Advanced Nonlinear Studies **13** (2013), xxx–xxx

Pullback Attractors for 2D Navier-Stokes Equations with Delays and Their Regularity*

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Received in revised form 09 March 2012
Communicated by Peter Kloeden

Abstract

In this paper we obtain some results on the existence of solution, and of pullback attractors, for a 2D Navier-Stokes model with finite delay studied in [4] and [6]. Actually, we prove a result of existence and uniqueness of solution under less restrictive assumptions than in [4]. More precisely, we remove a condition on square integrable control of the memory terms, which allows us to consider a bigger class of delay terms (for instance, just under a measurability condition on the delay function leading the delayed time). After that, we deal with dynamical systems in suitable phase spaces within two metrics, the L^2 norm and the H^1 norm. Moreover, we prove that under these assumptions, pullback attractors not only of fixed bounded sets but also of a set of tempered universes do exist. Finally, from comparison results of attractors we establish relations among them, and under suitable additional assumptions we conclude that these families of attractors are in fact the same object.

2010 Mathematics Subject Classification. 35B41, 35Q30, 37L30.
Key words. 2D Navier-Stokes equations; delay terms; pullback attractors.

*This work has been partially supported by Ministerio de Ciencia e Innovación (Spain) under project MTM2008-00088, and the Consejería de Innovación, Ciencia y Empresa (Junta de Andalucía), Proyecto de Excelencia P07-FQM-02468. J.G.-L. is a fellow of Programa de FPU del Ministerio de Educación (Spain).

†Deceased on January, 27th 2012. The first two coauthors dedicate this paper to the memory of their coauthor José Real.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with smooth enough boundary $\partial\Omega$, and consider an arbitrary initial time $\tau \in \mathbb{R}$, and the following functional Navier-Stokes problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f(t) + g(t, u_t) \quad \text{in } \Omega \times (\tau, \infty), \\ \operatorname{div} u = 0 \quad \text{in } \Omega \times (\tau, \infty), \\ u = 0 \quad \text{on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau + s) = \phi(x, s), \quad x \in \Omega, \ s \in [-h, 0], \end{cases} \tag{1.1}$$

where we assume that $\nu > 0$ is the kinematic viscosity, $u = (u_1, u_2)$ is the velocity field of the fluid, p is the pressure, f is a non-delayed external force field, g is another external force containing some hereditary characteristics, and $\phi(x, s - \tau)$ is the initial datum in the interval of time $[\tau - h, \tau]$, where $h > 0$ is the time of memory effect. For each $t \geq \tau$, we denote by u_t the function defined on $[-h, 0]$ by the relation $u_t(s) = u(t + s)$, $s \in [-h, 0]$.

The study of Navier-Stokes models including delay terms –existence, uniqueness, stationary solutions, exponential decay, and other asymptotic properties such as the existence of attractors– was initiated in the references [4, 5, 6], and after that, many different questions and models have been addressed (e.g., cf. [17, 21, 19, 14, 20, 11, 15, 16] among others).

However, to our knowledge, in all finite delay frameworks the assumptions for the delay terms used to involve estimates in L^2 spaces, which in turn means some restrictive conditions on the operators and on the function driving the delay time. As long as the solution for the problem (without delay) in dimension two is continuous in time, it seems natural to develop a theory just considering a phase space only requiring continuity in time. In this sense, we are able to remove an assumption on the L^2 estimates for the delay terms (e.g. cf. conditions (IV) and (V) in [4, 5, 6, 10]).

The goal of this paper is to generalize the conditions on the delay terms in the model by allowing just continuous (in time) spaces, which will require less restrictive conditions on the involved delay operators. Actually, we will provide a simple example where the delay function leading the delayed time is just measurable, instead of the usual assumption of being $\rho \in C^1$, with derivative $\rho'(t) \leq \rho_* < 1$. Observe that even we do not require any continuity on ρ. (cf. Example 2.1 and Remark 2.1 below, for more details).

The contents of the paper are structured as follows. In Section 2 we obtain a result on the existence, uniqueness, and regularity of the solution to (1.1). Our method to prove existence of solution in this new framework requires more technicalities than in previous papers, namely, an energy method for continuous functions.

In Section 3 we recall, for the sake of completeness, the necessary abstract theory in order to construct pullback attractors for a dynamical system associated to the problem via the solution operator. Actually, we provide results on the existence of minimal pullback attractors for two possible choices of the attracted universes, namely, the standard one of fixed bounded sets, and secondly, one given by a tempered condition. We conclude this section with several results comparing two families of attractors associated to the same process but with different phase spaces and/or universes.

Section 4 is devoted to prove the existence of pullback attractors in the L^2 norm (in the above senses) for weak solutions of the problem (1.1), via asymptotic compactness, and using an energy method which relies strongly on the energy equality associated to the problem.

The main results of the paper are given in Section 5. There, we strengthen the regularity of solutions and a second energy equality for them, in order to obtain additional attraction, namely, in the H^1 norm instead of L^2 as in Section 4. Different families of universes (tempered and non-tempered) are introduced. Now, a second (and more involved) energy method is employed to prove asymptotic compactness in the new metric. We finish analyzing the relationship among all these families. Actually, we are able to prove that under suitable assumptions, in fact all these objects coincide.

2 Existence and uniqueness of solution

In this section we prove existence, uniqueness, and regularity of solution to problem (1.1).

To start with, we consider the usual spaces in the variational theory of Navier-Stokes equations: H, the closure of $\mathcal{V} = \{u \in (C_0^\infty(\Omega))^2 : \operatorname{div} u = 0\}$ in $(L^2(\Omega))^2$ with norm $|\cdot|$, and inner product $(\cdot,\cdot)$, and V, the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^2$ with norm $\|\cdot\|$, and inner product $((\cdot,\cdot))$.

