Minimal time of controllability of two parabolic equations with disjoint control and coupling domains

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Abstract

We consider two parabolic equations coupled by a matrix $A(x) = q(x)A_0$, where A_0 is a Jordan block of order 1, and controlled by a single localized function, or by a single boundary control. The support of the coupling coefficient, q, and the control domain may be disjoint. We exhibit an explicit minimal time of null-controllability, $T_0(q) \in [0, +\infty]$.

Résumé

On considère deux équations paraboliques couplées par une matrice $A(x) = q(x)A_0$, où A_0 est un bloc de Jordan d'ordre 1, et contrôlées par un seul contrôle localisé en espace ou frontière. Le support du coefficient de couplage, q, et du contrôle peuvent être disjoints. Nous mettons en évidence un temps minimal de contrôlabilité à 0, $T_0(q) \in [0, +\infty]$.

Version française abrégée

L'objet de cette note est d'étudier la contrôlabilité à zéro du système parabolique (1). Il est connu (voir par exemple [15], [4] et [12]) que si $\operatorname{Supp} q \cap \omega \neq \emptyset$ le système (1) avec $B \neq 0$ et $C = 0$ est contrôlable à zéro en tout temps $T > 0$. Lorsque $\operatorname{Supp} q \cap \omega = \emptyset$, seuls quelques résultats ont été obtenus (voir [13], [8] et [2, 3], [14], [1], [9]). Dans [14], [1] et [9], les auteurs établissent la contrôlabilité à zéro en tout temps $T > 0$ dans le cas où le couplage q est positif. Dans cette note, on établit, en notant $\varphi_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx)$ et $I_k(q) = \int_0^\pi q(x)\, \varphi_k^2(x)\, dx$, que, aussi bien dans le cas de la contrôlabilité interne que dans celui de la contrôlabilité par le bord, il peut exister un temps minimal de contrôle $T_0(q) > 0$.

On montre plus précisément le résultat suivant.

Théorème 0.1. *Supposons que $I_k(q) \neq 0$ pour tout $k \geq 1$ et soit*

$$T_0(q) = T_0 := \limsup_{k\to+\infty} \frac{\log \frac{1}{|I_k|}}{k^2} \in [0, \infty].$$

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1. ***Contrôlabilité interne*** $(B \neq 0$ *et* $C = 0)$. *Soit* $\omega = (a, b)$ *avec* $0 < a < b < \pi$. *Pour tout* $T > T_0$, *le système (1) est contrôlable à zéro au temps* T. *Sous l'hypothèse* $\operatorname{Supp} q \subset (0, a)$ *ou* $\operatorname{Supp} q \subset (b, \pi)$, *pour tout* $T < T_0$ *le système (1) n'est pas contrôlable à zéro au temps* T.
2. ***Contrôlabilité par le bord*** $(B = 0$ *et* $C \neq 0)$. *Si* $T > T_0$, *le système (1) est contrôlable à zéro au temps* T. *Pour tout* $T < T_0$, *le système (1) n'est pas contrôlable à zéro au temps* T.

Remarque 0.1. *La condition* $I_k(q) \neq 0$ *pour tout* $k \geq 1$ *est nécessaire et suffisante pour la contrôlabilité approchée frontière* $(B = 0)$ *du système (1) (voir [5]). Elle est aussi nécessaire et suffisante pour la contrôlabilité approchée interne* $(C = 0)$ *du même système sous l'hypothèse géométrique (A1) (voir [8]).*

On peut alors se demander s'il peut arriver que $T_0(q) > 0$. En fait, on a:

Théorème 0.2. *Pour tout* $\tau \in [0, +\infty]$, *il existe* $q \in L^\infty(0, \pi)$ *tel que* $T_0(q) = \tau$.

On notera que si $\int_0^\pi q(x)\, dx \neq 0$ alors $T_0(q) = 0$. C'est en particulier le cas dans [14]. Noter que pour tout $\tau \in [0, \infty]$, il existe $q \in L^\infty(0, \pi)$ tel que $\operatorname{Supp} q = [0, \pi]$ et $T_0(q) = \tau$. Pour une telle fonction q le résultat de contrôlabilité frontière à zéro n'a pas lieu pour $T < \tau$.

