Stabbing simplices of point sets with k-flats

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Abstract

Let S be a set of n points in $\mathbb{R}^d$ in general position. A set $\mathcal{H}$ of k-flats is called an m_k-stabber of S if the relative interior of any m-simplex with vertices in S is intersected by at least one element of $\mathcal{H}$. In this paper we give lower and upper bounds on the size of minimum m_k-stabbers of point sets in $\mathbb{R}^d$. We study mainly m_k-stabbers in the plane and in $\mathbb{R}^3$.

Introduction

A set $\{x_0, \dots, x_k\} \in \mathbb{R}^d$ of $k+1$ points is called *linearly independent* if there is no linear combination $\lambda_0 x_0 + \cdots + \lambda_k x_k = \mathbf{0}$ in which at least one $\lambda_i \neq 0$, $k \leq d$. A set of points $\{x_0, \dots, x_k\} \in \mathbb{R}^d$ is called *affinely independent* if the set $\{x_1 - x_0, \dots, x_k - x_0\}$ is linearly independent, $k \geq 1$. A set consisting of a single point will also be considered as affinely independent. An affine combination of a set of $k+1$ affinely independent points in $\mathbb{R}^d$ is a linear combination $\lambda_0 x_0 + \cdots + \lambda_k x_k$ of $x_0, \dots, x_k$ such that $\lambda_0 + \cdots + \lambda_k = 1$. The set of affine combinations of $k+1$ affinely independent points of $\mathbb{R}^d$ is called a k-flat. In particular, a $(d-1)$-flat of $\mathbb{R}^d$ will be called a *hyperplane*. Following our intuition, a 1-flat is a line, and a 2-flat is a plane. A k-flat is isomorphic to the k-dimensional space $\mathbb{R}^k$. A point set S in $\mathbb{R}^d$ is in general position if any subset of S with at most $d+1$ elements is affinely independent.

An *m-simplex* of $\mathbb{R}^d$ is the convex hull of a set of $m+1$ affinely independent points in $\mathbb{R}^d$, $m \leq d$. For example, in $\mathbb{R}^2$ a 0-simplex is a point, a 1-simplex is a segment and a 2-simplex is a triangle.

Given a k-flat h and an m-simplex $\mathcal{P}$ of $\mathbb{R}^d$, we say that h stabs $\mathcal{P}$ if h intersects the relative interior of $\mathcal{P}$. A set of k-flats $\mathcal{H}$ is called an *m_k-stabber* of a set of points S if every m-simplex induced by the elements of S is stabbed by at least one element of $\mathcal{H}$.

For example, in the plane a set of points Q (respectively lines) is a 3_0-stabber (respectively a 3_1-stabber) of point set S if any triangle with vertices in S contains an element of Q in its interior (respectively is intersected by a line in Q).

In this paper we study the following problem: Given two integers $k < m$, $k < d$, and $m \leq d+1$, and a set S of n points in $\mathbb{R}^d$ in general position, how many k-flats are needed to stab all the m-simplices generated by the elements of S?

In this paper we focus mainly on stabbing all the r-simplices of point sets on the plane or in $\mathbb{R}^3$, and give some results for higher dimensions.

The problem of finding sets of points that stab all the triangles of a point set has been studied by Katchalsky and Meir [4], and independently by Czyzowicz, Kranakis, and Urrutia [2]. They prove that for any point set S in the plane in general position with n elements, such that its convex hull has c elements, the set of triangles of S can be stabbed with exactly $2n - c - 2$ points, and that such a bound is tight, as any triangulation of S has $2n-c-2$ triangles. Stabbers for other convex holes, such as quadrilaterals and pentagons, have also been studied [1].

Given a point set S in $\mathbb{R}^d$, we define $f_k^m(S)$ as the size of the smallest m_k-stabber of S. We define $f_k^m(n)$ as the largest value that a m_k-stabber can have over all the point sets S of $\mathbb{R}^d$ with n elements. With this terminology, the preceding result in the plane translates to $f_0^2(S) = 2n - c - 2$.