We will use $\|\cdot\|_*$ for the norm in V' and $\langle\cdot,\cdot\rangle$ for the duality product between V' and V. We consider every element $h \in H$ as an element of V', given by the equality $\langle h, v\rangle = (h, v)$ for all $v \in V$. It follows that $V \subset H \subset V'$, where the injections are dense and continuous, and, in fact, compact.

Define the operator $A : V \to V'$ as $\langle Au, v\rangle = ((u, v))$ for all $u, v \in V$. Let us denote $D(A) = \{u \in V : Au \in H\}$. By the regularity of $\partial\Omega$, one has that $D(A) = (H^2(\Omega))^2 \cap V$, and $Au = -P\Delta u$ for all $u \in D(A)$ is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^2$ onto H). On $D(A)$ we consider the norm $|\cdot|_{D(A)}$ defined by $|u|_{D(A)} = |Au|$. Observe that on $D(A)$ the norms $\|\cdot\|_{(H^2(\Omega))^2}$ and $|\cdot|_{D(A)}$ are equivalent (see [7] or [24]), and $D(A)$ is compactly and densely injected in V. Let us define

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx,$$

for every functions $u, v, w : \Omega \to \mathbb{R}^2$ for which the right-hand side is well defined. In particular, b makes sense for all $u, v, w \in V$, and is a continuous trilinear form on $V \times V \times V$.

Some useful properties concerning b that we will use in the next sections are the following (see [22] or [23]): $b(u, v, w) = -b(u, w, v)$ for all $u, v, w \in V$, which also implies that $b(u, v, v) = 0$ for all $u, v \in V$. Moreover,

$$|b(u, v, w)| \leq 2^{-1/2}|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2} \quad \forall\, u, v, w \in V, \tag{2.1}$$

and there exists a constant $C_1 > 0$, only dependent on Ω, such that

$$|b(u, v, w)| \leq C_1|u|^{1/2}|Au|^{1/2}\|v\||w| \quad \forall\, u \in D(A),\ v \in V,\ w \in H, \text{ and} \tag{2.2}$$

$$|b(u, v, w)| \leq C_1|Au|\|v\||w| \quad \forall\, u \in D(A),\ v \in V,\ w \in H. \tag{2.3}$$

In fact, (2.1) is a slight improvement of Lemma 3.3 in [24, p.291] (see [10]).

Now, we establish some appropriate assumptions on the term in (1.1) containing the delay.

Let us denote $C_H = C([-h, 0]; H)$, the space of continuous functions from $[-h, 0]$ into H, with the norm $|\varphi|_{C_H} = \max_{s\in[-h,0]} |\varphi(s)|$.

Let us establish some conditions on the delay operator in (1.1). Suppose that g is well defined as $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$, and it satisfies the following assumptions:

(I) for all $\xi \in C_H$, the function $\mathbb{R} \ni t \mapsto g(t, \xi) \in (L^2(\Omega))^2$ is measurable,

(II) $g(t, 0) = 0$, for all $t \in \mathbb{R}$,

(III) there exists $L_g > 0$ such that for all $t \in \mathbb{R}$, and for all $\xi, \eta \in C_H$, $|g(t, \xi) - g(t, \eta)| \leq L_g|\xi - \eta|_{C_H}$.

Observe that $(I)-(III)$ imply that given $T>\tau$ and $u\in C([\tau-h,T];H)$, the function $g_u:[\tau,T]\to(L^2(\Omega))^2$ defined by $g_u(t)=g(t,u_t)$ for all $t\in[\tau,T]$, is measurable and, in fact, belongs to $L^\infty(\tau,T;(L^2(\Omega))^2)$.

It is worth pointing out that any condition involving L^2 norms of the memory term in g is assumed (e.g. cf. conditions (IV) and (V) in [4, 5, 6, 10]).

Example 2.1 Consider a globally Lipschitz function $G:H\to(L^2(\Omega))^2$, with Lipschitz constant $L_G>0$, and such that $G(0)=0$, and a measurable function $\rho:\mathbb{R}\to[0,h]$. Then, it is not difficult to check that the operator $g:\mathbb{R}\times C_H\to(L^2(\Omega))^2$, defined by

$$\mathbb{R}\times C_H\ni(t,\xi)\mapsto g(t,\xi):=G(\xi(-\rho(t)))$$

satisfies the assumptions (I)–(III) given above.

Remark 2.1 (a) Observe that the only assumption on ρ is that it is measurable, in contrast with the usual conditions appearing in the previous literature, i.e., C^1, with derivative $\rho'(t)\le\rho_*<1$ (e.g., cf. [10]).

(b) The example above can be generalized in several senses. The most immediate generalization is to take into account more than one delay term in the problem. Namely, consider m measurable functions $\rho_i:\mathbb{R}\to[0,h]$ for $i=1$ to m, a measurable mapping $G:\mathbb{R}\times H^m\to(L^2(\Omega))^2$ such that $G(t,\cdot)$ is globally Lipschitz in H^m uniformly with respect to time, and with $G(t,0)=0$ for all $t\in\mathbb{R}$. Then, consider $g:\mathbb{R}\times C_H\to(L^2(\Omega))^2$ given by $g(t,\xi):=G(t,\xi(-\rho_1(t)),\ldots,\xi(-\rho_m(t)))$. This operator g also satisfies conditions (I)–(III).

Assume that $\phi\in C_H$, and $f\in L^2_{loc}(\mathbb{R};V')$.

Definition 2.1 *A weak solution of (1.1) is a function $u\in C([\tau-h,\infty);H)$ such that $u\in L^2(\tau,T;V)$ for all $T>\tau$, with $u(t)=\phi(t-\tau)$ for all $t\in[\tau-h,\tau]$, and such that for all $v\in V$,*

$$\frac{d}{dt}(u(t),v)+\nu\langle Au(t),v\rangle+b(u(t),u(t),v)=\langle f(t),v\rangle+(g(t,u_t),v), \tag{2.4}$$

where the equation must be understood in the sense of $\mathcal{D}'(\tau,\infty)$.