1. Main results and comments

Let $T > 0$ an $\omega = (a, b) \subset (0, \pi)$ be fixed and let us consider the following control problem:

$$\begin{cases} y_t - y_{xx} + q(x) A_0 y = B u 1_\omega & \text{in } Q_T := (0, \pi) \times (0, T), \\ y(0, \cdot) = C v, \quad y(\pi, \cdot) = 0 \text{ on } (0, T), & y(\cdot, 0) = y^0 \text{ in } (0, \pi), \end{cases} \tag{1}$$

where $A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^2)$, $B = \begin{pmatrix} 0 \\ b \end{pmatrix}$ and $C = \begin{pmatrix} 0 \\ c \end{pmatrix}$ are vectors of $\mathbb{R}^2$, $q \in L^\infty(0, \pi)$, y^0 is the initial datum and $u \in L^2(Q_T)$ and $v \in L^2(0, T)$ are the control functions. We will consider two different issues: distributed control (i.e. $C = 0$, $B \neq 0$) and boundary control (i.e., $C \neq 0$, $B = 0$). In each case we ask if for every $y^0 \in L^2(0, \pi; \mathbb{R}^2)$ (resp. $y^0 \in H^{-1}(0, \pi; \mathbb{R}^2)$) there exists u (resp. v) such that the solution y of (1) satisfies $y(T) = 0$ in $(0, \pi)$. In the sequel, we set $\varphi_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx)$ for $x \in (0, \pi)$ and $k \geq 1$. To the function $q \in L^\infty(0, \pi)$ we associate the sequence $\{I_k(q)\}_{k \geq 1}$ and the number $T_0(q)$ defined by

$$I_k(q) = \int_0^\pi q(x) \varphi_k^2(x)\, dx, \quad T_0(q) = T_0 := \limsup_{k \to +\infty} \frac{\log \frac{1}{|I_k(q)|}}{k^2} \in [0, \infty]. \tag{2}$$

It is well known (see for instance [15], [4] and [12]) that when $\operatorname{Supp} q \cap \omega \neq \emptyset$ the internal null-controllability result for System (1) $(B \neq 0,\ C = 0)$ is valid for any time $T > 0$. When $\operatorname{Supp} q \cap \omega = \emptyset$, only a few results are known (see [13], [8] and [2, 3], [14], [1], [9]). In [14], [1] and [9], the authors prove the internal null-controllability of System (1) for all time $T > 0$ in the case where the coupling coefficient $q \not\equiv 0$ is non-negative.

Throughout this paper and in some situations, we are going to consider the following geometrical assumption

Assumption (A1): The function q satisfies $\operatorname{Supp} q \subset (0, a)$ or $\operatorname{Supp} q \subset (b, \pi)$.

Remark 1.1. *Concerning the boundary controllability problem for System (1)* $(B \neq 0,\ C = 0)$, *the first results were proved in [2, 3] for particular coupling matrices. In [5], it is proved that System (1) is boundary approximate controllable at any time* $T > 0$ *if and only if* $I_k(q) \neq 0$ *for all* $k \geq 1$. *When* $\int_0^\pi q(x)\, dx \neq 0$, *in [5] it is also proved that condition* $I_k(q) \neq 0$ *for any* $k \geq 1$ *characterizes boundary null-controllability property for System (1) at any time* $T > 0$.

On the other hand, under Assumption (A1), System (1) is distributed approximately controllable at any time $T > 0$ *if and only if* $I_k(q) \neq 0$ *for all* $k \geq 1$ *(see [8]).*

The objective of this Note is to give a complete answer about the null-controllability properties of System (1) in the boundary case and under the geometrical assumption (A1) in the distributed case. One has:

Theorem 1.1. *Assume $I_k(q) \neq 0$ for all $k \geq 1$ and consider T_0 given in (2). Then,*

1. ***Internal controllability*** *($B \neq 0$ et $C = 0$). If $T > T_0$, System (1) is null-controllable at time T. Under the geometrical assumption (A1), for any $T < T_0$, System (1) is not null-controllable at time T.*
2. ***Boundary controllability*** *($B = 0$ et $C \neq 0$). If $T > T_0$, System (1) is null-controllable at time T. For any $T < T_0$, System (1) is not null-controllable at time T.*

Theorem 1.1 asserts that there is a minimal control time for both boundary and internal controllability. It remains to check that there exist functions $q \in L^\infty(0, \pi)$ for which $T_0(q) > 0$. Indeed, the following result shows that T_0 can be any non negative real number or even $+\infty$.