In this paper we show upper and lower bounds for $f_k^m(n)$, for point sets on the plane and $\mathbb{R}^3$ that are tight up to a constant.

1 Well-separated sets and generalized ham-sandwich cuts

A family of convex sets $C_0, \dots, C_k$ of $\mathbb{R}^d$, $k \leq d$, is *well separated* if for any choice of $x_i \in C_i$, the set of points $\{x_0, \dots, x_k\}$ is affinely independent. A family $P_0, \dots, P_k$ of finite point sets in $\mathbb{R}^d$ is *well separated* if their convex hulls $Conv(P_0), \dots, Conv(P_k)$ are well separated, $k \leq d$.

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Notice that in particular, this implies that for any pair of different indexes $i, j \leq k$, the convex hulls of P_i and P_j do not intersect.

Let $P_0, \ldots, P_k$ be a family of pairwise disjoint finite point sets in $\mathbb{R}^d$, $k \leq d$. Given positive integers $a_i \leq |P_i|$, an $(a_0, \ldots, a_k)$*-cut* is a hyperplane h for which $h \cap P_i \neq \emptyset$ and $|h^+ \cap P_i| = a_i$, $0 \leq i \leq d$, where h^+ is the half-space bounded below by h.

The following result by Steiger and Zao will be useful to us:

Theorem 1 ([8]) *Let $P_0, \ldots, P_d$ be well-separated point sets in $\mathbb{R}^d$ such that $P_0 \cup \cdots \cup P_d$ is in general position, and let $a_0, \ldots, a_d$ be positive integers such that $a_i \leq |P_i|$. Then there is a unique $(a_0, \ldots, a_d)$-cut of $P_0, \ldots, P_d$.*

Finally, we recall the almost folklore result about ham-sandwich cuts:

Theorem 2 (Ham-sandwich theorem, [9]) *Every d finite sets in $\mathbb{R}^d$ can be simultaneously bisected by a hyperplane. A hyperplane h bisects a finite set P if each of the open half-spaces defined by h contains at most $\lfloor \frac{|P|}{2} \rfloor$ points of P.*

Now we have all the tools that we will use to give an algorithm for constructing a worst-case optimal m_k-stabber for point sets in the plane and in the 3-dimensional space.

2 Stabbers in the plane

We start by studying the following problem: How many lines are necessary to stab the set of line triangles (segments) determined by triples (pairs) of elements of a point set S in the plane? In our previous terminology, determine lower and upper bounds for $f_1^1(n)$ and $f_1^2(n)$ for point sets on the plane. We first prove:

Theorem 3 *For any set S of n points in the plane $r \leq f_1^2(S) \leq \lceil \frac{n}{4} \rceil$, where r is the smallest number such that $\frac{n}{2} \leq 2 + 2 + 3 + \cdots + r$. Our bounds are tight.*

Proof. The lower bound follows from the fact that r lines, no three of which intersect at a point, divide the plane into $2 + 2 + 3 + \cdots + r$ convex regions, and if a set of r lines stabs all of the triangles with vertices in S, then S has at most two elements in each of these regions.

We now prove that $f_1^2(S) \leq \lceil \frac{n}{4} \rceil$. To this end, we now show how to obtain a set $\mathcal{H}$ with $\lceil \frac{n}{4} \rceil$ straight-lines that is a 2_1-stabber of S. To prove our result, it is sufficient to find a set $\mathcal{H}$ with $\lceil \frac{n}{4} \rceil$ lines such that any cell of the arrangement generated by $\mathcal{H}$ contains at most two elements of S.

First, we put in $\mathcal{H}$ one straight line ℓ that separates S into two subsets of size $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ respectively. Refer to these sets as S_1 and S_2. Clearly, these sets are well separated, and by Theorem 1 we can find a $(2, 2)$-cut for S_1 and S_2. This cut is a line ℓ_1 that leaves two elements x_1, x_2 of S_1 and two elements x_1', x_2' of S_2 above it. These points are separated from the remaining points of S by ℓ and ℓ_1, if ℓ_1 contains some element of S_1 or S_2 move it slightly. Thus any triangle containing at least one vertex in $\{x_1, x_2, x_1', x_2'\}$ is intersected by ℓ or ℓ_1.