Remark 2.2 If u is a weak solution of (1.1), then from (2.4) we deduce that for any $T>\tau$, one has $u'\in L^2(\tau,T;V')$, and the following energy equality holds:

$$|u(t)|^2+2\nu\int_s^t\|u(r)\|^2\,dr=|u(s)|^2+2\int_s^t[\langle f(r),u(r)\rangle+(g(r,u_r),u(r))]\,dr\quad\forall\,\tau\le s\le t.$$

A notion of more regular solution is also suitable for problem (1.1).

Definition 2.2 *A strong solution of (1.1) is a weak solution u of (1.1) such that $u\in L^2(\tau,T;D(A))\cap L^\infty(\tau,T;V)$ for all $T>\tau$.*

Remark 2.3 If $f\in L^2_{loc}(\mathbb{R};(L^2(\Omega))^2)$ and u is a strong solution of (1.1), then $u'\in L^2(\tau,T;H)$ for all $T>\tau$, and so $u\in C([\tau,\infty);V)$. In this case the following energy equality holds:

$$\begin{aligned}&\|u(t)\|^2+2\nu\int_s^t|Au(r)|^2\,dr+2\int_s^t b(u(r),u(r),Au(r))\,dr\\ =\;&\|u(s)\|^2+2\int_s^t(f(r)+g(r,u_r),Au(r))\,dr\quad\forall\,\tau\le s\le t.\end{aligned}\tag{2.5}$$

Let us denote $\lambda_1 = \inf_{v\in V\setminus\{0\}} \|v\|^2/|v|^2 > 0$ the first eigenvalue of the Stokes operator A.

Concerning the existence and uniqueness of weak solution for (1.1), we have the following result, which improves, in the case of initial data $\phi \in C_H$ and dimension two, Theorem 2.1 in [4] (see also [10, Th.2.3]). In fact, in the theorem below, we neither assume hypotheses (IV) nor (V) of [4].

Theorem 2.1 *Let $f \in L^2_{loc}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying (I)–(III), be given. Then, for each $\tau \in \mathbb{R}$ and $\phi \in C_H$, there exists a unique weak solution $u = u(\cdot; \tau, \phi)$ of (1.1). Moreover, if $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$, then*

(a) $u \in C([\tau + \varepsilon, T]; V) \cap L^2(\tau + \varepsilon, T; D(A))$ for all $T > \tau + \varepsilon > \tau$.

(b) If $\phi(0) \in V$, in fact u is a strong solution of (1.1).

Proof. The uniqueness of solution can be obtained in the following way. Consider two weak solutions of (1.1), u and v, with the same initial data, and denote $w = u - v$. We note that by (2.1),

$$\begin{aligned} |b(u(s), u(s), w(s)) - b(v(s), v(s), w(s))| &= |b(w(s), u(s), w(s))| \\ &\le 2^{-1/2}|w(s)|\|w(s)\|\|u(s)\|. \end{aligned}$$

Then, from the equation satisfied by w and the energy equality, we obtain for all $t \ge \tau$ that

$$\begin{aligned} &|w(t)|^2 + 2\nu \int_\tau^t \|w(s)\|^2\, ds \\ = \;& -2\int_\tau^t b(w(s), u(s), w(s))\, ds + 2\int_\tau^t (g(s, u_s) - g(s, v_s), w(s))\, ds \\ \le \;& 2^{1/2}\int_\tau^t |w(s)|\|w(s)\|\|u(s)\|\, ds + 2L_g \int_\tau^t |w_s|_{C_H}|w(s)|\, ds. \end{aligned} \tag{2.6}$$

Observe that $w(\theta) = 0$ if $\tau - h \le \theta \le \tau$, and therefore, $|w_s|_{C_H} = \max_{r\in[\tau,s]} |w(r)|$ for $\tau \le s$. So, from (2.6), using Young inequality, we deduce

$$\begin{aligned} &|w(t)|^2 + 2\nu \int_\tau^t \|w(s)\|^2 ds \\ \le \;& 2^{1/2}\int_\tau^t |w(s)|\|w(s)\|\|u(s)\|\, ds + 2L_g \int_\tau^t \max_{r\in[\tau,s]} |w(r)||w(s)|\, ds \\ \le \;& \nu \int_\tau^t \|w(s)\|^2\, ds + \frac{1}{2\nu}\int_\tau^t \|u(s)\|^2|w(s)|^2\, ds + 2L_g \int_\tau^t \max_{r\in[\tau,s]} |w(r)|^2\, ds, \end{aligned}$$

for all $t \ge \tau$, and therefore,

$$\max_{r\in[\tau,t]} |w(r)|^2 \le \left(\frac{1}{2\nu} + 2L_g\right)\int_\tau^t (1 + \|u(s)\|^2) \max_{r\in[\tau,s]} |w(r)|^2\, ds,$$

for all $t \ge \tau$. Thus, using Gronwall lemma, we finish the proof of uniqueness.

For the existence, we split the proof in two steps.

Step 1: Galerkin scheme. A priori estimates. Let us consider $\{w_j\} \subset V$, the Hilbert basis of H of all the normalized eigenfunctions of the Stokes operator A. Denote $V_m = \mathrm{span}[w_1, \ldots, w_m]$ and consider the projector P_m of H onto V_m given by $P_m v = \sum_{j=1}^m (v, w_j)w_j$, for all $v \in H$. Observe that by the choice of the basis $\{w_j\}$, the restriction $P_{m|_V}$ of P_m to V belongs to $\mathcal{L}(V)$, and $\|P_{m|_V}\|_{\mathcal{L}(V)} \le 1$ for all $m \ge 1$.

