Stability results for neutral stochastic functional differential equations via fixed point methods

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Abstract

In this paper we prove some results on the mean square asymptotic stability of a class of neutral stochastic differential systems with variable delays by using a contraction mapping principle. Namely, a necessary and sufficient condition ensuring the asymptotic stability is proved. The assumption does not require neither boundedness or differentiability of the delay functions, nor do they ask for a fixed sign on the coefficient functions. In particular, the results improve some previous ones proved by Guo et al. (2017). Finally, an example is exhibited to illustrate the effectiveness of the proposed results.

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1 Introduction

Liapunov's direct method has long been viewed the main classical method to study stability problems in several areas of differential equations. The success of Liapunov's direct method depends on finding a suitable Liapunov function or Liapunov functional. However, it may be difficult to look for such a good Liapunov functional for some classes of stochastic delay differential equations. Therefore, an alternative approach may be explored to overcome such difficulties.

It was proposed in (Burton, 2004–2006; Burton & Zhang, 2004) to use a fixed point method to study the stability problem for deterministic systems. Luo (2007) and Appleby (2008) have applied this method to deal with stability problems for stochastic delay differential equations, and afterwards, several types of stochastic delay differential equations are investigated by using fixed point methods. For example, Sakthivel & Luo (2009a, 2009b) investigate the asymptotic stability of nonlinear impulsive stochastic differential equations and impulsive stochastic partial differential equations with infinite delays by means of the fixed point theory. On the other hand, Luo (2008, 2010) firstly considers the exponential stability for stochastic partial differential equations with delays by the fixed point method. Zhou & Zhong (2010) study the exponential p-stability of neutral stochastic differential equations with multiple delays. Pinto & Sepúlveda (2011) deal with H-asymptotic stability by the fixed point method in neutral nonlinear differential equations with delay. It turns out that the fixed point method is becoming a powerful technique in dealing with stability problems for deterministic and stochastic differential equations with delays. Moreover, it possesses the advantage that it can yield the existence, uniqueness and stability criteria of the considered system in one step.

In this paper, we address the mean square asymptotic stability for neutral stochastic functional differential equations with variable delays. An asymptotic mean square stability theorem with a necessary and sufficient condition is proved. Some well-known results are improved and generalised. More precisely, our model contains as a particular case the one analyzed in (Guo, Chao & Jun, 2017), and therefore we ensure the validity of those results, despite the proof in (Guo et al., 2017) is not completely correct (see Remark 3.1 in Section 3 below for more details). This paper is organized as follows. In Section 2 we describe our model and recall the basic preliminary results which are necessary for our analysis. In Section 3, we prove the main result about mean-square asymptotic stability. Finally, in Section 4 an example is analyzed in order to test our abstract results as well as to highlight that the results in (Guo et al., 2017) cannot be applied to our model.

2 Statement of the problem and preliminaries

In this paper, we consider the following class of neutral stochastic differential systems with variable delays,

$$d\left[u_i(t)-\sum_{j=1}^{n}q_{ij}(t)u_j(t-\tau_j(t))\right]=\left[\sum_{j=1}^{n}a_{ij}(t)u_j(t)+\sum_{j=1}^{n}b_{ij}(t)f_j(u_j(t))\right.$$
$$\left.+\sum_{j=1}^{n}c_{ij}(t)g_j(u_j(t-\delta_j(t)))\right]dt+\sum_{j=1}^{n}\sigma_{ij}(u_j(t))dw_j(t),t\geq t_0, \tag{1}$$

for $i=1,2,3,...,n,$, which can be written in a vector–matrix form as follows:

$$d[u(t)-Q(t)u(t-\tau(t))]=[A(t)u(t)+B(t)f(u(t))+C(t)g(u(t-\delta(t))]dt$$
$$+\sigma(u(t))dw(t),t\geq t_0, \tag{2}$$

where $u(t)=[u_1(t),u_2(t),..,u_n(t)]^T\in\mathbb{R}^n$, and $a_{ij},b_{ij},\ c_{ij},\ q_{ij}\in C(\mathbb{R}^+,\mathbb{R})$, are continuous functions, $A(t)=(a_{ij}(t))_{n\times n}$, $B(t)=(b_{ij}(t))_{n\times n}$, $Q(t)=(q_{ij}(t))_{n\times n}$, are real matrices and $\sigma(\cdot)=(\sigma_{ij}(\cdot))_{n\times n}$ is the diffusion coefficient matrix, $f(u(t))=[f_1(u_1(t)),f_2(u_2(t)),...,f_n(u_n(t))]^T\in\mathbb{R}^n$, $g(u(t))=[g_1(u_1(t)),g_2(u_2(t)),...,g_n(u_n(t))]^T\in\mathbb{R}^n$.

Let $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},\mathbb{P}\right)$ be a complete filtered probability space and let $w(t)=[w_1(t),w_2(t),...,w_n(t)]^T$ be an $n-$dimensional Brownian motion on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},\mathbb{P})$ such that $\{\mathcal{F}_t\}_{t\geq0}$ is the natural filtration of $w(t)$ (i.e $\mathcal{F}_t$ is the completion of $\sigma\{w(s):0\leq s\leq t\}$). Here $C(S_1,S_2)$ denotes the set of all continuous functions $\varphi:S_1\to S_2$ with the supremum norm $\|.\|$.

Denote by $u(t)=u(t;s,\varphi)=(u_1(t;s,\varphi_1),...,u_n(t;s,\varphi_2))^T\in\mathbb{R}^n$ the solution to (1) with the intial condition

$$u_i(s)=\varphi_i(s)\ \text{ for } s\in[m(t_0),t_0],\ \text{ for each } t_0\geq0, \tag{3}$$

where

$$\begin{gathered}m_j(t_0)=\min\{\inf\{t-\tau_j(t),t\geq t_0\},\inf\{t-\delta_j(t),t\geq t_0\}\},\\ m(t_0)=\min\{m_j(t_0),1\leq j\leq n\}.\end{gathered} \tag{4}$$

and $\varphi_i(\cdot)\in C([m(t_0),t_0],\mathbb{R})$, and $s\to\varphi(s)=(\varphi_1(s),...,\varphi_n(s))^T\in\mathbb{R}^n$ belongs to the space $C([m(t_0),t_0],\mathbb{R}^n)$, with the norm defined by $\|\varphi\|=\sum_{i=1}^{n}\sup_{m(t_0)\leq s\leq t_0}|\varphi_i(s)|$. Finally, $\mathbb{E}$ will denote expectation. Before proceeding, we firstly introduce some assumptions to be imposed later on

(A1) The delay functions $\tau_j,\delta_j\in C(\mathbb{R}^+,\mathbb{R}^+)$, with $t-\delta_j(t)\to\infty$ and $t-\tau_j(t)\to\infty$ as $t\to\infty$ for $j=1,2,...,n$.

