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Continuation of Gerver's supereight choreography

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Abstract

In [6] we developed a continuation technique for periodic orbits in reversible systems having some first integrals and corresponding symmetries. One of the applications was the continuation of Gerver's supereight choreography when one or several of the masses are varied. In this note we give a more complete description of the families of periodic orbits which can be obtained in this way.

1 Symmetries of the N–body problem

Symmetries form one of the key ingredients in understanding the dynamics of the N-body problem. In this section we briefly describe these symmetries; since it is our aim to do continuation under a change of the masses we also include symmetries in which the mass parameters are involved.

We denote by $\mathbf{x} = (\mathbf{q}_1, \dots, \mathbf{q}_N, \mathbf{p}_1, \dots, \mathbf{p}_N)$ and $\mathbf{m} = (m_1, \dots, m_N)$ the state space vector and the mass vector, respectively; $\mathbf{q}_j \in \mathbf{R}^n$ is the position for the j-th body, $\mathbf{p}_j \in \mathbf{R}^n$ its momentum, and m_j its mass. The equations of motion of the N-body problem take the form

$$\dot{\mathbf{q}}_j = \frac{\partial H_{\mathbf{m}}}{\partial \mathbf{p}_j}(\mathbf{x}), \qquad \dot{\mathbf{p}}_j = -\frac{\partial H_{\mathbf{m}}}{\partial \mathbf{q}_j}(\mathbf{x}), \qquad j = 1, \dots, N, \tag{1}$$

where the Hamiltonian $H_{\mathbf{m}}$ is given by

$$H_{\mathbf{m}}(\mathbf{x}) = \sum_{j=1}^{N} \frac{1}{2m_j} \|\mathbf{p}_j\|^2 - \sum_{1 \le i < j \le N} \frac{m_i m_j}{\|\mathbf{q}_i - \mathbf{q}_j\|}; \tag{2}$$

we have explicitly indicated the dependance on the mass parameter $\mathbf{m}$. For brevity we will indicate the system (1) with the choice $\mathbf{m}$ of the mass parameters as the system $X_{\mathbf{m}}$.

The elements of the Euclidean group $E(n)$ can be identified with pairs $(Q, \mathbf{b})$, where Q is an $n \times n$ orthogonal matrix and $\mathbf{b} \in \mathbf{R}^n$. We define a symplectic action of $E(n)$ on the phase space by

$$\Psi_{Q,\mathbf{b}}(\mathbf{x}) := (Q\mathbf{q}_1 + \mathbf{b}, \ldots, Q\mathbf{q}_N + \mathbf{b}, Q\mathbf{p}_1, \ldots, Q\mathbf{p}_N), \quad \forall (Q, \mathbf{b}) \in E(n).$$

If $\mathbf{x}(t)$ is a solution of system $X_{\mathbf{m}}$ then so is $\Psi_{Q,\mathbf{b}}(\mathbf{x}(t))$. These symmetries are related to the first integrals of (1), namely the components of the total linear and angular momenta.

For every real number $\lambda \neq 0$ we define a scaling transformation Λ_λ by

$$\Lambda_\lambda(\mathbf{x}) := (\lambda^{-2}\mathbf{q}_1, \ldots, \lambda^{-2}\mathbf{q}_N, \lambda\mathbf{p}_1, \ldots, \lambda\mathbf{p}_N);$$

the system $X_{\mathbf{m}}$ has the property that if $\mathbf{x}(t)$ is a solution, then so is $\Lambda_\lambda\mathbf{x}(\lambda^3 t)$. In a similar way we define for each $\mu > 0$ the transformation

$$\Phi_\mu(\mathbf{x}) := (\mu^{1/3}\mathbf{q}_1, \ldots, \mu^{1/3}\mathbf{q}_N, \mu^{4/3}\mathbf{p}_1, \ldots, \mu^{4/3}\mathbf{p}_N);$$

it has the property that if $\mathbf{x}(t)$ is a solution of system $X_{\mathbf{m}}$, then $\Phi_\mu(\mathbf{x}(t))$ is a solution of system $X_{\mu\mathbf{m}}$.

Finally we can define an action of the symmetric group S_N (that is the group of all permutations of the set $\{1, \ldots, N\}$) on the phase space by

$$\Sigma_\sigma(\mathbf{x}) := (\mathbf{q}_{\sigma(1)}, \ldots, \mathbf{q}_{\sigma(N)}, \mathbf{p}_{\sigma(1)}, \ldots, \mathbf{p}_{\sigma(N)}), \qquad \forall \sigma \in S_N.$$

If $\mathbf{x}(t)$ is a solution of system $X_{\mathbf{m}}$, then $\Sigma_\sigma(\mathbf{x}(t))$ is a solution of system $X_{\sigma(\mathbf{m})}$, with $\sigma(\mathbf{m}) = (m_{\sigma(1)}, \ldots, m_{\sigma(N)})$. We also recall the notation for cycles: if $\{a_1, \ldots, a_r\}$ is any subset of $\{1, \ldots, N\}$, then we denote by $(a_1, \ldots, a_r)$ the permutation σ given by $\sigma(a_j) = a_{j+1}$ for $j = 1, \ldots, r-1$, $\sigma(a_r) = a_1$ and $\sigma(k) = k$ for all $k \notin \{a_1, \ldots, a_r\}$. Every permutation can be written as a composition of cycles.