Define also

$$u^m(t) = \sum_{j=1}^{m} \alpha_{m,j}(t) w_j,$$

where the upper script m will be used instead of (m) for short, since no confusion is possible with powers of u, and where the coefficients $\alpha_{m,j}$ are required to satisfy the system

$$\begin{aligned} & \frac{d}{dt}(u^m(t), w_j) + \nu((u^m(t), w_j)) + b(u^m(t), u^m(t), w_j) \\ = \quad & \langle f(t), w_j \rangle + (g(t, u_t^m), w_j), \quad \text{a.e. } t > \tau, \quad 1 \le j \le m, \end{aligned} \tag{2.7}$$

and the initial condition

$$u^m(\tau + s) = P_m \phi(s) \quad \forall\, s \in [-h, 0]. \tag{2.8}$$

The above system of ordinary functional differential equations with finite delay fulfills the conditions for existence and uniqueness of local solution (see for example [12]).

Next, we will deduce a priori estimates that in particular assure that the solutions u^m do exist for all time $t \in [\tau - h, \infty)$.

Multiplying in (2.7) by $\alpha_{m,j}(t)$, and summing from $j = 1$ to $j = m$, we obtain

$$\begin{aligned} \frac{d}{dt}|u^m(t)|^2 + 2\nu \|u^m(t)\|^2 &= 2\langle f(t), u^m(t) \rangle + 2(g(t, u_t^m), u^m(t)) \\ &\le \nu \|u^m(t)\|^2 + \nu^{-1} \|f(t)\|_*^2 + 2L_g |u_t^m|_{C_H}^2, \quad \text{a.e. } t > \tau. \end{aligned}$$

Hence,

$$|u^m(t)|^2 + \nu \int_\tau^t \|u^m(s)\|^2\, ds \le |\phi(0)|^2 + \int_\tau^t (\nu^{-1} \|f(s)\|_*^2 + 2L_g |u_s^m|_{C_H}^2)\, ds \quad \forall\, t \ge \tau. \tag{2.9}$$

From this inequality, in particular one deduces that

$$|u_t^m|_{C_H}^2 \le |\phi|_{C_H}^2 + \int_\tau^t (\nu^{-1} \|f(s)\|_*^2 + 2L_g |u_s^m|_{C_H}^2)\, ds \quad \forall\, t \ge \tau,$$

and therefore, by Gronwall lemma we have

$$|u_t^m|_{C_H}^2 \le e^{2L_g(t-\tau)} \Big(|\phi|_{C_H}^2 + \nu^{-1} \int_\tau^t \|f(s)\|_*^2\, ds \Big),$$

for all $t \ge \tau$, and any $m \ge 1$.

Then, by (2.9), we deduce that for each $T > \tau$ and $R > 0$, there exists a positive constant $C(\tau, T, R)$, depending on the constants of the problem ν, L_g and f, and on τ, T and R, such that for all $m \ge 1$

$$|u_t^m|_{C_H}^2 + \|u^m\|_{L^2(\tau,T;V)}^2 \le C(\tau, T, R) \quad \forall\, t \in [\tau, T],\ |\phi|_{C_H} \le R. \tag{2.10}$$

In particular, this implies that

$$\{u^m\} \text{ is bounded in } L^\infty(\tau - h, T; H) \cap L^2(\tau, T; V) \quad \forall\, T > \tau. \tag{2.11}$$

From (2.1), (2.7), and because of the choice of the basis, we obtain

$$\|(u^m)'(t)\|_* \le \nu \|u^m(t)\| + 2^{-1/2} |u^m(t)| \|u^m(t)\| + \|f(t)\|_* + \lambda_1^{-1/2} |g(t, u_t^m)|, \quad \text{a.e. } t > \tau,$$

which combined with (II), (III), (2.10) and (2.11), implies that

$$\{(u^m)'\} \text{ is bounded in } L^2(\tau, T; V') \quad \forall\, T > \tau. \tag{2.12}$$

Step 2: Energy method and compactness results. Now, we combine some well-known compactness results with an energy method to pass to the limit in a subsequence of $\{u^m\}$ to obtain a solution of (1.1). First we observe that

$$u^m{}_{|_{[\tau-h,\tau]}} = P_m\phi \to \phi \quad \text{in } C_H. \tag{2.13}$$

From the assumptions on the operator g and Step 1 we deduce, using the Compactness Theorem 5.1 [13, p.58] and Lemma 1.2 [24, p.260], that there exist a subsequence (which we relabel the same) $\{u^m\}$, a function $u \in C([\tau - h, \infty); H)$, with $u_{|_{[\tau-h,\tau]}} = \phi$, $u \in L^2(\tau, T; V)$ and $u' \in L^2(\tau, T; V')$ for all $T > \tau$, and an element $\xi \in L^\infty(\tau, T; (L^2(\Omega))^2)$ for all $T > \tau$, such that

$$\begin{cases} u^m \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(\tau, T; H), \\ u^m \rightharpoonup u & \text{weakly in } L^2(\tau, T; V), \\ (u^m)' \rightharpoonup u' & \text{weakly in } L^2(\tau, T; V'), \\ u^m \to u & \text{strongly in } L^2(\tau, T; H), \\ g(\cdot, u^m_\cdot) \overset{*}{\rightharpoonup} \xi & \text{weakly-star in } L^\infty(\tau, T; (L^2(\Omega))^2), \end{cases} \tag{2.14}$$

for all $T > \tau$. Using (2.14) we can also assume that

$$u^m(t) \to u(t) \quad \text{in } H \quad \text{a.e. } t \in (\tau, \infty), \tag{2.15}$$

which nevertheless is not enough to deduce that $\xi(\cdot) = g(\cdot, u_\cdot)$. However, we can obtain convergence for all $t \geq \tau$ with a little more effort and in a more general sense. Observe that

$$u^m(t) - u^m(s) = \int_s^t (u^m)'(r)\, dr \quad \text{in } V' \quad \forall\, s, t \in [\tau, \infty),$$

and by (2.12) we have that $\{u^m\}$ is equi-continuous on $[\tau, T]$ with values in V', for all $T > \tau$.