(A2) there exist nonnegative constants α_j such that for all $x, y \in \mathbb{R}$,

$$|f_j(x) - f_j(y)| \le \alpha_j |x - y|, \ j = 1, 2, ..., n. \tag{5}$$

(A3) there exist nonnegative constants β_j such that for all $x, y \in \mathbb{R}$,

$$|g_j(x) - g_j(y)| \le \beta_j |x - y|, \ j = 1, 2, ..., n. \tag{6}$$

(A4) there exist nonnegative constants L_{ij} such that for all $x, y \in \mathbb{R}$,

$$|\sigma_{ij}(x) - \sigma_{ij}(y)| \le L_{ij} |x - y|, \ i, j = 1, 2, ..., n. \tag{7}$$

Throughout this paper, we always assume that

$$f_j(0) = g_j(0) = \sigma_{ik}(0) = 0, \ \text{for } i, j, k = 1, 2, \ldots, n \tag{8}$$

thereby, problem (1) admits the trivial equilibrium $u = 0$.

Very recently, Guo et al. published in (2017) related results on the solutions of a particular case of (1). More precisely, the following result was established.

Theorem A. *Suppose that assumptions (A1)–(A4) hold and that there exist positive scalars a_i such that, for all $t \ge 0$,*

$$\sum_{i=1}^{n} \left\{ \left[\sum_{j=1}^{n} \left(|q_{ij}(t)| + \int_0^t e^{-a_i(t-s)} |\overline{a_{ij}}(s)| \, ds \right.\right.\right. \\ + \int_0^t e^{-a_i(t-s)} a_i |q_{ij}(s)| \, ds + \int_0^t e^{-a_i(t-s)} |b_{ij}(s)| \alpha_j ds \\ \left.\left.\left. + \int_0^t e^{-a_i(t-s)} |c_{ij}(s)| \beta_j ds \right) \right]^2 + \frac{2}{a_i} \sum_{j=1}^{n} L_{ij}^2 \right\} \le \gamma < \frac{1}{2}, \tag{9}$$

where $\overline{a_{ij}}(t) = a_{ij}(t) (i \ne j)$, $\overline{a_{ii}}(t) = a_{ii}(t) + a_i$. Then, for any $\varphi \in C([m(0), 0], \mathbb{R}^n)$, there exists a unique global solution $u(t, 0, \varphi)$. Moreover, the zero solution is mean-square asymptotically stable.

Our objective here is to generalize Theorem A to the general case of equation (1) by proving a necessary and sufficient condition for the asymptotic stability of the zero solution. We also provide an example to illustrate our results.

For each $t_0 \ge 0$ and $\varphi \in C([m(t_0), t_0], \mathbb{R}^n)$ fixed, we define $X_{\varphi_{i,t_0}}^{l_i}$ as the following space of stochastic processes

$$X_{\varphi_{i,t_0}}^{l_i} = \{ u_i(t, \omega) : [m(t_0), \infty) \times \Omega \to \mathbb{R} / \ u_i(t, .) = \varphi_i(t) \ \text{for } t \in [m(t_0), t_0], \\ \|u_i\|_{X_{\varphi_{i,t_0}}^{l_i}} \le l_i \text{ for } t \ge t_0 \text{ and } \mathbb{E} |u_i(t)|^2 \to 0 \text{ as } t \to \infty \},$$

where $\|u_i(t, \omega)\|_{X_{\varphi_{i,t_0}}^{l_i}} = \left(\mathbb{E} \left(\sup_{t \ge m(t_0)} |u_i(t)|^2 \right) \right)^{1/2}$.

Now, we denote $X^l_{\varphi,t_0} = X^{l_1}_{\varphi_{1,t_0}} \times X^{l_2}_{\varphi_2,t_0} ... \times X^{l_n}_{\varphi_{n,t_0}}$, which can be rewritten as

$$X^l_{\varphi,t_0} = \{u(t,\omega) : [m(t_0), \infty) \times \Omega \to \mathbb{R}^n / \ u(t,.) = \varphi(t) \ \text{ for } t \in [m(t_0), t_0],$$
$$\|u\|_X \leq l \text{ for } t \geq t_0 \text{ and } \mathbb{E}\sum_{i=1}^{n} |u_i(t)|^2 \to 0 \text{ as } \ t \to \infty\},$$

where $\|u\|_X := \left\{ \sum_{i=1}^{n} \mathbb{E} \left(\sup_{t \geq m(t_0)} |u_i(t)|^2 \right) \right\}^{\frac{1}{2}}$. It is easy to check that X^l_{φ,t_0} is a complete metric space with metric induced by the norm $\| \cdot \|_X$.

When no confusion is possible we will not write X^l_{φ,t_0},$X^{l_i}_{\varphi_{i,t_0}}$ but X^l_{φ},$X^{l_i}_{\varphi_i}$ respectively, and we will also omit the random parameter ω.

Let us know recall the definitions of stability that will be used in the next section.

Definition 2.1: The zero solution of the system (1) is said to be :

i) stable if for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\varphi \in C([m(t_0), t_0], \mathbb{R}^n)$ and $\|\varphi\| < \delta$ imply $\mathbb{E}\sum_{i=1}^{n} |u_i(t, t_0, \varphi)|^2 < \varepsilon$ for $t \geq t_0$.

ii) asymptotically stable if the zero solution is stable and for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\varphi \in C([m(t_0), t_0], \mathbb{R}^n)$ and $\|\varphi\| < \delta$ imply $\mathbb{E}\sum_{i=1}^{n} |u_i(t, t_0, \varphi)|^2 \to 0$ as $t \to \infty$.

To prove our main result we will use a classical contraction mapping principle. We recall it below for the readers convenience.

Theorem 2.1 (see Smart, 1974) *Let $\mathcal{H}$ be a contraction operator on a complete metric space X, then there exists a unique point $x^* \in X$ such that $\mathcal{H}(x^*) = x^*$.*

3 Main Results

Our purpose here is to extend the work carried out in (Guo et al., 2017) by providing a necessary and sufficient condition for asymptotic stability of the zero solution of equation (1). Zhang (2004, 2005) was the first to establish necessary and sufficient condition for the stability of solutions of functional differential equation by the fixed point theory. The necessity of condition (12) below for the main stability result was first established in (Zhang, 2004). It is well known that studying the stability of an equation using a fixed point technique involves the construction of a suitable fixed point mapping. This can be an arduous task. Thus, to construct our mapping $\mathcal{P}$, we begin by transforming (1) into a more tractable, but equivalent, equation, which we will then invert to obtain an equivalent integral equation from which we derive a fixed point mapping. After that, we use a suitable complete metric space X^l_{φ} defined above, which may depend on the initial condition φ. Using the contraction mapping principle, we obtain a fixed point for this mapping and hence a solution for (1), which in addition is mean square asymptotically stable.

