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Extensions of Rubio de Francia's extrapolation theorem

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Abstract

One of the main results in modern harmonic analysis is the extrapolation theorem of J.L. Rubio de Francia for A_p weights. In this paper we discuss some recent extensions of this result. We present a new approach that, among other things, allows us to obtain estimates in rearrangement-invariant Banach function spaces as well as weighted modular inequalities. We also extend this extrapolation technique to the context of A_∞ weights. We apply the obtained results to

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the dyadic square function. Fractional integrals, singular integral operators and their commutators with bounded mean oscillation functions are also considered. We present an extension of the classical results of Boyd and Lorentz-Shimogaki to a wider class of operators and also to weighted and vector-valued estimates. Finally, the same kind of ideas leads us to extrapolate within the context of an appropriate class of non A_∞ weights and this can be used to prove a conjecture proposed by E. Sawyer.

CONTENTS

1. Introduction

The purpose of this paper is to give a survey of some of the basic ideas and results developed in recent work on extrapolation [11, 15, 10, 12, 13] (see also [28, 29]). We believe that this circle of ideas should be very useful in studying some open problems in harmonic analysis.

The extrapolation we are considering concerns the theory of A_p weights. There is another theory of extrapolation more closely related to interpolation and having as its seminal result Yano's theorem. We refer the reader to [31] for a very interesting paper relating both fields.

The pioneering result for the theory of extrapolation with Muckenhoupt weights is the extrapolation theorem of Rubio de Francia:

Theorem 1.1 (Rubio de Francia, [44])

Let T be any operator acting a priori in some reasonable class of functions. Suppose that for some $1 \leq p_0 < \infty$, and every $w \in A_{p_0}$,

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0}\, w(x)\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p_0}\, w(x)\, dx, \qquad (1.1)$$

where the constant C only depends upon the A_{p_0} constant of w. Then for all $1 < p < \infty$ and for all $w \in A_p$,

$$\int_{\mathbb{R}^n} |Tf(x)|^{p}\, w(x)\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p}\, w(x)\, dx, \qquad (1.2)$$

where the constant C depends upon the A_p constant of w.

Rubio de Francia's theorem is one of the most beautiful and useful results in harmonic analysis. Roughly speaking the theorem says that it is enough to have a good initial starting point from where to "extrapolate" in order to get the whole range of scales. Observe that no assumption on the operator T is assumed, although in the original sources [44, 17] and [18] it is assumed that T is sublinear. If we further impose this condition, we can interpolate and obtain the following very useful consequence.

Corollary 1.2

Let T be a sublinear such that T is of weak type $(1,1)$ with respect to any weight $w \in A_1$, i.e., $T : L^1(w) \longrightarrow L^{1,\infty}(w)$ for all $w \in A_1$. Then for any $1 < p < \infty$, T is bounded on $L^p(w)$ for any $w \in A_p$, that is

$$\int_{\mathbb{R}^n} |Tf(x)|^{p}\, w(x)\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p}\, w(x)\, dx. \qquad (1.3)$$

These two extrapolation theorems are by now classical. The original proof of Rubio de Francia [43, 44] was based on "abstract" methods. By this we mean that the proof was not constructive; instead it worked by first "lifting" the problem to a related one involving vector-valued inequalities. In fact there is an anecdote that Rubio de Francia discovered the theorem while riding the bus to his office, without making any computations. Shortly thereafter, J. García-Cuerva [17] found an interesting constructive proof that later appeared in the monograph [18]. This proof was simplified more recently by J. Duoandikoetxea in [16].

The same family of ideas allowed Rubio de Francia to give a different proof of P. Jones' factorization theorem (see [8], and also B. Jawerth [21] and E. Hernández [20]).

Even though the constructive proofs mentioned above are interesting, they have the drawback that the proofs require multiple steps —for instance, the proof of García-Cuerva has two cases, depending on whether $p < p_0$ or $p > p_0$. In this paper we survey a new approach to this theorem that has the advantage that it requires a single step and can be extended to many different contexts, including rearrangement invariant Banach function spaces (RIQBFS in the sequel), and modular estimates, see [13]. We get all this extrapolating from A_p weights (as in the Rubio de Francia's theorem) and also from the bigger class of A_∞ weights (see [11, 15]).

This approach has two additional important features. First, vector-valued inequalities arise very naturally and without additional work. Second, the method is flexible enough that it allows us to consider more general bases that somehow behave, at least from the point of view of L^p for $p > 1$, as the basis formed by cubes. An important special case is the basis formed by rectangles that is associated to the strong maximal operator.

In certain cases we can push our method further to derive results for weights that are not in A_∞. In 1985, E. Sawyer, in a very interesting and frequently overlooked paper [45], proved the following weighted weak-type inequality for the Hardy-Littlewood maximal function on $\mathbb{R}$: for all $u, v \in A_1$,

$$u\,v\Big\{x \in \mathbb{R} : \frac{M(fv)(x)}{v(x)} > t\Big\} \le \frac{C}{t}\int_{\mathbb{R}} |f(x)|u(x)v(x)\,dx. \tag{1.4}$$

This estimate is a highly non-trivial extension of the classical weak $(1,1)$ inequality for the maximal operator. One difficulty is that the product $u\,v$ need not be locally integrable: for instance, $u(x) = v(x) = |x|^{-1/2} \in A_1$ on the real line, but $u(x)\,v(x) = |x|^{-1} \notin L^1_{\text{loc}}(\mathbb{R})$.

One motivation for studying inequality (1.4) stems from the fact that it yields a new proof of the boundedness of the maximal operator on $L^p(w)$, $w \in A_p$, assuming the factorization theorem of A_p weights. In his paper, Sawyer conjectured that the same inequality held for the Hilbert transform. We give a positive answer to this conjecture in [12]. We do so by showing that these inequalities extend to the Hardy-Littlewood maximal function and Calderón-Zygmund singular integrals in $\mathbb{R}^n$, and hold for a larger class of weights. We also extend to higher dimensions some related and interesting results of Muckenhoupt and Wheeden in [35]. Our proof uses some of the extrapolation ideas we are going to describe below.

In Section 5 we present the essential ideas for proving all the extrapolation results mentioned above. The proofs all have a common scheme and are based on the use of the so called Rubio de Francia algorithm to construct two operators:

$$\mathcal{R}_M h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|^k}, \qquad \mathcal{R}_{S_w} h(x) = \sum_{k=0}^{\infty} \frac{S_w^k h(x)}{2^k \|S_w\|^k},$$

where M denotes the Hardy-Littlewood maximal operator and S_w is in some sense a "dual" operator defined by

$$S_w h(x) = \frac{M(h\,w)(x)}{w(x)}.$$

The relevance of this operator can already be seen from (1.4). We adapt the algorithm to each situation by choosing the "norms" $\|M\|$ and $\|S\|$. See Section 5 for more details.

The remainder of this paper is organized as follows. In Section 2 we describe in greater detail the questions we are interested in and some of the consequences of our work. In Section 3 we give the exact statements of the main extrapolation results. Some needed background can be found in Section 4. In Section 5 we sketch the steps common to the proofs of the extrapolation results. Finally, in Section 6 we give a number of applications of our extrapolation results.

2. The main questions

In our work on extrapolation, we have addressed the following questions.

- **Is it possible to deduce an extrapolation result if in the initial extrapolation hypothesis** (1.1) **we consider A_∞ weights?**

  This question is strongly motivated by the following inequality due to Coifman [6, 7]: for $0 < p < \infty$ and $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |Tf(x)|^p \, w(x)\, dx \le C \int_{\mathbb{R}^n} Mf(x)^p \, w(x)\, dx, \tag{2.1}$$

  where T is any singular integral operator with standard kernel and M is the Hardy-Littlewood maximal function. Throughout this paper, these inequalities and similar ones will be understood in the sense that they hold for any "nice" function f (that is, $f \in L^\infty_c, C^\infty_0, \dots$) such that the lefthand side is finite and the constant C depends on the A_∞ constant of w.

  We showed in [11] that if (2.1) holds for one specific exponent $p = p_0$, $0 < p_0 < \infty$, and for all $w \in A_\infty$, then it holds for the whole range $0 < p < \infty$ and all $w \in A_\infty$. In other words, all the information contained in (2.1) is encoded in the corresponding estimates where the exponent is fixed; in applications some interesting cases are $p = 1$ or $p = 2$. In [30] this result is used to show that the classical Hörmander condition for a singular integral operator is not sufficient to guarantee Coifman's inequality (2.1).

Another useful result obtained in [11] states that it is enough to work with the best and smallest class of weights, A_1, provided the initial estimates hold for an open interval of exponents p close to 0. That is, if (2.1) holds for all $w \in A_1$ and for all $0 < p < p_0$ for some p_0, then one recovers the whole range of exponents $0 < p < \infty$ and $w \in A_\infty$.

In [11], we use this approach to give a proof of Coifman's estimate (2.1) that does not use good-λ inequalities. The proof is straightforward. We start with the following inequality due to Andrei Lerner [26]:

$$\int_{\mathbb{R}^n} |Tf(x)|\, w(x)\, dx \leq C \int_{\mathbb{R}^n} Mf(x)\, Mw(x)\, dx, \qquad \forall\, w \geq 0.$$

The proof of this estimate is interesting because it completely avoids the use of good-λ inequalities; instead, it involves decomposing the weight in an appropriate way. If we look at Lerner's proof more closely we can see immediately that for **every** $0 < p \leq 1$,

$$\int_{\mathbb{R}^n} |Tf(x)|^p\, w\, dx \leq C \int_{\mathbb{R}^n} Mf(x)^p\, Mw(x)\, dx, \qquad \forall\, w \geq 0.$$

Then, for every $w \in A_1$ (which means that $Mw \leq Cw$ a.e.) we obtain that (2.1) holds for all $0 < p \leq 1$. As mentioned above, by extrapolation (see Theorem 3.2 below) we derive the whole range $0 < p < \infty$ and all weights in A_∞.

- **Can we derive weak type estimates from a strong type estimate?**

This is one of the main points in [11]. Let T and M be two given operators (here we do not assumed linearity, sublinearity; T and M are defined in some reasonable class of functions). Assume that there exists $0 < p_0 < \infty$ such that for all $w \in A_\infty$

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0}\, w(x)\, dx \leq C \int_{\mathbb{R}^n} |Mf(x)|^{p_0}\, w(x)\, dx. \tag{2.2}$$

As mentioned in the previous question, this yields that the same estimate holds for all $0 < p < \infty$ and all $w \in A_\infty$. Furthermore, the corresponding weak type estimate,

$$\|Tf\|_{L^{p,\infty}(w)} \leq C\, \|Mf\|_{L^{p,\infty}(w)},$$

holds for any $0 < p < \infty$ and for any $w \in A_\infty$. (Again this inequality is understood in the sense that it holds for any function such that the lefthand side is finite.)