We repeat this recursively on $S_1 \setminus \{x_1, x_2\}$ and $S_2 \setminus \{x_1', x_2'\}$, $\lceil \frac{n}{4} \rceil - 2$ times, obtaining a set of $\lceil \frac{n}{4} \rceil$ lines (including ℓ and ℓ_1) that is a 2_1-stabber of S.

To show that our bound is tight, let S be a set of n points in the plane in convex position labeled $p_0, p_1, \ldots, p_{n-1}$ in counterclockwise order, with n even. Consider the set of $\frac{n}{2}$ triangles $p_i p_{i+1} p_{i+2}$, i even, and $i \leq \frac{n}{2}$ for every even value of i. Observe that any two of these triangles intersect at exactly one vertex, for $n \geq 6$.

We claim that any straight line stabs at most 2 of these triangles. This is true since every such triangle has exactly two edges of the convex hull of P_n; then any straight line stabbing one such triangle stabs at least one edge of the convex hull. Any straight line stabs at most two edges of the convex hull, so then it can stab at most two of such triangles. Thus to stab all of those triangles we need at least one straight line for every two triangles. Observe next that if the elements of S are in convex position, then to stab the edges of the convex hull of S we need at last $\lceil \frac{n}{4} \rceil$ lines. Thus our upper bound is tight. □

The key property of $\mathcal{H}$ in the proof of this result is the fact that any cell of the arrangement induced by $\mathcal{H}$ contains at most two elements of S. Observe that if instead of finding a $(2, 2)$-cut for S_1 and S_2, we use $(m-1, m-1)$-cuts, then we obtain a set of lines such that any cell of the arrangement generated by them contains at most $m - 1$ elements of S. An m island of S is a subset S' of S with exactly m elements and such that the convex hull of S' contains no element of $S \setminus S'$ in its convex hull. The next result follows easily from the proof of Theorem 3.

Theorem 4 *For any set S of n points in the plane, $\lceil \frac{n}{2(m-1)} \rceil$ lines are always sufficient and sometimes necessary to stab all of the m islands of S.*

In particular, for $m = 2$, this proves that $f_1^1(S) \leq \lceil \frac{n}{2} \rceil$; that is, to stab the segments of S we need at most $\lceil \frac{n}{2} \rceil$ lines. It is easy to see that the bound $\lceil \frac{n}{2(m-1)} \rceil$ in Theorem 4 is achieved for point sets in convex position.

We close this section by observing that the problem of calculating $f_1^1(S)$ is equivalent to calculating the minimum number of lines such that in each face of the arrangement defined by these lines, there is at most one element of S. This problem is known as the shattering problem and its decision version is known to be NP-complete [5].

3 Stabbers in the space

We consider now the problem of stabbing the tetrahedra induced by an n point set S in $\mathbb{R}^3$ using sets of planes. Our approach is similar to that of the previous section.

For the lower bound we will use the following set of n points in the plane. Let $f\colon \mathbb{R} \to \mathbb{R}^3$, $f(a) = (a, a^2, a^3)$. $f(\mathbb{R})$ is known as the *momentum curve* in $\mathbb{R}^3$. Clearly, any plane in $\mathbb{R}^3$ intersects $f(\mathbb{R})$ in at most 3 points. Let $C_n = \{p_0, \dots, p_{n-1}\}$ with $p_i = f(i)$. Since any plane intersects $f(\mathbb{R})$ in at most 3 points, C_n is in general position. Given any segment p_ip_j, let $f([i,j])$ be the curve containing the points $f(x)$, with $i \le x \le j$. We call $f([i,j])$ the *shadow* of the segment p_ip_j in the momentum curve. The following observation will be useful.

