# A non-local perturbation of the logistic equation in $\mathbb{R}^N$

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**Abstract**

A logistic equation in the whole space is considered. In this problem, a non-local perturbation is included. We establish a new sub-supersolution method for general nonlocal elliptic equations and, consequently, we obtain the existence of positive solutions of a nonlocal logistic equation.

*Keywords:* Non-local term, logistic equation, sub-supersolution method.
*2010 MSC:* 35J15, 35J60.

## 1. Introduction

In this paper we are interested in studying a kind of equations whose model is

$$-\Delta u + u = K(x)u\left(\lambda - u^p + \alpha \int_{\mathbf{R}^N} M(x,y)g(u(y))dy\right) \quad \text{in } \mathbb{R}^N, \qquad (1)$$

where $N \geq 1$, $\lambda, \alpha \in \mathbb{R}$, $p > 0$ and $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous, $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $g(s) > 0$ for $s > 0$. $K$ is a regular function such that there exist $\beta > 1, k > 0$ satisfying

$$0 < K(x) \leq \frac{k}{1 + |x|^\beta}, \quad x \in \mathbb{R}^N. \qquad (2)$$

With respect to the kernel $M$, we assume that $M \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$ is positive and

$$\mathcal{M}(x) := \int_{\mathbf{R}^N} M(x,y)dy \in L^\infty(\mathbb{R}^N).$$

Equation (1) has been extensively studied in the local case, that is, for $\alpha = 0$. In such case, and assuming also $K \equiv 1$, it is known that there exists a unique positive solution for $\lambda > 1$ while no positive solution exists when $\lambda \leq 1$.

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Moreover, in the case of existence, the solution must be a constant, see [9] and references therein.

However, the inclusion of a non-local term has not been studied in detail yet. In [1], the authors used suitable weighted Sobolev spaces and abstract perturbation results to study a similar equation to (1), specifically the elliptic nonlocal Fisher-KPP equation

$$-\Delta u = \mu u \left( 1 - \int_{\mathbf{R}^N} \phi(x-y)u(y)dy \right) \quad \text{in } \mathbf{R}^N,$$

where the positive kernel $\phi \in L^1(\mathbf{R}^N)$ and $\int_{\mathbf{R}^N} \phi(x)dx = 1$. In this case, the authors proved that for $\mu \in (0, \mu_0]$ for some $\mu_0$, the only bounded non-negative classical solutions of the problem are $u \equiv 0$ and $u \equiv 1$. For the parabolic Fisher-KPP equation, some results about travelling waves and periodic nonconstant solutions in the unidimensional case are obtained in [6] and [12].

The bifurcation method was employed in [4] to study a equation related to (1). Indeed, in [4] the following problem was studied

$$\begin{cases} -\Delta u = u \left( \lambda f(x) - \int_{\mathbf{R}^N} M(x,y)|u(y)|^\gamma dy \right) & \text{in } \mathbf{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{3}$$

Under several conditions on $f$ and $M$, and in the radially symmetric framework, the authors proved the existence of an unbounded continuum of positive solutions bifurcating from the trivial solution at the principal eigenvalue associated to (3), that is,

$$\begin{cases} -\Delta u = \lambda f(x)u & \text{in } \mathbf{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases}$$

Finally, the time-dependent problem associated to (1) is analyzed in [7] when $K \equiv 1$, $g(s) = s$, $\alpha < 0$ and $M(x,y) = m(x-y)$ with $\int_{\mathbf{R}^N} m(x)dx = 1$. The authors used the sub-supersolution method to prove the existence and uniqueness of positive solution and also the global stability of the positive uniform steady-state solution of (1) under restrictions on the parameters of the problem. We remark that in [7] the stationary problem is not analyzed.

In this paper, we use the sub-supersolution method to study (1). Firstly, we would like to recall that for a local equation

$$-\Delta u = f(x,u) \quad \text{in } \mathbf{R}^N, \tag{4}$$

it is well known from the paper of Ako and Kusano [3] that if there exist a bounded supersolution $\overline{u}$ and a bounded subsolution $\underline{u}$ of (4) such that $\underline{u} \leq \overline{u}$ in $\mathbf{R}^N$, then (4) possesses at least an entire solution $u \in [\underline{u}, \overline{u}]$. In their proof, they built a sequence of solutions $u_R$ of the problem

$$-\Delta u = f(x,u) \quad \text{in } B_R, \qquad u = \phi \quad \text{on } \partial B_R,$$

where $B_R := \{x \in \mathbb{R}^N : |x| < R\}$ and $\phi$ is a regular function such that $\underline{u} \le \phi \le \overline{u}$ in $\mathbb{R}^N$. By means of the elliptic regularity theory and appropriate estimates, they can pass to the limit and conclude that

$$u(x) = \lim_{R \to \infty} u_R(x)$$

is the required solution in $\mathbb{R}^N$.