Now, we can state our main result.

Theorem 3.1. *Suppose that assumptions (A1)–(A4) hold, and there exist continuous functions* $a_i : [t_0, \infty) \to \mathbb{R}$ *such that for* $t \geq t_0$

$$\liminf_{t\to\infty} \int_{t_0}^{t} a_i\left(s\right) ds > -\infty, \quad i = 1, ..., n \tag{10}$$

$$\begin{aligned}\sum_{i=1}^{n} \Bigg\{ \Bigg[\sum_{j=1}^{n} \Bigg(&|q_{ij}(t)| + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \left|\overline{a_{ij}}(s)\right| ds \\ &+ \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \left|a_i\left(s\right)\right| \left|q_{ij}(s)\right| ds + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \left|b_{ij}(s)\right| \alpha_j ds \\ &+ \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \left|c_{ij}(s)\right| \beta_j ds \Bigg) \Bigg]^2 \\ &+ 4\sum_{j=1}^{n} \int_{t_0}^{t} L_{ij}^2 e^{-2\int_s^t a_i(\xi)d\xi} \Bigg\} \leq \gamma < \frac{1}{2},\end{aligned} \tag{11}$$

where $\overline{a_{ij}}(t) = a_{ij}(t)(i \neq j)$, $\overline{a_{ii}}(t) = a_{ii}\left(t\right) + a_i\left(t\right)$. *Then for any* $\varphi \in C\left(\left[m\left(t_0\right), t_0\right], \mathbb{R}^n\right)$ *there exists a unique global solution* $u\left(t, t_0, \varphi\right)$. *Moreover, the zero solution is mean-square asymptotically stable if and only if*

$$\int_{t_0}^{t} a_i\left(s\right) ds \to \infty \text{ as } t \to \infty. \tag{12}$$

Proof: Set

$$M_i = \sup_{t \geq t_0} \left\{ e^{-\int_{t_0}^{t} a_i(s)ds} \right\}, \tag{13}$$

which is well defined thanks to (10). Suppose also that (12) holds.

We now re-write equation (1) in an equivalent form. For this end, we use the variation of parameter formula and rewrite the equation in an integral from which we derive a contracting mapping.

We rewrite (1) as

$$\begin{aligned} & d\left[u_i(t)-\sum_{j=1}^{n} q_{ij}(t)u_j(t-\tau_j(t))\right] \\ = & \left[-a_i(t)\,u_i(t)+\sum_{j=1}^{n}\overline{a_{ij}}(t)u_j(t)\right. \\ & \left.+\sum_{j=1}^{n} b_{ij}(t)f_j(u_j(t))+\sum_{j=1}^{n} c_{ij}(t)g_j(u_j(t-\delta_j(t)))\right]dt \\ & +\sum_{j=1}^{n}\sigma_{ij}(u_j(t))dw_j(t)\,, t\geq t_0, \end{aligned} \tag{14}$$

with the intial condition $u_i(t)=\varphi_i(t)$ for $t\in[m(t_0),t_0]$.

Multiplying both sides of (14) by $e^{\int_0^t a_i(\xi)d\xi}$ and integrating from t_0 to t,

$$\begin{aligned} & \int_{t_0}^{t}\left[e^{\int_0^s a_i(\xi)d\xi}u_i(s)\right]' ds=\int_{t_0}^{t} e^{\int_0^s a_i(\xi)d\xi}\left\{d\left(\sum_{j=1}^{n} q_{ij}(s)u_j(s-\tau_j(s))\right)\right. \\ & \left.+\sum_{j=1}^{n}\overline{a_{ij}}(s)u_j(s)+\sum_{j=1}^{n} b_{ij}(s)f_j(u_j(s))+\sum_{j=1}^{n} c_{ij}(s)g_j(u_j(s-\delta_j(s)))\right\}ds \\ & +\int_{t_0}^{t} e^{\int_0^s a_i(\xi)d\xi}\sum_{j=1}^{n}\sigma_{ij}(u_j(s))dw_j(s). \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned} & e^{\int_0^t a_i(\xi)d\xi}u_i(t)-e^{\int_0^{t_0} a_i(\xi)d\xi}u_i(t_0)=\int_{t_0}^{t} e^{\int_0^s a_i(\xi)d\xi}\left\{d\left(\sum_{j=1}^{n} q_{ij}(s)u_j(s-\tau_j(s))\right)\right. \\ & \left.+\sum_{j=1}^{n}\overline{a_{ij}}(s)u_j(s)+\sum_{j=1}^{n} b_{ij}(s)f_j(u_j(s))+\sum_{j=1}^{n} c_{ij}(s)g_j(u_j(s-\delta_j(s)))\right\}ds \\ & +\int_{t_0}^{t} e^{\int_0^s a_i(\xi)d\xi}\sum_{j=1}^{n}\sigma_{ij}(u_j(s))dw_j(s). \end{aligned}$$

Dividing both sides of the above equation by $e^{\int_0^t a_i(\xi)d\xi}$, we obtain

$$\begin{aligned} & u_i(t)=e^{-\int_{t_0}^t a_i(\xi)d\xi}u_i(t_0)+\int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi}\left\{d\left(\sum_{j=1}^{n} q_{ij}(s)u_j(s-\tau_j(s))\right)\right. \\ & \left.+\sum_{j=1}^{n}\overline{a_{ij}}(s)u_j(s)+\sum_{j=1}^{n} b_{ij}(s)f_j(u_j(s))+\sum_{j=1}^{n} c_{ij}(s)g_j(u_j(s-\delta_j(s)))\right\}ds \\ & +\int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}\sigma_{ij}(u_j(s))dw_j(s). \end{aligned}$$

Performing now an integration by parts, we have for $t \geq t_0$, $i = 1, 2, ..., n$,

$$\begin{aligned}
& u_i(t) \\
= & \left[\varphi_i(t_0) - \left(\sum_{j=1}^{n} q_{ij}(t_0)\varphi_j(t_0 - \tau_j(t_0))\right)\right] e^{-\int_{t_0}^{t} a_i(\xi)d\xi} \\
& + \left(\sum_{j=1}^{n} q_{ij}(t)u_j(t - \tau_j(t))\right) - \int_{t_0}^{t} a_i(s)\, e^{-\int_s^t a_i(\xi)d\xi} \left(\sum_{j=1}^{n} q_{ij}(s)u_j(s - \tau_j(s))\right) ds \\
& + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} \overline{a_{ij}}(s)u_j(s)\, ds + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} b_{ij}(s)f_j(u_j(s))\, ds \\
& + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} c_{ij}(s)g_j(u_j(s - \delta_j(s)))\, ds \\
& + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} \sigma_{ij}(u_j(s))dw_j(s). \qquad (15)
\end{aligned}$$

