}\ [0,T],\ n_j\to\infty,\ \forall\Phi\in\hat{H}.\tag{3.27}$$

Proof. Let $\{w_n(t)\}_{n\geqslant1}$ be a sequence of weak solutions to problem (3.6) such that $w_n(t)\in X$ for all $t\geqslant0$. For a given $T>0$, we integrate (3.16) over $[0,T]$ and obtain

$$\|w_n(t)\|^2+\delta\int_0^T\|\nabla w_n(t)\|^2\mathrm{d}t\lesssim\|w_n(0)\|^2+T(\|g\|^2+\|f\|^2),\ t>0,\tag{3.28}$$

which implies that

$$\{w_n(t)\}_{n\geqslant1}\ \text{ is bounded in }L^2([0,T];\hat{V})\cap L^\infty([0,T];\hat{H}).\tag{3.29}$$

We thus conclude from equation (3.6), Lemma 3.2 and the embedding $L^2([0,T];\hat{V}^*)\hookrightarrow L^{4/3}([0,T];\hat{V}^*)$ that

$$\begin{aligned}\Big\|\frac{\mathrm{d}w_n(t)}{\mathrm{d}t}\Big\|_{L^{4/3}([0,T];\hat{V}^*)}\leqslant&\|Aw_n(t)\|_{L^{4/3}([0,T];\hat{V}^*)}+\|B(w_n(t),w_n(t))\|_{L^{4/3}([0,T];\hat{V}^*)}\\ &+\|N(w_n(t))\|_{L^{4/3}([0,T];\hat{V}^*)}+\|F\|_{L^{4/3}([0,T];\hat{V}^*)}\\ \lesssim&\|w_n(t)\|_{L^2([0,T];\hat{V})}+\|w_n(t)\|^{1/2}_{L^\infty([0,T];\hat{H})}\|w_n(t)\|^{3/2}_{L^2([0,T];\hat{V})}\\ &+\|w_n(t)\|_{L^2([0,T];\hat{V})}+T\|F\|.\end{aligned}\tag{3.30}$$

Since that $F\in\hat{H}$, we see from (3.29) that the right hand side of (3.30) is bounded by a constant independent of n. Assertion (1) of Lemma 3.4 is proved.

We next prove assertion (2) of Lemma 3.4. In fact, from assertion (1) of this lemma, we see that there exists some $w(t) \in L^\infty([0,T];\hat{H}) \cap L^2([0,T];\hat{V})$ with $\frac{\mathrm{d}w(t)}{\mathrm{d}t} \in L^{4/3}([0,T];\hat{V}^*)$ and a subsequence $\{w_{n_j}(t)\}_{n_j \geqslant 1}$ of $\{w_n(t)\}_{n \geqslant 1}$, such that

$$w_{n_j}(t) \rightharpoonup w(t) \text{ weakly star in } L^\infty([0,T];\hat{H}) \text{ as } n_j \to \infty, \tag{3.31}$$

$$w_{n_j}(t) \rightharpoonup w(t) \text{ weakly in } L^2([0,T];\hat{V}) \text{ as } n_j \to \infty, \tag{3.32}$$

$$\frac{\mathrm{d}w_{n_j}(t)}{\mathrm{d}t} \rightharpoonup \frac{\mathrm{d}w(t)}{\mathrm{d}t} \text{ weakly in } L^{4/3}([0,T];\hat{V}^*) \text{ as } n_j \to \infty. \tag{3.33}$$

Now we denote by E the Banach space consisting of functions $\Phi \in L^2([0,T];\hat{V})$ and $\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} \in L^{4/3}([0,T];\hat{V}^*)$, and endow E with the norm

$$\|\Phi\|_E = \|\Phi\|_{L^2([0,T];\hat{V})} + \|\Phi\|_{L^{4/3}([0,T];\hat{V}^*)}.$$

Then we have (cf. [23], Theorem 2.3.1)

$$E \subset \mathcal{C}([0,T];\hat{H}_{\mathrm{w}}) \text{ and } E \hookrightarrow L^2([0,T];\hat{H}) \text{ with compact embedding.} \tag{3.34}$$

From (3.31)-(3.34) we conclude that

$$w_{n_j}(t) \to w(t) \text{ in } \mathcal{C}([0,T];\hat{H}_{\mathrm{w}}),\ n_j \to \infty. \tag{3.35}$$

Next we check that $w(t)$ is a weak solution of (3.6). Firstly, by (3.32) and definitions of the operators A and $N(\cdot)$ we obviously have

$$Aw_{n_j}(t) \rightharpoonup Aw(t) \text{ weakly in } L^2([0,T];\hat{V}^*),\ n_j \to \infty, \tag{3.36}$$

$$N(w_{n_j}(t)) \rightharpoonup N(w(t)) \text{ weakly in } L^2([0,T];\hat{V}^*),\ n_j \to \infty. \tag{3.37}$$

