Fixed Point Theory, 14(2013), No. 1, 53-58
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

DENSITY OF THE SET OF RENORMINGS IN c_0 WITHOUT ASYMPTOTICALLY ISOMETRIC COPIES OF c_0 AND FAILING TO HAVE THE FIXED POINT PROPERTY

A. BARRERA-CUEVAS AND M. A. JAPÓN

Dedicated to K. Goebel on the occasion of his retirement, and to L. Ciric, W.A. Kirk and I. A. Rus on the occasion of their 75th birthday

Departamento de Análisis Matemático
Facultad de Matemáticas, Universidad de Sevilla
Tarfia s/n, 41012, Sevilla, España
E-mail: japon@us.es, gauss_alf@hotmail.com

Abstract. It is proved that the family of all equivalent norms in c_0 without asymptotically isometric copies of c_0 and failing to have the fixed point property is dense in the set of all renormings of c_0.
Key Words and Phrases: Asymptotically isometric copies, fixed point theory, renorming theory, nonexpansive mappings.
2010 Mathematics Subject Classification: 46B03, 46B20, 47H09, 47H10.

1. Introduction

A Banach space is said to have the fixed point property (FPP) if every nonexpansive mapping defined from a closed convex bounded subset into itself has a fixed point. Recall that a mapping defined on a subset of a Banach space is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all x, y in the domain of T. It is well-known that the geometry of the Banach space plays a fundamental role to assure the FPP (see the monographs [8], [10] and the references therein). It is also classical that the sequence Banach spaces ℓ_1 and c_0 fail to have the FPP. In 1997, P. Dowling, C. Lennard and B. Turett [2] proved that a Banach space fails to have the FPP whenever this space contains a sequence which generates either an asymptotically isometric copy of c_0 or an asymptotically isometric copy of ℓ_1 (see also Chapter 9 in [10]). In the last years, asymptotically isometric copies of c_0 and ℓ_1 have become very helpful in the development of the fixed point theory for nonexpansive mappings. Actually, many classes of nonreflexive Banach spaces are known to fail the FPP because it is proved that they contain one of such copies (see for instance [3], [5], [6] or [13]).

It is noteworthy that the non-expansiveness condition strongly depends on the given norm in the Banach space X. If T is nonexpansive for a norm $\|\cdot\|$, T may fail

The authors are partially supported by DGES, Grant MTM-2012-34847-C02-01 and Junta de Andalucía, Grant FQM-127.

to be nonexpansive for another equivalent norm $|\cdot|$ in X. In fact, it was proved by P.K. Lin [12] that ℓ_1 can be renormed to have the fixed point property for nonexpansive mappings and it is still an open problem whether there exists an equivalent norm in c_0 satisfying such condition. On the other hand, not all equivalent norms in c_0 contain an asymptotically isometric copy of c_0 [2]. In fact, in [9] a dense family of equivalent norms in c_0 failing to have an asymptotically isometric copy of c_0 is given. In [1] T. Domínguez-Benavides proved that the set of all equivalent norms in c_0 which fail to have the FPP is also dense in the set of all renormings of c_0. The main object of this paper is to unify these two results. That is, we will prove that for every equivalent norm in c_0, we can obtain a renorming of c_0 as close as the first one and we like and which fails both, to have the FPP and to have an asymptotically isometric copy of c_0, which improves the density results given for the Banach space c_0 in [1] and [9].

2. Preliminaries

In this section we introduce the definitions, notation and the known results that we will use in the proof of the main theorem.

Definition 2.1. *A Banach space $(X, \|\cdot\|)$ is said to have an asymptotically isometric copy of c_0 if there are a null sequence (ε_n) in $(0,1)$ and a sequence (x_n) in X so that*

$$\sup_{n\in\mathbb{N}}(1-\varepsilon_n)|t_n| \leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \leq \sup_{n\in\mathbb{N}}|t_n|,$$

for all $(t_n) \in c_0$. In this case we say that X contains a c_0-a.i. copy.