Theorem 1.2. *For any $\tau \in [0, +\infty]$, there exists $q \in L^\infty(0, \pi)$ such that $T_0(q) = \tau$.*

At this level, some consequences of Theorems 1.1 and 1.2 must be stressed. The null-controllability property of System (1) depends on the coupling function q. This dependence is described by the asymptotic behavior of $I_k(q)$. Observe that, when $I_k(q) \neq 0$ for any $k \geq 1$, System (1) is approximately controllable at any positive time. Nevertheless, the corresponding null-controllability property could fail at a given $T > 0$ or even at any positive time. We have already pointed out this fact for the boundary controllability of this kind of systems (see [6]). But, to our knowledge, this fact is new for internal controllability by L^2-controls supported in space. In [5] it is shown that if $\int_0^\pi q(x)\,dx \neq 0$, then $T_0(q) = 0$. This is the case in [14]. Observe that for any $\tau \in [0, \infty]$, there exists $q \in L^\infty(0, \pi)$ such that $\operatorname{Supp} q = [0, \pi]$ and $T_0(q) = \tau$. For this function q the boundary null-controllability result fails when $T < \tau$. This Note is part of the results on null-controllability for System (1) which will be developed in [7], a forthcoming work of the authors.

2. Tools for the proofs. Reduction to a problem of moments

Let us consider the operator $L := -\frac{d^2}{dx^2} Id + q(x)A_0 : D(L) \subset L^2(0, \pi; \mathbb{R}^2) \longrightarrow L^2(0, \pi; \mathbb{R}^2)$ with domain $D(L) = H^2(0, \pi; \mathbb{R}^2) \cap H_0^1(0, \pi; \mathbb{R}^2)$. We will assume in the sequel that $I_k(q) \neq 0$ for all $k \geq 1$. In this case, direct computations provide that the family $\mathcal{B} = \left\{ \Phi_{k,1} := \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}, \Phi_{k,2} := \begin{pmatrix} \psi_k \\ \frac{1}{I_k}\varphi_k \end{pmatrix} : k \geq 1 \right\}$ is a basis of root vectors (generalized eigenfunctions) of the operator $(L, D(L))$ in $L^2(0, \pi; \mathbb{R}^2)$. The family $\mathcal{B}^* = \left\{ \Phi^*_{k,1} := \begin{pmatrix} \varphi_k \\ I_k \psi_k \end{pmatrix}, \Phi^*_{k,2} := \begin{pmatrix} 0 \\ I_k \varphi_k \end{pmatrix} : k \geq 1 \right\}$ is biorthogonal to $\mathcal{B}$ and satisfies $\left(L^* - k^2 I_d\right) \Phi^*_{k,1} = \Phi^*_{k,2}$ and $\left(L^* - k^2 I_d\right) \Phi^*_{k,2} = 0$, for $k \geq 1$. With the notation $h_k(x) = 1 - q(x)/I_k$, the function ψ_k is given by:

$$\psi_k(x) = \alpha_k \varphi_k(x) - \frac{1}{k} \int_0^x \sin(k(x-\xi)) h_k(\xi) \varphi_k(\xi)\, d\xi, \quad \alpha_k = \frac{1}{k} \int_0^\pi \int_0^x \sin(k(x-\xi)) h_k(\xi) \varphi_k(\xi) \varphi_k(x)\, d\xi\, dx. \tag{3}$$

With the previous notation, one has:

Lemma 2.1. *There exists a constant $C > 0$ such that*

$$|I_k \alpha_k| \leq \frac{C}{k}, \quad \|I_k \psi_k\|_{L^\infty(0,\pi)} \leq \frac{C}{k}, \quad \|I_k \psi_k'\|_{L^\infty(0,\pi)} \leq C, \quad \forall k \geq 1. \tag{4}$$