Use (15) to define the operator $\mathcal{P} : X_{\varphi}^{l} \to X_{\varphi}^{l}$ by

$$(\mathcal{P}u)(t) := \left[(\mathcal{P}_1 u_1)(t), (\mathcal{P}_2 u_2)(t), ..., (\mathcal{P}_n u_n)(t)\right]^T \in X_{\varphi}^{l},$$

where $\mathcal{P}_i : X_{\varphi_i}^{l_i} \to X_{\varphi_i}^{l_i}$ by $(\mathcal{P}_i u_i)(t) = \varphi_i(t)$ for $t \in [m(t_0), t_0]$ and for $t \geq t_0$, where $\mathcal{P}_i(u_i) : [m(t_0), +\infty) \to \mathbb{R}$ $(i = 1, 2, ..., n)$ is defined as follows:

$$\begin{aligned}
& (\mathcal{P}_i u_i)(t) \\
= & \left[\varphi_i(t_0) - \left(\sum_{j=1}^{n} q_{ij}(t_0)\varphi_j(t_0 - \tau_j(t_0))\right)\right] e^{-\int_{t_0}^{t} a_i(\xi)d\xi} \\
& + \left(\sum_{j=1}^{n} q_{ij}(t)u_j(t - \tau_j(t))\right) - \int_{t_0}^{t} a_i(s)\, e^{-\int_s^t a_i(\xi)d\xi} \left(\sum_{j=1}^{n} q_{ij}(s)u_j(s - \tau_j(s))\right) ds \\
& + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} \overline{a_{ij}}(s)u_j(s)\, ds + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} b_{ij}(s)f_j(u_j(s))\, ds \\
& + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} c_{ij}(s)g_j(u_j(s - \delta_j(s)))\, ds + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} \sigma_{ij}(u_j(s))dw_j(s) \\
= & \sum_{m=1}^{7} Q_{im}(t), \qquad (16)
\end{aligned}$$

where,

$$Q_{i1}(t) = \left[\varphi_i(t_0) - \left(\sum_{j=1}^{n} q_{ij}(t_0)\varphi_j(t_0 - \tau_j(t_0)) \right) \right] e^{-\int_{t_0}^{t} a_i(\xi)d\xi},$$

$$Q_{i2}(t) = \sum_{j=1}^{n} q_{ij}(t) u_j(t - \tau_j(t)),$$

$$Q_{i3}(t) = \int_{t_0}^{t} a_i(s) e^{-\int_s^t a_i(\xi)d\xi} \left(\sum_{j=1}^{n} q_{ij}(s) u_j(s - \tau_j(s)) \right) ds,$$

$$Q_{i4}(t) = \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} \overline{a_{ij}}(s) u_j(s)\, ds,$$

$$Q_{i5}(t) = \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} b_{ij}(s) f_j(u_j(s))\, ds,$$

$$Q_{i6}(t) = \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} c_{ij}(s) g_j(u_j(s - \delta_j(s)))\, ds,$$

$$Q_{i7}(t) = \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} \sigma_{ij}(u_j(s)) dw_j(s).$$

Now we split the rest of our proof into three steps.

First step: We prove that $\mathcal{P}\left(X_{\varphi}^{l}\right) \subset X_{\varphi}^{l}$. First we show the mean square continuity of $\mathcal{P}$ on $[t_0, \infty)$. For $u_i \in X_{\varphi_i}^{l_i}$, it is necessary to show that $\mathcal{P}_i(u_i) \in X_{\varphi_i}^{l_i}$. It is clear that $\mathcal{P}_i$ is continuous on $[m(t_0), t_0]$. For fixed time $t \geq t_0$, each $i \in \{1, 2, 3, ..., n\}, u_i \in X_{\varphi_i}^{l_i}$, and $|\varepsilon|$ be sufficiently small, we then have

$$E \left| (\mathcal{P}_i(u_i))(t + \varepsilon) - (\mathcal{P}_i(u_i))(t) \right|^2 \leq 7 \sum_{m=1}^{7} \mathbb{E} \left| Q_{im}(t + \varepsilon) - Q_{im}(t) \right|^2. \quad (17)$$

We must prove the mean square continuity of $\mathcal{P}_i$ on $[t_0, \infty[$.

It is easy to obtain that

$$\mathbb{E} \left| Q_{im}(t + r) - Q_{im}(t) \right|^2 \to 0, \text{ as } r \to 0, \; i = 1, 2, ..., 6.$$

As for the last term,

$$\begin{aligned}
&\mathbb{E} \left| Q_{i7}(t + \varepsilon) - Q_{i7}(t) \right|^2 \\
&= \mathbb{E} \left| \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \left(e^{-\int_t^{t+\varepsilon} a_i(\xi)d\xi} - 1 \right) \sum_{j=1}^{n} \sigma_{ij}(u_j(s)) dw_j(s) \right. \\
&\left. + \int_{t}^{t+\varepsilon} e^{-\int_s^{t+\varepsilon} a_i(\xi)d\xi} \sum_{j=1}^{n} \sigma_{ij}(u_j(s)) dw_j(s) \right|^2 \\
&\leq 2\mathbb{E} \left| \sum_{j=1}^{n} \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \left(e^{-\int_t^{t+\varepsilon} a_i(\xi)d\xi} - 1 \right) \sigma_{ij}(u_j(s)) dw_j(s) \right|^2
\end{aligned}$$

$$+2\mathbb{E}\left|\sum_{j=1}^{n}\int_{t}^{t+\varepsilon} e^{-\int_{s}^{t+\varepsilon} a_i(\xi)d\xi}\sigma_{ij}(u_j(s))dw_j\left(s\right)\right|^2$$
$$\leq 2\mathbb{E}\left(\sum_{j=1}^{n}\int_{t_0}^{t} e^{-2\int_{s}^{t} a_i(\xi)d\xi}\ \left(e^{-\int_{t}^{t+\varepsilon} a_i(\xi)d\xi}\ -1\right)^2\sigma_{ij}^2(u_j(s))ds\right)$$
$$+2\mathbb{E}\left(\sum_{j=1}^{n}\int_{t}^{t+\varepsilon} e^{-2\int_{s}^{t+\varepsilon} a_i(\xi)d\xi}\sigma_{ij}^2(u_j(s))ds\right)\to 0$$

as $\varepsilon\to\infty$. Thus, $\mathcal{P}_i$ ($i=1,2,...,n$) is mean square continuous on $[t_0,\infty)$. Then $\mathcal{P}$ is indeed mean square continuous on $[t_0,\infty)$.