This result should be compared with the fact that there is no analog in the context of Rubio de Francia's theorem. Indeed, this theorem roughly says that a **strong** initial estimate yields a **strong** conclusion for each $p > 1$; if we further assume that T is sublinear then a **weak** $(1, 1)$ initial estimate yields a **strong** type (p, p) result for each $p > 1$. However, in general

$$\boxed{\textbf{STRONG} \not\Rightarrow \textbf{WEAK}}$$

That is the case for $M^2 = M \circ M$, where M is the Hardy-Littlewood maximal function. By Muckenhoupt's theorem, M^2 is bounded on $L^p(w)$ for all $1 < p < \infty$ and all $w \in A_p$ (since M is). However, M^2 is not bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ since the following $L \log L$ type estimate

$$\begin{aligned} &\big|\{x \in \mathbb{R}^n : M^2 f(x) > \lambda\}\big| \\ &\leq C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda}\right)\right) dx, \end{aligned} \tag{2.3}$$

holds as the right end-point estimate for M^2.

- **Can we derive estimates for function spaces other than the Lebesgue L^p or Marcinkiewicz $L^{p,\infty}$ spaces?**

This question was addressed in [15]. In that paper it is shown that if (2.2) holds, for any rearrangement invariant Banach function space $\mathbb{X}$ with finite upper Boyd index $q_{\mathbb{X}} < \infty$, we have for all $w \in A_\infty$ that

$$\|Tf\|_{\mathbb{X}(w)} \leq C \, \|Mf\|_{\mathbb{X}(w)}.$$

More generally, $\mathbb{X}$ can be any rearrangement invariant quasi-Banach function space with upper Boyd index $q_{\mathbb{X}} < \infty$ such that $\mathbb{X}$ is p-convex for some $0 < p \leq 1$, or equivalently, $\mathbb{X}^r$ is Banach for some $r \geq 1$. Examples of spaces to which this result can be applied include $L^{p,q}$, $L^p (\log L)^\alpha$, $L^{p,q} (\log L)^\alpha$; we refer to [15] for further examples, such as generalized Lorentz or Marcinkiewicz spaces, and for a variety of applications such as those in the context of multilinear estimates.

In [10] we derived estimates of this type for variable $L^{p(\cdot)}$ spaces, that is, Banach function spaces modeled on Lebesgue space in which the exponent $p(\cdot)$ is a function. This turned out to be very useful in solving and giving an unified approach to some open problems in this theory. Note that variable $L^{p(\cdot)}$ spaces are not rearrangement invariant and so the theory developed in [15] cannot be applied to them.

- **Can we derive estimates of modular type?**

Our examination of this question in [15] was strongly motivated by the fact that there are operators for which the natural endpoint space is not $L^{1,\infty}$: for example, the operator $M^2 = M \circ M$ mentioned above in (2.3). This operator arises naturally in the study of commutators,

$$[b,T]f(x) = b(x)\,Tf(x) - T(b\,f)(x),$$

where T is a singular integral with standard kernel and $b \in \mathrm{BMO}$. The maximal operator associated to this operator is $M^2 = M \circ M$; this can be seen, for instance, by the following estimate from [39]:

$$\int_{\mathbb{R}^n} |[b,T]f(x)|^p\,w\,dx \le C \int_{\mathbb{R}^n} M^2 f(x)^p\,w(x)\,dx, \tag{2.4}$$

for all $w \in A_\infty$ and any function f such that the lefthand side is finite. This control of the commutator $[b,T]$ by M^2 suggests that $[b,T]$ satisfies a modular inequality like (2.3). This was already obtained in [38], where it was shown that for $w \in A_\infty$,

$$\begin{aligned} &\sup_{\lambda>0} \varphi(\lambda)\,w\{y \in \mathbb{R}^n : \big|[b,T]f(y)\big| > \lambda\} \\ &\le C\,\sup_{\lambda>0} \varphi(\lambda)\,w\{y \in \mathbb{R} : M^2 f(y) > \lambda\}, \end{aligned} \tag{2.5}$$

where $\varphi(\lambda) = \lambda/(1+\log^+\frac{1}{\lambda})$. From that estimate one can obtain

$$\big|\{y \in \mathbb{R}^n : \big|[b,T]f(y)\big| > \lambda\}\big| \le C_{\|b\|_{\mathrm{BMO}}} \int_{\mathbb{R}^n} \phi\left(\frac{|f(x)|}{\lambda}\right)\,dx.$$

In [15] we showed that this kind of modular estimates can be obtained by extrapolation: there is a general extrapolation result that allows one to pass from (2.4) to (2.5). Given a non-negative increasing function with $\phi(0+) = 0$ and $\phi(\infty) = \infty$ such that ϕ is doubling, if we assume again that (2.2) holds, then for all $w \in A_\infty$,

$$\int_{\mathbb{R}^n} \phi\big(Tf(x)\big)\,w(x)\,dx \le C \int_{\mathbb{R}^n} \phi\big(Mf(x)\big)\,w(x)\,dx, \tag{2.6}$$

and similarly,

$$\sup_\lambda \phi(\lambda)\,w\{x : |Tf(x)| > \lambda\} \le C\,\sup_\lambda \phi(\lambda)\,w\{x : Mf(x) > \lambda\}.$$

- **Can we extend Rubio de Francia's theorem to function spaces or to modular type estimates?**

In [15], the extrapolation result for A_∞ weights in [11] is extended to obtain estimates in function spaces and also of modular type. It is natural to wonder whether starting from A_p weights as in Rubio de Francia's theorem (see Theorem 1.1 above) one can get similar estimates. This question is considered in the forthcoming paper [13]; see Theorem 3.1 below. In this paper we also give a proof of Rubio de Francia's extrapolation theorem in which we do not need to distinguish between the cases $p > p_0$ and $p < p_0$ (see [18, 17, 16]). This approach allowed us to obtain the extensions of Rubio de Francia's theorem to the context of function spaces and to modular type estimates.

- **Can we work with non-A_∞ weights?**

As we mentioned above, Sawyer [45] proved (1.4) in $\mathbb{R}$ for the Hardy-Littlewood maximal function and conjectured that the analogous estimate held for the Hilbert transform. As the previous questions show, our extrapolation results combined with Coifman's inequality yield that T and M behave the same way in Lebesgue spaces, Lorentz spaces, RIQBFS and also in the sense of modular estimates and this can be done with respect to A_∞ weights.

However, as we noted before, in (1.4) for $u, v \in A_1$ it may happen that $u\,v$ is not locally integrable, and so $u\,v$ is not an A_∞ weight. Hence, all the previous results cannot be applied. We need an estimate of the form

$$\Big\| \frac{Tf}{v} \Big\|_{L^{1,\infty}(u\,v)} \leq C \Big\| \frac{Mf}{v} \Big\|_{L^{1,\infty}(u\,v)} \tag{2.7}$$

for all $u, v \in A_1$. If such result were true, then, taking $T = H$, by (1.4) we would prove the conjecture for the Hilbert transform. Moreover, if we could extend Sawyer's estimate (1.4) to $\mathbb{R}^n$ (for the Hardy-Littlewood maximal function or even for its dyadic version) then we would get, by extrapolation, that Sawyer's estimate (1.4) holds for any Calderón-Zygmund singular integral.

This problem was considered in [12], where we showed that if T and M are two operators satisfying (2.2), then (2.7) holds for all $u \in A_1$ and $v \in A_\infty$. (These hypotheses are weaker than we need, since above we only need $v \in A_1$.) This yields Sawyer's conjecture not only for the Hilbert transform but also for Calderón-Zygmund singular integrals in $\mathbb{R}^n$ (see Section 3.3).

- **What about vector-valued estimates?**

It is well known that vector-valued inequalities and weights are closely related. Indeed, Rubio de Francia's extrapolation theorem was originally proved in [44] using this connection. This connection extends to all the extrapolation results we have mentioned. Roughly speaking, every time we can extrapolate from an estimate like (2.2) we (automatically) obtain vector-valued inequalities. For instance, we get that

$$\Big\| \Big(\sum_j |Tf_j|^r \Big)^{1/r} \Big\|_{L^p(w)} \leq C \Big\| \Big(\sum_j |Mf_j|^r \Big)^{1/r} \Big\|_{L^p(w)} \tag{2.8}$$

for all $0 < p, r < \infty$ and all $w \in A_\infty$. There is also a version for function spaces:

$$\Big\| \Big(\sum_j (Tf_j)^p \Big)^{1/p} \Big\|_{\mathbb{X}(w)} \leq C \Big\| \Big(\sum_j |Mf_j|^p \Big)^{1/p} \Big\|_{\mathbb{X}(w)}.$$

We can also get vector-valued modular estimates: see Theorem 3.2. The same can be done starting with A_p weights. To prove vector-valued inequalities, we change our point of view and work with pairs of functions instead of operators. This leads to the following question.

- **Do we really need to work with operators?**

In [11] (see also [15, 14]), we observed that in the extrapolation results we were considering, the operators do not need to appear explicitly since they play no role. Instead, we could work with pairs of functions. This means that we can consider (2.2) as an inequality for pairs of functions of the form $(|Tf|, |Mf|)$ with f in some class of nice functions.

Let $\mathcal{F}$ denote a family of ordered pairs of non-negative measurable functions (f, g). In place of (2.2) we assume that there is $0 < p_0 < \infty$ such that for all $w \in A_\infty$

$$\int_{\mathbb{R}^n} f(x)^{p_0}\, w(x)\, dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0}\, w(x)\, dx, \qquad (f, g) \in \mathcal{F}, \tag{2.9}$$

and we always mean that (2.9) holds for any $(f, g) \in \mathcal{F}$ such that the lefthand side is finite, and that the constant C depends only upon the A_∞ constant of w.

In [11] we showed that starting from (2.9) one can extrapolate, and the same estimate holds for the full range of exponents $0 < p < \infty$ and for all $w \in A_\infty$ with constant C depending only upon p and the A_∞ constant of w. Similarly, all of the results mentioned above can be restated in this form; see Section 3 for the precise statements.

This approach immediately yields vector-valued inequalities. For example, to prove (2.8), define the new family $\mathcal{F}_q$ to consist of the pairs of functions (F_q, G_q), where

$$F_q(x) = \Big(\sum_j |Tf_j|^q\Big)^{1/q}, \qquad G_q(x) = \Big(\sum_j (Mf_j)^q\Big)^{1/q},$$

and apply extrapolation to these pairs.

- **Can we go beyond the standard basis formed by cubes?**

  Many of the results we have mentioned can be extended to the context of more general bases $\mathcal{B}$ with associated maximal functions

$$\mathcal{M}f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{\mathcal{B}} |f(y)|\, dy, \qquad \text{if } x \in \bigcup_{B \in \mathcal{B}} B,$$

  and $\mathcal{M}f(x) = 0$ otherwise. (Recall that a basis is a collection of open sets $B \subset \mathbb{R}^n$, such as the set of all cubes.) The class of weights $A_{p,\mathcal{B}}$ with respect to $\mathcal{B}$ is defined by replacing the cubes by the sets of the basis in the classical A_p condition (see Section 4).