Introduce the backward adjoint problem associated with System (1):

$$\begin{cases} -\theta_t - \theta_{xx} + q(x) A_0^* \theta = 0 & \text{in } Q_T, \\ \theta(0, \cdot) = \theta(\pi, \cdot) = 0 \text{ on } (0, T), & \theta(\cdot, T) = \theta^0 \text{ in } (0, \pi), \end{cases} \tag{5}$$

where $\theta^0 \in H^{-1}(0,\pi;\mathbb{R}^2)$. If y is the solution of System (1) associated with $y^0 \in L^2(0,\pi;\mathbb{R}^2)$ ($y^0 \in H^{-1}(0,\pi;\mathbb{R}^2)$ for the boundary problem) $u \in L^2(Q_T)$ and $v \in L^2(0,T)$, then it can be easily checked that $y(T) = 0$ in Ω if and only if

$$\iint_{Q_T} u 1_\omega B^*\theta \, dx \, dt + \int_0^T v(t) C^*\theta_x(0,t) \, dt = -\langle y^0, \theta(\cdot,0)\rangle_{H^{-1},H_0^1}, \quad \forall \theta^0 \in H_0^1(0,\pi;\mathbb{R}^2).$$

For all $k \geq 1$, if $\theta^0 = \Phi^*_{k,1}$, then $\theta_{k,1}(\cdot,t) = e^{-k^2(T-t)}\Phi^*_{k,1} - (T-t)e^{-k^2(T-t)}\Phi^*_{k,2}$ is the associated solution of (5) and if $\theta^0 = \Phi^*_{k,2}$, the associated solution of (5) is $\theta_{k,2}(\cdot,t) = e^{-k^2(T-t)}\Phi^*_{k,2}$. Thus:

- For $C = 0$ (internal controllability), we seek a control in the form $u(x,t) = f(x)\gamma(t)$. Let $f_{k,1} := \int_\omega f(x)\varphi_k(x)\,dx$ and $f_{k,2} := \int_\omega f(x)\psi_k(x)\,dx$ for all $k \geq 1$. Assuming that a function f can be found such that $f_{k,1} \neq 0$ for all $k \geq 1$ and proceeding as in [6] and [11] we reduce the null-controllability issue to the following problem of moments:

$$\begin{cases} \displaystyle\int_0^T e^{-k^2 t}\gamma(T-t)\,dt = -\frac{e^{-k^2T}}{bI_k f_{k,1}}\int_0^\pi y^0\cdot\Phi^*_{k,2}\,dx := M_{k,1}(y^0), \\ \displaystyle\int_0^T te^{-k^2 t}\gamma(T-t)\,dt = \frac{e^{-k^2T}}{bI_k f_{k,1}}\int_0^\pi y^0\cdot\left(\Phi^*_{k,1} - \left(T + \frac{f_{k,2}}{f_{k,1}}\right)\Phi^*_{k,2}\right)dx := M_{k,2}(y^0), \ \forall k \geq 1. \end{cases} \tag{6}$$

- For $B = 0$ (boundary controllability), we get in the same way the problem of moments:

$$\begin{cases} \displaystyle\int_0^T e^{-k^2 t}v(T-t)\,dt = -\frac{e^{-k^2T}}{cI_k\varphi_k'(0)}\langle y^0, \Phi^*_{k,2}\rangle_{H^{-1},H_0^1} := \widetilde{M}_{k,1}(y^0), \\ \displaystyle\int_0^T te^{-k^2 t}v(T-t)t\,dt = \frac{e^{-k^2T}}{cI_k\varphi_k'(0)}\langle y^0, \Phi^*_{k,1} - \left(T + \frac{\psi_k'(0)}{\varphi_k'(0)}\right)\Phi^*_{k,2}\rangle_{H^{-1},H_0^1} := \widetilde{M}_{k,2}(y^0), \ \forall k \geq 1. \end{cases} \tag{7}$$

3. Internal null-controllability

Let us take $T > T_0$. In view of the relations (6), we first build a function $f \in L^2(0,\pi)$ such that $f_{k,1} \neq 0$ for all $k \geq 1$, where $f_{k,1} = \displaystyle\int_\omega f(x)\varphi_k(x)\,dx$.