All the symmetry operators $\Psi_{Q,\mathbf{0}}$, Λ_λ, Φ_μ and Σ_σ commute with each other.

2 Theoretical results about periodic orbits

In the past many different numerical methods have been used to calculate families of periodic orbits of the N-body problem. One of the main problems comes from the fact that by applying the symmetry operators above to a periodic orbit one obtains a new periodic orbit; therefore, periodic orbits typically belong to multi-parameter families of such orbits (compare with the cylinder theorem — see [3]), and it is not always easy to devellop a good strategy to calculate appropriate representants from such families. In [4] and [6] the authors, in collaboration with E.J. Doedel, have worked out an approach which in combination with boundary value packages such as AUTO (see [1]) appears to be very effective in handling this type of problem.

The basic concept in [4] is that of a *normal periodic orbit*, with normality defined by a geometric condition involving the monodromy matrix and the first integrals. The main result then says that normal periodic orbits belong to families which can be obtained (both theoretically and numerically) by solving an appropriate regular boundary value problem (BVP); this BVP involves, next to certain phase conditions, an adapted set of system equations obtained from the original one by adding artificial parameters multiplied by the gradients of the first integrals (see [4] for more details).

In [6] this approach was adapted to take advantage of reversibility properties of the system. A *reversor* is a linear operator R on the phase space which anti-commutes with the vectorfield. For example, $\Sigma_\sigma \circ \Psi_{Q,\mathbf{0}} \circ \Lambda_{-1}$ is a reversor for (1) on condition that $\sigma(\mathbf{m}) = \mathbf{m}$. An R-symmetric solution is then a solution whose orbit is invariant under R; an orbit is R-symmetric and periodic if and only if it has exactly two intersection points with the fixed point subspace $\mathrm{Fix}(R) = \{\mathbf{x} \mid R(\mathbf{x}) = \mathbf{x}\}$. Again, under an appropriate normality condition, a regular BVP can be set up to calculate families of R-symmetric periodic orbits. Examples of such BVP can be found in [5] and [6].

When at a particular orbit along a family the normality condition is not satisfied then bifurcation can occur; in such case AUTO allows to switch branches and to resume continuation of the bifurcating family.

3 Gerver's supereight

Gerver's supereight is a planar *choreographic* solution for 4 bodies with equal masses, i.e. a solution where the 4 bodies follow the same curve at equal time intervals. More precisely, $\Sigma_{(1\,2\,3\,4)}(\mathbf{x}(t)) = \mathbf{x}(t + T/4)$, where T is the period of the solution. Figure 1 shows the orbit and the position of the bodies at two different times, namely $t = 0$ and $t = T/8$. A closer analysis shows that the corresponding periodic orbit is invariant with respect to several reversors. To be more precise, let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the canonical basis in $\mathbf{R}^2$ (from now on we restrict to $n = 2$), and let S be the reflection in $\mathbf{R}^2$ with respect to the $\mathbf{e}_1$-axis: $S\mathbf{e}_1 = \mathbf{e}_1$ and $S\mathbf{e}_2 = -\mathbf{e}_2$. For each $\sigma \in S_4$, let $R_\sigma^+ := \Sigma_\sigma \circ \Psi_{S,\mathbf{0}} \circ \Lambda_{-1}$ and $R_\sigma^- := \Sigma_\sigma \circ \Psi_{-S,\mathbf{0}} \circ \Lambda_{-1}$. It is not hard to see that for the supereight as shown in Fig. 1 we have $\mathbf{x}(0) \in \mathrm{Fix}(R^+_{(1\,3)}) \cap \mathrm{Fix}(R^-_{(2\,4)})$, $\mathbf{x}(T/8) \in \mathrm{Fix}(R^+_{(1\,2)(3\,4)}) \cap \mathrm{Fix}(R^-_{(1\,4)(2\,3)})$, $\mathbf{x}(T/4) \in \mathrm{Fix}(R^+_{(2\,4)}) \cap \mathrm{Fix}(R^-_{(1\,3)})$ and $\mathbf{x}(3T/8) \in \mathrm{Fix}(R^+_{(1\,4)(2\,3)}) \cap \mathrm{Fix}(R^-_{(1\,2)(3\,4)})$. At $t = T/2$ we have again $\mathbf{x}(T/2) \in \mathrm{Fix}(R^+_{(1\,3)}) \cap \mathrm{Fix}(R^-_{(2\,4)})$, and then the sequence repeats. This implies that Gerver's supereight can be considered as an R-symmetric periodic orbit, with R any of the reversors $R_\sigma^\pm$ appearing in the foregoing.

