Witness Bar Visibility

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Abstract

Bar visibility graphs were introduced in the seventies as a model for some VLSI layout problems. They have been also studied since then by the graph drawing community, and recently several generalizations and restricted versions have been proposed.

We introduce a generalization, *witness-bar visibility graphs*, and we prove that this class encompasses all the bar-visibility variations considered so far. In addition, we show that many classes of graphs are contained in this family, including in particular all planar graphs, interval graphs, circular arc graphs and permutation graphs.

1 Introduction and preliminary definitions

Given a set S of disjoint horizontal line segments in the plane (called *bars* hereafter) we say that G is a *bar-visibility graph* if there is a bijection between S and the vertices of G, and an edge between two of these if and only if there is a vertical segment (called *line of sight*) between the corresponding bars that does not intersect any other bar. We also say that S is a *bar visibility representation* (or a *bar visibility drawing*) of G.

Bar visibility graphs were introduced by Garey, Johnson and So [13] as a modeling tool for digital circuit design (see also [16]). These representations are also a useful tool for displaying diagrams that convey visual information on relations among data, which is why many variations of these graphs have been considered by the graph drawing community [6, 7, 8, 9, 12, 14, 15].

We need some definitions before we can pose precisely our problem; we use standard terminology as in [5]. We call *v-segment* any vertical segment. We call ε*-segment* any axis aligned rectangle having width $\varepsilon > 0$ (intuitively, a *thick* vertical segment). Let s and t be two horizontal bars. We say that a v-segment connects s and t if its endpoints are in s and t. We say that an ε-segment connects s and t if its horizontal sides are contained in s and t.

Let S be a set of non-overlapping horizontal segments (bars). Two bars $s, t \in S$ are *visible* if, and only if, there is a v-segment connecting s and t intersecting no other segment in S, and we say that s and t are ε*-visible* if, and only if there is an ε–segment connecting s and t intersecting no other segment in S.

With the preceding definition, bar visibility graphs as defined in the first paragraph of this section take as nodes a set of disjoint bars, and there is an edge between two nodes if and only if the corresponding bars are visible (this is also called a *strong visibility representation* of the graph [17]). If instead of visibility we require ε-visibility, then we get bar ε-visibility graphs or, equivalently, an ε*-visibility representation* of the graph. The latter have been characterized as those graphs that admit a planar embedding with all cutpoints on the exterior face [17, 18].

A graph G is a *weak bar visibility graph* if its nodes can be put in bijection with a set of disjoint bars and the nodes corresponding to every edge in G are ε-visible (note that not every ε-visibility need be an edge). This family of graphs is exactly the class of all planar graphs [10].

Finally, we say that G is a *bar* k*-visibility graph* if there is a bijection between a set of bars S and the vertices of G, and an edge between two of these if and only if there is a v-segment joining the corresponding bars that intersects at most k other bars. This generalization has been introduced in recent years [8, 12].

In this paper we introduce a stronger generalization, *witness-bar visibility graphs*, and we prove that this representation approach encompasses all the bar-visibility variations considered so far. In addition, we show that many classes of graphs are contained in this family, including in particular all planar graphs, interval graphs, circular arc graphs and permutation graphs.

For the definition of witness-bar visibility graphs we consider, in addition to the set S of bars that are in correspondence one-to-one with the vertices of the

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graph being constructed, a set of green bars that "favor" visibility, and a set of red bars that "obstruct" visibility. Green bars act as *positive witnesses* while red bars correspond to *negative witnesses*. The bars from S neither favor nor obstruct visibilities.

For the ease of description it is useful to consider also purple bars that obstruct visibility in a slightly different way than red bars.

Definition 1 Let S, S_G, S_P and S_R be four sets of horizontal segments (*bars*, *green-bars*, *purple-bars*, and *red-bars*, respectively) such that any two elements in $S \cup S_G \cup S_R$ are disjoint. We define:

1. The *green-bar visibility graph* of S with respect to S_G has one vertex for every element in S, and two bars $s, t \in S$ are adjacent if and only if there is an ε-segment connecting s and t that *crosses at least one green bar.*

2. The *purple-bar visibility graph* of S with respect to S_P has one vertex for every element in S, and two bars $s, t \in S$ are adjacent if and only if there is an ε-segment connecting s and t that *does not cross any purple bar.*

3. The *witness-bar visibility graph* of S with respect to S_G and S_R has one vertex for every element in S, and two bars $s, t \in S$ are adjacent if and only if there is an ε-segment connecting s and t that *crosses strictly more green bars than red bars.*

The class of green, purple and witness-bar visibility graph are denoted, respectively, by $\mathcal{GBG}$, $\mathcal{PBG}$ and $\mathcal{WBG}$.