  We restrict our attention to the following class of bases: $\mathcal{B}$ is a **Muckenhoupt basis** if for each p, $1 < p < \infty$, and for every $w \in A_{p,\mathcal{B}}$, the maximal operator $\mathcal{M}$ is bounded on $L^p(w)$, that is,

$$\int_{\mathbb{R}^n} \mathcal{M}f(x)^p\, w(x)\, dx \le C \int_{\mathbb{R}^n} |f(x)|^p\, w(x)\, dx. \tag{2.10}$$

  These bases were introduced and characterized in [36]. Three immediate examples of Muckenhoupt bases are $\mathcal{D}$, the set of dyadic cubes in $\mathbb{R}^n$; $\mathcal{Q}$, the set of all cubes in $\mathbb{R}^n$ whose sides are parallel to the coordinate axes, and $\mathcal{R}$, the set of all rectangles (i.e., parallelepipeds) in $\mathbb{R}^n$ whose sides are parallel to the coordinate axes—see [16]. One advantage of these bases is that by using them we avoid any direct appeal to the underlying geometry: the relevant properties are derived from (2.10), and we do not use covering lemmas of any sort.

  In [11, 13] we showed that many of our extrapolation results can be extended to Muckenhoupt bases. In some cases (namely for RIBFS and for modular inequalities) we need to assume that the basis satisfies an openness property (that is, $w \in A_{p,\mathcal{B}}$, for $1 < p < \infty$, implies $w \in A_{p-\varepsilon,\mathcal{B}}$ for some $\varepsilon > 0$). In the case of the extrapolation in L^p with $w \in A_p$ there is no need to assume this.

3. Main results

In this section we give the precise statements of all the extrapolation results discussed above.

Let $\mathcal{F}$ be a family of ordered pairs of non-negative measurable functions (f,g). Recall that if we say that for some p_0, $0 < p_0 < \infty$, and $w \in A_\infty$ (or $w \in A_{p_0}$),

$$\int_{\mathbb{R}^n} f(x)^{p_0}\, w(x)\, dx \le C \int_{\mathbb{R}^n} g(x)^{p_0}\, w(x)\, dx, \qquad (f,g) \in \mathcal{F},$$

we always mean that this estimate holds for any $(f,g) \in \mathcal{F}$ such that the lefthand side is finite, and that the constant C depends only upon p and the A_∞ (or A_p) constant of w. We use the same convention when instead of the L^p norm we use any other norm, quasi-norm or any modular inequality. Throughout this section we will state our results for pairs of functions instead of for operators.

3.1 Extrapolation from A_p weights

We first present Rubio de Francia's extrapolation theorem and its extensions. Here, in place of writing specific operators, we state our result using pairs of functions.

Theorem 3.1

Let $\mathcal{F}$ be a family such that for some $1 \le p_0 < \infty$, and for every $w \in A_{p_0}$,

$$\int_{\mathbb{R}^n} f(x)^{p_0}\, w(x)\, dx \le C \int_{\mathbb{R}^n} g(x)^{p_0}\, w(x)\, dx, \qquad (f,g) \in \mathcal{F}. \tag{3.1}$$

Then, for all $(f,g) \in \mathcal{F}$ and all $\{(f_j,g_j)\}_j \subset \mathcal{F}$ we have the following estimates:

(a) **Lebesgue spaces** [44, 17]**:** *For all $1 < p, q < \infty$ and for every $w \in A_p$,*

$$\|f\|_{L^p(w)} \le C\, \|g\|_{L^p(w)},$$
$$\Big\| \Big(\sum_j (f_j)^q \Big)^{1/q} \Big\|_{L^p(w)} \le C \Big\| \Big(\sum_j (g_j)^q \Big)^{1/q} \Big\|_{L^p(w)}.$$

(b) **Rearrangement invariant Banach function spaces** [13]**:** *Let $\mathbb{X}$ be a RIBFS such that $1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$. Then for all $1 < q < \infty$ and for every $w \in A_{p_{\mathbb{X}}}$,*

$$\|f\|_{\mathbb{X}(w)} \le C\, \|g\|_{\mathbb{X}(w)},$$
$$\Big\| \Big(\sum_j (f_j)^q \Big)^{1/q} \Big\|_{\mathbb{X}(w)} \le C \Big\| \Big(\sum_j (g_j)^q \Big)^{1/q} \Big\|_{\mathbb{X}(w)}.$$

(c) **Modular inequalities** [13]**:** *Let $\phi \in \Phi$ be a convex function such that $1 < i_\phi \le I_\phi < \infty$ (i.e., ϕ, $\overline{\phi} \in \Delta_2$). Then for all $1 < q < \infty$ and for every $w \in A_{i_\phi}$,*

$$\int_{\mathbb{R}^n} \phi\big(f(x)\big)\, w(x)\, dx \ \le\ C \int_{\mathbb{R}^n} \phi\big(g(x)\big)\, w(x)\, dx,$$

$$\int_{\mathbb{R}^n} \phi\bigg(\Big(\sum_j f_j(x)^q\Big)^{1/q}\bigg)\, w(x)\, dx \ \le\ C \int_{\mathbb{R}^n} \phi\bigg(\Big(\sum_j g_j(x)^q\Big)^{1/q}\bigg)\, w(x)\, dx.$$

Furthermore, for $\mathbb{X}$ as before one can also get that $\phi(f)$ is controlled by $\phi(g)$ on $\mathbb{X}(w)$.

3.2 Extrapolation from A_∞ weights

Theorem 3.2

Let $\mathcal{F}$ be a family such that for some $0 < p_0 < \infty$, and for every $w \in A_\infty$,

$$\int_{\mathbb{R}^n} f(x)^{p_0}\, w(x)\, dx \le C \int_{\mathbb{R}^n} g(x)^{p_0}\, w(x)\, dx, \qquad (f,g) \in \mathcal{F}, \tag{3.2}$$

or equivalently, for some $0 < p_0 < \infty$, and for every $0 < p < p_0$ the following estimate holds for every $w \in A_1$:

$$\int_{\mathbb{R}^n} f(x)^{p}\, w(x)\, dx \le C \int_{\mathbb{R}^n} g(x)^{p}\, w(x)\, dx, \qquad (f,g) \in \mathcal{F}.$$

Then, for all $(f,g) \in \mathcal{F}$ and all $\{(f_j,g_j)\}_j \subset \mathcal{F}$ we have the following estimates:

(a) **Lebesgue spaces** [11]**:** *For all $0 < p, q < \infty$ and for every $w \in A_\infty$,*

$$\|f\|_{L^p(w)} \ \le\ C\, \|g\|_{L^p(w)},$$

$$\Big\| \Big(\sum_j (f_j)^q\Big)^{1/q} \Big\|_{L^p(w)} \ \le\ C \Big\| \Big(\sum_j (g_j)^q\Big)^{1/q} \Big\|_{L^p(w)}.$$

(b) **Rearrangement invariant quasi-Banach function spaces** [15]**:** *Let $\mathbb{X}$ be a RIQBFS such that $\mathbb{X}$ is p-convex for some $0 < p \le 1$ —equivalently $\mathbb{X}^r$ is a Banach space for some $r \ge 1$— and with upper Boyd index $q_{\mathbb{X}} < \infty$. Then for all $0 < q < \infty$ and for every $w \in A_\infty$*

$$\|f\|_{\mathbb{X}(w)} \ \le\ C\, \|g\|_{\mathbb{X}(w)},$$

$$\Big\| \Big(\sum_j (f_j)^q\Big)^{1/q} \Big\|_{\mathbb{X}(w)} \ \le\ C \Big\| \Big(\sum_j (g_j)^q\Big)^{1/q} \Big\|_{\mathbb{X}(w)}.$$

(c) **Modular inequalities** [15]**:** *Let $\phi \in \Phi$ with $\phi \in \Delta_2$ and suppose that there exist exponents $0 < r, s < \infty$ such that $\phi(t^r)^s$ is quasi-convex. Then for all $0 < q < \infty$ and for every $w \in A_\infty$,*

$$\int_{\mathbb{R}^n} \phi\big(f(x)\big)\, w(x)\, dx \;\leq\; C \int_{\mathbb{R}^n} \phi\big(g(x)\big)\, w(x)\, dx,$$

$$\int_{\mathbb{R}^n} \phi\bigg(\Big(\sum_j f_j(x)^q\Big)^{1/q}\bigg)\, w(x)\, dx \;\leq\; C \int_{\mathbb{R}^n} \phi\bigg(\Big(\sum_j g_j(x)^q\Big)^{1/q}\bigg)\, w(x)\, dx.$$

Furthermore, for $\mathbb{X}$ as before one can also get that $\phi(f)$ is controlled by $\phi(g)$ on $\mathbb{X}(w)$. In particular, if $\mathbb{X} = L^{1,\infty}$, then for all $w \in A_\infty$ the following weak-type modular inequalities hold:

$$\sup_\lambda \phi(\lambda)\, w\{x : f(x) > \lambda\} \leq C \sup_\lambda \phi(\lambda)\, w\{x : g(x) > \lambda\},$$

$$\sup_\lambda \phi(\lambda)\, w\bigg\{x : \Big(\sum_j f_j(x)^q\Big)^{1/q} > \lambda\bigg\}$$

$$\leq C \sup_\lambda \phi(\lambda)\, w\bigg\{x : \Big(\sum_j g_j(x)^q\Big)^{1/q} > \lambda\bigg\}.$$

3.3 Extrapolation and non-A_∞ weights

Theorem 3.3 ([12])

Let $\mathcal{F}$ be a family such that for some p_0, $0 < p_0 < \infty$, and every $w \in A_\infty$,

$$\int_{\mathbb{R}^n} f(x)^{p_0} w(x)\, dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0} w(x)\, dx, \qquad (f,g) \in \mathcal{F}. \tag{3.3}$$

Then for all weights $u \in A_1$ and $v \in A_\infty$,

$$\Big\| \frac{f}{v} \Big\|_{L^{1,\infty}(uv)} \leq C \Big\| \frac{g}{v} \Big\|_{L^{1,\infty}(uv)}, \qquad (f,g) \in \mathcal{F}. \tag{3.4}$$

As mentioned before, the point of this result is that the product of two A_1 weights need not be locally integrable.

Theorem 3.3 allows us to prove the following result which was conjectured by E. Sawyer [45].

Theorem 3.4 ([12])

If $u \in A_1$, and $v \in A_1$ or $v \in A_\infty(u)$, then there is a constant C such that for all $t > 0$,

$$u\,v\Big\{x \in \mathbb{R}^n : \frac{M(f\,v)(x)}{v(x)} > t\Big\} \le \frac{C}{t}\int_{\mathbb{R}^n} |f(x)| u(x)\,v(x)\,dx, \qquad (3.5)$$

$$u\,v\Big\{x \in \mathbb{R}^n : \frac{|T(f\,v)(x)|}{v(x)} > t\Big\} \le \frac{C}{t}\int_{\mathbb{R}^n} |f(x)|\,u(x)\,v(x)\,dx, \qquad (3.6)$$

where M is the Hardy-Littlewood maximal operator and T is any Calderón-Zygmund singular integral.

Inequality (3.5) was obtained in $\mathbb{R}$ in [45] and he conjectured (3.6) for the Hilbert transform. Here we extend (3.5) to $\mathbb{R}^n$ and we show (3.6) not only for the Hilbert transform but also for any Calderón-Zygmund singular integral. As before, we can also prove vector-valued extensions of these estimates (see [12]).