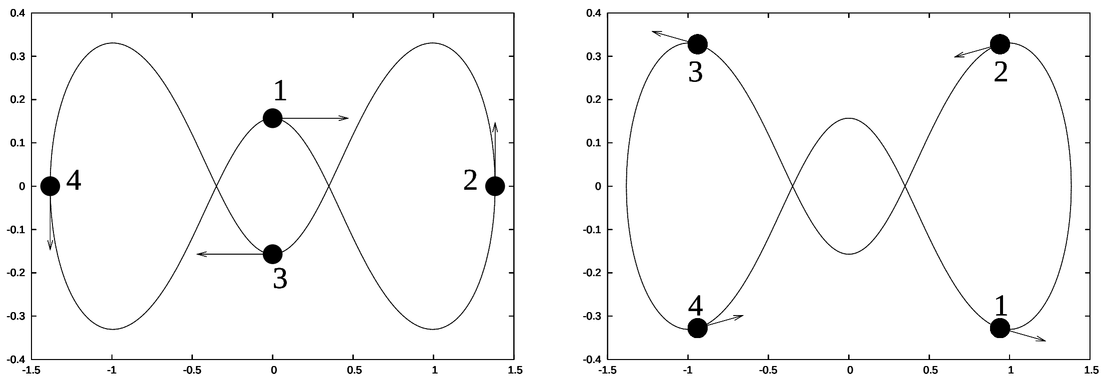

Figure 1: Gerver's supereight, with the positions and momenta of the four bodies at $t = 0$ and $t = T/8$; the particular supereight shown here has period $T = 2\pi$, and all masses are equal to 1. Initial data were taken from [2].

4 Continuation when varying the masses of two of the bodies

In this section we describe the results of the continuation of Gerver's supereight when we vary the masses of two of the four bodies; more precisely, we consider the cases $\mathbf{m} = (m, m, 1, 1)$, $\mathbf{m} = (m, 1, m, 1)$, $\mathbf{m} = (m, 1, 1, m)$, $\mathbf{m} = (1, m, m, 1)$, $\mathbf{m} = (1, m, 1, m)$ and $\mathbf{m} = (1, 1, m, m)$ with $m > 0$, doing continuation in the parameter m starting at Gerver's supereight for $m = 1$. It is easily seen that we can restrict to the cases $\mathbf{m} = (m, 1, m, 1)$ and $\mathbf{m} = (m, 1, 1, m)$ with $m \geq 1$; indeed, applying $\Phi_{1/m}$ and $\Sigma_{(1\,2\,3\,4)}$ to the families found this way we cover all the other cases. A further consequence of this is that if we detect a bifurcation at $m = m_0$, then there will be a corresponding bifurcation at $m = 1/m_0$. All the calculations were done using AUTO [1]; initial data for all solutions can be found at `http://www.maia.ub.es/~malmaraz`.

4.1 Varying the mass of two non-consecutive bodies: $\mathbf{m} = (m, 1, m, 1)$

Setting $\mathbf{m} = (m, 1, m, 1)$ and using the approach of [4] one can show numerically that the supereight is a normal periodic orbit and hence can be continued in the parameter m. Working out this continuation bifurcation points are detected for increasing values of m at $m = \mu_1 = 1.403682\ldots$, $m = \mu_2 = 1.459246\ldots$ and $m = \mu_3 = 3.945835\ldots$; for decreasing values of m we have corresponding bifurcation points at $m = 1/\mu_1 = 0.712412\ldots$, $m = 1/\mu_2 = 0.685285\ldots$ and $m = 1/\mu_3 = 0.253431\ldots$.

For each m the system $X_{(m,1,m,1)}$ has the reversors $R^{\pm}_{(1\,3)}$ and $R^{\pm}_{(2\,4)}$. Setting R equal to any of these reversors the supereight appears also to be normal as a R-symmetric periodic orbit, and hence we can use the approach of [6] to continue the supereight as a R-symmetric periodic orbit. This will give the same continuation family as before; some of the orbits along this family can be seen in [6, Fig. 5]. All the solutions along

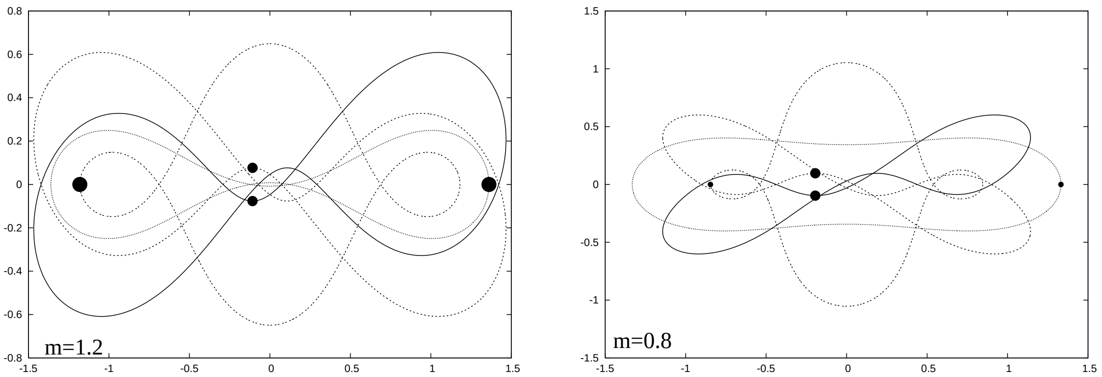

Figure 2: Some solutions of $X_{(m,1,m,1)}$ from the family of $R^{+}_{(2\,4)}$-symmetric periodic solutions bifurcating at $m = \mu_1$. The initial data shown belong to $\mathrm{Fix}(R^{+}_{(2\,4)}$.

this family have the full symmetry, i.e. they are R-symmetric for $R = R^{\pm}_{(1\,3)}$ and also for $R = R^{\pm}_{(2\,4)}$. However, when we fix some R and calculate the continuation we will detect only those bifurcation points where some other branch of R-symmetric periodic orbits bifurcates. This way we can determine the symmetries of the solutions along the bifurcating branches.