Since the injection of V into H is compact, the injection of H into V' is compact too. So, from (2.11) and the equi-continuity in V', by the Ascoli-Arzelà theorem and (2.14), we have that (again, up to a subsequence)

$$u^m \to u \quad \text{in } C([\tau, T]; V') \quad \forall\, T > \tau. \tag{2.16}$$

This, jointly with (2.11), allows us to claim that for any sequence $\{t_m\} \subset [\tau, \infty)$, with $t_m \to t$, one has

$$u^m(t_m) \rightharpoonup u(t) \quad \text{weakly in } H, \tag{2.17}$$

where we have used (2.16) in order to identify which is the weak limit.

Our goal now is to prove that in fact

$$u^m \to u \quad \text{in } C([\tau, T]; H) \quad \forall\, T > \tau. \tag{2.18}$$

If it were not so, then, taking into account that $u \in C([\tau, \infty); H)$, there would exist $T > \tau$, $\varepsilon_0 > 0$, a value $t_0 \in [\tau, T]$, and subsequences (relabelled the same) $\{u^m\}$ and $\{t_m\} \subset [\tau, T]$, with $\lim_{m\to\infty} t_m = t_0$, such that

$$|u^m(t_m) - u(t_0)| \geq \varepsilon_0 \quad \forall\, m \geq 1.$$

To prove that this is absurd, we will use an energy method.

Observe that the following energy inequality holds for all u^m:

$$\begin{aligned} & \frac{1}{2}|u^m(t)|^2 + \frac{\nu}{2}\int_s^t \|u^m(r)\|^2\,dr \\ \leq\ & \frac{1}{2}|u^m(s)|^2 + \int_s^t \langle f(r), u^m(r)\rangle\,dr + C(t-s) \quad \forall\, \tau \leq s \leq t \leq T, \end{aligned} \tag{2.19}$$

where $C = \frac{D}{2\nu\lambda_1}$ and D corresponds to the upper bound

$$\int_s^t |g(r, u_r^m)|^2\,dr \leq D(t-s) \quad \forall\, \tau \leq s \leq t \leq T,$$

by (II), (III) and (2.10). On the other hand, observe that by (2.14), passing to the limit in (2.7), we have that $u \in C([\tau, T]; H)$ is a solution of a similar problem to (1.1), namely,

$$\frac{d}{dt}(u(t), v) + \nu((u(t), v)) + b(u(t), u(t), v) = \langle f(t), v\rangle + (\xi(t), v) \quad \forall\, v \in V,$$

fulfilled with the initial datum $u(\tau) = \phi(0)$. Therefore, it satisfies the energy equality

$$|u(t)|^2 + 2\nu\int_s^t \|u(r)\|^2\,dr = |u(s)|^2 + 2\int_s^t (\langle f(r), u(r)\rangle + (\xi(r), u(r)))\,dr \quad \forall\, \tau \leq s \leq t \leq T.$$

On other hand, from the last convergence in (2.14) we deduce that

$$\begin{aligned} \int_s^t |\xi(r)|^2\,dr &\leq \liminf_{m\to\infty}\int_s^t |g(r, u_r^m)|^2\,dr \\ &\leq D(t-s) \quad \forall\, \tau \leq s \leq t \leq T. \end{aligned}$$

So, u also satisfies inequality (2.19) with the same constant C.

Now, consider the functions J_m, $J : [\tau, T] \to \mathbb{R}$ defined by

$$J_m(t) = \frac{1}{2}|u^m(t)|^2 - \int_\tau^t \langle f(r), u^m(r)\rangle\,dr - Ct,$$

$$J(t) = \frac{1}{2}|u(t)|^2 - \int_\tau^t \langle f(r), u(r)\rangle\,dr - Ct,$$

with C the constant given in (2.19). From (2.19) and the analogous inequality for u, it is clear that J_m and J are non-increasing (and continuous) functions. Moreover, by (2.14) and (2.15),

$$J_m(t) \to J(t) \quad \text{a.e. } t \in [\tau, T]. \tag{2.20}$$

Now we are ready to prove that

$$u^m(t_m) \to u(t_0) \quad \text{in } H. \tag{2.21}$$

Firstly, recall from (2.17) that

$$u^m(t_m) \rightharpoonup u(t_0) \quad \text{weakly in } H. \tag{2.22}$$

So, we have that $|u(t_0)| \leq \liminf_{m\to\infty} |u^m(t_m)|$. Therefore, if we show that

$$\limsup_{m\to\infty} |u^m(t_m)| \leq |u(t_0)|, \tag{2.23}$$

we obtain that $\lim_{m\to\infty} |u^m(t_m)| = |u(t_0)|$, which jointly with (2.22) implies (2.21).