Next, we verify that $\|\mathcal{P}\left(u\right)\|_X\leq l$. Let φ be a small bounded initial function with $\|\varphi\|<\delta$, where we choose $\delta>0$, $(\delta<l)$ such that

$$2\delta\sum_{i=1}^{n}\left(1+\sum_{j=1}^{n}\left|q_{ij}(t_0)\right|\right)^2 M_i^2\leq l^2\left(1-4\gamma\right). \tag{18}$$

Let $u\in X_{\varphi}^{l}$, then $\|u\|_X\leq l$. Since f, g, σ, satisfy a Lipschitz condition, it follows from (16), condition (11) and L^p-Doob inequality that

$$\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t\geq m(t_0)}\left|\left(\mathcal{P}_iu_i\right)(t)\right|^2\right]$$
$$\leq 2\sum_{i=1}^{n}\left[\left|\varphi_i(t_0)\right|+\left(\sum_{j=1}^{n}\left|q_{ij}(t_0)\right|\left|\varphi_j(t_0-\tau_j\left(t_0\right))\right|\right)\right]^2 e^{-2\int_{t_0}^{t} a_i(\xi)d\xi}$$
$$+4\sum_{i=1}^{n}\left\{\mathbb{E}\sup_{t\geq t_0}\left[\left(\sum_{j=1}^{n}\left|q_{ij}(t)\right|\left|u_j(t-\tau_j\left(t\right))\right|\right)\right.\right.$$
$$+\int_{t_0}^{t}\left|a_i\left(s\right)\right|e^{-\int_{s}^{t} a_i(\xi)d\xi}\left(\sum_{j=1}^{n}\left|q_{ij}(s)\right|\left|u_j(s-\tau_j\left(s\right))\right|\right)ds$$
$$+\int_{t_0}^{t} e^{-\int_{s}^{t} a_i(\xi)d\xi}\sum_{j=1}^{n}\left|\overline{a_{ij}}(s)\right|\left|u_j\left(s\right)\right|ds+\int_{t_0}^{t} e^{-\int_{s}^{t} a_i(\xi)d\xi}\sum_{j=1}^{n}\left|b_{ij}(s)\right|\left|f_j\left(u_j(s)\right)\right|ds$$
$$\left.+\int_{t_0}^{t} e^{-\int_{s}^{t} a_i(\xi)d\xi}\sum_{j=1}^{n}\left|c_{ij}(s)\right|\left|g_j\left(u_j(s-\delta_j\left(s\right))\right)\right|ds\right]^2$$
$$+4\sum_{i=1}^{n}\mathbb{E}\sup_{t\geq t_0}\left[\int_{t_0}^{t} e^{-\int_{s}^{t} a_i(\xi)d\xi}\sum_{j=1}^{n}\left|\sigma_{ij}(x_j(s))\right|dw_j\left(s\right)\right]^2.$$

Therefore,

$$\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t\geq m(t_0)}\left|\left(\mathcal{P}_iu_i\right)(t)\right|^2\right]$$

$$\leq 2\sum_{i=1}^{n}|\varphi_i(t_0)|^2\left[1+\sum_{j=1}^{n}|q_{ij}(t_0)|\right]^2 e^{-2\int_{t_0}^{t}a_i(\xi)d\xi}$$

$$+4\left[\sum_{i=1}^{n}\left(\mathbb{E}\sup_{s\geq m(t_0)}|u_j(s))|^2\right)\right]\left\{\sum_{i=1}^{n}\sup_{t\geq t_0}\left[\left(\sum_{j=1}^{n}|q_{ij}(t)|\right)\right.\right.$$

$$+\int_{t_0}^{t}|a_i(s)|\,e^{-\int_s^t a_i(\xi)d\xi}\left(\sum_{j=1}^{n}|q_{ij}(s)|\right)ds+\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}|\overline{a_{ij}}(s)|\,ds$$

$$\left.+\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}|b_{ij}(s)|\,\alpha_j ds+\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}|c_{ij}(s)|\,\beta_j ds\right]^2$$

$$\left.+4\sum_{j=1}^{n}\int_{t_0}^{t}L_{ij}^2 e^{-2\int_s^t a_i(\xi)d\xi}\right\}$$

$$\leq 2\sum_{i=1}^{n}|\varphi_i(t_0)|^2\left[1+\sum_{j=1}^{n}|q_{ij}(t_0)|\right]^2 e^{-2\int_{t_0}^{t}a_i(\xi)d\xi}+4\gamma\sum_{i=1}^{n}\left(\mathbb{E}\sup_{s\geq m(t_0)}|u_j(s))|^2\right)$$

$$\leq 2\delta\sum_{i=1}^{n}\left(1+\sum_{j=1}^{n}|q_{ij}(t_0)|\right)^2 e^{-2\int_{t_0}^{t}a_i(\xi)d\xi}+4\gamma l^2.$$

By applying (18), we see that $\sum_{i=1}^{n}\left(\mathbb{E}\sup_{t\geq m(t_0)}|(\mathcal{P}_i u_i)(t)|^2\right)\leq l^2(1-4\gamma)+4\gamma l^2=l^2$. Hence, $\|\mathcal{P}u\|_X\leq l$ for $t\in[m(t_0),\infty)$ because $\|\mathcal{P}u\|_X=\|\varphi\|\leq l$ for $t\in[m(t_0),t_0]$.

We will prove that $\mathbb{E}\sum_{i=1}^{n}|(\mathcal{P}_i(u_i))(t)|^2\to 0$ as $t\to\infty$. Indeed, $\mathbb{E}|u_i(t)|^2\to 0$ as $t\to\infty$. Then, for any $\varepsilon>0$, there exists $T_1>0$, such that $t\geq T_1$ we have $\mathbb{E}|u_i(t)|^2<\varepsilon$, for $i=1,2,...,n$. Hence

$$\mathbb{E}|Q_{i7}(t)|^2\leq\mathbb{E}\int_{t_0}^{T_1}e^{-2\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}\sigma_{ij}^2(u_j(s))ds$$

$$+\mathbb{E}\int_{T_1}^{t}e^{-2\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}\left|\sigma_{ij}^2(u_j(s))\right|ds$$

$$\leq\sum_{j=1}^{n}L_{ij}^2\mathbb{E}\left(\sup_{s>m(t_0)}|u_j(s)|\right)^2 e^{-2\int_{T_1}^{t}a_i(\xi)d\xi}\left(\int_{t_0}^{T_1}e^{-2\int_s^{T_1}a_i(\xi)d\xi}ds\right)$$

$$+\sum_{j=1}^{n}L_{ij}^2\varepsilon\left(\int_{T_1}^{t}e^{-2\int_s^t a_i(\xi)d\xi}ds\right).$$

By using condition (12), there is $T_2\geq T_1$ such that when $t\geq T_2$ we have

$$\sum_{j=1}^{n}L_{ij}^2\mathbb{E}\left(\sup_{s>m(t_0)}|u_j(s)|\right)^2 e^{-2\int_{T_1}^{t}a_i(\xi)d\xi}\left(\int_{t_0}^{T_1}e^{-2\int_s^{T_1}a_i(\xi)d\xi}ds\right)\leq(1-\gamma)\varepsilon.$$