Given $(X, \|\cdot\|)$ a Banach space we define by $\mathcal{P}(X)$ the set of all equivalent norms on X endowed with the metric [7]:

$$\rho(p, q) = \sup\{|p(x) - q(x)| : \|x\| \leq 1\}, \quad \text{if } p, q \in \mathcal{P}(X).$$

We consider the Banach space c_0 endowed with its usual norm given by $\|x\|_\infty = \sup_n |t_n|$ whenever $x = (t_n)_n \in c_0$.

Similarly to [9], we set:

$$\mathcal{P}_{FPP}(c_0) := \{p \in \mathcal{P}(c_0) : \ (c_0, p) \text{ has the FPP }\},$$

and

$$\mathcal{P}_0(c_0) := \{p \in \mathcal{P}(c_0) : (c_0, p) \text{ fails to have a } c_0\text{-a.i. copy }\}.$$

We know that $\mathcal{P}_{FPP}(c_0) \subset \mathcal{P}_0(c_0)$ [2] although it is an open problem whether $\mathcal{P}_{FPP}(c_0)$ could be nonempty. On the other hand, $\mathcal{P}_0(c_0) \setminus \mathcal{P}_{FPP}(c_0) \neq \emptyset$. Indeed, let us consider the following equivalent norm on c_0:

$$|x|_\infty := \sup_n |t_n| + \sum_{n=1}^{\infty} \frac{|t_n|}{2^n}$$

if $x = (t_n)_n \in c_0$. It can be proved that $(c_0, |\cdot|_\infty)$ fails to have both, an asymptotically isometric copy of c_0 [2], and the FPP [5].

We recall James' Lemma:

Lemma 2.2. [11] *Let $(X, \|\cdot\|)$ be a Banach space which contains a subspace isomorphic to c_0. Then for any positive number $\delta > 0$ there is a sequence (y_n) in X such that*

$$(1-\delta)\sup_n |t_n| < \left\| \sum_n t_n y_n \right\| \leq \sup_n |t_n|$$

for all sequences $(t_n) \in c_0$.

In [1] T. Domínguez-Benavides proved that $\mathcal{P}(c_0) \setminus \mathcal{P}_{FPP}(c_0)$ is dense in $\mathcal{P}(c_0)$. The main object of this manuscript is to prove that $\mathcal{P}_0(c_0) \setminus \mathcal{P}_{FPP}(c_0)$ is also dense in $\mathcal{P}(c_0)$, that is, that for every equivalent norm p in c_0 we are able to find another equivalent norm q as close as p as we like in the ρ metric such that q fails to have both, the FPP and to have an asymptotically isometric copy of c_0. Notice that $\mathcal{P}_0(c_0) \setminus \mathcal{P}_{FPP}(c_0)$ is a set of equivalent norms of c_0 which is strictly contained in $\mathcal{P}(c_0) \setminus \mathcal{P}_{FPP}(c_0)$.

3. Main result

Theorem 3.1. *The set $\mathcal{P}_0(c_0) \setminus \mathcal{P}_{FPP}(c_0)$ is dense in $\mathcal{P}(c_0)$.*

Proof. Let p be any norm in $\mathcal{P}(c_0)$ and $\epsilon > 0$. Take a constant b such that $p(x) \leq b\|x\|_\infty$ for all $x \in c_0$ and δ a positive number such that

$$b\,\frac{\delta}{1-\delta} < \frac{\epsilon}{2}$$

Using Lemma 2.2 we obtain a sequence (y_n) in c_0 such that

$$(1-\delta)\sup_n |t_n| \leq p\left(\sum_n t_n y_n\right) \leq \sup_n |t_n|$$

for all $(t_n) \in c_0$.

Moreover, following James' proof (see [11], Lemma 2.2) se can achieve that the sequence $(y_n)_n$ is finitely supported and that $supp(y_n) \cap supp(y_m) = \emptyset$ si $n \neq m$, where by $supp(y)$ we denote the support of the vector y. Moreover, from the proof of Lemma 2.2 in [11] we can also deduce that $\|y_n\|_\infty = \|y_m\|_\infty$ if $n, m \in \mathbb{N}$.