For the nonlinear term $B(w_{n_j}(t), w_{n_j}(t))$, we can use the same derivation as that as [35] to obtain

$$B(w_{n_j}(t), w_{n_j}(t)) \rightharpoonup B(w(t), w(t)) \text{ weakly in } L^{4/3}([0,T];\hat{V}^*),\ n_j \to \infty. \tag{3.38}$$

Since for each $n_j \geqslant 1$ we have

$$\frac{\mathrm{d}w_{n_j}(t)}{\mathrm{d}t} + Aw_{n_j}(t) + B(w_{n_j}(t), w_{n_j}(t)) + N(w_{n_j}(t)) = F, \text{ a.e. for } t > 0 \tag{3.39}$$

in $L^{4/3}([0,T];\hat{V}^*)$, we therefore conclude from (3.33) and (3.36)-(3.39) that

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} + Aw(t) + B(w(t), w(t)) + N(w(t)) = F, \text{ a.e. for } t > 0 \tag{3.40}$$

in $L^{4/3}([0,T];\hat{V}^*)$. Since $L^{4/3}[0,T] \subset \mathcal{D}'[0,T]$, we see that $w(t)$ satisfies (3.6) in the sense of distributions in $\mathcal{D}'(0,T;\hat{V}^*)$.

Secondly, we check that $w(t)$ matches the energy inequality in the sense of (3.9). In fact (3.34) and (3.35) imply

$$w_{n_j}(t) \longrightarrow w(t) \text{ strongly in } L^2([0,T];\hat{H}),\ n_j \to \infty, \tag{3.41}$$

$$\|w_{n_j}(t)\|^2 \longrightarrow \|w(t)\|^2 \text{ for a.e. } t \in [0,T],\ n_j \to \infty. \tag{3.42}$$

Consider a given $\phi(\cdot) \in \mathcal{C}_0^\infty([0,T])$ with $\phi(t) \geqslant 0$. Obviously, $\|w_{n_j}(t)\|^2\phi'(t) \in L^1([0,T])$. $\|w_{n_j}(t)\|^2\phi'(t)$ has essential upper bound due to (3.31), thus it possesses an integrable dominated function. (3.42) and the Lebesgue Dominated Convergence Theorem imply

$$\lim_{n_j\to\infty}\int_0^T \|w_{n_j}(t)\|^2\phi'(t)\mathrm{d}t = \int_0^T \|w(t)\|^2\phi'(t)\mathrm{d}t. \tag{3.43}$$

Also, according to the lower semicontinuity of norm and (3.32), we have

$$\int_0^T \|w(t)\|_{\hat{V}}^2\phi(t)\mathrm{d}t \leqslant \lim_{n_j\to\infty}\int_0^T \|w_{n_j}(t)\|_{\hat{V}}^2\phi(t)\mathrm{d}t. \tag{3.44}$$

Now Lemma 3.3 shows that $\langle Aw(t), w(t)\rangle + \langle N(w(t)), w(t)\rangle$ is equivalent to $\|w(t)\|_{\hat{V}}^2$. Hence,

$$\begin{aligned}&\int_0^T \langle Aw(t), w(t)\rangle\phi(t)\mathrm{d}t + \int_0^T \langle N(w(t)), w(t)\rangle\phi(t)\mathrm{d}t\\ &\leqslant \liminf_{n_j\to\infty}\int_0^T \big(\langle Aw_{n_j}(t), w_{n_j}(t)\rangle + \langle N(w_{n_j}(t)), w_{n_j}(t)\rangle\big)\phi(t)\mathrm{d}t.\end{aligned} \tag{3.45}$$

Since $\{w_{n_j}(t)\}_{n_j\geqslant 1}$ is a sequence of weak solutions satisfying

$$\begin{aligned}&-\frac{1}{2}\int_0^T \|w_{n_j}(t)\|^2\phi'(t)\mathrm{d}t + \int_0^T \langle Aw_{n_j}(t), w_{n_j}(t)\rangle\phi(t)\mathrm{d}t + \int_0^T \langle N(w_{n_j}(t)), w_{n_j}(t)\rangle\phi(t)\mathrm{d}t\\ &\leqslant \int_0^T (F, w_{n_j}(t))\phi(t)\mathrm{d}t,\ \forall\phi \in \mathcal{C}_0^\infty([0,T]) \text{ with } \phi \geqslant 0,\end{aligned} \tag{3.46}$$

we pass to the limit in (3.46), using (3.35), (3.43)-(3.45) and conclude that $w(t)$ satisfies the energy inequality (3.9). The proof of Lemma 3.4 is completed. □