We give a brief sketch of the proof of this conjecture.

Step 1: We establish (3.5) with the dyadic Hardy-Littlewood maximal function M_d in place of M. This requires two cases; if $v \in A_\infty(u)$ we use a Calderón-Zygmund decomposition; if $v \in A_1$, we use a more subtle decomposition argument based on the ideas in [45] for the case $n = 1$.

Step 2: We extrapolate from the classical inequality

$$\|Mf\|_{L^p(w)} \le C\,\|M_d f\|_{L^p(w)}, \qquad 0 < p < \infty, \quad w \in A_\infty.$$

Thus Theorem 3.3 and **Step 1** yield (3.5).

Step 3: We again use Theorem 3.3 to extrapolate from Coifman's estimate (2.1); thus (3.5) implies (3.6).

4. Preliminaries

In this section we present some definitions and results which provide a useful preliminary to the results given in the previous section.

4.1 Muckenhoupt weights

We begin with some basic facts about the A_p theory of weights necessary for our results; for complete information we refer the reader to [16, 18]. The Hardy-Littlewood maximal function in $\mathbb{R}^n$ is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where the cubes $Q \subset \mathbb{R}^n$ have their sides parallel to the coordinate axes. This operator is bounded on L^p for every $1 < p \le \infty$ and it maps L^1 into

$L^{1,\infty}$. One can change the underlying measure in the Lebesgue spaces by introducing a weight w, i.e., a non-negative, locally integrable function. The estimates of M on weighted Lebesgue spaces $L^p(w) = L^p(w(x)\,dx)$ are governed by the Muckenhoupt weights, which are defined as follows: we say that $w \in A_p$, $1 \le p < \infty$, if there exists a constant C such that for every cube $Q \subset \mathbb{R}^n$ we have

$$\Big(\frac{1}{|Q|}\int_Q w(x)\,dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\,dx\Big)^{p-1} \le C,$$

when $1 < p < \infty$, and, for $p = 1$,

$$\frac{1}{|Q|}\int_Q w(x)\,dx \le C\,w(x), \qquad \text{for a.e. } x \in Q.$$

This latter condition can be rewritten in terms of the Hardy-Littlewood maximal function: $w \in A_1$ if and only if $Mw(x) \le C\,w(x)$ for a.e. $x \in \mathbb{R}^n$. The class A_∞ is defined as $A_\infty = \bigcup_{p\ge1} A_p$.

Muckenhoupt [33] proved that the weights for the weighted norm inequalities for the Hardy-Littlewood maximal function are characterized by the A_p classes: M maps $L^1(w)$ into $L^{1,\infty}(w)$ if and only if $w \in A_1$ and M is bounded on $L^p(w), 1 < p < \infty$, if and only if $w \in A_p$.

4.2 Basics on function spaces

We collect several basic facts about rearrangement invariant quasi-Banach function spaces (RIQBFS). We start with Banach function spaces; for a complete account we refer the reader to [1]. Let (Ω, Σ, μ) be a σ-finite non-atomic measure space. Let $\mathcal{M}$ denote the set of measurable functions. Given a Banach function norm ρ we define the Banach function space $\mathbb{X} = \mathbb{X}(\rho)$ as

$$\mathbb{X} = \big\{f \in \mathcal{M} :\ \|f\|_{\mathbb{X}} = \rho(|f|) < \infty\big\}.$$

The associate space of $\mathbb{X}$ is a function space $\mathbb{X}'$ such that the following generalized Hölder's inequality holds:

$$\int_\Omega |f\,g|\,d\mu \le \|f\|_{\mathbb{X}}\,\|g\|_{\mathbb{X}'}.$$

A Banach function norm ρ is rearrangement invariant if $\rho(f) = \rho(g)$ for every pair of functions f, g which are equimeasurable, that is, $\mu_f = \mu_g$ (where μ_f is the distribution function of f). In this case, we say that the Banach function space $\mathbb{X} = \mathbb{X}(\rho)$ is rearrangement invariant. It follows that $\mathbb{X}'$ is also rearrangement invariant. The decreasing rearrangement of f is the function f^* defined on $[0,\infty)$ by

$$f^*(t) = \inf\big\{\lambda \ge 0 :\ \mu_f(\lambda) \le t\big\}, \qquad t \ge 0.$$

The main property of f^* is that it is equimeasurable with f, that is,

$$\mu\{x \in \Omega : |f(x)| > \lambda\} = \big|\{t \in \mathbb{R}^+ : f^*(t) > \lambda\}\big|.$$

This allows one to obtain a representation of $\mathbb{X}$ on the measure space $(\mathbb{R}^+, dt)$. That is, there exists a RIBFS $\overline{\mathbb{X}}$ over $(\mathbb{R}^+, dt)$ such that $f \in \mathbb{X}$ if and only if $f^* \in \overline{\mathbb{X}}$, and in this case $\|f\|_{\mathbb{X}} = \|f^*\|_{\overline{\mathbb{X}}}$ (the Luxemburg representation theorem, see [1, p. 62]). Furthermore, the associate space $\mathbb{X}'$ of $\mathbb{X}$ is represented in the same way by the associate space $\overline{\mathbb{X}}'$ of $\overline{\mathbb{X}}$, and so $\|f\|_{\mathbb{X}'} = \|f^*\|_{\overline{\mathbb{X}}'}$.

From now on let $\mathbb{X}$ be a rearrangement invariant Banach function space (RIBFS) in $(\mathbb{R}^n, dx)$ and let $\overline{\mathbb{X}}$ be its corresponding RIBFS in $(\mathbb{R}^+, t)$.

The Boyd indices $p_{\mathbb{X}}$, $q_{\mathbb{X}}$ of $\mathbb{X}$ give information about the localization of $\mathbb{X}$ in terms of interpolation properties. Roughly speaking, $\mathbb{X}$ is an interpolation space between $L^{p_{\mathbb{X}}-\varepsilon}$ and $L^{q_{\mathbb{X}}+\varepsilon}$. (The definition involves the norm of the dilation operator in $\mathbb{X}$; see [1, Chapter 3] for more details.) One has that $1 \le p_{\mathbb{X}} \le q_{\mathbb{X}} \le \infty$. The relationship between the Boyd indices of $\mathbb{X}$ and $\mathbb{X}'$ is the following: $p_{\mathbb{X}'} = (q_{\mathbb{X}})'$ and $q_{\mathbb{X}'} = (p_{\mathbb{X}})'$, where, as usual, p and p' are conjugate exponents. As expected, the spaces L^p, $L^{p,q}$, $L^p(\log L)^\alpha$ have Boyd indices $p_{\mathbb{X}} = q_{\mathbb{X}} = p$.

We now consider weighted versions of these spaces. Given $w \in A_\infty$ on $\mathbb{R}^n$, we use the standard notation $w(E) = \int_E w(x)\,dx$. The distribution function and the decreasing rearrangement with respect to w are given by

$$w_f(\lambda) = w\{x \in \mathbb{R}^n : |f(x)| > \lambda\}; \qquad f^*_w(t) = \inf\{\lambda \ge 0 : w_f(\lambda) \le t\}.$$

Define the weighted version of the space $\mathbb{X}$ by

$$\mathbb{X}(w) = \{f \in \mathcal{M} : \|f^*_w\|_{\overline{\mathbb{X}}} < \infty\},$$

and the norm associated to it by $\|f\|_{\mathbb{X}(w)} = \|f^*_w\|_{\overline{\mathbb{X}}}$. By construction $\mathbb{X}(w)$ is a Banach function space built over $\mathcal{M}(\mathbb{R}^n, w(x)\,dx)$. The same procedure with the associate spaces shows that the associate space $\mathbb{X}(w)'$ coincides with the weighted space $\mathbb{X}'(w)$.

Given a Banach function space $\mathbb{X}$, for each $0 < r < \infty$, as in [22], we define

$$\mathbb{X}^r = \{f \in \mathcal{M} : |f|^r \in \mathbb{X}\} = \{f \in \mathcal{M} : \|f\|_{\mathbb{X}^r} = \||f|^r\|_{\mathbb{X}}^{1/r}\}.$$

Note that this notation is natural for the Lebesgue spaces since L^r coincides with $(L^1)^r$. If $\mathbb{X}$ is a RIBFS and $r \ge 1$ then, $\mathbb{X}^r$ still is a RIBFS but, in general, for $0 < r < 1$, the space $\mathbb{X}^r$ is not necessarily a Banach space. Note that in the same way we can also define powers of weighted spaces and we have $(\mathbb{X}(w))^r = \mathbb{X}^r(w)$.

We will work with spaces $\mathbb{X}$ so that $\mathbb{X} = \mathbb{Y}^s$ for some RIBFS $\mathbb{Y}$ and some $0 < s < \infty$. In this case the space $\mathbb{X}$ is a rearrangement invariant quasi-Banach space (RIQBFS in the sequel), see [19, 32] for more details. We note that another equivalent approach consists in first introducing the quasi-Banach case and then restricting attention to those RIQBFS for which a large power is a Banach space. This latter property turns out to be equivalent to the fact that the RIQBFS $\mathbb{X}$ is p-convex for some $0 < p \leq 1$.

In (b) of Theorem 3.2 we restricted ourselves to the case of $\mathbb{X}$ p-convex with $q_{\mathbb{X}} < \infty$. As we have just mentioned, this means that $\mathbb{X}^r$ is a Banach space (with $r = 1/p$). Thus, by the Lorentz-Shimogaki Theorem (see [27, 48] and [1, p. 54]) $q_{\mathbb{X}} < \infty$ is equivalent to the boundedness of the Hardy-Littlewood maximal function on $(\mathbb{X}^r)'$.

Some examples of RIQBFS are Lebesgue spaces, classical Lorentz spaces, Lorentz Λ-spaces, Orlicz spaces, and Marcinkiewicz spaces; see [15] for more details. In some of these examples, the Boyd indices can be computed very easily, for instance if $\mathbb{X}$ is L^p, $L^{p,q}$, $L^p(\log L)^\alpha$ or $L^{p,q}(\log L)^\alpha$ (where $0 < p < \infty$, $0 < q \leq \infty$, $\alpha \in \mathbb{R}$), then $p_{\mathbb{X}} = q_{\mathbb{X}} = p$. In this cases, it is trivial to calculate the powers of $\mathbb{X}$:

$$(L^{p,q})^r = L^{p\,r,q\,r}, \qquad (L^{p,q}(\log L)^\alpha)^r = L^{p\,r,q\,r}(\log L)^\alpha.$$

The same observation applies to $L^p = L^{p,p}$ and

$$L^p(\log L)^\alpha = L^{p,p}(\log L)^\alpha.$$

4.3 Basics on modular inequalities

We introduce some notation and terminology which are taken from [24] and [42]. Let Φ be the set of functions $\phi : [0,\infty) \longrightarrow [0,\infty)$ which are nonnegative, increasing and such that $\phi(0^+) = 0$ and $\phi(\infty) = \infty$. If $\phi \in \Phi$ is convex we say that ϕ is a Young function. An N-function (from *nice Young function*) ϕ is a Young function such that

$$\lim_{t\to 0^+} \frac{\phi(t)}{t} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \frac{\phi(t)}{t} = \infty.$$

The function $\phi \in \Phi$ is said to be quasi-convex if it is equivalent to a convex function. We say that $\phi \in \Phi$ satisfies the Δ_2 condition, denoted by $\phi \in \Delta_2$, if ϕ is doubling, that is, if

$$\phi(2\,t) \leq C\,\phi(t), \qquad t \geq 0.$$

Given $\phi \in \Phi$ one can define a complementary function $\overline{\phi}$ such that Young's inequality holds:

$$s\,t \leq \phi(s) + \overline{\phi}(t), \qquad s,t \geq 0. \tag{4.1}$$

If ϕ is an N-function, then $\overline{\phi}$ is an N-function as well.