Now, observe that the case $t_0 = \tau$ follows directly from (2.13) and (2.19) with $s = \tau$. So, we may assume that $t_0 > \tau$. This is important, since we will approach this value t_0 from the left by a sequence $\{\tilde{t}_k\}$, i.e., $\lim_{k\to\infty} \tilde{t}_k \nearrow t_0$, being $\{\tilde{t}_k\}$ values where (2.20) holds. Since $J(\cdot)$ is continuous at t_0, for any $\varepsilon > 0$ there is k_ε such that $|J(\tilde{t}_k) - J(t_0)| < \varepsilon/2$ for all $k \geq k_\varepsilon$. On other hand, taking $m \geq m(k_\varepsilon)$ such that $t_m > \tilde{t}_{k_\varepsilon}$, as J_m is non-increasing and for all $\tilde{t}_k$ the convergence (2.20) holds, one has that

$$J_m(t_m) - J(t_0) \leq |J_m(\tilde{t}_{k_\varepsilon}) - J(\tilde{t}_{k_\varepsilon})| + |J(\tilde{t}_{k_\varepsilon}) - J(t_0)|,$$

and obviously, taking $m \geq m'(k_\varepsilon)$, it is possible to obtain $|J_m(\tilde{t}_{k_\varepsilon}) - J(\tilde{t}_{k_\varepsilon})| < \varepsilon/2$. It can also be deduced from (2.14) that

$$\int_\tau^{t_m} \langle f(r), u^m(r)\rangle\, dr \to \int_\tau^{t_0} \langle f(r), u(r)\rangle\, dr,$$

so we conclude that (2.23) holds. Thus, (2.21) and finally (2.18) are also true, as we wanted to check.

This also implies, thanks to (2.13), that $u_t^m \to u_t$ in C_H for all $t \geq \tau$. Therefore, we identify the weak limit ξ from (2.14), and indeed, from the above convergence and since g satisfies (III), we have that $g(\cdot, u^m_\cdot) \to g(\cdot, u_\cdot)$ in $L^2(\tau, T; (L^2(\Omega))^2)$ for all $T > \tau$. Thus, we can pass to the limit finally in (2.7) concluding that u solves (1.1).

Finally, the regularity in (a) and (b) is a consequence of well-known regularity results and the fact that, if $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$, then the function $\hat{f}$ defined by $\hat{f}(t) = f(t) + g(t, u_t)$, $t > \tau$, belongs to $L^2_{loc}(\tau, \infty; (L^2(\Omega))^2)$.

Remark 2.4 Observe that by the uniqueness of the weak solution of (1.1), the convergences in (2.14) hold for the entire sequence $\{u^m\}$ of the Galerkin approximations defined by (2.7) and (2.8).

We also have the following result on continuity of solutions with respect to the initial datum ϕ.

Proposition 2.1 *Let $f \in L^2_{loc}(\mathbb{R}; V')$, $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying (I)–(III), $\tau \in \mathbb{R}$, and ϕ, $\psi \in C_H$, be given. Let us denote $u = u(\cdot; \tau, \phi)$ and $v = v(\cdot; \tau, \psi)$ the corresponding weak solutions of (1.1). Then, the following estimate holds:*

$$|u_t - v_t|^2_{C_H} \leq |\phi - \psi|^2_{C_H} \exp\left\{ \int_\tau^t ((2\nu)^{-1}\|u(s)\|^2 + 2L_g)\, ds \right\} \quad \forall\, t \geq \tau.$$

Proof. Let us denote $w = u - v$. Analogously to the obtention of (2.6) in the proof of uniqueness of weak solution of (1.1), we obtain that

$$\begin{aligned} &|w(t)|^2 + 2\nu \int_\tau^t \|w(s)\|^2\, ds \\ \leq\ & |\phi(0) - \psi(0)|^2 + 2^{1/2} \int_\tau^t |w(s)|\|w(s)\|\|u(s)\|\, ds + 2L_g \int_\tau^t |w_s|_{C_H} |w(s)|\, ds \quad \forall\, t \geq \tau. \end{aligned}$$

So, we deduce that

$$\begin{aligned} &|w(t)|^2 + 2\nu \int_\tau^t \|w(s)\|^2\, ds \\ \leq\ & |\phi(0) - \psi(0)|^2 + 2^{1/2} \int_\tau^t |w_s|_{C_H} \|w(s)\|\|u(s)\|\, ds + 2L_g \int_\tau^t |w_s|^2_{C_H}\, ds \\ \leq\ & |\phi(0) - \psi(0)|^2 + \nu \int_\tau^t \|w(s)\|^2\, ds + \int_\tau^t ((2\nu)^{-1}\|u(s)\|^2 + 2L_g)|w_s|^2_{C_H}\, ds \quad \forall\, t \geq \tau, \end{aligned}$$

and in particular, we have that

$$|w(t)|^2 \leq |\phi(0) - \psi(0)|^2 + \int_\tau^t ((2\nu)^{-1}\|u(s)\|^2 + 2L_g)|w_s|^2_{C_H}\, ds \quad \forall\, t \geq \tau. \tag{2.24}$$

Taking into account that $|w(\tau+s)|^2 \leq |\phi - \psi|^2_{C_H}$ for all $s \in [-h, 0]$, from (2.24), we deduce that

$$|w_t|^2_{C_H} \leq |\phi - \psi|^2_{C_H} + \int_\tau^t ((2\nu)^{-1}\|u(s)\|^2 + 2L_g)|w_s|^2_{C_H}\, ds \quad \forall\, t \geq \tau.$$

From this inequality and Gronwall lemma, we can conclude the result.

3 Abstract results on minimal pullback attractors

In this section we recall briefly some results from [9] about the existence of minimal pullback attractors (see also [2, 3, 18]). In particular, we consider the process U being closed (see Definition 3.1 below). Consider given a metric space (X, d_X), and let us denote $\mathbb{R}^2_d = \{(t,\tau) \in \mathbb{R}^2 : \tau \leq t\}$. A process U on X is a mapping $\mathbb{R}^2_d \times X \ni (t,\tau,x) \mapsto U(t,\tau)x \in X$ such that $U(\tau,\tau)x = x$ for any $(\tau, x) \in \mathbb{R} \times X$, and $U(t,r)(U(r,\tau)x) = U(t,\tau)x$ for any $\tau \leq r \leq t$ and all $x \in X$.