Theorem 3.1. *The translation semigroup $\{T(t)\}_{t\geqslant 0}$ possesses a trajectory attractor $\mathcal{A}_{\hat{H}}^{\mathrm{tr}}$ in $\mathcal{M}_{\mathrm{loc}}^+ \subset \mathcal{F}_{\mathrm{loc}}^+$ satisfying*

$$\mathcal{A}_{\hat{H}}^{\mathrm{tr}} = \Pi_+\mathcal{K}_{\hat{H}} = \Big\{w(t)|_{[0,+\infty)}\Big|w(t) \in \mathcal{K}_{\hat{H}}\Big\} \subset \mathcal{T}_{\hat{H}}^+,$$

where $\mathcal{K}_{\hat{H}} = \{w(t)|w(t)$ is a weak solution of (3.6) on $(-\infty,+\infty)$ and $w(t) \in X$ for all $t \in (-\infty,+\infty)$ $\}$ and $\Pi_+w(t) = w(t)$ if $t \in [0,+\infty)$.

Proof. According to [9, Theorem 7.4], we only need to prove that $\mathcal{T}_{\hat{H}}^+$ is compact in $\mathcal{M}_{\mathrm{loc}}^+$. In fact, $\mathcal{T}_{\hat{H}}^+ \subset \mathcal{M}_{\mathrm{loc}}^+$ is clear. Now pick any sequence $\{w_n(t)\}_{n\geqslant 1}$ in $\mathcal{T}_{\hat{H}}^+$. By Lemma 3.4, there exists a subsequence $\{w_n^{(1)}(t)\}_{n\geqslant 1}$ of $\{w_n(t)\}_{n\geqslant 1}$ which converges to some $w^{(1)}(t) \in \mathcal{T}_{\hat{H}}^+$ in $\mathcal{C}([0,1];X_{\mathrm{w}})$ as $n \to \infty$. Similarly, there exists a subsequence $\{w_n^{(2)}(t)\}_{n\geqslant 1}$ of $\{w_n^{(1)}(t)\}_{n\geqslant 1}$ which converges to some $w^{(2)}(t) \in \mathcal{T}_{\hat{H}}^+$ in $\mathcal{C}([0,2];X_{\mathrm{w}})$ as $n \to \infty$, and we have $w^{(1)}(t) = w^{(2)}(t)$ on $[0,1]$. Analogously, there exists a subsequence $\{w_n^{(k)}(t)\}_{n,k\geqslant 1}$ of $\{w_n^{(k-1)}(t)\}_{n,k\geqslant 1}$ converging to some $w^{(k)}(t) \in \mathcal{T}_{\hat{H}}^+$ in $\mathcal{C}([0,k];X_{\mathrm{w}})$ as $n \to \infty$, and we have $w^{(k)}(t) = w^{(k-1)}(t)$ on $[0,k-1]$. By using the diagonal procedure, we can extract a subsequence $\{w^{(n_k)}(t)\}_{n_k\geqslant 1}$ of $\{w_n(t)\}_{n\geqslant 1}$ and some $w(t) \in \mathcal{T}_{\hat{H}}^+$ such that

$$w^{(n_k)}(t) \to w(t) \text{ in } \mathcal{C}_{\mathrm{loc}}([0,+\infty);X_{\mathrm{w}}),\ n_k \to \infty.$$

The proof of Theorem 3.1 is completed. □

3.4 The trajectory statistical solutions and Liouville type equation

We have defined the corresponding spaces $\mathcal{F}_{\text{loc}}^{+}$, $\mathcal{M}_{\text{loc}}^{+}$ and the trajectory space $\mathcal{T}_{\hat{H}}^{+}$, as well as its topology Θ_{loc}^{+} for equation (3.6) of the 3D incompressible magneto-micropolar fluids. The definition of trajectory statistical solutions for equation (3.6) is then the same as Definition 2.3. To state the existence and Liouville type equation for the statistical solutions, we write equations (3.6) as

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = \mathcal{F}(w),\ t > 0,$$

where $\mathcal{F}(w) = Aw(t) + B(w(t), w(t)) + N(w(t)) + F$. Note that $\mathcal{F}(w) \in \hat{V}^*$ for each $w \in \hat{V}$.

Definition 3.3. *We define the class $\mathcal{T}$ of test functions to be the set of real-valued functionals $\Upsilon = \Upsilon(w)$ on $\hat{H}$ that are bounded on bounded subset of $\hat{H}$ and satisfy*

(1) *for any $w \in \hat{V}$, the Fréchet derivative $\Upsilon'(w)$ exists: for each $w \in \hat{V}$ there exists an element $\Upsilon'(w)$ such that*

$$\frac{|\Upsilon(w+v) - \Upsilon(w) - (\Upsilon'(w), v)|}{\|v\|_{\hat{V}}} \longrightarrow 0 \ \textit{as}\ \|v\|_{\hat{V}} \to 0,\ \ v \in \hat{V};$$

(2) *$\Upsilon'(w) \in \hat{V}$ for all $w \in \hat{V}$, and the mapping $w \longmapsto \Upsilon'(w)$ is continuous and bounded as a functional from $\hat{V}$ to $\hat{V}$;*

(3) *for every global weak solution $w(t)$ of equation* (3.6)*, there holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon(w(t)) = \langle \Upsilon'(w(t)), \mathcal{F}(w(t))\rangle,\ \ t > 0.$$

Similar to the example given after Definition 2.5, we also can construct the cylindrical test functions satisfying Definition 3.3. The details are omitted here.