When we work with weak sub and supersolutions, we need to require conditions on the function $f$. For instance, in [13] a condition of $f$ is imposed for any $R$ large. Specifically, they assumed that for $R$ large

$$|f(x,t)| \le |f_R(x)| + h_R(|t|), \quad \text{in } \Omega_R = \Omega \cap B_R,$$

for functions $f_R \in L^q(\Omega_R)$ and increasing $h_R$. Note that the nonlocal problem (1) can not be restricted to $B_R$ in its current form because the integral term needs the definition of $u$ over $\mathbb{R}^N$. Moreover, in both papers ([3] and [13]), the following fact is crucial: a solution $u_R$ in $B_R$ is also solution in $B_r$ for $r \le R$. However, these arguments do not work for equations having a non-local term in the non-linear term.

Here, we overcome this difficulty and present a general sub-supersolution method that is valid for general equations of the form (see [15] for a related problem)

$$-\text{div}(h(x)\nabla u) + q(x)u = K(x)f(x,u,Bu) \quad \text{in } \mathbb{R}^N, \tag{5}$$

where $Bu$ is a nonlocal operator and $h$, $q$, $f$ and $B$ verify some hypotheses detailed in Section 2. We will use the function $K$ with the boundness condition (2) that allows to approach an adequate functional setting with compactness in $\mathbb{R}^N$, see [5] for the treatment of other problem with the same idea.

The first step to establish the method is the study of the linear problem

$$-\text{div}(h(x)\nabla u) + q(x)u = K(x)f(x) \quad \text{in } \mathbb{R}^N, \tag{6}$$

and the eigenvalue problem

$$-\text{div}(h(x)\nabla u) + q(x)u = \lambda K(x)u \quad \text{in } \mathbb{R}^N. \tag{7}$$

We prove the existence and uniqueness of solution for the first one and the existence of a principal eigenvalue, an eigenvalue with a positive eigenfunction associated to it and denoted by $\mu_1$, of the second one. Then, we can prove the existence of solution of (5) if there exists a pair of sub-supersolutions and a boundness condition for $f$ holds (see (15)).

Once the method is established, we study in detail (1). The main results can be summarized as follows:

1. Assume $\alpha = 0$. Then, (1) possesses at least a positive solution if and only if $\lambda > \mu_1$.
2. Assume
$$\lim_{s \to \infty} \frac{g(s)}{s^p} = 0.$$

(a) If $\alpha > 0$. Then, if $\lambda > \mu_1$ there exists at least a positive solution of (1).

(b) If $\alpha < 0$. Then, there exists $\mu_0 > \mu_1$ such that for $\lambda > \mu_0$ there exists at least a positive solution of (1). Moreover, (1) does not possess positive solutions for $\lambda \leq \mu_1$.

An outline of the paper is as follows: In Section 2, we describe our functional setting, we study the linear and the eigenvalue problems (6) and (7) and we establish the sub-supersolution method. In Section 3, we apply these results to problem (1).

## 2. The sub-supersolutions method

### 2.1. The functional setting

**Definition 2.1.** Let $\Omega \subset \mathbb{R}^N$ be a domain (eventually, $\Omega = \mathbb{R}^N$).

1. Let $w$ a weight function, i.e, a measurable, positive and finite a.e. $x \in \Omega$ function. We define the weighted Lebesgue space
$$L^2(\Omega; w) := \{u \in L^2_{\mathrm{loc}}(\Omega) : \int_\Omega w(x)|u(x)|^2 dx < \infty\}.$$

2. Let $v_0, v_1$ weight functions. We define the weighted Sobolev space
$$W^{1,2}(\Omega; v_0, v_1) := \{u \in L^2(\Omega; v_0) : \nabla u \in (L^2(\Omega; v_1))^N\}.$$

These spaces are Hilbert spaces with the respective norms
$$\|u\|_{2,w} = \left[\int_\Omega w(x)|u(x)|^2 \, dx\right]^{\frac{1}{2}},$$

$$\|u\|_{1,2,v_0,v_1} = \left[\int_\Omega v_0(x)|u(x)|^2 \, dx + \int_\Omega v_1(x)|\nabla u(x)|^2 \, dx\right]^{\frac{1}{2}}.$$