Lemma 3.1. *There exists $f \in L^2(0,\pi)$ such that $Supp\, f \subset \omega$ and for all $\varepsilon > 0$ one has $\inf_{k\geq 1} f_{k,1}e^{\varepsilon k^2} > 0$.*

Sketch of the proof. Let $f = 1_{(a_0,b_0)}$ with $(a_0,b_0) \subset \omega$ and $r_1 := \frac{b_0-a_0}{2\pi}$, $r_2 := \frac{b_0+a_0}{2\pi} \notin \mathbb{Q}$. Then,

$$f_{k,1} = \int_{a_0}^{b_0} f(x)\varphi_k(x)dx = \frac{2\sqrt{2}}{k\sqrt{\pi}}\sin(\pi k r_1)\sin(\pi k r_2) \neq 0, \quad \forall k \geq 1.$$

If r_1 and r_2 are algebraic numbers of order $d \geq 2$, using diophantine approximations it can be proved that $|f_{k,1}|^2 \underset{k\to\infty}{\sim} \frac{1}{2\pi}\frac{c}{k^{4d-2}}$. ■

Now from the results in [11], the family $\left\{e_{k,1} = e^{-k^2t}, e_{k,2} = te^{-k^2t}\right\}_{k\geq 1}$ admits a biorthogonal family $\{q_{k,1}, q_{k,2}\}_{k\geq 1}$ in $L^2(0,T)$, i.e.,

$$\int_0^T e_{k,r}q_{j,s}(t) = \delta_{kj}\delta_{rs}, \quad \forall k,j \geq 1, \quad 1 \leq r,s \leq 2, \tag{8}$$

which moreover satisfies that for every $\varepsilon > 0$ there exists $C_{\varepsilon,T} > 0$ such that $\|(q_{k,1}, q_{k,2})\|_{L^2(0,T)} \leq C_{\varepsilon,T}e^{\varepsilon k^2}$ for any $k \geq 1$.

Looking for $\gamma \in L^2(0,T)$ in the form $\gamma(T-t) = \sum_{k\geq 1} \left(\gamma_k^1 q_{k,1}(t) + \gamma_k^2 q_{k,2}(t)\right)$ and using (8), we see that γ satisfies (6) if and only if

$$\gamma_k^1 = M_{k,1}(y^0) \quad \text{and} \quad \gamma_k^2 = M_{k,2}(y^0), \quad \forall k \geq 1.$$

Taking into account Lemma 3.1, inequalities (4) and the definition of $T_0 = T_0(q)$ in (2), we get that for all $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ (independent of k) such that

$$|\gamma_k^1| + |\gamma_k^2| \leq C_\varepsilon e^{-k^2(T-T_0-2\varepsilon)}|y^0|, \quad \forall k \geq 1.$$

Taking for instance $\varepsilon = (T-T_0)/4$, the previous inequality ensures that the series which defines γ converges in $L^2(0,T)$. This gives the proof of the internal null-controllability of System (1) if $T > T_0(q)$.

Assume now that $T \in (0, T_0(q))$ and, in particular $I_k(q) \to 0$. We will prove that (1) is not null-controllable at time T by contradiction. Indeed, System (1) is null-controllable at time T if and only if there exists $C > 0$ such that any solution θ of the adjoint problem (5) satisfies the observability inequality:

$$\|\theta(0)\|^2_{L^2(0,\pi;\mathbb{R}^2)} \leq C \int_0^T \int_\omega |\theta_2|^2 \, dx \, dt, \quad \forall \theta^0 \in L^2\left(0,\pi;\mathbb{R}^2\right). \tag{9}$$

Let us fix $k \geq 1$. For $\theta^0 = a_k \Phi_{k,1}^* + b_k \Phi_{k,2}^*$ with $(a_k, b_k)_{k\geq 1} \subset \mathbb{R}^2$, the previous inequality reads as $A_{k,1} \leq C A_{k,2}$, with

$$\begin{aligned} A_{k,1} &:= e^{-2k^2 T}\left\{|a_k|^2\left(1 + I_k^2|\psi_k|^2 + T^2 I_k^2\right) + |b_k|^2 I_k^2 - 2a_k b_k T I_k^2\right\}, \\ A_{k,2} &:= I_k^2 \int_0^T \int_\omega e^{-2k^2 t}\left|a_k \psi_k(x) + (b_k - t a_k)\varphi_k(x)\right|^2 \, dx. \end{aligned}$$