Observation 1 *Let $p_i, p_j \in C_n$. Then any plane intersecting the relative interior of the segment p_ip_j intersects its shadow.*

Consider the set $\{t_0, \dots, t_{\lfloor \frac{n-1}{3} \rfloor}\}$ of tetrahedra, such that the vertex set of t_i is $\{p_{3i}, p_{3i+1}, p_{3i+2}, p_{3i+3}\}$. Observe that the interiors of these tetrahedra are pairwise disjoint, and that t_i and t_{i+1} share exactly one vertex, namely p_{3i+3}.

Observation 2 *If a plane intersects the interior of a tetrahedron, then it intersects the relative interior of one of its edges.*

We define the *shadow* of t_i as the shadow of the segment $p_{3i}p_{3i+3}$ which is in fact the union of the shadows of the edges of t_i. Then the relative interiors of the shadows of $\{t_0, \dots, t_{\lfloor \frac{n-1}{3} \rfloor}\}$ are pairwise disjoint. The next lemma follows now directly from Observations 1 and 2:

Lemma 5 *If a plane stabs t_i, then it intersects the relative interior of its shadow.*

Since any plane intersects the momentum curve at most three times, it follows that any plane intersects at most three elements of $\{t_0, \dots, t_{\lfloor \frac{n-1}{3} \rfloor}\}$.

By Observation 1, any plane stabbing t_i must intersect its shadow; thus any any plane stabs at most three elements of $\{t_0, \dots, t_{\lfloor \frac{n-1}{3} \rfloor}\}$.

Thus we have:

Lemma 6 $f_2^3(n) \ge \lceil \frac{n-1}{9} \rceil$.

We will now prove that $f_2^3(n) \le \lceil \frac{n}{9} \rceil + 2$. To this end, we are going to give an algorithm that splits any point set S into a a family of well-separated families of subsets of S.

First find two parallel planes π_1 and π_2 such that they split S into three subsets S_1, S_2 and S_3 such that $||S_i| - |S_j|| \le 1$, $i \ne j \le 3$. That is, each S_i contains $\lfloor \frac{n}{3} \rfloor$ or $\lceil \frac{n}{3} \rceil$ elements.

We now apply the ham-sandwich theorem in $\mathbb{R}^3$ to find a plane π_3 that simultaneously bisects S_1, S_2 and S_3. Let S_i^+ be the subset of S_i that lies above π_3 and S_i^- the one that lies below π_3, with $i = 1, 2, 3$. This gives us two disjoint well-separated families $\mathcal{F} = \{S_1^+, S_2^-, S_3^+\}$ and $\mathcal{F}' = \{S_1^-, S_2^+, S_3^-\}$. Similarly to the algorithm in the previous section, iteratively find $(3,3,3)$-cuts for $\mathcal{F}$, removing the nine points above the cut until each set in $\mathcal{F}$ contains at most three points, and adding the cut planes to the 3_2-stabber. Then repeat the same process for $\mathcal{F}'$.

It is now easy to see that the set of planes we obtain has at most $\lceil \frac{n}{9} \rceil + 2$ elements, and that they form a stabber of all the tetrahedra with vertices in S. Thus we have proved:

Theorem 7 $\lceil \frac{n-1}{9} \rceil \le f_2^3(n) \le \lceil \frac{n}{9} \rceil + 2$.

Observe that using similar arguments we can prove that $f_2^2(n) \approx n/6$ (stabbing triangles with planes) and $f_2^1(n) \approx n/3$ (stabbing segments with planes).

3.1 Stabbing 2-simplices of point sets with lines

For the problem of stabbing triangles with lines, it is easy to see that in some instances, we need at least $\frac{2n-4}{2}$ lines. Indeed, consider any set of points S in $\mathbb{R}^3$ with n points in convex position. Observe that the convex hull of S contains exactly $2n-4$ triangles, and that any line intersects the interior of at most two of these triangles. Thus we have proved that in $\mathbb{R}^3$, $f_1^2(n) \ge n - 2$.