To clarify the relation between the symmetry of a family bifurcating at $m = m_0$ and the symmetry of the corresponding family bifurcating at $m = 1/m_0$ let s be the permutation $(1\,2\,3\,4)$. One can easily verify that if $\mathbf{x}(t)$ is a $R^{\pm}_{\sigma}$-symmetric solution of $X_{(m,1,m,1)}$ then $\Sigma_s(\Phi_{1/m}(\mathbf{x}(t)))$ is a $R^{\pm}_{s\sigma s^{-1}}$-symmetric solution of $X_{(1/m,1,1/m,1)}$. Since $s(1\,3)s^{-1} = (2\,4)$ and $s(2\,4)s^{-1} = (1\,3)$ it follows that if we have at $m = m_0$ a bifurcation of a branch of $R^{\pm}_{(1\,3)}$-symmetric (respectively $R^{\pm}_{(2\,4)}$-symmetric) periodic orbits, then we will have at $m = 1/m_0$ the bifurcation of a branch of $R^{\pm}_{(2\,4)}$-symmetric (respectively $R^{\pm}_{(1\,3)}$-symmetric) periodic orbits.

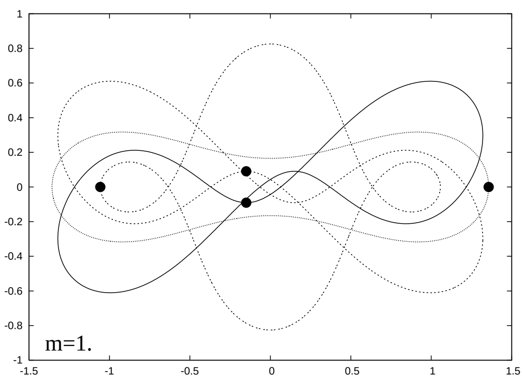

Figure 3: The special solution along the family shown in Fig. 2 for $m = 1$; this solution is $R^{+}_{(2\,4)}$-symmetric.

For $R = R^+_{(1\,3)}$ we find along the main branch only one bifurcation point, namely at $m = 1/\mu_1$; some of the solutions along the bifurcating branch of $R^+_{(1\,3)}$-symmetric periodic orbits are shown in [6, Fig. 6]. By the remark above this implies that for $R = R^+_{(2\,4)}$ we have just one bifurcation point at $m = \mu_1$; some of the solutions on the bifurcating branch of $R^+_{(2\,4)}$-symmetric periodic orbits are shown in Fig. 2. It follows from the calculations that the family of $R^+_{(1\,3)}$-symmetric periodic orbits bifurcating at $m = 1/\mu_1$ passes the value $m = 1$ (corresponding to four equal masses). By the symmetry also the branch of $R^+_{(2\,4)}$-symmetric periodic orbits bifurcating at $m = \mu_1$ passes $m = 1$. The corresponding orbit is shown in Fig. 3; except for a relabelling of the bodies this is also the orbit for $m = 1$ on the $R^+_{(1\,3)}$-symmetric family bifurcating at $m = 1/\mu_1$. This same orbit will also appear along other families discussed further on.