An illustration of the three types of graphs is shown in Figure 1 (on a black and white printer, node-bars appear as thin lines, red bars as thick dark lines, purple lines as thick lines colored light grey, and the green lines are seen as thick striped lines).

This work is devoted to the study of the classes of graphs that can be represented via green, purple or bar-visibility graphs and its properties. We start by considering the classes $\mathcal{GBG}$ and $\mathcal{PBG}$, which will be proved to be subclasses of $\mathcal{WBG}$. Then we will enumerate classes of graphs that are contained in $\mathcal{WBG}$, as well as properties of this class related to planarity.

The terminology *witness-bar visibility graphs* is inspired by the concept of *witness proximity graphs*, which focuses on deciding neighborliness relations among points in a finite set according to the presence of some positive and/or negative witness points, a topic that has been studied in recent years [1, 2, 3, 4, 11].

2 The subclasses $\mathcal{GBG}$ and $\mathcal{PBG}$

In this section we study the classes $\mathcal{GBG}$ and $\mathcal{PBG}$ and its relationships with other graph classes. The

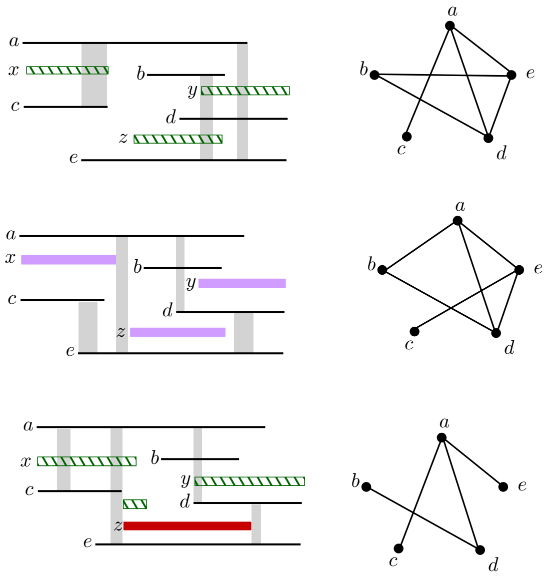

Figure 1: Examples of graphs in the families $\mathcal{GBG}$ (top), $\mathcal{PBG}$ (center) and $\mathcal{WBG}$ (bottom).

interest of this is made clear by our first result:

Lemma 2 *The class of graphs $\mathcal{WBG}$ contains strictly the classes $\mathcal{GBG}$ and $\mathcal{PBG}$.*

An *interval graph* is the intersection graph of a set of (closed) intervals on the real line.

Theorem 3 *Let G be a graph. If G is an interval graph, then $G \in \mathcal{GBG}$. The reverse is in general false.*

The graph class inclusion in Theorem 3 is strict, as one can show that C_4, which is not an interval graph, is in $\mathcal{GBG}$. However, graphs with induced cycles of length greater than 5 are not in $\mathcal{GBG}$.

Proposition 4 *If the girth of a graph in $\mathcal{GBG}$ is finite, then it is at most four.*

As a consequence of the previous result, it follows that the green-bar visibility graph class does not contain any of the bar-visibility classes described in the introduction of this paper, because C_n can be represented as weak/ε/strong bar visibilty graph for every $n \geq 3$ [17].

Note that even although one may think that the classes $\mathcal{GBG}$ and $\mathcal{PBG}$ are related by complementation, possibly by switching purple and green bar coloring, but it is not the case. For example the union of two disjoint triangular cycles is in $\mathcal{GBG}$, as seen in the preceding section, but its complement is $K_{3,3}$, which is not in $\mathcal{PBG}$, a fact that we will see below.

On the positive side, let us see that interval graphs admit a purple-bar visibility representation and prove a useful lemma.

Theorem 5 *If G is an interval graph, then $G \in \mathcal{PBG}$.*

Lemma 6 *Let G be a triangle-free graph. If $G \in \mathcal{PBG}$ then G is a planar graph.*

Proof. The idea is given in Figure 2. □

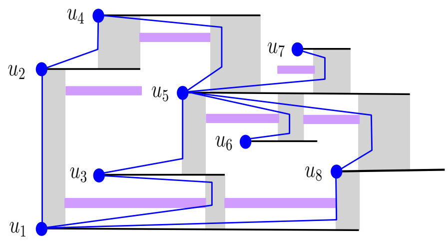

Figure 2: A purple-bar visibility representation of a triangle-free graph and the corresponding construction of its planar embedding.