Definition 3.1 *A process U on X is said to be closed if for any $\tau \leq t$, and any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ and $U(t,\tau)x_n \to y \in X$, then $U(t,\tau)x = y$.*

Let us denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

Definition 3.2 *We say that a process U on X is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.*

Denote

$$\Lambda(\widehat{D}_0, t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t,\tau)D_0(\tau)}^X \qquad \forall\, t \in \mathbb{R},$$

where $\overline{\{\cdots\}}^X$ is the closure in X. We denote by $\mathrm{dist}_X(O_1, O_2)$ the Hausdorff semi-distance in X between two sets O_1 and O_2, defined as

$$\mathrm{dist}_X(O_1, O_2) = \sup_{x \in O_1} \inf_{y \in O_2} d_X(x,y) \quad \text{for } O_1, O_2 \subset X.$$

Let $\mathcal{D}$ be a given nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 3.3 *A process U on X is said to be pullback $\mathcal{D}$-asymptotically compact if it is pullback $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$. It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for the process U on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \widehat{D}) \leq t$ such that*

$$U(t,\tau)D(\tau) \subset D_0(t) \quad \forall\, \tau \leq \tau_0(t, \widehat{D}).$$

Remark 3.1 Observe that if $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for the process U on X, and U is pullback $\widehat{D}_0$-asymptotically compact, then U is also $\mathcal{D}$-asymptotically compact.

With the above definitions, we may establish the main result of this section (cf. [9, Th.3.11]).

Theorem 3.1 *Consider a closed process* $U : \mathbb{R}^2_d \times X \to X$*, a universe* $\mathcal{D}$ *in* $\mathcal{P}(X)$*, and a family* $\widehat{D_0} = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ *which is pullback* $\mathcal{D}$*-absorbing for* U*, and assume also that* U *is pullback* $\widehat{D}_0$*-asymptotically compact. Then, the family* $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ *defined by* $\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda_X(\widehat{D}, t)}^X$*, has the following properties:*

(a) for any $t \in \mathbb{R}$*, the set* $\mathcal{A}_{\mathcal{D}}(t)$ *is a nonempty compact subset of* X*, and* $\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda_X(\widehat{D}_0, t)$*,*

(b) $\mathcal{A}_{\mathcal{D}}$ *is pullback* $\mathcal{D}$*-attracting, i.e.,* $\lim\limits_{\tau\to-\infty} \mathrm{dist}_X(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0$ *for all* $\widehat{D} \in \mathcal{D}$*, and any* $t \in \mathbb{R}$*,*

(c) $\mathcal{A}_{\mathcal{D}}$ *is invariant, i.e.,* $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ *for all* $\tau \le t$*,*

(d) if $\widehat{D}_0 \in \mathcal{D}$*, then* $\mathcal{A}_{\mathcal{D}}(t) = \Lambda_X(\widehat{D}_0, t) \subset \overline{D_0(t)}^X$ *for all* $t \in \mathbb{R}$*.*

The family $\mathcal{A}_{\mathcal{D}}$ *is minimal in the sense that if* $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ *is a family of closed sets such that for any* $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$*,* $\lim_{\tau\to-\infty} \mathrm{dist}_X(U(t,\tau)D(\tau), C(t)) = 0$*, then* $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$*.*

Remark 3.2 Under the assumptions of Theorem 3.1, the family $\mathcal{A}_{\mathcal{D}}$ is called the minimal pullback $\mathcal{D}$-attractor for the process U. If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (b)–(c).

A sufficient condition for $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ is to have that $\widehat{D}_0 \in \mathcal{D}$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and the family $\mathcal{D}$ is inclusion-closed (i.e., if $\widehat{D} \in \mathcal{D}$, and $\widehat{D'} = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all t, then $\widehat{D'} \in \mathcal{D}$).

We will denote by $\mathcal{D}_F(X)$ the universe of fixed nonempty bounded subsets of X, i.e., the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X.

Now, it is easy to conclude the following result.

Corollary 3.1 *[cf. [9, Cor.3.13]] Under the assumptions of Theorem 3.1, if the universe* $\mathcal{D}$ *contains the universe* $\mathcal{D}_F(X)$*, then both attractors,* $\mathcal{A}_{\mathcal{D}_F(X)}$ *and* $\mathcal{A}_{\mathcal{D}}$*, exist, and* $\mathcal{A}_{\mathcal{D}_F(X)}(t) \subset \mathcal{A}_{\mathcal{D}}(t)$ *for all* $t \in \mathbb{R}$*.*

Remark 3.3 It can be proved (see [18]) that, under the assumptions of the preceding corollary, if for some $T \in \mathbb{R}$, the set $\cup_{t\le T} D_0(t)$ is a bounded subset of X, then $\mathcal{A}_{\mathcal{D}_F(X)}(t) = \mathcal{A}_{\mathcal{D}}(t)$ for all $t \le T$.

Now, and since it will be useful below, we establish an abstract result (cf. [9, Th.3.15]) that allows us to compare two attractors for a process under appropriate assumptions.

Theorem 3.2 *Let* $\{(X_i, d_{X_i})\}_{i=1,2}$ *be two metric spaces such that* $X_1 \subset X_2$ *with continuous injection, and for* $i = 1, 2$*, let* $\mathcal{D}_i$ *be a universe in* $\mathcal{P}(X_i)$*, with* $\mathcal{D}_1 \subset \mathcal{D}_2$*. Assume that we have a map* U *that acts as a process in both cases, i.e.,* $U : \mathbb{R}^2_d \times X_i \to X_i$ *for* $i = 1, 2$ *is a process.*

For each $t \in \mathbb{R}$*, let us denote*

$$\mathcal{A}_i(t) = \overline{\bigcup_{\widehat{D}_i\in\mathcal{D}_i} \Lambda_i(\widehat{D}_i, t)}^{X_i}, \quad i = 1, 2,$$

where the subscript i in the symbol of the omega-limit set Λ_i *is used to denote the dependence of the respective topology. Then,* $\mathcal{A}_1(t) \subset \mathcal{A}_2(t)$ *for all* $t \in \mathbb{R}$*. Suppose moreover that the two following conditions are satisfied:*