Now, we will use the following expression of $\psi_k(x)$ deduced from (3):

$$\begin{cases} \psi_k(x) = \tau_k(x)\varphi_k(x) + g_k(x), \quad \tau_k(x) = \alpha_k + \dfrac{1}{2kI_k}\displaystyle\int_0^x \sin(2k\xi) q(\xi)\, d\xi; \\ g_k(x) = -\dfrac{1}{k}\displaystyle\int_0^x \sin(k(x-\xi))\varphi_k(\xi)\, d\xi - \dfrac{\sqrt{\pi/2}}{kI_k}\displaystyle\int_0^x q(\xi)\varphi_k^2(\xi)\, d\xi \cos(kx). \end{cases}$$

If we assume that $\operatorname{Supp} q \cap \omega = \emptyset$, then the function τ_k is constant on $\omega = (a,b)$ and, thanks to Lemma 2.1, $\tau_k I_k \to 0$ uniformly on $(0,\pi)$. Moreover, if $\operatorname{Supp} q \subset (0,a)$ or $\operatorname{Supp} q \subset (b,\pi)$, then $\|g_k\|_{L^\infty(\omega)} \leq C/k$ for any $k \geq 1$. Indeed, for $x \in \omega$ one has:

$$g_k(x) = \begin{cases} -\dfrac{1}{k}\displaystyle\int_0^x \sin(k(x-\xi))\varphi_k(\xi)\, d\xi - \dfrac{\sqrt{\pi/2}}{k}\cos(kx) & \text{if} \quad \operatorname{Supp} q \subset (0,a), \\ -\dfrac{1}{k}\displaystyle\int_0^x \sin(k(x-\xi))\varphi_k(\xi)\, d\xi & \text{if} \quad \operatorname{Supp} q \subset (b,\pi). \end{cases}$$

Thus, in this case, we can choose $a_k = 1$ and $b_k = -\tau_k$, to get:

$$A_{k,2} = I_k^2 \int_0^T \int_\omega e^{-2k^2 t}\left|g_k(x) - t\varphi_k(x)\right|^2 \, dx\, dt \leq C I_k^2.$$

On the other hand, using again that $\tau_k I_k \to 0$, we also deduce the existence of $k_0 \geq 1$ such that

$$A_{k,1} = e^{-2k^2 T}\left\{1 + I_k^2|\psi_k|^2 + T^2 I_k^2 + \tau_k^2 I_k^2 + 2T\tau_k I_k^2\right\} \geq \frac{1}{2}e^{-2k^2 T}, \quad \forall k \geq k_0.$$

Inequality (9) leads to $1 \leq C e^{2k^2 T} I_k^2$ for all $k \geq k_0$. From the definition of T_0 in (2), there exists a subsequence of $\{I_k\}_{k\geq k_0}$ (still denoted by $\{I_k\}_{k\geq k_0}$) satisfying: for any $\varepsilon > 0$ there is $k_1(\varepsilon) \geq 1$ such that $I_k^2 \leq e^{-2k^2(T_0-\varepsilon)}$ for all $k \geq k_1(\varepsilon)$. In particular, $1 \leq C e^{2k^2(T-T_0+\varepsilon)}$ for any $k \geq k_1(\varepsilon)$. Taking $\varepsilon = (T_0-T)/2 > 0$, the previous inequality provides a contradiction and completes the proof of Theorem 1.1 for internal controllability.

4. Boundary null-controllability

We assume here that $B = 0$ and we have to solve the problem of moments (7). Using the previous arguments, it is not difficult to see that $v(T-t) = \sum_{k\geq 1}\left(\widetilde{M}_{k,1}(y^0)q_{k,1}(t) + \widetilde{M}_{k,2}(y^0)q_{k,2}(t)\right)$ is a formal solution of (7). Using the estimates (3), (4) and the definition of $T_0(q)$, it can be also checked that $v \in L^2(0,T)$ when $T > T_0(q)$. This finalizes the positive part of point 2 in Theorem 1.1.