Nevertheless the previous result cannot be extended to a characterization of the class $\mathcal{PBG}$:

Theorem 7 *1. $K_{3,3} \notin \mathcal{PBG}$.*

2. $K_n \in \mathcal{PBG}$, $\forall n$.

3. To admit a purple-bar visibility representation is not inherited by subgraphs.

Proposition 8 *There are nonplanar graphs with triangles that do not admit a purple-bar visibility representation.*

An example of these graphs is given in Figure 3.

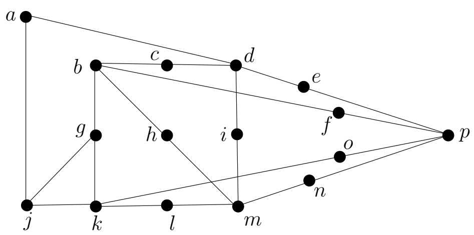

Figure 3: A nonplanar graph G with a triangle ($\triangle gjk$), which does not admit a purple-bar visibility representation.

The class $\mathcal{PBG}$ generalizes the *classical* bar-visibility representations:

Theorem 9 *Every graph G that can be represented as strong/ε/weak bar visibility graph admits as well a purple-bar visibility representation.*

Corollary 10 *Every graph G that can be represented as strong/ε/weak bar visibility graph admits as well a witness-bar visibility representation.*

3 The class $\mathcal{WBG}$ of witness-bar visibility graphs

Witness bar visibility also generalizes k-bar visibility:

Theorem 11 *Every graph G that can be represented as a bar k-visibility graph admits as well a witness-bar visibility representation.*

The idea of the proof is given in Figure 4.

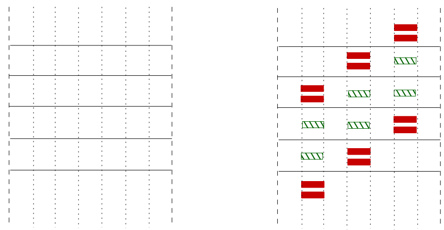

Figure 4: Assume we have to deal with bar 1-visibility, and consider a stack of 5 bars in a strip (left). We subdivide the strip into 7 slabs (right). In the second bar we mimic 1-visibility using $\mathcal{WBG}$-visibility for the 3 lowest bars. In the fourth bar we do the same for the 3 intermediate bars, and in the sixth slab for the 3 highest bars.

Theorem 12 *The circular arc graphs and permutation graphs are contained in $\mathcal{WBG}$*

Given a graph G, let $\widetilde{G}$ be the graph resulting from subdividing once every edge in G.

Lemma 13 *Let G be a graph. If $\widetilde{G} \in \mathcal{WBG}$ then G is a planar graph.*

Lemma 14 *$\widetilde{K}_{3,3} \notin \mathcal{WBG}$ and $\widetilde{K}^c_{3,3} \in \mathcal{WBG}$.*

As a consequence of the preceding lemma we immediately obtain the following result:

Theorem 15 *The class of the graphs that admit a witness-bar visibility representation is not closed under complementation.*

We conclude this section with another result on the class $\mathcal{WBG}$, that discards the possibility of characterizing the class by forbidden minors:

Theorem 16 *The property of admitting a witness-bar visibility representation is not inherited by subgraphs.*

Proof. We know that $K_6 \in \mathcal{WBG}$ from Theorem 7 and Lemma 2. On the other hand $\widetilde{K}_{3,3}$ is a subdivision of a subgraph of K_6, but we know from Lemma 14 that $\widetilde{K}_{3,3}$ is not in $\mathcal{WBG}$. This settles the claim. □

4 Concluding remarks

Let us summarize the properties we have proved for the class $\mathcal{WBG}$ of witness-bar visibility graphs:

- Every graph G that can be represented as strong/ε/weak bar visibility graph admits as well a witness-bar visibility representation.
- Every graph G that can be represented as a bar k-visibility graph admits as well a witness-bar visibility representation.
- The class of interval graphs is contained in the class $\mathcal{WBG}$.
- If G is a circular arc graph or a permutation graph then $G \in \mathcal{WBG}$.
- The class of the graphs that admit a witness-bar visibility representation is not closed under complementation.
- The property of admitting a witness-bar visibility representation is not inherited by subgraphs, which discards the possibility of characterizing the graph class $\mathcal{WBG}$ by forbidden minors.

We conclude that the graph class $\mathcal{WBG}$ is very rich and encompasses many other classes. However, to obtain a characterization or a recognition algorithm appear to be quite challenging problems.

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