By condition (11) we have $\mathbb{E}\left|Q_{i7}\left(t\right)\right|^{2}\leq\gamma\varepsilon+\left(1-\gamma\right)\varepsilon=\varepsilon$. Thus $\mathbb{E}\left(\left|Q_{i7}\left(s\right)\right|^{2}\right)\rightarrow 0$ as $t\rightarrow\infty$. Similarly, we can show that $\mathbb{E}\left(\left|Q_{im}\left(s\right)\right|^{2}\right)\rightarrow 0$ $(m=1,2,...,7)$ as $t\rightarrow\infty$. This implies $\mathbb{E}\left|\left(\mathcal{P}_{i}\left(u_{i}\right)\right)\left(t\right)\right|^{2}\rightarrow 0$ as $t\rightarrow\infty$, and hence, $\mathcal{P}_{i}\left(X_{\varphi_{i}}^{l_{i}}\right)\subset X_{\varphi_{i}}^{l_{i}}$, for $i=1,2,...,n$. Then $\mathcal{P}\left(X_{\varphi}^{l}\right)\subset X_{\varphi}^{l}$.

Second step: Now we will show that $\mathcal{P}$ has a unique fixed point u in X_{φ}^{l}. For any $u=\left(u_{1},u_{2},...,u_{n}\right)^{T}\in X_{\varphi}^{l}, y=\left(y_{1},y_{2},...,y_{n}\right)^{T}\in X_{\varphi}^{l}$, we have

$$
\begin{aligned}
&\mathbb{E}\left(\sum_{i=1}^{n}\sup_{t\geq m(t_0)}\left|\left(\mathcal{P}_{i}u_{i}\right)(t)-\left(\mathcal{P}_{i}y_{i}\right)(t)\right|^{2}\right)\\
&\leq\mathbb{E}\left(\sum_{i=1}^{n}\sup_{t\geq t_0}\left|\sum_{j=1}^{n}q_{ij}(t)\left[u_j(t-\tau_j(t))-y_j(t-\tau_j(t))\right]\right.\right.\\
&-\int_{t_0}^{t}a_i(s)\,e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}q_{ij}(s)\left[u_j(s-\tau_j(s))-y_j(s-\tau_j(s))\right]ds\\
&+\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}\overline{a_{ij}}(s)\left[u_j(s)-y_j(s)\right]ds\\
&+\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}b_{ij}(s)\left[f_j(u_j(s))-f_j(y_j(s))\right]ds\\
&+\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}c_{ij}(s)\left[g_j(u_j(s-\delta_j(s))-g_j(y_j(s-\delta_j(s))))\right]ds\\
&\left.\left.+\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}\left[\sigma_{ij}(u_j(s))-\sigma_{ij}(y_j(s))\right]dw_j(s)\right|\right)^{2}.
\end{aligned}
$$

By using the Doob L^p-inequality (see Karatzas & Shreve, 1991),

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t\geq t_0}\left|\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}\left[\sigma_{ij}(u_j(s))-\sigma_{ij}(y_j(s))\right]dw_j(s)\right|\right]^{2}\\
\leq\quad & 4\mathbb{E}\sum_{i=1}^{n}\sum_{j=1}^{n}\sup_{t\geq t_0}\left(\int_{t_0}^{t}e^{-2\int_s^t a_i(\xi)d\xi}\left|\sigma_{ij}(u_j(s))-\sigma_{ij}(y_j(s))\right|^{2}ds\right)\\
\leq\quad & 4\sum_{i=1}^{n}\sum_{j=1}^{n}L_{ij}^{2}\sup_{t\geq t_0}\left(\int_{t_0}^{t}e^{-2\int_s^t a_i(\xi)d\xi}\mathbb{E}\sum_{j=1}^{n}\left(\sup_{s\geq m(t_0)}\left|u_j(s))-y_j(s))\right|^{2}\right)ds\right).
\end{aligned}
$$

Then,

$$
\left\{\mathbb{E}\sum_{i=1}^{n}\sup_{t\geq m(t_0)}\left|\left(\mathcal{P}_{i}u_{i}\right)(t)-\left(\mathcal{P}_{i}y_{i}\right)(t)\right|^{2}\right\}^{\frac{1}{2}}
$$

$$\leq \sqrt{2}\left\{\left[\mathbb{E}\sum_{i=1}^{n}\left(\sup_{t\geq m(t_0)}\left|u_i\left(t\right)-y_i\left(t\right)\right|^2\right)\right]\right\}^{\frac{1}{2}}$$
$$\times\left\{\sum_{i=1}^{n}\sup_{t\geq t_0}\left[\sum_{j=1}^{n}\left(\left|q_{ij}(t)\right|+\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\left|\overline{a_{ij}}(s)\right|ds\right.\right.\right.$$
$$\left.\left.+\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\left|q_{ij}(s)\right|\left|a_i\left(s\right)\right|ds+\left|b_{ij}(s)\right|\alpha_j ds+\left|c_{ij}(s)\right|\beta_j\right)ds\right)\right]^2$$
$$\left.+4\sum_{j=1}^{n}\int_{t_0}^{t}L_{ij}^2 e^{-2\int_s^t a_i(\xi)d\xi}\right\}^{\frac{1}{2}}.$$

By condition (11), $\mathcal{P}$ is a contraction mapping with constant $\sqrt{2\gamma}$. Thanks to the contraction mapping principle (Smart, 1974, p. 2), we deduce that $\mathcal{P}: X_\varphi^l \to X_\varphi^l$ possesses a unique fixed point $u(t) = (u_1(t), u_2(t), ..., u_n(t))$ in X_φ^l, which is the unique solution of (1) with $u(s) = \varphi(s)$ on $s \in [m(t_0), t_0]$ and $\mathbb{E}\sum_{i=1}^{n}|u_i(t,t_0,\varphi)|^2 \to 0$ as $t \to \infty$.

Referring to (Burton, 2006; Dib, Maroun & Raffoul, 2005; Raffoul, 2004), except for the fixed point method, we know of another way to prove that solutions of (1) are stable. Let $\varepsilon > 0$ be given such that $0 < \varepsilon < l$. Replacing l by ε in X_φ^l, we obtain that there is $\delta > 0$ such that $\|\varphi\| < \delta$ implies that the unique solution u of (1) with $u = \varphi$ on $[m(t_0), t_0]$ satisfies $\mathbb{E}\sum_{i=1}^{n}|u_i(t,t_0,\varphi)|^2 < \varepsilon$ for all $t \geq m(t_0)$. Moreover $\mathbb{E}\sum_{i=1}^{n}|u_i(t,t_0,\varphi)|^2 \to 0$ as $t \to \infty$. This also shows that the zero solution of (1) is asymptotically stable if (12) holds.