The following facts are well known:

1. If $w \in L^1(\mathbb{R}^N)$, then $L^\infty(\mathbb{R}^N) \subset L^2(\mathbb{R}^N; w)$.

2. If $w_1, w_2$ are weight functions and there exits $M > 0$ such that $w_1(x) \leq M w_2(x)$ a.e. $x \in \Omega$ then
$$L^2(\mathbb{R}^N; w_2) \subset L^2(\mathbb{R}^N; w_1).$$

3. (cf. [14]) If there exist positive constants, $c$, $C$ such that
$$c \leq w(x), \ v_0(x), \ v_1(x) \leq C \qquad \text{a.e } x \in \Omega,$$

then $L^2(\Omega; w)$ and $W^{1,2}(\Omega; v_0, v_1)$ are isometrically isomorphic to $L^2(\Omega)$ and $W^{1,2}(\Omega)$ and hence, if $\Omega$ is bounded, the classic Rellich Theorem assures the compact embedding
$$W^{1,2}(\Omega; v_0, v_1) \hookrightarrow L^2(\Omega; w).$$

When $\Omega = \mathbb{R}^N$ a number of conditions, that we will detail in our particular case, on $v_0$, $v_1$ and $w$ must be fulfilled (see pag. 289 of [14]) to reach this compact embedding.

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*2.2. The linear problem*

We first consider the following linear problem

$$-\mathrm{div}(h(x)\nabla u) + q(x)u = K(x)f(x) \quad \text{in } \mathbb{R}^N, \tag{8}$$

where $h$, $q$ are weight functions, $K$ verifies (2) and $f \in L^2(\mathbb{R}^N; K)$. We will suppose furthermore the following hypotheses

(H1) For some $n \in \mathbb{N}$,

$$\exists k_0 : \ h(x) \le k_0 q(x) \quad \forall |x| > n.$$

(H2) For some $n \in \mathbb{N}$, there exists a measurable function $b_1$ such that

$$b_1(x) \le h(y) \quad \forall |x| > n, \ \forall y : |y - x| < 1,$$

and

$$\lim_{n \to \infty} \sup_{|x| > n} \frac{1}{(1 + (|x| - 1)^\beta) b_1(x)} = 0.$$

(H3) There exists $C > 0$ such that

$$K(x) \le C q(x) \ a.e. \ x \in \mathbb{R}^N. \tag{9}$$

The hypotheses (H1) and (H2) are those obtained by putting

$$w(x) = K(x), \ v_0(x) = q(x), \ v_1(x) = h(x), \ r(x) = 1, \ b_0(x) = \frac{k}{1 + (|x| - 1)^\beta}$$

to apply Theorem 18.7 of [14], see also page 289, and they assure the compact embedding

$$W^{1,2}(\mathbb{R}^N; q, h) \hookrightarrow L^2(\mathbb{R}^N; K). \tag{10}$$

Observe that these hypotheses are verified in particular if $h$ and $q$ are constant.

**Definition 2.2.** *We say that $u \in W^{1,2}(\mathbb{R}^N; q, h)$ is a weak solution of (8) if $\forall v \in W^{1,2}(\mathbb{R}^N; q, h)$,*

$$\int_{\mathbb{R}^N} h(x)\nabla u(x) \cdot \nabla v(x)\, dx + \int_{\mathbb{R}^N} q(x)u(x)v(x)\, dx = \int_{\mathbb{R}^N} K(x)f(x)v(x)\, dx. \tag{11}$$

Note that $v \in L^2(\mathbb{R}^N; q)$ and (H3) imply

$$L^2(\mathbb{R}^N; q) \subset L^2(\mathbb{R}^N; K)$$

and hence the right hand of the equality (11) is well defined.

We study the linear problem (8). The following result provides us the existence and uniqueness of solution of (8) as well as the compactness of the solution operator.