Let us define $Y := \overline{span\langle y_n\rangle}$ which is a subspace of c_0. We define in the subspace Y the equivalent norm $|y| := \sup_n |t_n|$ whenever $y = \sum_n t_n y_n$. According to the above inequalities we have

$$(1-\delta)|y| \leq p(y) \leq |y|$$

for all $y \in Y$.

Since Y is a closed subspace of c_0, applying Lemma 2.2 in [1], we can extend the norm $|\cdot|$ to the whole Banach space c_0 keeping the same equivalent constants, that is, there exists a norm q in c_0 such that $q(y) = |y|$ whenever $y \in Y$ and

$$(1-\delta)q(x) \leq p(x) \leq q(x),$$

for all $x \in c_0$.

Using now Theorem 4.1 in [9], for every $\lambda > 0$, the norm $q + \lambda|\cdot|_\infty$ fails to have an asymptotically isometric copy of c_0, where $|\cdot|_\infty$ was defined in Section 2. Let us prove that $(c_0, q + \lambda|\cdot|_\infty)$ also fails to have the FPP for every $\lambda > 0$:

For $x = \sum_n t_n e_n \in c_0$, we set

$$\eta(x) := \sum_n \frac{|t_n|}{2^n}.$$

With this notation $|x|_\infty = \|x\|_\infty + \eta(x)$ for all $x \in c_0$. Notice that $\eta(ax + by) = |a|\eta(x) + |b|\eta(y)$ if x, y are two vectors in c_0 with disjoint supports and $a, b \in \mathbb{R}$. Since the sequence (y_n) is bounded and disjointly supported, we can check that $\lim_n \eta(y_n) = 0$. Hence, without loss of generality, we can extract a subsequence$(y_{n_k})_k \subset (y_n)_n$ such that $\eta(y_{n_{(k+1)}}) \le \eta(y_{n_k})$ for all $k \in \mathbb{N}$.

Define the set

$$C := \{\sum_{k=1}^\infty t_k y_{n_k} : 0 \le t_k \le 1\},$$

which is a closed convex bounded subset of c_0 and it is contained in Y. Let us consider the mapping $T : C \to C$ given by

$$T(\sum_{k=1}^\infty t_k y_{n_k}) = y_1 + \sum_{k=1}^\infty t_n y_{n_{(k+1)}}$$

It is easy to check that T is fixed point free since $(y_{n_k})_k$ is a basic sequence. Let us prove that T is nonexpansive for the norm $q + \lambda|\cdot|_\infty$.

Firstly, T is nonexpansive for the q norm: set $x = \sum_{k=1}^\infty t_k y_{n_k}$, $y = \sum_{k=1}^\infty s_k y_{n_k}$ two vectors in C.

$$q(Tx - Ty) = |Tx - Ty| = \sup_k |t_k - s_k| = |x - y| = q(x - y).$$

On the other hand, T is also nonexpansive for the $|\cdot|_\infty$ norm. Indeed:

$$|Tx - Ty|_\infty = \left|\sum_{k=1}^\infty (t_k - s_k) y_{n_{(k+1)}}\right|_\infty$$

$$= \left\|\sum_{k=1}^\infty (t_k - s_k) y_{n_{(k+1)}}\right\|_\infty + \eta\left(\sum_{k=1}^\infty (t_k - s_k) y_{n_{(k+1)}}\right)$$

$$= \sup_k |t_k - s_k| \|y_{n_{(k+1)}}\|_\infty + \sum_{k=1}^\infty |t_k - s_k| \eta(y_{n_{(k+1)}})$$

$$\le \sup_k |t_k - s_k| \|y_{n_k}\|_\infty + \sum_{k=1}^\infty |t_k - s_k| \eta(y_{n_k})$$

$$= \left\|\sum_{k=1}^\infty (t_k - s_k) y_{n_k}\right\|_\infty + \eta\left(\sum_{k=1}^\infty (t_k - s_k) y_{n_k}\right)$$

$$= \left| \sum_{k=1}^{\infty} (t_k - s_k) y_{n_k} \right|_{\infty} = |x - y|_{\infty}.$$