Third step: We will prove that the zero solution of (1) is mean-square asymptotically stable. Let $\varepsilon > 0$ be given and choose $\delta > 0$ ($\delta < \varepsilon$) satisfying

$$4\delta\sum_{i=1}^{n}\left[1+\sum_{j=1}^{n}\left|q_{ij}(t_0)\right|\right]^2 M_i^2 + 2\gamma\varepsilon < (1-2\gamma)\,\varepsilon, \tag{19}$$

where γ is the left hand side of (11). If $u(t) = u(t,t_0,\varphi)$ is a solution of (1) with the initial condition (3) satisfying $\|\varphi\|^2 < \delta$, then $u(t) = (\mathcal{P}u)(t)$ as defined in (16). We claim that $\mathbb{E}\sum_{i=1}^{n}|u_i(t)|^2 < \varepsilon$ for all $t \geq t_0$. Notice that $\mathbb{E}\sum_{i=1}^{n}|u_i(t)|^2 < \varepsilon$ on $t \in [m(t_0), t_0]$, we suppose that there exists $t^* > t_0$ such that $\mathbb{E}\sum_{i=1}^{n}|u_i(t^*)|^2 = \varepsilon$ and $\mathbb{E}\sum_{i=1}^{n}|u_i(t)|^2 < \varepsilon$ for $m(t_0) \leq t \leq t^*$. Then, it follows from (19) and (16) that

$$\mathbb{E}\sum_{i=1}^{n}|u_i(t^*)|^2 \le 4\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t_0)|^2\left[1+\sum_{j=1}^{n}|q_{ij}(t_0)|\right]^2 e^{-2\int_{t_0}^{t^*}a_i(\xi)d\xi}$$
$$+2\varepsilon\sum_{i=1}^{n}\left\{\left[\sum_{j=1}^{n}\left(|q_{ij}(t^*)|+\int_{t_0}^{t^*}e^{-\int_s^{t^*}a_i(\xi)d\xi}|\overline{a_{ij}}(s)|\,ds\right.\right.\right.$$
$$\left.\left.\left.+\int_{t_0}^{t^*}e^{-\int_s^{t^*}a_i(\xi)d\xi}|q_{ij}(s)|\,|a_i(s)|\,ds\right)\right]^2+L_{ij}^2\int_{t_0}^{t^*}e^{-2\int_s^{t^*}a_i(\xi)d\xi}ds\right\}$$
$$\le 4\delta\sum_{i=1}^{n}\left[1+\sum_{j=1}^{n}|q_{ij}(t_0)|\right]^2 M_i+2\gamma\varepsilon<(1-2\gamma)\varepsilon+2\gamma\varepsilon=\varepsilon,$$

which contradicts that $\mathbb{E}\sum_{i=1}^{n}|u_i(t^*)|^2=\varepsilon$. Thus $\mathbb{E}\sum_{i=1}^{n}|u_i(t)|^2<\varepsilon$ for all $t\ge t_0$, and the zero solution of (1) is stable. This shows that the zero solution of (1) is asymptotically stable if (12) holds.

Conversely, we suppose that (12) fails. For each i fixed, $i\in\{1,2,...,n\}$. From (10), there exists a sequence $\{t_n\}$ with $t_n\to\infty$ as $n\to\infty$ such that $\lim_{n\to\infty}\int_0^{t_n}a_i(s)ds=\xi_i$ for some $\xi_i\in\mathbb{R}$. We may also choose a positive constant J_i satisfying

$$-J_i\le\int_0^{t_n}a_i(s)ds\le J_i, \tag{20}$$

for all $n\ge 1$. To simplify the expression, we define

$$F_i(s):=\sum_{j=1}^{n}\left[|\overline{a_{ij}}(s)|+|q_{ij}(s)a_i(s)|+|b_{ij}(s)|\,\alpha_j+|c_{ij}(s)|\,\beta_j\right],$$

for all $s\ge 0$. From (11), we have

$$\int_0^{t_n}e^{-\int_s^{t_n}a_i(\xi)d\xi}F_i(s)\,ds\le\sqrt{\gamma}, \tag{21}$$

wich implies that

$$\int_0^{t_n}e^{\int_0^{s}a_i(\xi)d\xi}F_i(s)\,ds\le\sqrt{\gamma}e^{\int_0^{t_n}a_i(\xi)d\xi}\le\sqrt{\gamma}e^{M_i}. \tag{22}$$

The sequence $\left\{\int_0^{t_n}e^{\int_0^{s}a_i(\xi)d\xi}F_i(s)\,ds\right\}$ is bounded, so there exists a convergent subsequence. For brevity of notation, we may assume that

$$\lim_{n\to\infty}\int_0^{t_n}e^{\int_0^{s}a_i(\xi)d\xi}F_i(s)\,ds=\theta_i, \tag{23}$$

for some $\theta_i \in \mathbb{R}^+$ and choose a positive integer m large enough that

$$\int_{t_m}^{t_n} e^{\int_0^s a_i(\xi)d\xi} F_i(s)\, ds \le \frac{\delta_0}{8M_i}, \tag{24}$$

for all $n \ge m$, where $\delta_0 > 0$ satisfies

$$2\delta_0^2 M_i^2 e^{2J_i} \left(1 + \sum_{j=1}^{n} |q_{ij}(t_m)|\right)^2 \le (1 - 4\gamma).$$

Now we consider the solution $u_i(t) = u_i(t, t_m, \varphi)$ of (1) with $\|\varphi_i(t_m)\| = \delta_0$ and $\|\varphi_i(s)\| \le \delta_0$ for $s < t_m$. If we replace l_i by 1 in the proof of $\|\mathcal{P}_i(u_i)\|_X \le l_i$, we have $\mathbb{E}|u_i(t)|^2 < 1$ for $t \ge t_m$. We may choose φ_i so that

$$G_i(t_m) := \varphi_i(t_m) - \sum_{j=1}^{n} q_{ij}(t_m)\varphi_j(t_m - \tau_j(t_m)) \ge \frac{\delta_0}{2}. \tag{25}$$

It follows from $(16), (24)$ and (25) with $u_i(t) = (\mathcal{P}u_i)(t)$ that for $n \ge m$,

$$\begin{aligned}
&\mathbb{E}\left|u_i(t_n) - \sum_{j=1}^{n} q_{ij}(t_n)u_i(t_n - \tau_j(t_n))\right|^2 \\
&\ge G_i^2(t_m)\, e^{-2\int_{t_m}^{t_n} a_i(\xi)d\xi} - 2G_i(t_m)\, e^{-\int_{t_m}^{t_n} a_i(\xi)d\xi} \int_{t_m}^{t_n} e^{-\int_s^{t_n} a_i(\xi)d\xi} F_i(s)\, ds \\
&\ge \frac{\delta_0}{2} e^{-2\int_{t_m}^{t_n} a_i(\xi)d\xi} \left(\frac{\delta_0}{2} - 2M_i \int_{t_m}^{t_n} e^{\int_0^s a_i(\xi)d\xi} F_i(s)\, ds\right) \ge \frac{\delta_0^2}{8} e^{-2M_i} > 0. \qquad (26)
\end{aligned}$$