Associated with ϕ are the dilation indices i_ϕ and I_ϕ (see [25, 24]) that satisfy $0 \le i_\phi \le I_\phi \le \infty$. If ϕ is quasi-convex, then $i_\phi \ge 1$ and if ϕ is an N-function, then we have that $i_{\overline{\phi}} = (I_\phi)'$ and $I_{\overline{\phi}} = (i_\phi)'$. The dilation indices provide information about the localization of ϕ in the scale of powers t^p. Indeed, if $0 < i_\phi \le I_\phi < \infty$, given ε small enough, we have for all $t \ge 0$ that

$$\begin{aligned} \phi(\lambda\, t) &\le C_\varepsilon\, \lambda^{I_\phi+\varepsilon}\, \phi(t), \qquad \text{for} \quad \lambda \ge 1, \\ \phi(\lambda\, t) &\le C_\varepsilon\, \lambda^{i_\phi-\varepsilon}\, \phi(t), \qquad \text{for} \quad \lambda \le 1. \end{aligned}$$

It is clear then that $\phi \in \Delta_2$ if and only if $I_\phi < \infty$. So if ϕ is an N-function, then $1 < i_\phi \le I_\phi < \infty$ if and only if ϕ, $\overline{\phi} \in \Delta_2$.

Remark 4.1 We would like to stress the analogy between the hypotheses of Theorem 3.2 parts (b) and (c). That $\mathbb{X}^r$ is a Banach space for some $r \ge 1$ and that $\phi(t^r)^s$ is quasi-convex for some $0 < r, s < \infty$ play the same role. Indeed, in both proofs these properties are used to ensure the existence of a dual space and a complementary function which allow one to make a duality argument. On the other hand, in (b) one assumes that $q_{\mathbb{X}} < \infty$ and in (c) one assumes that $\phi \in \Delta_2$ which, as mentioned, means $I_\phi < \infty$. Thus, in both cases we are assuming the finiteness of the upper indices. In the proofs, these conditions are used to assure that the Hardy-Littlewood maximal function is bounded on the dual of $\mathbb{X}^r$ and also that it satisfies a modular inequality with respect to the complementary function of $\phi(t^r)^s$.

Some examples to which these results can be applied are $\phi(t) = t^p$, $\phi(t) = t^p\,(1 + \log^+ t)^\alpha$, $\phi(t) = t^p\,(1 + \log^+ t)^\alpha\,(1 + \log^+ \log^+ t)^\beta$ with $0 < p < \infty$ and $\alpha, \beta \in \mathbb{R}$. In all these cases $i_\phi = I_\phi = p$ and $\phi(t^r)$ is quasi-convex for r large enough.

5. About the proofs

The proofs of all of the extrapolation results follow a common scheme that has to be adapted to each situation. The central idea is to use appropriate versions of the Rubio de Francia algorithm associated to the Hardy-Littlewood maximal function and the "dual" operator $S_w f = M(f\,w)/w$.

Consider, for instance, the case of extrapolation to $L^p(w)$ with $w \in A_p$. We have that M is bounded on $L^p(w)$ by Muckenhoupt's theorem. Since $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$, S_w is bounded on $L^{p'}(w)$. These two facts are enough to apply the Rubio de Francia algorithm and define the new operators

$$\mathcal{R}_M h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|^k_{L^p(w)}}, \qquad \mathcal{R}_{S_w} h(x) = \sum_{k=0}^{\infty} \frac{S_w^k h(x)}{2^k \|S_w\|^k_{L^{p'}(w)}}.$$

These have the following properties:

$\mathcal{R}_M$	$\mathcal{R}_{S_w}$
$0 \le h(x) \le \mathcal{R}_M h(x)$	$0 \le h(x) \le \mathcal{R}_{S_w} h(x)$
$\|\mathcal{R}_M h\|_{L^p(w)} \le 2\,\|h\|_{L^p(w)}$	$\|\mathcal{R}_S h\|_{L^{p'}(w)} \le 2\,\|h\|_{L^{p'}(w)}$
$\mathcal{R}_M h \in A_1$	$\mathcal{R}_{S_w} h \cdot w \in A_1$

Similar constructions can also be done in RIQBFS or with modular inequalities. In this case M is associated with $\mathbb{X}$ or ϕ and S_w is associated with $\mathbb{X}'$ or $\overline{\phi}$. Given these operators, one then uses a duality argument $L^p - L^{p'}$, $\mathbb{X} - \mathbb{X}'$ or $\phi - \overline{\phi}$.

We emphasize that in Theorem 3.1 these ideas work without difficulty. However, in Theorem 3.2 we are considering spaces L^p with $p < 1$, RIQBFS $\mathbb{X}$ and $\phi \in \Phi$, so duality cannot be used directly. But in the case of L^p (resp. $\mathbb{X}$) we have that $(L^p)^r$ (resp. $\mathbb{X}^r$) is a Banach space for some $r \ge 1$ and consequently we can use this change in the scale to apply duality arguments. Roughly the same approach is possible if ϕ is such that $\phi(t^r)^s$ is quasi-convex for some r, s.

6. Some applications

We now present some applications of these extrapolation results. In our first group of applications, we show that in some cases there is a natural choice of p_0 for which it is easier to prove (3.2). For instance, in the case of the dyadic square function we show (3.2) with the natural exponent $p_0 = 2$ for the family $\mathcal{F}$ consisting of the pairs $(|f|, S_d f)$ with $f \in C_c^\infty$ (see Proposition 6.2). We also consider certain potential operators generalizing the fractional integrals and commutators of the fractional integrals with BMO functions. Using a technique of discretizing the operator (see [37, 40, 46] and [47]) we show that these operators are controlled by a maximal function adapted to the situation. This control is written in terms of (3.2) with $p_0 = 1$ (see Propositions 6.5 and 6.8).

The fact that $p_0 = 1$ is crucial in the proofs since this allows us to take advantage of the discretized operators.

We also present an extension of the classical results by Boyd and Lorentz-Shimogaki for Calderón-Zygmund singular integrals and the Hardy-Littlewood maximal function. Finally, we give natural end-point estimates for the commutators of Calderón-Zygmund singular integrals with functions in BMO.

6.1 Dyadic square function

Given the set $\mathcal{D}$ of dyadic cubes in $\mathbb{R}^n$, for each $m \in \mathbb{Z}$, let $\mathcal{D}_m = \{Q \in \mathcal{D} : \ell(Q) = 2^m\}$. For each $Q \in \mathcal{D}$, let $\tilde{Q}$ denote the dyadic parent of Q: if $Q \in \mathcal{D}_m$, the unique cube $\tilde{Q} \in \mathcal{D}_{m+1}$ such that $Q \subset \tilde{Q}$. Given a function f, let $f_Q = |Q|^{-1} \int_Q f(x)\, dx$. The dyadic square function S_d is defined by

$$S_d f(x) = \left(\sum_{Q \in \mathcal{D}} |f_Q - f_{\tilde{Q}}|^2 \chi_Q(x) \right)^{1/2}.$$

In what follows let A_∞^d denote the Muckenhoupt basis given by dyadic cubes. In [11] we showed that Theorem 3.2 part (a) holds for any Muckenhoupt basis and in particular for the one given by dyadic cubes.

The following result seems to be known (at least for $p = 2$, see [4]), but an explicit proof does not appear in the literature.

Theorem 6.1

If $w \in A_\infty^d$, then for all $f \in C_c^\infty$ and $0 < p < \infty$,

$$\int_{\mathbb{R}^n} |f(x)|^p\, w(x)\, dx \le C \int_{\mathbb{R}^n} S_d f(x)^p\, w(x)\, dx. \tag{6.1}$$

We obtain this by giving an elementary proof (using ideas from [3]) when $p = 2$, which is the natural exponent for square functions. We actually prove a more general result, at least for the case $p = 2$, since in our proof we only use the fact that w is a dyadic doubling weight: $w(\tilde{Q}) \le c\, w(Q)$ for any dyadic cube Q.

Proposition 6.2

If $w \in A_\infty^d$, then for all $f \in C_c^\infty$

$$\int_{\mathbb{R}^n} |f(x)|^2\, w(x)\, dx \le C \int_{\mathbb{R}^n} S_d f(x)^2\, w(x)\, dx.$$

This result plus Theorem 3.2 part (a) applied to the family of pairs $(|f|, S_d f)$ with $f \in C_c^\infty$ yield Theorem 6.1. We remark that our proof does not use a good-λ inequality.

Lemma 6.3

Given a sequence $\{\lambda_Q\}_{Q\in\mathcal{D}}$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then for any $m \in \mathbb{Z}$,

$$\sum_{Q\in\mathcal{D}_m} \lambda_{\tilde{Q}}(f_Q - f_{\tilde{Q}})^2 = \sum_{Q\in\mathcal{D}_m} \lambda_{\tilde{Q}}(f_Q^2 - f_{\tilde{Q}}^2).$$

Proof. The desired equality follows from a straightforward computation:

$$\begin{aligned}\sum_{Q\in\mathcal{D}_m} \lambda_{\tilde{Q}}(f_Q - f_{\tilde{Q}})^2 &= \sum_{P\in\mathcal{D}_{m+1}} \sum_{\substack{Q\in\mathcal{D}_m \\ \tilde{Q}=P}} \lambda_P(f_Q - f_P)^2 \\ &= \sum_{P\in\mathcal{D}_{m+1}} \lambda_P \sum_{\substack{Q\in\mathcal{D}_m \\ \tilde{Q}=P}} (f_Q^2 - 2f_Qf_P + f_P^2).\end{aligned}$$

For a fixed cube $P \in \mathcal{D}_{m+1}$,

$$\sum_{\substack{Q\in\mathcal{D}_m \\ \tilde{Q}=P}} 2f_Qf_P = \frac{2^{n+1}f_P}{|P|} \sum_{\substack{Q\in\mathcal{D}_m \\ \tilde{Q}=P}} \int_Q f(x)\,dx = 2^{n+1}f_P^2.$$

