Abstract Voronoi diagrams

Rolf Klein*

Universität Bonn, Institut für Informatik I

Abstract

Abstract Voronoi diagrams are a unifying framework that covers many types of concrete Voronoi diagrams. This talk reports on the state of the art, including recent progress.

Introduction

Concrete Voronoi diagrams [1] are mostly defined in terms of sites and distance, and both concepts can vary greatly. Abstract Voronoi diagrams [6] are built on what most concrete diagrams have in common: a system of simple bisecting curves $J(p, q) = J(q, p)$, where p, q are just indices from a set S of n elements. Each curve $J(p, q)$ divides the plane into two domains, $D(p, q)$ and $D(q, p)$. The *abstract Voronoi region* of p with respect to S is defined by

$$\mathrm{VR}(p, S) := \bigcap_{q \in S \setminus \{p\}} D(p, q)$$

and the *abstract Voronoi diagram* of S is just the plane minus all Voronoi regions.

An interesting question is what properties to require of the curves $J(p, q)$. They should be as weak as possible for generality, but strong enough to ensure that useful "Voronoi" structures result from the above definitions. It turns out [7] that the following are sufficient.

(A1) Each curve $J(p, q)$, where $p \neq q$, is unbounded. After stereographic projection to the sphere, it can be completed to a closed Jordan curve through the north pole.

For any three indices p, q, r in S, and $S' := \{p, q, r\}$,

(A2) each Voronoi region $\mathrm{VR}(p, S')$ is path-wise connected,

(A3) each point of the plane belongs to the closure of a Voronoi region $\mathrm{VR}(p, S')$.

*Email: rolf.klein@uni-bonn.de. This work was supported by the European Science Foundation (ESF) in the EUROCORES collaborative research project EuroGIGA/VORONOI.

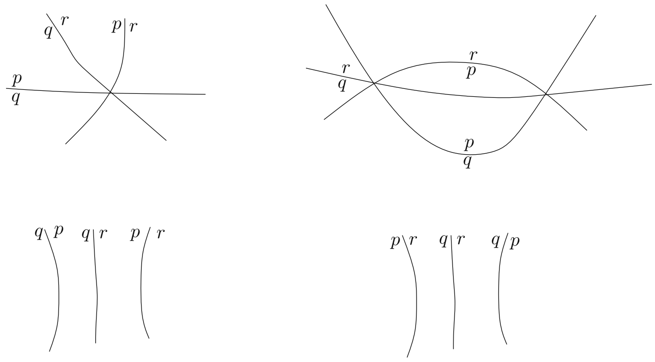

Figure 1: Admissible curve systems

Informally, if the bisecting curves are unbounded and behave decently, and if any triplet $J(p, q), J(p, r), J(q, r)$ is situated as shown in Figure 1, the AVD theory applies.

1 Results

This means that structural results and efficient algorithms become available without further effort [7].

Theorem 1 *$V(S)$ is a planar graph of complexity $O(n)$. It can be constructed in an expected number of $O(n \log n)$ many steps.*

If we replace Axiom A2 by the more general requirement

(A2') Each Voronoi region $\mathrm{VR}(p, S')$ has at most s connected components

(and assume that any two curves intersect only finitely often), the above result can be generalized as follows [3].

Theorem 2 *Abstract Voronoi diagrams with disconnected regions can be computed in an expected number of*

$$O\left(s^2 n \sum_{j=3}^{n} \frac{m_j}{j}\right)$$

steps, where m_j denotes the average number of faces per region in all AVDs of j sites from S.

One can extend the definition of abstract Voronoi diagrams to orders $k > 1$ by defining

$$\mathrm{VR}^k(P,S) := \bigcap_{p \in P,\ q \in S \setminus P} D(p,q).$$

For order $k = n-1$, the resulting AVDs are trees [9] of linear size. In the general case the following complexity result holds. Here we assume that all curves are in general position, and that the standard Voronoi-regions are non-empty.

Theorem 3 *The abstract order-k Voronoi diagram $V^k(S)$ has at most $2k(n-k)$ many faces. This bound can be achieved.*

2 Conclusion

Open is the case of closed bisecting curves.

References

[1] F. Aurenhammer, R. Klein, and D.-T. Lee, *Voronoi Diagrams and Delaunay Triangulations*, World Scientific Publishing Company, to appear August 2013.

[2] C. Bohler, P. Cheilaris, R. Klein, C.H. Liu, E. Papadopoulou, and M. Zavershynskyi, On the complexity of higher order abstract Voronoi diagrams, to be presented at ICALP' 13.

[3] C. Bohler and R. Klein, Abstract Voronoi diagrams with disconnected regions, Bonn 2013, manuscript.

[4] C. Bohler and R. Klein, Point sites with individual distance functions, Bonn 2012, manuscript.

[5] A. G. Corbalan, M. Mazon, and T. Recio, Geometry of bisectors for strictly convex distance functions, *International Journal of Computational Geometry and Applications* **6(1)** (1996), 45–58.

[6] R. Klein, *Concrete and abstract Voronoi diagrams*, Lecture Notes in Computer Science, 400, Springer-Verlag, 1987.

[7] R. Klein, E. Langetepe, and Z. Nilforoushan, Abstract Voronoi diagrams revisited, *Computational Geometry: Theory and Applications* **42(9)** (2009), 885–902.

[8] R. Klein, K. Mehlhorn, and S. Meiser, Randomized incremental construction of abstract Voronoi diagrams, *Computational Geometry: Theory and Applications* **3** (1993), 157–184.

[9] K. Mehlhorn, S. Meiser, and R. Rasch, Furthest site abstract Voronoi diagrams, *International Journal of Computational Geometry and Applications* **11(6)** (2001), 583–616.