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**Lemma 2.3.** *Assume that $f \in L^2(\mathbb{R}^N; K)$, (H1), (H2) and (H3). Then, there exists a unique weak solution $u$ of (8). Moreover, if we define the map*

$$T : L^2(\mathbb{R}^N; K) \mapsto L^2(\mathbb{R}^N; K), \quad f \mapsto T(f) := u,$$

*where $u$ is the unique solution of (8), then $T$ is a linear and compact operator.*

PROOF. Define the bilinear map $a : W^{1,2}(\mathbb{R}^N; q, h) \times W^{1,2}(\mathbb{R}^N; q, h) \mapsto \mathbb{R}$ as

$$a(u, v) := \int_{\mathbb{R}^N} h(x)\nabla u(x) \cdot \nabla v(x)dx + \int_{\mathbb{R}^N} q(x)u(x)v(x)dx,$$

and $F : W^{1,2}(\mathbb{R}^N; q, h) \mapsto \mathbb{R}$

$$F(v) := \int_{\mathbb{R}^N} K(x)f(x)v(x)dx.$$

It is clear that $a$ is coercive and continuous. Moreover, $F$ is continuous because

$$|F(v)| \leq \int_{\mathbb{R}^N} K^{1/2}(x)|f(x)|K(x)^{1/2}|v(x)|\, dx$$

$$\leq \|f(x)\|_{2,K}\|v(x)\|_{2,K} \leq \|f(x)\|_{2,K}\|v\|_{1,2,q,h}$$

and then the existence and uniqueness of solution of (8) follow by Lax-Milgram Theorem.

Hence, $T$ is well-defined and is compact by (10). □

Now, we analyze the following eigenvalue problem

$$-\mathrm{div}(h(x)\nabla u) + q(x)u = \lambda K(x)u \quad \text{in } \mathbb{R}^N. \tag{12}$$

**Proposition 2.4.**

1. *There exists the principal eigenvalue of (12), $\mu_1 \in \mathbb{R}$, $\mu_1 > 0$, which has associated a unique positive, up to multiplicative constants, eigenfunction $\varphi_1 \in W^{1,2}(\mathbb{R}^N; q, h)$.*
2. *If there exist positive constants, $\underline{\Lambda}$, $\overline{\Lambda}$ such that*

$$h(x) \geq \underline{\Lambda}, \ q(x) \leq \overline{\Lambda}, \quad \forall x \in \mathbb{R}^N, \tag{13}$$

   *then, $\varphi_1 \in L^\infty(\mathbb{R}^N)$.*

PROOF. Observe that $\lambda$ is an eigenvalue of (12) if and only if $1/\lambda$ is an eigenvalue of $T$. Now, the first result follows because $T$ is compact, self-adjoint and positive. It remains to prove that $\varphi_1 \in L^\infty(\mathbb{R}^N)$ in the second case. For that, we employ the apriori estimates of Theorem 8.17 in [11]. Observe that $\varphi_1$ verifies

$$-\mathrm{div}(h(x)\nabla\varphi_1) + (q(x) - \mu_1 K(x))\varphi_1 = 0 \quad \text{in } \mathbb{R}^N.$$

By (13), we have that $q(x) - \mu_1 K(x) \in L^\infty(\mathbb{R}^N)$ and the operator is strictly elliptic, and hence we get

$$\varphi_1(x) \leq \sup_{y \in B_1(x)} |\varphi_1(y)| \leq C\|\varphi_1\|_{L^2(B_2(x))} \leq C\|\varphi_1\|_{L^2(\mathbb{R}^N)} \leq C$$

where $B_i(x)$ is the ball of $\mathbb{R}^N$ centered on $x$ with radius $i$ and $C$ depends on the data of the problem. This concludes the proof. □

*2.3. The nonlinear problem. The sub-supersolution method*

We consider the problem

$$-\operatorname{div}(h(x)\nabla u) + q(x)u = K(x)f(x, u, Bu) \quad \text{in } \mathbb{R}^N, \tag{14}$$

where $h$, $q$ are weight functions, $K$ verifies (2) and $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function, $B$ defined by

$$B(u)(x) =: \alpha \int_{\mathbf{R}^N} M(x, y)g(u(y))dy, \quad \alpha \geq 0,$$

with $g : \mathbb{R} \mapsto \mathbb{R}$ a continuous and positive function and $M \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$ is positive and

$$\mathcal{M}(x) := \int_{\mathbf{R}^N} M(x, y)dy \in L^\infty(\mathbb{R}^N).$$

**Definition 2.5.** *We say that $u \in W^{1,2}(\mathbb{R}^N; q, h)$ is a weak solution of (14) if $\forall v \in W^{1,2}(\mathbb{R}^N; q, h)$,*

$$\int_{\mathbf{R}^N} h(x)\nabla u(x) \cdot \nabla v(x)\, dx + \int_{\mathbf{R}^N} q(x)u(x)v(x)\, dx$$

$$= \int_{\mathbf{R}^N} K(x)f(x, u(x), (Bu)(x))v(x)\, dx.$$

**Definition 2.6.** *We say that $\underline{u}, \overline{u} \in W^{1,2}(\mathbb{R}^N; q, h)$ is a pair of sub-supersolution of (14) if*