At this stage, we have proved that equation (3.6) possesses a nonempty trajectory space $\mathcal{T}_{\hat{H}}^{\text{tr}}$ which is metrizable with the topology Θ_{loc}^{+}, and the translation semigroup $\{T(t)\}_{t\geqslant 0}$ possesses a trajectory attractor $\mathcal{A}_{\hat{H}}^{\text{tr}} \subseteq \mathcal{T}_{\hat{H}}^{\text{tr}}$ in the topology Θ_{loc}^{+}. Thanks to the abstract results of Theorems 2.1 and 2.2, we have

Theorem 3.2.

(1) *Let $v \in \mathcal{A}_{\hat{H}}^{\text{tr}}$, then there exists a trajectory statistical solution ρ_v on $\mathcal{T}_{\hat{H}}^{\text{tr}}$ of equation* (3.6)*, which satisfies the following invariant property*

$$\int_{\mathcal{T}_{\hat{H}}^{\text{tr}}} \Gamma(T(t)w)\mathrm{d}\rho_v(w) = \int_{\mathcal{T}_{\hat{H}}^{\text{tr}}} \Gamma(w)\mathrm{d}\rho_v(w) = \int_{\mathcal{A}_{\hat{H}}^{\text{tr}}} \Gamma(w)\mathrm{d}\rho_v(w),\ \ \forall\, t \geqslant 0, \tag{3.47}$$

for any $\Gamma \in \mathcal{C}(\mathcal{T}_{\hat{H}}^{\text{tr}})$.

(2) *Define* $\mu_t = T(t)\rho_v$ *by set*

$$T(t)\rho_v(E) = \mu_t(E) = \rho_v(T(t)^{-1}E)$$

for every set $E \subseteq \mathcal{T}_{\hat{H}}^{\mathrm{tr}}$ *that is* ρ_v*-measurable. Then* μ_t *satisfies the following Liouville type equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}_{\hat{H}}^{\mathrm{tr}}} \Upsilon(w)\mathrm{d}\mu_t(w) = \int_{\mathcal{T}_{\hat{H}}^{\mathrm{tr}}} \langle \Upsilon'(w), \mathcal{F}(w)\rangle \mathrm{d}\mu_t(w), \tag{3.48}$$

for all test functions $\Upsilon \in \mathcal{T}$.

Remark 3.1. *The trajectory statistical solution* ρ_v *obtained in Theorem 3.2(1) is supported by the trajectory attractor* $\mathcal{A}_{\hat{H}}^{\mathrm{tr}}$. *We see that* $\mathcal{A}_{\hat{H}}^{\mathrm{tr}}$ *consists of all bounded, complete trajectories of equation* (3.6) *and* $\mathcal{A}_{\hat{H}}^{\mathrm{tr}}$ *attracts any subset of* $\mathcal{T}_{\hat{H}}^{\mathrm{tr}}$ *in the topology* $\Theta_{\mathrm{loc}}^{+}$. *Notice that* $\Theta_{\mathrm{loc}}^{+}$ *is indeed a weak topology of* $\mathcal{C}_{\mathrm{loc}}([0,+\infty);\hat{H})$. *If we can prove that* $\mathcal{A}_{\hat{H}}^{\mathrm{tr}}$ *is compact in the strong topology* $\Theta_{\mathrm{s,loc}}^{+}$ *of* $\mathcal{C}_{\mathrm{loc}}([0,+\infty);\hat{H})$ *and* $\mathcal{A}_{\hat{H}}^{\mathrm{tr}}$ *attracts any subset of* $\mathcal{T}_{\hat{H}}^{\mathrm{tr}}$ *in the strong topology* $\Theta_{\mathrm{s,loc}}^{+}$ *as well, then we call* $\mathcal{A}_{\hat{H}}^{\mathrm{tr}}$ *the strong compact strong trajectory attractor (cf. [27]). With this strong compact strong trajectory attractor (if it exists), we can also construct the trajectory statistical solution which will be called the strong trajectory statistical solution. Chepyzhov, Vishik and Zelik proved the existence of strong compact strong trajectory attractors for dissipative Euler equations in [7]. Very recently, Zhao, Song and Caraballo proved that the dissipative Euler equations addressed in [7] possesses strong trajectory statistical solutions (see [46]).*

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