The above shows that T is nonexpansive for the norm $q + \lambda|\cdot|_\infty$ for every $\lambda > 0$. Consider $0 < \lambda < \epsilon/4$ and take $x \in c_0$ with $\|x\|_\infty \leq 1$:

$$\begin{aligned} |p(x) - q(x) - \lambda|x|_\infty| &\leq |p(x) - q(x)| + \lambda|x|_\infty \\ &\leq \tfrac{1}{1-\delta} p(x) - p(x) + \lambda|x|_\infty \\ &\leq \tfrac{\delta}{1-\delta} b\|x\|_\infty + 2\lambda\|x\|_\infty \\ &< \epsilon \end{aligned}$$

which implies that $\rho(p, q + \lambda|\cdot|_\infty) < \epsilon$ and this finishes the proof.

Remark 3.2. A classical way to measure the distance between two isomorphic Banach spaces is to consider the Banach-Mazur distance defined by

$$d(X, Y) = \inf\{\|T\|\|T^{-1}\| : \ T : X \to Y \ \text{ isomorphism } \}.$$

From the proof of Theorem 3.1 and by using the identity operator between c_0 endowed with two equivalent norms, we can derive that for every renorming of c_0 we can find an isomorphic Banach space without the FPP, failing to have an asymptotically isometric copy of c_0 and whose Banach-Mazur distance is as close to one as we like.

References

[1] T. Domínguez Benavides, *Distortion and stability of the fixed point property for non-expansive mappings*, Nonlinear Anal., **75**(2012), 3229-3234.

[2] P.N. Dowling, C.J. Lennard, B. Turett, *Asymptotically perturbed norms of classical sequence spaces with applications to fixed point theory,* Proceedings of Workshop on Fixed Point Theory (Kazimierz Dolny, 1997), Ann. Univ. Mariae Curie-Sklodowska Sect. A, **51**(1997), no. 2, 67–98.

[3] P.N. Dowling, C.J. Lennard, *Every nonreflexive subspace of $L_1[0,1]$ fails the fixed point property*, Proc. Amer. Math. Soc., **125**(1997), 443-446.

[4] P.N. Dowling, C.J. Lennard, B. Turett, *Asymptotically Isometric Copies of c_0 in Banach Spaces*, J. Math. Anal. Appl., **219**(1998), 377-391.

[5] P.N. Dowling, P.K. Lin, B. Turett, *Failure of the FPP inside an asymptotically isometric-free copy of c_0,* Nonlinear Anal., **73**(2010), 1175-1179.

[6] P.N. Dowling, N. Randrianantoanina, *Spaces of compact operators on a Hilbert space with the fixed point property*, J. Funct. Anal., **168**(1)(1999), 111-120.

[7] M. Fabian, L. Zajicek, V. Zizler, *On residuality of the set of rotund norms on a Banach space*, Math. Ann., **258**(1982), 349-351.

[8] K. Goebel, W. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, 1990.

[9] C. Hernández-Linares, M. Japón, C. Lennard, *Renormings failing to have asymptotically isometric copies of ℓ_1 or c_0,* Nonlinear Anal. (to appear) http://dx.doi.org/10.1016/j.na.2012.09.006.

[10] W.A. Kirk, B. Sims (Ed.), *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, 2001.

[11] R.C. James, *Uniformly non-square Banach spaces,* Ann. of Math., **80**(1964), no. 2, 542-550.

[12] P.K. Lin, *There is an equivalent norm on ℓ_1 that has the fixed point property*, Nonlinear Anal., **68**(2008), no. 8, 2303-2308.

[13] H. Pfitzner, *A note on asymptotically isometric copies of ℓ_1 and c_0*, Proc. Amer. Math. Soc., **129**(5)(2001), 1367-1373.

Received: August 30, 2012; Accepted: October 31, 2012.