If $T < T_0(q)$, we again reason by contradiction. In this case, the observability inequality for a solution θ to (5) is:

$$\|\theta(0)\|^2_{H^1_0(0,\pi;\mathbb{R}^2)} \leq C\int_0^T \left|\frac{\partial \theta_2}{\partial x}(0,t)\right|^2 dt, \quad \forall \theta^0 \in H^1_0(0,\pi;\mathbb{R}^2).$$

For $\theta^0 = \Phi^*_{k,1} - (\psi'(0)/k)\,\Phi^*_{k,2}$ this inequality gives:

$$e^{-2k^2T}\left\{k^2 + I_k^2\|\psi_k\|^2_{H^1_0(0,\pi)} + \left[T^2k^2 + \psi_k'(0)^2\right]I_k^2 + 2T\psi_k'(0)I_k^2 k\right\} \leq k^2 I_k^2 \int_0^T e^{-2k^2t}\, t^2\, dt \leq Ck^2 I_k^2.$$

Then, as in previous computations and using once more (4), we infer the existence of $k_2 \geq 1$ such that $1 \leq Ce^{2k^2T}I_k^2$ for any $k \geq k_2$. As previously, this gives a contradiction with the definition of $T_0(q)$ and ends the proof of Theorem 1.1.

References

[1] F. Alabau-Boussouira, *Insensitizing exact controls for the scalar wave equation and exact controllability of 2-coupled cascade systems of PDE's by a single control*, Math. Control Signals Systems **26** (2014), no. 1, 1–46.

[2] F. Alabau-Boussouira, M. Léautaud, *Indirect controllability of locally coupled systems under geometric conditions*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 7-8, 395–400.

[3] F. Alabau-Boussouira, M. Léautaud, *Indirect controllability of locally coupled wave-type systems and applications*, J. Math. Pures Appl. (9) **99** (2013), no. 5, 544–576.

[4] F. Ammar Khodja, A. Benabdallah, C. Dupaix, *Null-controllability of some reaction-diffusion systems with one control force*, J. Math. Anal. Appl. **320** (2006), no. 2, 928–943.

[5] F. Ammar Khodja, A. Benabdallah, M. González-Burgos, L. de Teresa, *Controllability of some systems of parabolic equations*, Proceedings of the II Encuentro RSME - SMM (Málaga, 2012), http://personal.us.es/manoloburgos/es/conferencias/

[6] F. Ammar Khodja, A. Benabdallah, M. González-Burgos, L. de Teresa, *A new relation between the condensation index of complex sequences and the null controllability of parabolic systems*, C. R. Math. Acad. Sci. Paris **351** (2013), no. 19–20, 743–746.

[7] F. Ammar Khodja, A. Benabdallah, M. González-Burgos, L. de Teresa, *Controllability of parabolic systems with disjoint control and coupling domains*, in preparation.

[8] F. Boyer and G. Olive, *Approximate controllability conditions for some linear 1D parabolic systems with space-dependent coefficients*, to appear in Maths Control and Related Fields, http://hal.archives-ouvertes.fr/hal-00848709

[9] B. Dehman, J. Le Rousseau and M. Léautaud, *Controllability of two coupled wave equations on a compact manifold* To appear in A.R.M.A., http://hal.archives-ouvertes.fr/hal-00686967

[10] H.O. Fattorini, D. L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rational Mech. Anal., **43** (1971), 272–292.

[11] E. Fernández-Cara, M. González-Burgos, L. de Teresa, *Boundary controllability of parabolic coupled equations*, J. Funct. Anal. **259** (2010), no. 7, 1720–1758.

[12] M. González-Burgos, L. de Teresa, *Controllability results for cascade systems of m coupled parabolic PDEs by one controll force*, Port. Math. **67** (2010), no. 1, 91–113.

[13] O. Kavian, L. de Teresa, *Unique continuation principle for systems of parabolic equations*, ESAIM Control Optim. Calc. Var. **16** (2010), no. 2, 247–274.

[14] L. Rosier, , L. de Teresa, *Exact controllability of a cascade system of conservative equations* C. R. Math. Acad. Sci. Paris **349** (2011), no. 5-6, 291–296.

[15] L. de Teresa, *Insensitizing controls for a semilinear heat equation*, Comm. Partial Differential Equations **25** (2000), no. 1–2, 39–72.