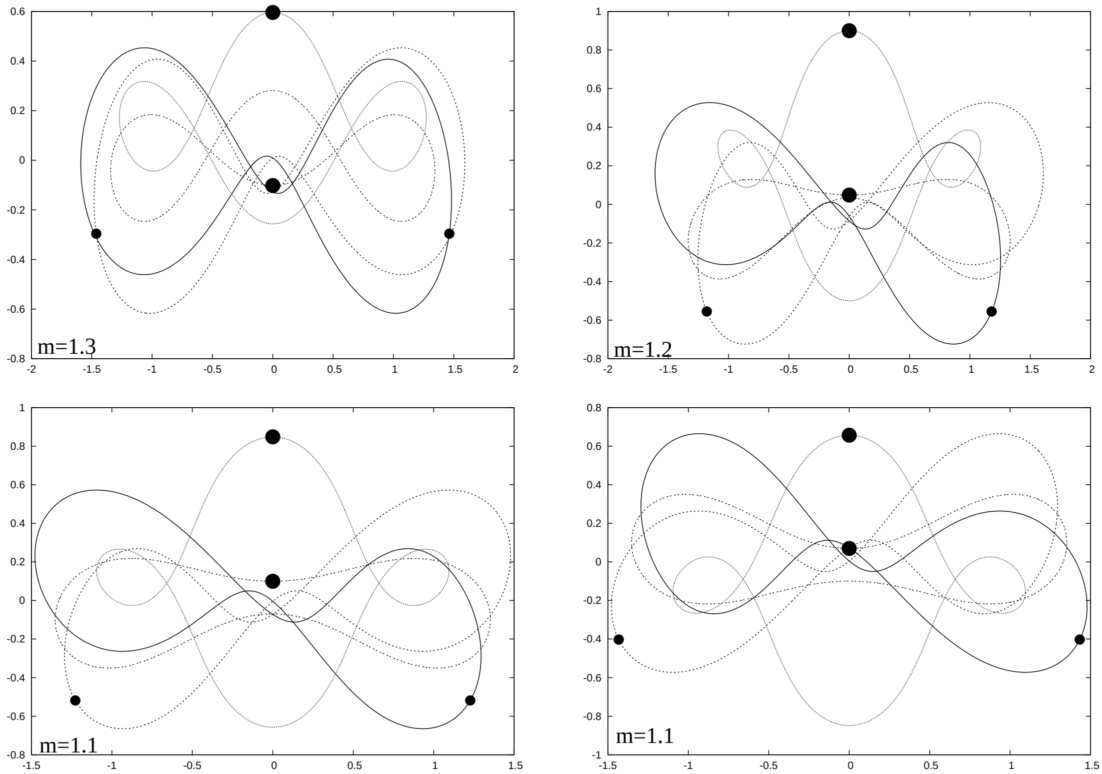

Figure 4: Some solutions from the family of $R^-_{(2\,4)}$-symmetric periodic orbits connecting the two bifurcation points along the family of periodic orbits shown in Fig. 2. The initial data shown belong to Fix($R^-_{(2\,4)}$).

Continuation of the supereight as a R-symmetric periodic orbit with $R = R^-_{(2\,4)}$ gives bifurcation points at $m = \mu_1, \mu_2, \mu_3, 1/\mu_2$ and $1/\mu_3$. The branch of $R^-_{(2\,4)}$-symmetric periodic orbits bifurcating at $m = \mu_1$ coincides with the branch of $R^+_{(2\,4)}$-symmetric periodic orbits bifurcating at the same point $m = \mu_1$ and shown in Fig. 2, i.e. the solutions along this branch are both $R^+_{(2\,4)}$-symmetric and $R^-_{(2\,4)}$-symmetric. Two solutions along this family are not normal as a $R^-_{(2\,4)}$-symmetric solution (for m =1.393362 and m =1.064269)

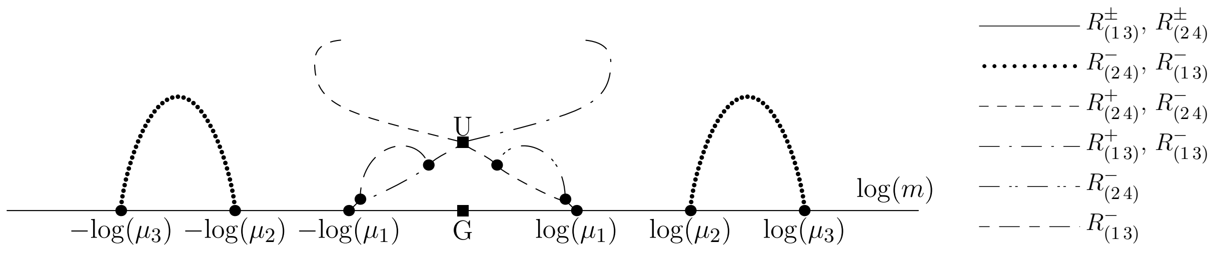

Figure 5: Schematic bifurcation diagram for the system $X_{(m,1,m,1)}$. G is Gerver's supereight, U is the special solution shown in Fig. 3.

and a new family of $R^-_{(2\,4)}$-symmetric periodic orbits appears connecting both bifurcation points; some of the solutions along this family of $R^-_{(2\,4)}$-symmetric orbits are shown in Fig. 4. By the symmetry the branch bifurcating at $m = 1/\mu_1$ is both $R^+_{(1\,3)}$-symmetric and $R^-_{(1\,3)}$-symmetric, and shows two bifurcation points which are connected by a branch of $R^-_{(1\,3)}$-symmetric periodic orbits.

The families of periodic orbits bifurcating at $m = \mu_2, \mu_2, 1/\mu_2$ and $1/\mu_3$ are both $R^-_{(2\,4)}$-symmetric and $R^-_{(1\,3)}$-symmetric. Actually, it appears from the calculations that the branches bifurcating at $m = \mu_2$ and $m = \mu_3$ connect to each other; some solutions along this connection are shown in [6, Fig. 7]. In the same way there is a connection between $m = 1/\mu_2$ and $m = 1/\mu_3$. A summarizing bifurcation diagram is sketched in Fig. 5.

4.2 Varying the mass of two consecutive bodies: $\mathbf{m} = (m, 1, 1, m)$

Now we set $\mathbf{m} = (m, 1, 1, m)$; again we use m as the continuation parameter. Numerical continuation of the supereight as a (normal) periodic orbit using the approach of [4] gives a family of periodic orbits some of which are displayed in Fig. 6. Along this family we find two bifurcation points, namely at $m = \mu_4 = 1.24875\ldots$ and at $m = 1/\mu_4$. Some of the solutions along the family bifurcating at $m = \mu_4$ are shown in Fig. 7; the family bifurcating at $m = 1/\mu_4$ can be obtained by applying the operator $\Sigma_{(1\,3)(2\,4)} \circ \Phi_{1/m}$ to the family bifurcating at $m = \mu_4$.