If the zero solution of (1) is mean square asymptotically stable, then $\mathbb{E}|u_i(t)|^2 = \mathbb{E}|u_i(t, t_m, \varphi)|^2 \to 0$ as $t \to \infty$. Since $t_n - \tau_j(t_n) \to \infty$ as $n \to \infty$, for $j = 1, 2, ..., n$ and condition (11) holds, we have

$$\mathbb{E}\left|u_i(t_n) - \sum_{j=1}^{n} q_{ij}(t_n)u_i(t_n - \tau_j(t_n))\right|^2 \to 0,$$

as $n \to \infty$, which contradicts (26). Hence condition (12) is necessary in order that (1) has a solution $\mathbb{E}|u_i(t, t_0, \varphi)|^2 \to 0$ as $t \to \infty$. The proof is complete.

Remark 3.1: When $a_i(t) = a_i$ and a_i are positive scalars, then Theorem 3.1 becomes Theorem A, which was recently stated in (Guo et al., 2017). Therefore, paper (Guo et al., 2017) is a particular case of ours. But we would like to emphasize that the proof in (Guo et al., 2017) is not completely correct since they claim that the spaces denoted by X or X^n with the norm $\|u_i\|_{[0,t]} = \left\{\mathbb{E}(\sup_{s\in[0,t]} |u_i(s,\omega)|^2\right\}^{\frac{1}{2}}$ are Banach spaces and they use this fact in the proof

(Guo et al., 2017, p. 1557), but this statement is not correct. However, in our investigation we use a different space which is indeed a complete metric space.

Remark 3.2: It follows from the first part of the proof of Theorem 3.1 that the zero solution of (1) is stable under (11). Moreover, Theorem 3.1 still holds if (11) is satisfied for $t \geq t_\rho$ for some $t_\rho \in \mathbb{R}^+$.

4 Example

In this section, we analyze an example to illustrate two facts. On the one hand, we will show how to apply our main result in this paper, Theorem 3.1. On the other hand and most importantly, we will highlight the real interest and importance of our result because the previous theory developed by Guo et al. (2017) cannot be applied to this example.

Exemple: 4.1 Consider the following two-dimensional stochastic delay differential equation

$$\begin{aligned} d\left[x(t) - Q\left(t\right)x(t-\tau\left(t\right))\right] &= \left[A\left(t\right)x(t) + B\left(t\right)x(t-\tau\left(t\right)\right]dt \\ &+ G\left(t\right)x(t-\tau\left(t\right))dw\left(t\right), t \geq 0, \end{aligned} \tag{27}$$

Where

$$Q(t) = \begin{pmatrix} -\dfrac{\sin t}{8} & 0 \\ 0.025 & \dfrac{3\sin t}{20} \end{pmatrix}, A\left(t\right) = \begin{pmatrix} -\dfrac{0.112}{t+1} & 0 \\ 0 & -\dfrac{0.125}{t+1} \end{pmatrix}$$

$$B\left(t\right) = \begin{pmatrix} -\dfrac{0.112}{20\left(t+1\right)^2} & 0 \\ -\dfrac{5\times 10^{-3}}{\left(t+1\right)^2} & -\dfrac{3\times 10^{-2}}{4\left(t+1\right)^2} \end{pmatrix}, G\left(t\right) = \sqrt{\frac{0.01}{2\left(t+1\right)}}I.$$

By straightforward computations, we can check that condition (11) in Theorem 3.1 holds true, where $\tau \in C(\mathbb{R}^+, \mathbb{R}^+)$ is an arbitrary continuous functions which satisfies $t - \tau\left(t\right) \to \infty$ and $t \to \infty$ and choosing $a_1\left(t\right) = \frac{0.112}{t+1}, a_2\left(t\right) = \frac{0.125}{t+1}$, we obtain that

$$\begin{aligned} &\sum_{i=1}^{2}\left\{\left[\sum_{j=1}^{2}\left(\left|q_{ij}(t)\right| + \int_0^t e^{-\int_t^s a_i(\xi)d\xi}\left|\overline{a_{ij}}(s)\right|ds\right.\right.\right. \\ &\left.\left.+\int_0^t e^{-\int_t^s a_i(u)du}\left|a_i\left(s\right)\right|\left|q_{ij}(s)\right|ds + \int_0^t e^{-\int_t^s a_i(u)du}\left|b_{ij}(s)\right|ds\right)\right]^2 \\ &\left.+ 4\sum_{j=1}^{2}\int_0^t\left|g_{ij}^2\left(s\right)\right|e^{-2\int_t^s a_i(\xi)d\xi}ds\right\} < 0.2925 + 0.17 < \frac{1}{2}, \end{aligned} \tag{28}$$

and since $\int_0^t a_1(s)\,ds = \int_0^t \frac{0.112}{s+1} ds = 0.112 \ln(t+1) \to \infty$ and $\int_0^t a_2(s)\,ds$ $= \int_0^t \frac{0.125}{s+1} ds = 0.125 \ln(t+1) \to \infty$ as $t \to \infty$. It is easy to see that all the conditions of Theorem 3.1 hold for $\gamma \simeq 0.4625 < 0.5$. Thus, Theorem 3.1 implies that the zero solution of (27) is asymptotically stable. ■

Remark 4.1: Observe that Example 4.1 cannot be analyzed by applying Theorem A (see also Theorem 3.1 in Guo et al., 2017). Indeed, in order to apply Theorem A, we need to check that there exist positive constants a_1, a_2 such that (9) holds. However, notice that, for any (fixed) $a_1 > 0$, if we set $\overline{a_{11}}(t) = a_1 - \frac{0.112}{t+1}$, we have that there exists $T_0 > 0$ such that $\overline{a_{11}}(t) > \frac{3a_1}{4}$ for all $t \geq T_0$. Consequently for one of the integrals appearing in (9) we deduce, for $t > T_0$,

$$\begin{aligned}\int_0^t e^{-a_1(t-s)} \left| a_1 - \frac{0.112}{s+1} \right| ds &\geq \int_{T_0}^t e^{-a_1(t-s)} \left| a_1 - \frac{0.112}{s+1} \right| ds \\ &> \frac{3}{4} a_1 \int_{T_0}^t e^{-a_1(t-s)} ds \\ &= \frac{3}{4} \left[1 - e^{-a_1(t-T_0)} \right].\end{aligned}$$

Then, it is clear that there exists $T_1 \geq T_0$ such that for $t \geq T_1$,

$$\frac{3}{4} \left[1 - e^{-a_1(t-T_0)} \right] > \frac{1}{2},$$

which implies that (9) cannot hold true.

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