1. *$\underline{u} \leq \overline{u}$ in $\mathbb{R}^N$,*
2. *$\forall v \in W^{1,2}(\mathbb{R}^N; q, h), v \geq 0, \forall w : \underline{u}(x) \leq w(x) \leq \overline{u}(x)$ a.e. $x \in \mathbb{R}^N$*

$$\int_{\mathbf{R}^N} h(x)\nabla \overline{u}(x) \cdot \nabla v(x)\, dx + \int_{\mathbf{R}^N} q(x)\overline{u}(x)v(x)\, dx$$

$$\geq \int_{\mathbf{R}^N} K(x)f(x, \overline{u}(x), B(w)(x))v(x)\, dx,$$

*and*

$$\int_{\mathbf{R}^N} h(x)\nabla \underline{u}(x) \cdot \nabla v(x)\, dx + \int_{\mathbf{R}^N} q(x)\underline{u}(x)v(x)\, dx$$

$$\leq \int_{\mathbf{R}^N} K(x)f(x, \underline{u}(x), B(w)(x))v(x)\, dx.$$

The main result in this section reads as follows:

**Theorem 2.7.** *Suppose that the hypotheses (H1), (H2) and (H3) are verified. Assume that there exist a couple of sub-supersolution $\underline{u}, \overline{u}$ of (14) and a positive function $m \in L^2(\mathbb{R}^N; K)$ such that*

$$|f(x, t, B(w)(x))| \leq m(x), \tag{15}$$

*a.e. $x \in \mathbb{R}^N$, $t \in [\underline{u}(x), \overline{u}(x)]$ and $w \in [\underline{u}, \overline{u}]$. Then, there exists at least a solution $u$ of (14) such that*

$$u \in [\underline{u}, \overline{u}].$$

PROOF. Define the truncation operator $T : L^2(\mathbb{R}^N; K) \mapsto L^2(\mathbb{R}^N; K)$ by

$$Tw(x) := \begin{cases} \underline{u}(x) & \text{if } w(x) < \underline{u}(x), \\ w(x) & \text{if } \underline{u}(x) \leq w(x) \leq \overline{u}(x), \\ \overline{u}(x) & \text{if } w(x) > \overline{u}(x). \end{cases}$$

Consider now the operator $S : L^2(\mathbb{R}^N; K) \mapsto L^2(\mathbb{R}^N; K)$ defined by $S(w) := u$ where $u$ is the unique solution of

$$-\text{div}(h(x)\nabla u) + q(x)u = K(x)f(x, T(w), B(T(w))) \quad \text{in } \mathbb{R}^N. \tag{16}$$

Observe that $T(w) \in [\underline{u}, \overline{u}]$ and hence by (15) we get

$$f(x, T(w), B(T(w))) \in L^2(\mathbb{R}^N; K).$$

Thus, it is clear that $S$ is well defined, and compact by Lemma 2.3. Moreover, thanks to (15) there exists $R > 0$ such that

$$S(B_{L^2(\mathbf{R}^N;K)}(0, R)) \subset B_{L^2(\mathbf{R}^N;K)}(0, R),$$

where $B_{L^2(\mathbf{R}^N;K)}(0, R)$ is the ball of $L^2(\mathbb{R}^N; K)$ centered on 0 with radius $R$, and then by Schauder Theorem, there exists a fixed point $u$ of $S$, that is

$$-\text{div}(h(x)\nabla u) + q(x)u = K(x)f(x, T(u), B(T(u))) \quad \text{in } \mathbb{R}^N.$$

Now, we show that $u \in [\underline{u}, \overline{u}]$. Indeed, by Definition 2.6 with $w = T(u)$ we get in weak sense that

$$-\text{div}(h(x)\nabla\overline{u}) + q(x)\overline{u} \geq K(x)f(x, \overline{u}, B(T(u))) \quad \text{in } \mathbb{R}^N.$$

If we denote $w = \overline{u} - u$,

$$-\text{div}(h(x)\nabla w) + q(x)w \geq K(x)[f(x, \overline{u}, B(T(u))) - f(x, T(u), B(T(u)))],$$

taking as test function $w^-$, where $w^- = \min\{w, 0\}$, and taking into account that $w^- \leq 0$, we obtain

$$\int_{\mathbf{R}^N} h(x)|\nabla w^-(x)|^2 + \int_{\mathbf{R}^N} q(x)|w^-(x)|^2 \leq$$

$$\int_{\mathbf{R}^N} K(x)[f(x, \overline{u}(x), B(T(u))(x)) - f(x, T(u)(x), B(T(u))(x))]w^-(x) = 0.$$

Hence, since $h$ and $q$ are positive functions, we conclude that $w^- \equiv 0$ and then $w \geq 0$. This yields to $u \leq \overline{u}$.