For each value of m the system $X_{(m,1,1,m)}$ has the reversors $R^+_{(1\,4)(2\,3)}$ and $R^-_{(1\,4)(2\,3)}$. Setting R equal to either $R^+_{(1\,4)(2\,3)}$ or $R^-_{(1\,4)(2\,3)}$ it appears that the supereight is normal as a R-symmetric periodic orbits, and hence we can use the continuation scheme of [6] to calculate the family of R-symmetric periodic orbits to which the supereight belongs. Because of the uniqueness of the continuation each of these families coincides with the family of periodic orbits which we have obtained before and which is shown in Fig. 6. This implies that the orbits along this family are both $R^+_{(1\,4)(2\,3)}$-symmetric and $R^-_{(1\,4)(2\,3)}$-symmetric. In fact, the data for the solutions in Fig. 6 were obtained from the continuation with $R = R^+_{(14)(23)}$. Moreover, using $R^+_{(1\,4)(2\,3)}$-symmetric or $R^-_{(1\,4)(2\,3)}$-symmetric continuation

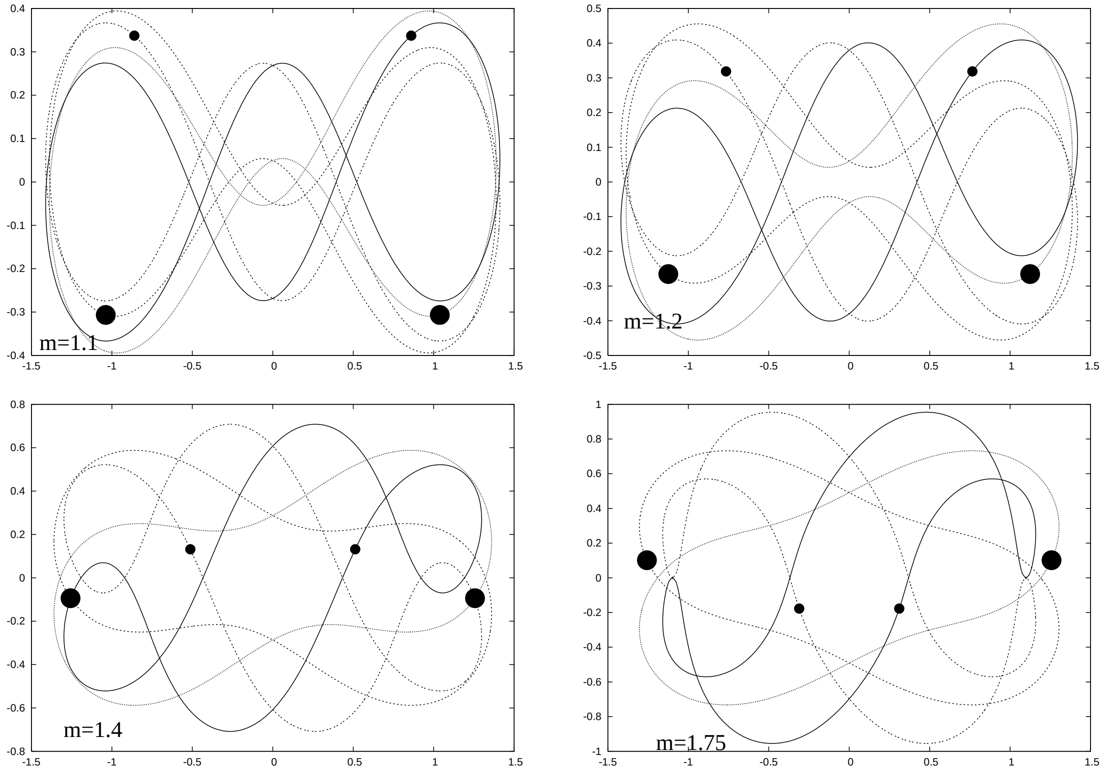

Figure 6: Some periodic orbits along the continuation of the supereight with $\mathbf{m} = (m, 1, 1, m)$. The initial data shown belong to $\mathrm{Fix}(R^-_{(1\,4)(2\,3)})$.

we find no bifurcation points. Therefore, the periodic orbits along the families which bifurcate at $m = \mu_4$ and $m = 1/\mu_4$ will not be invariant under either $R^+_{(1\,4)(2\,3)}$ or $R^-_{(1\,4)(2\,3)}$.

Finally we should remark that both the branch bifurcating at $m = \mu_4$ and the branch bifurcating at $m = 1/\mu_4$ cross the value $m = 1$; the corresponding periodic solutions of the 4-body problem with equal masses are the same as the one displayed in Fig. 3, except for an appropriate relabelling of the bodies.