Therefore,

$$\begin{aligned}\sum_{P\in\mathcal{D}_{m+1}} \lambda_P \sum_{\substack{Q\in\mathcal{D}_m \\ \tilde{Q}=P}} (f_Q^2 - 2f_Qf_P + f_P^2) &= \sum_{P\in\mathcal{D}_{m+1}} \lambda_P \sum_{\substack{Q\in\mathcal{D}_m \\ \tilde{Q}=P}} (f_Q^2 - f_P^2) \\ &= \sum_{Q\in\mathcal{D}_m} \lambda_{\tilde{Q}}(f_Q^2 - f_{\tilde{Q}}^2). \qquad \square\end{aligned}$$

Proof of Proposition 6.2 Fix f; by a straight-forward argument we may assume it is real-valued. Since w is a dyadic doubling weight and by Lemma 6.3 we have

$$\begin{aligned}\int_{\mathbb{R}^n} S_df(x)^2w(x)\,dx &= \sum_{Q\in\mathcal{D}} (f_Q - f_{\tilde{Q}})^2w(Q) \geq c\sum_{Q\in\mathcal{D}} (f_Q - f_{\tilde{Q}})^2w(\tilde{Q}) \\ &= c\sum_{Q\in\mathcal{D}} (f_Q^2 - f_{\tilde{Q}}^2)w(\tilde{Q}) \\ &= c\sum_{Q\in\mathcal{D}} \big(2^nw(Q)f_Q^2 - w(\tilde{Q})f_{\tilde{Q}}^2\big) \\ &\quad + c\sum_{Q\in\mathcal{D}} \big(w(\tilde{Q}) - 2^nw(Q)\big)f_Q^2 = W_1 + W_2.\end{aligned}$$

To complete the proof, we first claim that $W_2 \geq 0$, so that we may discard this term. To see this, for each $P \in \mathcal{D}$ we gather all the terms in W_2 in which $w(\tilde{Q})$ appears; we then get

$$\begin{aligned} W_2 &= \sum_{P\in\mathcal{D}} \left(\left(\sum_{\tilde{Q}=P} w(P) f_Q^2 \right) - 2^n w(P) f_P^2 \right) \\ &= \sum_{P\in\mathcal{D}} w(P) \sum_{\tilde{Q}=P} \left(f_Q^2 - f_P^2 \right). \end{aligned}$$

Arguing exactly as we did in the proof of Lemma 6.3, we see that for each P,

$$w(P) \sum_{\tilde{Q}=P} \left(f_Q^2 - f_P^2 \right) = w(P) \sum_{\tilde{Q}=P} \left(f_Q - f_P \right)^2 \geq 0.$$

It follows immediately that $W_2 \geq 0$. To estimate W_1, we set

$$\begin{aligned} f_m(x) &= \sum_{Q\in\mathcal{D}_m} f_Q \chi_Q(x) \\ A_m &= \sum_{Q\in\mathcal{D}_m} 2^n w(Q) f_Q^2 = 2^n \int_{\mathbb{R}^n} f_m(x)^2 w(x)\, dx. \end{aligned}$$

Since f is bounded and has compact support, it follows immediately from the dominated convergence theorem and the Lebesgue differentiation theorem that

$$W_1 = \sum_{-\infty}^{\infty} A_m - A_{m+1} = \lim_{N\to-\infty} A_N - \lim_{M\to+\infty} A_M = 2^n \int_{\mathbb{R}^n} f(x)^2\, w(x)\, dx.$$

This establishes the desired inequality. □

6.2 Fractional integrals and potential operators

Recall that for $0 < \alpha < n$, the fractional integral of order α (also known as the Riesz potential) is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,$$

and the closely related fractional maximal operator is defined by

$$M_\alpha f(x) = \sup_{Q\ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)|\, dy,$$

where Q is any cube with sides parallel to the coordinate axes. It is well known that for every x, $M_\alpha f(x) \leq C I_\alpha\big(|f|\big)(x)$. The converse pointwise inequality is false, but the following norm inequality is true.

Theorem 6.4

Given $0 < \alpha < n$*, then for every* $0 < p < \infty$ *and* $w \in A_\infty$*,*

$$\int_{\mathbb{R}^n} |I_\alpha f(x)|^p\, w(x)\, dx \le C \int_{\mathbb{R}^n} M_\alpha f(x)^p\, w(x)\, dx.$$

Theorem 6.4 is due to Muckenhoupt and Wheeden [34], who proved it using a good-λ inequality relating I_α and M_α.

Here we give an alternate proof that applies to more general potential operators,

$$T_\Psi f(x) = \int_{\mathbb{R}^n} \Psi(x-y) f(y)\, dy,$$

whose kernels satisfy a very mild size condition: there exist constants $\delta, c > 0$, and $0 \le \epsilon < 1$, such that for every $k \in \mathbb{Z}$

$$\sup_{2^k < |x| \le 2^{k+1}} \Psi(x) \le \frac{c}{2^{kn}} \int_{\delta(1-\epsilon)2^k < |y| \le \delta(1+\epsilon)2^{k+1}} \Psi(y)\, dy. \tag{6.2}$$

Examples of functions which satisfy (6.2) include functions Ψ which are radial and monotonic; more generally we can take Ψ which are essentially constant on annuli, that is, $\Psi(y) \le c\,\Psi(x)$ for $|y|/2 \le |x| \le 2|y|$.

Associated to the operator T_Ψ is the maximal operator $M_{\tilde\Psi}$ defined by

$$M_{\tilde\Psi} f(x) = \sup_{x \in Q} \frac{\tilde\Psi(\ell(Q))}{|Q|} \int_Q f(y)\, dy,$$

where $\tilde\Psi(t) = \int_{|z| \le t} \Psi(z)\, dz$.

Note that if $\Psi(x) = |x|^{\alpha - n}$, then (6.2) holds and the operators T_Ψ and $M_{\tilde\Psi}$ are I_α and M_α. The key will be the following result.

Proposition 6.5

Let Ψ *be a kernel satisfying condition* (6.2)*. Then for every weight* $w \in A_\infty$*,*

$$\int_{\mathbb{R}^n} |T_\Psi f(x)|\, w(x)\, dx \le C \int_{\mathbb{R}^n} M_{\tilde\Psi} f(x)\, w(x)\, dx. \tag{6.3}$$

In the proof of this result it is crucial that $p = 1$, since we perform a discretization of T_Ψ and then we change the order of integration. As a consequence of Theorem 3.2, by extrapolation we get the full range of exponents.

Theorem 6.6

Let Ψ be as above. Then for every $0 < p < \infty$ and $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |T_\Psi f(x)|^p\, w(x)\, dx \le C \int_{\mathbb{R}^n} M_{\tilde{\Psi}} f(x)^p\, w(x)\, dx.$$

Sketch of the Proof of Proposition 6.5 We follow [37] using a discretization method for T_Ψ developed in [40]. (This in turn is based on ideas from E. Sawyer and R. Wheeden in citesawyerwheeden, see also [47, 41].) We refer the reader to [37, 40] for full details.

For each $t > 0$, define

$$\overline{\Psi}(t) = \sup_{t<|x|\le 2t} \Psi(x), \qquad \underline{\Psi}(t) = \frac{1}{t^n} \int_{\delta(1-\epsilon)t<|y|\le 2\delta(1-\epsilon)t} \Psi(y)\, dy,$$

where δ, c, ϵ are the constants provided by condition (6.2). Without loss of generality we may assume $f \ge 0$. We discretize the operator T_Ψ to get

$$\begin{aligned} T_\Psi f(x) &= \sum_{k\in\mathbb{Z}} \int_{2^{-k-1}<|x-y|\le 2^{-k}} \Psi(x-y) f(y)\, dy \\ &\le \sum_{Q\in\mathcal{D}} \overline{\Psi}\left(\frac{\ell(Q)}{2}\right) \int_{3Q} f(y)\, dy\ \chi_Q(x). \end{aligned}$$

The next step is to replace the sum over all dyadic cubes by a sum over dyadic Calderón-Zygmund cubes. As in [40] (to be rigorous here we must restrict w to a fixed (big) cube; we skip these technical details), we fix $a > 2^n$ and for each $k \in \mathbb{Z}$ we take the collection $\{Q_{k,j}\}$ of disjoint maximal dyadic cubes such that

$$\begin{aligned} D_k &= \{x \in \mathbb{R}^n : M^d w(x) > a^k\} = \bigcup_j Q_{j,k}, \\ a^k &< \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w(x)\, dx \le 2^n a^k. \end{aligned}$$

Further, every dyadic cube which satisfies the first inequality is contained in a unique cube $Q_{k,j}$. Finally, if we define $E_{k,j} = Q_{k,j} \setminus D_{k+1}$, then $|Q_{k,j}| \approx |E_{k,j}|$. For each $k \in \mathbb{Z}$, define

$$\mathcal{C}^k = \Big\{ Q \in \mathcal{D} : a^k < \frac{1}{|Q|} \int_Q w(x)\, dx \le a^{k+1} \Big\}.$$

Then we have that

$$\begin{aligned}
&\int_{\mathbb{R}^n} T_\Psi f(x)\, w(x)\, dx \\
&\le C \sum_k \sum_{Q\in\mathcal{C}_k} \overline{\Psi}\left(\frac{\ell(Q)}{2}\right) \int_{3\,Q} f(y)\, dy \int_Q w(x)\, dx \\
&\le C \sum_{k,j} \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w(x)\, dx \\
&\qquad \times \sum_{\substack{Q\,\in\,\mathcal{D}\\ Q\,\subset\, Q_{k,j}}} |Q|\, \overline{\Psi}\left(\frac{\ell(Q)}{2}\right) \int_{3\,Q} f(y)\, dy \\
&\le C \sum_{k,j} \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w(x)\, dx\;\; \tilde{\Psi}\big(\delta\,(1+\epsilon)\,\ell(Q_{k,j})\big) \int_{3\,Q_{k,j}} f(y)\, dy,
\end{aligned}$$

where the last inequality will follow if we show that there is a constant C_Ψ such that for any dyadic cube P,

$$\begin{aligned}
&\sum_{\substack{Q\,\in\,\mathcal{D}\\ Q\,\subset\, P}} |Q|\, \overline{\Psi}\left(\frac{\ell(Q)}{2}\right) \int_{3\,Q} f(y)\, dy \qquad (6.4)\\
&\le C_\Psi\, \tilde{\Psi}\big(\delta\,(1+\epsilon)\,\ell(P)\big) \int_{3\,P} f(y)\, dy.
\end{aligned}$$

We show this estimate below. Since $w \in A_\infty$, we have that $w(Q_{k,j}) \approx w(E_{k,j})$. Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^n} T_\Psi f(x)\, w(x)\, dx \\
&\le C \sum_{k,j} \int_{E_{k,j}} \left(\frac{\tilde{\Psi}\big(\tau\,\ell(Q_{k,j})\big)}{|\tau\, Q_{k,j}|} \int_{\tau\, Q_{k,j}} f(y)\, dy \right) w(x)\, dx \\
&\le C \sum_{k,j} \int_{E_{k,j}} M_{\tilde{\Psi}} f(x)\, w(x)\, dx \le C \int_{\mathbb{R}^n} M_{\tilde{\Psi}} f(x)\, w(x)\, dx,
\end{aligned}$$

where $\tau = \max\{3, \delta\,(1+\epsilon)\}$. Note that we have also used that $\tilde{\Psi}$ is nondecreasing and that the sets $E_{k,j}$ are pairwise disjoint.