We now prove that $2n-5$ lines are always sufficient to stab all of the triangles of S. To prove this, first project the elements of S into the real plane, thus obtaining a point set S' in $\mathbb{R}^2$. It is known that if the convex hull of S' has c elements, it is always possible to find a set of points Q with at most $2n - c - 2$ points that stabs all of the triangles of S'; see [2, 4]. Each point $q \in Q$ generates a vertical line ℓ_q passing through q. Since each triangle T of S projects to a triangle T' of S', there is a line ℓ_q, $q \in Q$ that stabs T' and thus T. Since in the worst case, the convex hull of S' has three points, we have proved:

Theorem 8 *In $\mathbb{R}^3$, $f_1^2(n) \le 2n - 5$.*

3.2 Stabbing the 3-simplices of a point set in $\mathbb{R}^3$ with points

We now turn our attention to the problem of stabbing the set of tetrahedra generated by a point set S in $\mathbb{R}^3$ using sets of points. We will assume that no two elements of S lie on a line vertical to the real plane.

Consider again the momentum curve $f(a) = (a, a^2, a^3)$, and the set of points $C_n = \{p_0, \ldots, p_n\}$ as defined above. Let $0 \leq i, j \leq n$ such that $i + 1 < j$. Let $T_{i,j}$ be the tetrahedra with vertex set $\{p_i, p_{i+1}, p_j, p_{j+1}\}$. It is well known [3] that the set of tetrahedra $T_{i,j}$, with i and j as above, have disjoint interiors, and form a tetrahedralization of the convex hull of $C_n = \{p_0, \ldots, p_n\}$. Since any set of points that stabs the tetrahedra of S contains at least one point in each of these $T_{i,j}$, it follows that $f_0^3(n+1)$ is at least $\binom{n-1}{2}$.

We now prove an upper bound on $f_0^3(n+1)$. We proceed as follows: For every two elements $x_i, x_j \in S$, place a point $p_{i,j}$ slightly *above* the midpoint of the line segment joining x_i to x_j. For every $x_i \in S$, place a point p_i slightly *below* x_i.

We now prove that the set of points containing all the $p'_{i,j}s$ and the p_i's stabs all the tetrahedra of S.

Consider any four points x_i, x_j, x_k, x_ℓ of S. Let H be the convex hull of $\{x_i, x_j, x_k, x_\ell\}$. Two cases arise. In the first case, H projects to a convex quadrilateral in the plane. In this case, one edge of the convex hull of $\{x_i, x_j, x_k, x_\ell\}$ is not visible from below. Suppose w.l.o.g. that it is the edge joining x_k to x_ℓ. Then the point $p_{i,j}$ stabs H.

Suppose then that H projects to a triangle T on the plane. Then one vertex of H, say x_i, belongs to the interior of T. Two sub-cases arise. In the first sub-case, when we see H from below, x_i is not visible. In this case, the point p_i stabs the convex hull of $\{x_i, x_j, x_k, x_\ell\}$. In the remaining sub-case, $p_{i,j}$ stabs H.

Thus we have that we can always choose $\binom{n+1}{2} + n$ points that stab the tetrahedra of S. It is not hard to see that we can reduce the above number by 9 by observing that if the line segment joining x_i to x_j is not visible from below, and this line segment belongs to the convex hull of S; then the point $p_{i,j}$ is redundant. Also if a point x_i belongs to the convex hull of S and is visible from below, then p_i is redundant.

Thus we have proved:

Theorem 9 $\binom{n-1}{2} \leq f_0^3(n+1) \leq \binom{n+1}{2} + n - 9.$

4 Some thoughts on higher dimensions

Some of the results in this paper; e.g. Lemma 5 and Lemma 6, generalise easily to higher dimensions, yielding the following result:

Lemma 10 *In* $\mathbb{R}^d$, $\lceil \frac{n-1}{d^2} \rceil \leq f_{d-1}^d(n)$.

And in general:

Lemma 11 *In* $\mathbb{R}^d$, $\lceil \frac{n-1}{dm} \rceil \leq f_{d-1}^m(n)$.

To finish this paper, we conjecture:

Conjecture 1 $f_m^d(n) \approx \frac{n}{m \cdot d}$.

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