5 Continuation when varying the mass of one of the bodies

In this section we consider the continuation of periodic solutions for (1) with $\mathbf{m} = (1, m, 1, 1)$, taking as starting solution the supereight; in this case we have to do the calculations both for $m \geq 1$ and for $m \leq 1$, since we can no longer use $\Phi_{1/m}$ to go from one case to the other. For all values of m the system $X_{(1,m,1,1)}$ has the reversors $R^+_{(1\,3)}$ and $R^-_{(1\,3)}$.

One can check numerically that the supereight is normal as a periodic orbit, as a $R^+_{(1\,3)}$-symmetric periodic orbit, and as a $R^-_{(1\,3)}$-symmetric periodic orbit. Using the appropriate continuation schemes from [4] and [6] we find each time the same continuation family, shown as the solid swished curve in the bifurcation diagram of Fig. 8; this curve has fold

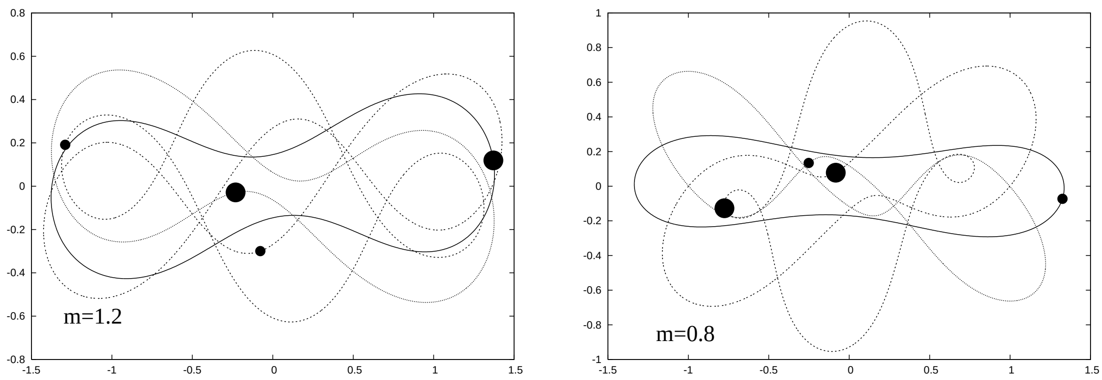

Figure 7: Some solutions of the system $X_{(m,1,1,m)}$ along the family bifurcating at $m = \mu_4$.

points at $m = 0.691818\ldots$ and $m = 1.13933\ldots$. The solutions along this family are both $R^+_{(1\,3)}$-symmetric and $R^-_{(1\,3)}$-symmetric. Some of these solutions are shown in Fig. 10. The family contains three different solutions for equal masses ($m = 1$): one is the supereight from which we started, the other two coincide up to a relabelling of the bodies with the solution shown in Fig. 3.

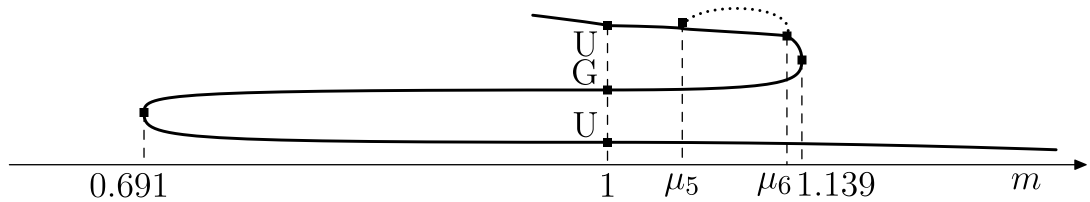

Figure 8: Schematic bifurcation diagram for $X_{(1,m,1,1)}$. There are three solutions for $m = 1$: the supereight G and two solutions U which, up to a relabelling of the bodies, coincide with the solution shown in Fig. 3.

The numerical calculations show that along the continuation family the solutions at $m = \mu_5 = 1.05652\ldots$ and at $m = \mu_6 = 1.13913\ldots$ are not normal as periodic solutions, and also not as $R^-_{(1\,3)}$-symmetric periodic solutions; they are normal as $R^+_{(1\,3)}$-symmetric periodic solutions. The bifurcation points $m = \mu_5$ and $m = \mu_6$ are connected by a new family of $R^-_{(1\,3)}$-symmetric periodic solutions; some of these solutions are shown in Fig. 9.

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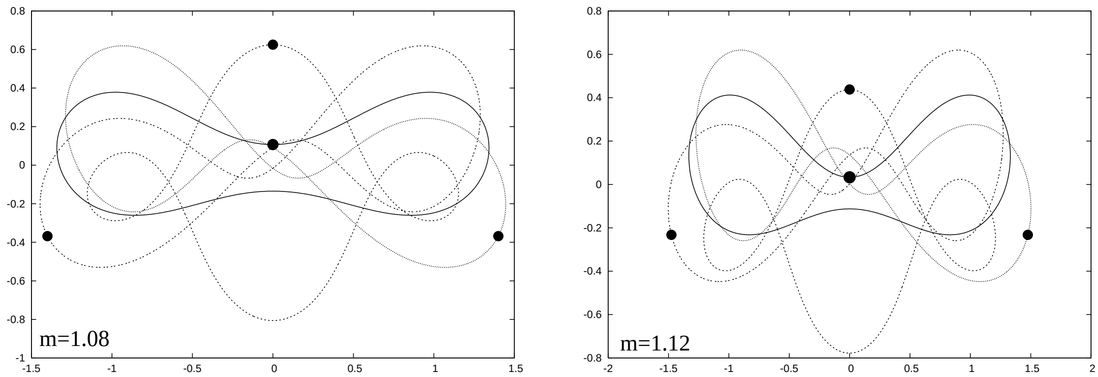

Figure 9: Some of the solutions for $X_{(1,m,1,1)}$ along the connection between the bifurcation points $m = \mu_5$ and $m = \mu_6$. The initial data shown belong to $\mathrm{Fix}(R^-_{(1\,3)})$.