In a similar way, we can show that $\underline{u} \leq u$ taking as test function $w^-$ being $w = u - \overline{u}$. This implies that $T(u) = u$ and so, $u$ is solution of (14). This completes the proof. $\qquad\square$

## 3. Application: the nonlocal logistic equation

In this Section, we study completely (1). Here $h \equiv q \equiv 1$ in the general formulation and it holds that

$$H^1(\mathbb{R}^N) := W^{1,2}(\mathbb{R}^N; 1, 1) \hookrightarrow L^2(\mathbb{R}^N; K). \tag{17}$$

Denote by $e$ the unique positive solution of the linear equation

$$-\Delta u + u = K(x) \quad \text{in } \mathbb{R}^N. \tag{18}$$

The following result will be useful along this section.

**Lemma 3.1.** *There exist $\gamma \in (0, \beta)$ and $C > 0$ such that*

$$e(x) \leq \frac{C}{1 + |x|^\gamma} \qquad x \in \mathbb{R}^N. \tag{19}$$

*As consequence, $e \in L^\infty(\mathbb{R}^N)$.*

PROOF. Observe that since $\beta > 1$, then the positive constants belong to $L^2(\mathbb{R}^N; K)$, and thus there exists a unique solution $e$ of (18) taking $f \equiv 1$ in Lemma 2.3.

Take now

$$\overline{u} := \frac{C}{1 + |x|^\gamma}.$$

Then, it is direct calculation that

$$-\Delta \overline{u} = C\gamma |x|^{\gamma-2}(1 + |x|^\gamma)^{-3}((N - 2 - \gamma)|x|^\gamma + N - 2 + \gamma),$$

and hence, $-\Delta \overline{u} + \overline{u} \geq K(x)$ provided of

$$C[(\gamma |x|^{\gamma-2}(1+|x|^\gamma)^{-3}((N-2-\gamma)|x|^\gamma+N-2+\gamma)+(1+|x|^\gamma)^{-1})] \geq k(1+|x|^\beta)^{-1}.$$

Observe that for $|x| \approx \infty$, this inequality is equivalent to

$$C[\gamma(N - 2 - \gamma)|x|^{\beta-\gamma-2} + \gamma(N - 2 + \gamma)|x|^{\beta-2\gamma-2} + |x|^{\beta-\gamma}] \geq k.$$

Since $\gamma \in (0, \beta)$, this inequality is true for $|x| > R_1$ with $R_1$ large enough.

On the other hand, for $|x| \approx 0$, the inequality is equivalent to

$$C[\gamma |x|^{\gamma-2}((N - 2 - \gamma)|x|^\gamma + (N - 2 + \gamma)) + 1] \geq k.$$

Hence, taking $\gamma \in (2 - N, 2)$ and $\gamma > 0$, the inequality holds for $|x| < R_2$ with $R_2$ small enough. Now, in the set $R_2 \leq |x| \leq R_1$ we take $C$ large enough. This concludes the result. $\qquad \square$

### 3.1. The local and nonlocal nonlinear problems

Our first result deals with the case $\alpha = 0$.

**Theorem 3.2.** *Assume $\alpha = 0$. Then, there exists at least a positive solution of (1) if and only if $\lambda > \mu_1$. If we denote this solution by $\theta_\lambda$, it holds that*

$$\theta_\lambda \leq \min\{\lambda^{1/p}, C(\lambda)e\}, \tag{20}$$

*where*

$$C(\lambda) = \lambda^{(p+1)/p} p \left(\frac{1}{p+1}\right)^{(p+1)/p}.$$

*Finally, given $\mu_1 < \gamma < \mu$ we can obtain $v$ and $w$ positive solutions of (1) for $\lambda = \gamma$ and $\lambda = \mu$, respectively such that*

$$v < w \quad in \ \mathbb{R}^N.$$

PROOF. First observe that for $\alpha = 0$, (1) is a local equation. We take as subsolution $\underline{u} = \varepsilon\varphi_1$ where $\varphi_1$ is a positive eigenfunction associated to $\mu_1$ such that

$$\|\varphi_1\|_\infty = 1$$

and $\varepsilon > 0$ a positive constant to be chosen. As supersolution we take $\overline{u} = Me$, where $M$ is a positive constant. Taking into account that

$$t(\lambda - t^p) \leq C(\lambda), \quad \forall t \geq 0, \tag{21}$$

it is easy to show that $\underline{u}$ and $\overline{u}$ verify the inequalities of the second condition of Definition 2.6 provided of

$$\varepsilon^p \varphi_1^p \leq \lambda - \mu_1 \quad \text{and} \quad C(\lambda) \leq M.$$