We now prove (6.4): if $\ell(P) = 2^{-i_0}$, then

$$\sum_{\substack{Q \in \mathcal{D} \\ Q \subset P}} |Q|\, \overline{\Psi}\left(\frac{\ell(Q)}{2}\right) \int_{3\,Q} f(y)\, dy$$

$$= \sum_{i=i_0}^{\infty} 2^{-in}\, \overline{\Psi}(2^{-i-1}) \sum_{\substack{Q \in \mathcal{D} : Q \subset P \\ \ell(Q) = 2^{-i}}} \int_{3\,Q} f(y)\, dy$$

$$\leq C \int_{3\,P} f(y)\, dy \sum_{i=i_0}^{\infty} 2^{-in} \overline{\Psi}(2^{-i-1})$$

$$\leq C \int_{3\,P} f(y)\, dy \sum_{i=i_0}^{\infty} 2^{-in} \underline{\Psi}(2^{-i-1}),$$

since the overlap is finite and where we have used (6.2), that is,

$$\overline{\Psi}(2^{-i}) \leq C_\Psi\, \underline{\Psi}(2^{-i}),$$

for all $i \in \mathbb{Z}$. We get the desired estimate by observing

$$\sum_{i=i_0}^{\infty} 2^{-in}\, \underline{\Psi}(2^{-i-1}) \leq C_{\delta,\epsilon} \int_{|y| \leq \delta\,(1+\epsilon)\,\ell(P)} \Psi(y)\, dy = C\, \tilde{\Psi}\big(\delta\,(1+\epsilon)\,\ell(P)\big). \qquad \square$$

6.3 Commutators of fractional integrals

The techniques used in the previous example can be exploited again to get estimates for commutators of fractional integrals. For $0 < \alpha < n$ and $b \in \mathrm{BMO}$, define

$$[b, I_\alpha] f(x) = b(x)\, I_\alpha b(x) - I_\alpha(b\, f)(x) = \int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x - y|^{n-\alpha}}\, f(y)\, dy.$$

These commutators were considered by S. Chanillo [5], who proved that if $1/p - 1/q = \alpha/n$, then $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$. A weighted version of this result also holds. For such pairs (p, q) we say that $w \in A_{p,q}$ if for all cubes Q,

$$\left(\frac{1}{|Q|} \int_Q w^q\, dx\right)^{1/q} \left(\frac{1}{|Q|} \int_Q w^{-p'}\, dx\right)^{1/p'} \leq C < \infty.$$

Note that if $w \in A_{pq}$, then $w \in A_\infty$.

Theorem 6.7

Fix $0 < \alpha < n$. *Suppose* $1 < p < n/\alpha$ *and* $1/p - 1/q = \alpha/n$. *Then for all* $w \in A_{pq}$,

$$\left(\int_{\mathbb{R}^n} |[b, I_\alpha]f|^q\, w^q\, dx\right)^{1/q} \le C \left(\int_{\mathbb{R}^n} |f|^p\, w^p\, dx\right)^{1/p}. \qquad (6.5)$$

A proof of this result was given in [9] which depended on the good-λ inequality relating the maximal function and the sharp maximal function.

Our proof instead relies on extrapolation and on the weighted norm inequalities for fractional Orlicz maximal operators which were also proved in [9]. Define the fractional Orlicz maximal operator as follows: given a Young function ψ, let

$$M_{\psi,\alpha}f(x) = \sup_{Q \ni x} |Q|^{\alpha/n}\, \|f\|_{\psi,Q}.$$

(For notation and basic facts about Orlicz spaces see below.) Let $\psi(t) = t\log(e+t)$. Then (6.5) is true with the commutator replaced by $M_{\psi,\alpha}$. To prove Theorem 6.7 we apply Theorem 3.2 to the family of pairs $(|[b, I_\alpha]f|, M_{\psi,\alpha}f)$, starting from the following result. We want to highlight that this approach yields vector-valued inequalities which are new.

Proposition 6.8

Given $0 < \alpha < n$, $b \in \mathrm{BMO}$ *and* $w \in A_\infty$, *then*

$$\int_{\mathbb{R}^n} \big|[b, I_\alpha]f(x)\big|\, w(x)\, dx \le C \int_{\mathbb{R}^n} M_{\psi,\alpha}f(x)\, w(x)\, dx.$$

The proof of Proposition 6.8 is taken from [11] and we have included it below for completeness. Again the fact that the exponent is 1 plays an important role in the proof.

Before beginning the proof, we need to state some definitions and basic facts about Orlicz spaces. For a complete information, see [42, 1]. Let $\psi : [0,\infty) \longrightarrow [0,\infty)$ be a Young function, that is, a continuous, convex, increasing function with $\psi(0) = 0$ and such that $\psi(t) \longrightarrow \infty$ as $t \to \infty$. The Orlicz space L_ψ is defined to be the set of measurable functions f such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \psi\left(\frac{|f(x)|}{\lambda}\right) dx < \infty.$$

The space L_ψ is a Banach function space when endowed with the Luxemburg norm

$$\|f\|_\psi = \|f\|_{L_\psi} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \psi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\Big\}.$$

Each Young function ψ has associated to it a complementary Young function $\overline{\psi}$. For example, if $\psi(t) = t^p$ for $1 < p < \infty$, then $L_\psi = L^p(\mu)$ and $\overline{\psi}(t) = t^{p'}$. Another classical example is given by $\psi(t) = t\,\log(e+t)$. In this case L_ψ is the Zygmund space $L\log L$. The complementary function, $\overline{\psi}(t) = t$ for $0 \le t \le 1$ and $\overline{\psi}(t) = \exp(t-1)$ for $t > 1$, gives the Zygmund space $\exp L$.

We also need a localized version of the Orlicz norm: for every Q, define

$$\|f\|_{\psi,Q} = \inf\Big\{\lambda > 0 : \frac{1}{|Q|}\int_Q \psi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\Big\}.$$

There is a generalized Hölder's inequality associated with these norms:

$$\frac{1}{|Q|}\int_Q |f(x)\,g(x)|\,dx \le 2\,\|f\|_{\psi,Q}\,\|g\|_{\overline{\psi},Q}. \tag{6.6}$$

Sketch of the Proof of Proposition 6.8 The proof of this proposition is similar to the one given for potential operators and uses ideas from [37] (see this reference and [11] for more details). Throughout the proof, let $\psi(t) = t\log(e+t)$. Fix f; without loss of generality we may assume that $f \ge 0$.

The first step of the proof is to discretize the commutator as we did before with the potential operators:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big|[b, I_\alpha]f(x)\big|\,w(x)\,dx \\
&\le \sum_{Q\in\mathcal{D}} \frac{\ell(Q)^\alpha}{|Q|}\int_Q\int_{3\,Q} |b(x)-b(y)|\,f(y)\,dy\,w(x)\,dx \\
&\le C\sum_{Q\in\mathcal{D}} \frac{\ell(Q)^\alpha}{|Q|}\int_Q |b(x)-b_Q|\,w(x)\,dx\int_{3\,Q} f(y)\,dy \\
&\quad +\; C\sum_{Q\in\mathcal{D}} \frac{\ell(Q)^\alpha}{|Q|}\int_{3\,Q} |b(y)-b_Q|\,f(y)\,dy\int_Q w(x)\,dx = C\,(A+B);
\end{aligned}$$

we will estimate each term separately. To estimate A, we use that w satisfies a reverse Hölder inequality with some exponent $\theta > 1$ (since $w \in A_\infty$):

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |b(x)-b_Q|\,w(x)\,dx \\
&\le \Big(\frac{1}{|Q|}\int_Q |b(x)-b_Q|^{\theta'}\,dx\Big)^{1/\theta'}\Big(\frac{1}{|Q|}\int_Q w(x)^\theta\,dx\Big)^{1/\theta} \\
&\le C\,\|b\|_{\mathrm{BMO}}\,\frac{1}{|Q|}\int_Q w(x)\,dx.
\end{aligned}$$

Therefore,

$$A \le C \sum_{Q\in\mathcal{D}} \frac{\ell(Q)^\alpha}{|Q|} \int_{3\,Q} f(y)\,dy \int_Q w(x)\,dx \le C \int_{\mathbb{R}^n} M_\alpha f(x)\,w(x)\,dx$$
$$\le C \int_{\mathbb{R}^n} M_{\psi,\alpha} f(x)\,w(x)\,dx,$$

where the last estimate was shown in [37].

To estimate B, we use the John-Nirenberg inequality: for every cube Q, we have $\|b-b_Q\|_{\exp L,Q} \le C\,\|b\|_{\mathrm{BMO}}$ and $\|b-b_Q\|_{\exp L,3Q} \le C\,\|b\|_{\mathrm{BMO}}$. As we noted above, the conjugate function of e^t-1 is $t\,\log(1+t)$. Hence, by the generalized Hölder's inequality for Orlicz spaces, for every cube Q,

$$\frac{1}{|Q|}\int_{3\,Q} |b(y)-b_Q|\,f(y)\,dy \le 2\,\|b-b_Q\|_{\exp L,3\,Q}\,\|f\|_{L\log L,3\,Q}$$
$$\le C\,\|b\|_{\mathrm{BMO}}\,\|f\|_{L\log L,3\,Q}.$$

Therefore, we conclude that

$$B \le C \sum_{Q\in\mathcal{D}} \ell(Q)^\alpha\,\|f\|_{L\log L,Q} \int_Q w(x)\,dx.$$

We next show that there is a constant C such that for any dyadic cube P,

$$\sum_{Q\in\mathcal{D}:Q\subset P} \ell(Q)^\alpha\,|Q|\,\|f\|_{L\log L,3\,Q} \le C\,\ell(P)^\alpha\,|P|\,\|f\|_{L\log L,3\,P}. \tag{6.7}$$

Using a characterization of the Orlicz norms in [42], for any $\lambda>0$,

$$\sum_{Q\in\mathcal{D}:Q\subset P} \ell(Q)^\alpha\,|Q|\,\|f\|_{L\log L,3\,Q}$$
$$\le C \sum_{Q\in\mathcal{D}:Q\subset P} \ell(Q)^\alpha\,|Q| \inf_{\lambda>0}\left\{\lambda + \frac{\lambda}{|3\,Q|}\int_{3\,Q}\psi\left(\frac{|f(x)|}{\lambda}\right)dx\right\}$$
$$\le C\,\lambda \sum_{Q\in\mathcal{D}:Q\subset P} \ell(Q)^\alpha \int_{3\,Q}\left(1+\psi\left(\frac{|f(x)|}{\lambda}\right)\right)dx$$
$$\le C\,\lambda\,\ell(P)^\alpha \int_{3\,P}\left(1+\psi\left(\frac{|f(x)|}{\lambda}\right)\right)dx$$
$$= C\,\ell(P)^\alpha\,|P|\left(\lambda + \frac{1}{|3\,P|}\int_{3\,P}\psi\left(\frac{|f(x)|}{\lambda}\right)dx\right),$$

where in the last inequality we used [37, Lemma 3.1]. This estimate holds for every $\lambda>0$ and so we can take the infimum over all λ to get (6.7).