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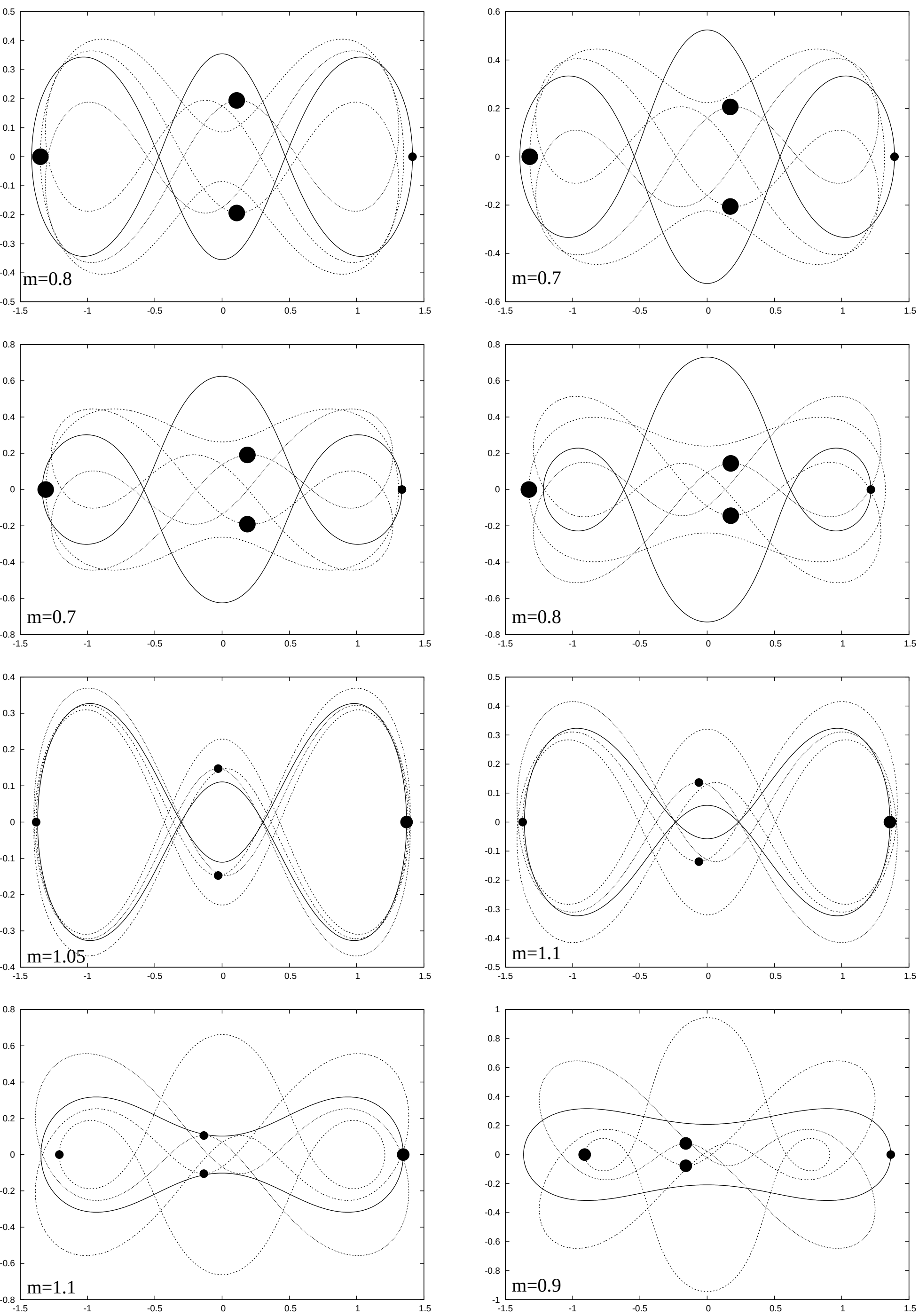

Figure 10: Some $R^+_{(1\,3)}$-symmetric periodic solutions along the continuation of the supereight for $\mathbf{m} = (1, m, 1, 1)$; the top pannels show solutions between the supereight and the fold at $m = 0.6918$, the second row displays solutions after passing the fold at $m = 0.6918$; solutions on the third row are from the section between the supereight and the fold at $m = 1.1393$, while the lower pannels display solutions after passing the fold at $m = 1.1393$. The orbits are also $R^-_{(1\,3)}$-symmetric, but the initial data shown are those belonging to $\mathrm{Fix}(R^+_{(1\,3)})$.