The above inequalities are true for $\varepsilon$ and $M$ small and large enough, respectively. Finally, observe that for $\varepsilon$ small enough, we get that

$$-\Delta(\overline{u} - \underline{u}) + (\overline{u} - \underline{u}) = K(x)(C(\lambda) - \mu_1\varepsilon\varphi_1) \geq 0,$$

whence $\underline{u} \leq \overline{u}$ in $\mathbb{R}^N$, the first condition of Definition 2.6. Hence $\underline{u}$ and $\overline{u}$ is a couple of sub-supersolutions. On the other hand, hypothesis (15) is verified because (21) implies

$$m(x) = C(\lambda) \in L^\infty(\mathbb{R}^N) \subset L^2(\mathbb{R}^N; K).$$

Hence, there exists at least a solution, that will be denoted by $\theta_\lambda$, of (1), and thus,

$$\varepsilon\varphi_1 \leq \theta_\lambda \leq C(\lambda)e. \tag{22}$$

Now, we claim that

$$\theta_\lambda \leq \lambda^{1/p}. \tag{23}$$

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Take $w = (\lambda^{1/p} - \theta_\lambda)$. Since $\theta_\lambda$ is a solution of (1) we have that

$$\int_{\mathbf{R}^N} \nabla\theta_\lambda \cdot \nabla v + \int_{\mathbf{R}^N} \theta_\lambda v = \int_{\mathbf{R}^N} K(x)\theta_\lambda(x)(\lambda - \theta_\lambda^p(x))v, \quad \forall v \in H^1(\mathbb{R})^N,$$

and then

$$-\int_{\mathbf{R}^N} \nabla(\lambda^{1/p} - \theta_\lambda) \cdot \nabla v + \int_{\mathbf{R}^N} \theta_\lambda v = \int_{\mathbf{R}^N} K(x)\theta_\lambda(x)(\lambda - \theta_\lambda^p(x))v, \ \forall v \in H^1(\mathbb{R})^N.$$

Observe that by (22), there exists $R > 0$ such that $\theta_\lambda < \lambda^{1/p}$ in $\mathbb{R}^N \setminus B(0, R)$. Hence, $w^- \in H^1(\mathbb{R}^N)$. Now, taking $w^-$ as test function in (1) we obtain

$$-\int_{\mathbf{R}^N} |\nabla w^-(x)|^2 + \int_{\mathbf{R}^N} \theta_\lambda(x)w^-(x) = \int_{\mathbf{R}^N} K(x)\theta_\lambda(x)(\lambda - \theta_\lambda^p(x))w^-(x) \geq 0,$$

and we conclude (23). Hence, we have proved (20).

On the other hand, taking $v = \varphi_1$ in the definition of solution of (1) we get that $\lambda > \mu_1$.

Finally, take $\mu_1 < \gamma < \mu$. Consider $\theta_\gamma$ the positive solution of (1) for $\lambda = \gamma$ found in the first part of the result. Then, $\underline{u} = \theta_\gamma$ and $\overline{u} = C(\mu)e$ is a pair of sub-supersolution of (1) for $\lambda = \mu$. This completes the proof. $\qquad\square$

**Remark 3.3.** *The uniqueness of positive solution of (1) with $\alpha = 0$ is a hard problems, see for instance [10], [9], [8] and [2].*

Now, we treat the case $\alpha \neq 0$. Our main result in this case is:

**Theorem 3.4.** *Assume*

$$\lim_{s \to \infty} \frac{g(s)}{s^p} = 0. \tag{24}$$

1. *Assume that $\alpha > 0$. Then, if $\lambda > \mu_1$ there exists at least a positive solution of (1).*
2. *Assume that $\alpha < 0$. Then, there exists $\mu_0 > \mu_1$ such that for $\lambda > \mu_0$ there exists at least a positive solution of (1). Moreover, (1) does not possess positive solutions for $\lambda \leq \mu_1$.*