We can now argue as we did in the proof of Proposition 6.5, replacing a sum over all dyadic cubes with a sum over Calderón-Zygmund cubes. With the same notation as there, we have that

$$\begin{aligned} B &\leq C \sum_{k,j} \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w(x)\,dx \sum_{Q\in\mathcal{D}:Q\subset Q_{k,j}} \ell(Q)^\alpha\,|Q|\,\|f\|_{L\log L,3\,Q} \\ &\leq C \sum_{k,j} \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w(x)\,dx\,\ell(Q_{k,j})^\alpha\,|Q_{k,j}|\,\|f\|_{L\log L,3\,Q_{k,j}} \\ &\leq C \sum_{k,j} \int_{E_{k,j}} M_{\psi,\alpha}f(x)\,w(x)\,dx \leq C \int_{\mathbb{R}^n} M_{\psi,\alpha}f(x)\,w(x)\,dx. \end{aligned}$$

Combining our estimates for A and B we get the desired result. □

6.4 Extensions of the theorems of Boyd and Lorentz-Shimogaki

We present an extension of the classical results of Boyd and Lorentz-Shimogaki to a wider class of operators and also to weighted and vector-valued estimates. These two theorems are basic in the theory of rearrangement invariant Banach function spaces. They characterize those RIBFS on which the Hilbert transform, in the case of Boyd, or the Hardy-Littlewood maximal function, in the case of Lorentz-Shimogaki, are bounded operators.

Theorem 6.9

Let $\mathbb{X}$ be a rearrangement invariant Banach function space associated to $(\mathbb{R}, dx)$, let H be the Hilbert transform and let M be the Hardy-Littlewood maximal function. Then,

- **[Boyd, 1967]** *H is bounded on $\mathbb{X}$ if and only if $1 < p_{\mathbb{X}} \leq q_{\mathbb{X}} < \infty$.*
- **[Lorentz, 1955; Shimogaki, 1965]** *M is bounded on $\mathbb{X}$ if and only if $p_{\mathbb{X}} > 1$.*

The proofs of these results (see [2, 27, 48] or [1, p. 154]) are based on the pointwise rearrangement inequalities

$$(Hf)^*(t) \leq C\left(\frac{1}{t}\int_0^t f^*(s)\,ds + \int_t^\infty f^*(s)\frac{ds}{s}\right)$$

and

$$(Mf)^*(t) \approx \frac{1}{t}\int_0^t f^*(s)\,ds,$$

$0 < t < \infty$, where f^* is the decreasing rearrangement of f. Observe that the righthand side of the second estimate is just the classical Hardy operator acting on f^* and that in the inequality for H we have the sum of the Hardy operator and its adjoint. By using the fact that the Hardy operator is bounded on $\mathbb{X}$ if and only if $p_{\mathbb{X}} > 1$, we get the restrictions on the Boyd indices that guarantee the boundedness of H and M. Both results were originally proved for Banach spaces, but they have been extended to the quasi-Banach case in [32] with the same restriction on the Boyd indices.

In [15] the following extension of these classical results is obtained.

Theorem 6.10 ([15])

Let T be a Calderón-Zygmund singular integral and let M be the Hardy-Littlewood maximal function. Let $\mathbb{X}$ be a RIQBFS which is p-convex for some $p > 0$ (equivalently, $\mathbb{X}^r$ is Banach for some $r \geq 1$).

(*i*) *If $1 < p_{\mathbb{X}} \leq \infty$, then M is bounded on $\mathbb{X}(w)$ for all $w \in A_{p_{\mathbb{X}}}$.*

(*ii*) *If $1 < p_{\mathbb{X}} \leq q_{\mathbb{X}} < \infty$, then for all $w \in A_{p_{\mathbb{X}}}$, T satisfies the following weighted inequality*

$$\|Tf\|_{\mathbb{X}(w)} \leq C\,\|f\|_{\mathbb{X}(w)}.$$

In particular, T is bounded on $\mathbb{X}$.

(*iii*) *If $1 < p_{\mathbb{X}} \leq q_{\mathbb{X}} < \infty$ we have that for all $1 < q < \infty$ and for all $w \in A_{p_{\mathbb{X}}}$, M satisfies*

$$\Big\| \Big(\sum_j (Mf_j)^q \Big)^{1/q} \Big\|_{\mathbb{X}(w)} \leq C \Big\| \Big(\sum_j |f_j|^q \Big)^{1/q} \Big\|_{\mathbb{X}(w)}. \qquad (6.8)$$

In particular, taking $w \equiv 1$, M satisfies the corresponding vector-valued inequalities on $\mathbb{X}$. Similarly, T verifies the same estimates.

Part (*i*) extends Lorentz-Shimogaki's result to the case of weighted RIBFS. Part (*ii*) generalizes Boyd's theorem to cover both more general operators and Muckenhoupt weights. Part (*iii*) extends both Boyd's and Lorentz-Shimogaki's results by establishing weighted (and unweighted) vector-valued estimates for M and T.

We also present a version of this result for modular inequalities.

Theorem 6.11 ([15])

Let T be a Calderón-Zygmund singular integral and let M be the Hardy-Littlewood maximal function. Let $\phi \in \Phi$ be such that ϕ is quasi-convex.

(*i*) *Let* $w \in A_{i_\phi}$. *If* $1 < i_\phi \le \infty$,

$$\int_{\mathbb{R}^n} \phi\big(Mf(x)\big)\, w(x)\, dx \le C \int_{\mathbb{R}^n} \phi\big(C\,|f(x)|\big)\, w(x)\, dx;$$

if $i_\phi = 1$,

$$\sup_\lambda \phi(\lambda)\, w\big\{x \in \mathbb{R}^n : Mf(x) > \lambda\big\} \le C \int_{\mathbb{R}^n} \phi\big(C\,|f(x)|\big)\, w(x)\, dx.$$

(*ii*) *Let* $\phi \in \Delta_2$ (*i.e.,* $I_\phi < \infty$) *and* $w \in A_{i_\phi}$. *If* $i_\phi > 1$,

$$\int_{\mathbb{R}^n} \phi\big(|Tf(x)|\big)\, w(x)\, dx \le C \int_{\mathbb{R}^n} \phi\big(|f(x)|\big)\, w(x)\, dx;$$

if $i_\phi = 1$,

$$\sup_\lambda \phi(\lambda)\, w\big\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\big\} \le C \int_{\mathbb{R}^n} \phi\big(|f(x)|\big)\, w(x)\, dx.$$

(*iii*) *Let* $\phi \in \Delta_2$ (*i.e.,* $I_\phi < \infty$), $w \in A_{i_\phi}$ *and* $1 < q < \infty$. *If* $i_\phi > 1$,

$$\int_{\mathbb{R}^n} \phi\bigg(\Big(\sum_j Mf_j(x)^q\Big)^{1/q}\bigg)\, w(x)\, dx$$
$$\le C \int_{\mathbb{R}^n} \phi\bigg(\Big(\sum_j |f_j(x)|^q\Big)^{1/q}\bigg)\, w(x)\, dx;$$

if $i_\phi = 1$,

$$\sup_\lambda \phi(\lambda)\, w\bigg\{x : \Big(\sum_j Mf_j(x)^q\Big)^{1/q} > \lambda\bigg\}$$
$$\le C \int_{\mathbb{R}^n} \phi\bigg(\Big(\sum_j |f_j(x)|^q\Big)^{1/q}\bigg)\, w(x)\, dx.$$

Similarly, T *satisfies the same estimates.*

Part (*i*), under slightly stronger hypotheses (i.e., ϕ, $\overline{\phi} \in \Delta_2$), was first proved in [23] (see also [24].) Conclusions (*ii*) and (*iii*) generalize some of the estimates obtained, by different methods, in [24, Chapters 1, 2]. We refer the reader to this book for a complete account of modular inequalities.

The proofs of both results follow a common path. In each, part (*i*) is proved directly with no use of extrapolation. Part (*ii*) in each result follows by extrapolation: Coifman's inequality (2.1) gives the starting estimates for the family of pairs $(|Tf|, Mf)$ with $f \in L^\infty_c(\mathbb{R}^n)$. Note that (3.2) holds for all $0 < p < \infty$. Thus, parts (b) and (c) in Theorem 3.2 yield that M controls T in RIQBFS and also in the modular sense. Then, by (i) we get the desired estimates in part (ii).

Part (iii) is again done by extrapolation: first it is shown in [15] that for any $1 < q < \infty$, for all $0 < p < \infty$ and all $w \in A_\infty$,

$$\Big\| \Big(\sum_j (Mf_j)^q \Big)^{1/q} \Big\|_{L^p(w)} \leq \big\| M\big(\|\{f_j\}_j\|_{\ell^q}\big) \big\|_{L^p(w)}.$$

This gives a new starting estimate in Theorem 3.2 (which holds for all $0 < p < \infty$) and so the vector-valued operator associated with M is controlled by M itself. Thus part (i) yields the desired vector-valued estimates for M. Since Theorem 3.2 yields that the vector-valued operator associated with T is controlled by the one associated with M, and we have already shown the desired estimates for M, we conclude that T satisfies the same inequalities.

6.5 Commutators of Singular Integrals

Let T be a Calderón-Zygmund singular integral and $b \in \mathrm{BMO}$. The first order commutator is defined as

$$C_b^1 f(x) = [b, T]f(x) = b(x)\, Tf(x) - T(b\, f)(x),$$

and, for $m \geq 2$, we define by induction the higher order commutators $C_b^m f(x) = [b, C_b^{m-1}]f(x)$. In [39] it is shown that

$$\int_{\mathbb{R}^n} |C_b^m f(x)|^p\, w(x)\, dx \leq C \int_{\mathbb{R}^n} M^{m+1} f(x)^p\, w(x)\, dx \tag{6.9}$$

for every $0 < p < \infty$ and $w \in A_\infty$, where M^{m+1} is the Hardy-Littlewood maximal function iterated $m+1$ times. Thus using Theorem 3.2, parts (b) and (c), we get that M^{m+1} controls C_b^m. Hence, we can find end-point estimates for C_b^m once we show them for M^{m+1}. This was done in [15] where the following result was proved.

Theorem 6.12 ([15])

Let $\varphi_m(t) = \dfrac{t}{(1+\log^+ t)^m}$ *and* $\psi_m(t) = t\,(1+\log^+ t)^m$. *Then*

$$C_b^m : L(\log L)^m \longrightarrow \widetilde{\mathbb{M}}_{\varphi_m}$$

and

$$\big|\{x \in \mathbb{R}^n : |C_b^m f(x)| > \lambda\}\big| \leq C \int_{\mathbb{R}^n} \psi_m\left(\frac{|f(x)|}{\lambda}\right) dx.$$

Similarly, both estimates hold for all $w \in A_1$.

In this later result, the Marcinkiewicz type space $\widetilde{\mathbb{M}}_{\varphi_m}$ is defined by the function quasi-norm

$$\|f\|_{\widetilde{\mathbb{M}}_{\varphi_m}} = \sup_t \varphi_m(t)\, f^*(t).$$

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