PROOF. 1.- Assume that $\alpha > 0$ and $\lambda > \mu_1$. It is clear that $\underline{u} = \theta_\lambda$ is subsolution of (1), where $\theta_\lambda$ is a positive solution of (1) with $\alpha = 0$. As supersolution, we take $\overline{u} = v$, where $v$ is a positive solution of (1) with $\alpha = 0$ with $\lambda = \mu$ to be chosen. By Theorem 3.2, we have that $\underline{u} \leq \overline{u}$ if $\mu > \lambda$. By (24), for any $\varepsilon > 0$ there exist $s_0 > 0$ and $R > 0$ such that

$$g(s) \leq \varepsilon s^p \quad s \geq s_0 \quad \text{and} \quad g(s) \leq R \quad s \in [0, s_0].$$

Then, take $w \in [\underline{u}, \overline{u}]$. We get by (23)

$$
\begin{aligned}
\int_{\mathbf{R}^N} M(x,y)g(w(y))dy \quad &\leq \varepsilon \int_{w \geq s_0} M(x,y)w^p(y)dy + R \int_{w < s_0} M(x,y)dy \\
&\leq \varepsilon \int_{w \geq s_0} M(x,y)\theta_\mu^p(y)dy + R \int_{w < s_0} M(x,y)dy \\
&\leq \varepsilon\mu \int_{w \geq s_0} M(x,y)dy + R \int_{w < s_0} M(x,y)dy \\
&\leq (\varepsilon\mu + R)\mathcal{M}(x).
\end{aligned}
\tag{25}
$$

Then, for $t \in [\underline{u}(x), \overline{u}(x)]$ and $w \in [\underline{u}, \overline{u}]$ we get using (15)

$$
\begin{aligned}
|f(x,t,B(w)(x))| \quad &= |t(\lambda - t^p + \alpha \int_{\mathbf{R}^N} M(x,y)g(w(y))dy)| \\
&\leq C(\lambda) + \overline{u}(x)\alpha \int_{\mathbf{R}^N} M(x,y)g(w(y))dy \\
&\leq C(\lambda) + \alpha(\varepsilon\mu + R)\mathcal{M}(x)v(x) \in L^2(\mathbf{R}^N; K).
\end{aligned}
$$

Now, we show that $\underline{u}, \overline{u}$ is sub-supersolution of (1). It is clear that $\underline{u} = \theta_\lambda$ is a subsolution due to $\alpha > 0$. On the other hand, $\overline{u} = v$ is a supersolution if

$$
\lambda + \alpha \int_{\mathbf{R}^N} M(x,y)g(w(y))dy \leq \mu \qquad \forall w \in [\underline{u}, \overline{u}].
$$

Using (25), $\overline{u}$ is a supersolution if

$$
\lambda + \alpha(\varepsilon\mu + R)\mathcal{M}(x) \leq \mu,
$$

or equivalently,

$$
\lambda + \alpha R \sup_{\mathbf{R}^N} \mathcal{M} \leq \mu(1 - \alpha\varepsilon \sup_{\mathbf{R}^N} \mathcal{M}).
$$

It is enough to take $\mu$ large and $\varepsilon$ small.

2.- Assume that $\alpha < 0$. Now, take $\underline{u} = \rho\varphi_1$, with $\rho$ a positive constant to be chosen and $\overline{u} = \theta_\lambda$. Since $\alpha < 0$, it is clear that $\overline{u}$ is supersolution. On the other hand, $\underline{u}$ is subsolution if

$$
(\rho\varphi_1)^p \leq \lambda - \mu_1 + \alpha \int_{\mathbf{R}^N} M(x,y)g(w(y))dy \qquad \forall w \in [\underline{u}, \overline{u}].
$$

Using (25) we get that

$$
\int_{\mathbf{R}^N} M(x,y)g(w(y))dy \leq (\varepsilon\lambda + R) \sup_{\mathbf{R}^N} \mathcal{M}.
$$

Hence,

$$
\lambda - \mu_1 + \alpha \int_{\mathbf{R}^N} M(x,y)g(w(y))dy \geq \lambda - \mu_1 + \alpha(\varepsilon\lambda + R) \sup_{\mathbf{R}^N} \mathcal{M}.
$$

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Then, there exists $\mu_0 > \mu_1$ such that for $\lambda \geq \mu_0$ we have that

$$\lambda - \mu_1 + \alpha \int_{\mathbf{R}^N} M(x,y)g(w(y))dy > 0.$$

Now, it suffices to take $\rho$ small so that $\underline{u}$ is subsolution of (1).

Again, taking $v = \varphi_1$ in the definition of solution of (1), we get that $\lambda > \mu_1$. This concludes the proof. $\square$

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