(i) $\mathcal{A}_1(t)$ *is a compact subset of* X_1 *for all* $t \in \mathbb{R}$,

(ii) for any $\widehat{D}_2 \in \mathcal{D}_2$ *and any* $t \in \mathbb{R}$, *there exist a family* $\widehat{D}_1 \in \mathcal{D}_1$ *and a* $t^*_{\widehat{D}_1} \le t$ *(both possibly depending on* t *and* $\widehat{D}_2$*), such that* U *is pullback* $\widehat{D}_1$*-asymptotically compact, and for any* $s \le t^*_{\widehat{D}_1}$ *there exists a* $\tau_s \le s$ *such that* $U(s,\tau)D_2(\tau) \subset D_1(s)$ *for all* $\tau \le \tau_s$.

Then, under all the conditions above, $\mathcal{A}_1(t) = \mathcal{A}_2(t)$ *for all* $t \in \mathbb{R}$.

Remark 3.4 In the preceding theorem, if instead of assumption (ii) we consider the following condition:

(ii') for any $\widehat{D}_2 \in \mathcal{D}_2$ and any sequence $\tau_n \to -\infty$, there exist another family $\widehat{D}_1 \in \mathcal{D}_1$ and another sequence $\tau'_n \to -\infty$ with $\tau'_n \ge \tau_n$ for all n, such that U is pullback $\widehat{D}_1$-asymptotically compact, and $U(\tau'_n, \tau_n)D_2(\tau_n) \subset D_1(\tau'_n)$ for all n,

then, with a similar proof, one can obtain that the equality $\mathcal{A}_2(t) = \mathcal{A}_1(t)$ for all $t \in \mathbb{R}$ also holds.

Observe that a sufficient condition for (ii') is that there exists $T > 0$ such that for any $\widehat{D}_2 \in \mathcal{D}_2$, there exists a $\widehat{D}_1 \in \mathcal{D}_1$ satisfying that U is pullback $\widehat{D}_1$-asymptotically compact, and $U(\tau + T, \tau)D_2(\tau) \subset D_1(\tau + T)$ for all $\tau \in \mathbb{R}$.

4 Existence of pullback attractors for the process associated to (1.1)

Now, by the previous results, we are able to define correctly a process U on C_H associated to (1.1), and to obtain the existence of minimal pullback attractors.

Proposition 4.1 *Let* $f \in L^2_{loc}(\mathbb{R}; V')$, *and* $g : \mathbb{R}\times C_H \to (L^2(\Omega))^2$ *satisfying (I)–(III), be given. Then, the bi-parametric family of maps* $U(t,\tau) : C_H \to C_H$, *with* $\tau \le t$, *given by*

$$U(t,\tau)\phi = u_t, \tag{4.1}$$

where $u = u(\cdot;\tau,\phi)$ *is the unique weak solution of (1.1), defines a continuous process on* C_H.

Proof. It is a consequence of Theorem 2.1 and Proposition 2.1.

Lemma 4.1 *Consider that the assumptions of Proposition 4.1 are satisfied and let* μ *be such that* $0 < \mu < 2\nu\lambda_1$. *Then, for any* $\phi \in C_H$, *the following estimates hold for the solution to (1.1) for all* $t \ge \tau$:

$$\begin{aligned} |u_t|^2_{C_H} &\le e^{\mu h} e^{-(\mu - 2e^{\mu h}L_g)(t-\tau)}|\phi|^2_{C_H} \\ &\quad + e^{\mu h}(2\nu - \mu\lambda_1^{-1})^{-1}\int_\tau^t e^{-(\mu-2e^{\mu h}L_g)(t-s)}\|f(s)\|^2_* \, ds, \end{aligned} \tag{4.2}$$

$$\nu\int_\tau^t \|u(s)\|^2\,ds \le |u(\tau)|^2 + \nu^{-1}\int_\tau^t \|f(s)\|^2_*\,ds + 2L_g\int_\tau^t |u_s|^2_{C_H}\,ds. \tag{4.3}$$

Proof. Take a μ such that $0 < \mu < 2\nu\lambda_1$. By the energy equality (see Remark 2.2), one has

$$\begin{aligned} &\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 \\ = \;& 2\langle f(t), u(t)\rangle + 2(g(t,u_t), u(t)) \\ \le \;& 2\|f(t)\|_*\|u(t)\| + 2L_g|u_t|_{C_H}|u(t)| \\ \le \;& (2\nu - \mu\lambda_1^{-1})\|u(t)\|^2 + (2\nu-\mu\lambda_1^{-1})^{-1}\|f(t)\|^2_* + 2L_g|u_t|^2_{C_H}, \quad \text{a.e. } t > \tau. \end{aligned}$$

Thus,

$$\frac{d}{dt}|u(t)|^2 + \mu|u(t)|^2 \leq (2\nu - \mu\lambda_1^{-1})^{-1}\|f(t)\|_*^2 + 2L_g|u_t|_{C_H}^2, \quad \text{a.e. } t > \tau,$$

and therefore,

$$e^{\mu t}|u(t)|^2 \leq e^{\mu\tau}|u(\tau)|^2 + \int_\tau^t e^{\mu s}((2\nu - \mu\lambda_1^{-1})^{-1}\|f(s)\|_*^2 + 2L_g|u_s|_{C_H}^2)\, ds \quad \forall\, t \geq \tau.$$

From this inequality, we deduce

$$e^{\mu t}|u_t|_{C_H}^2 \leq e^{\mu h}e^{\mu\tau}|\phi|_{C_H}^2 + e^{\mu h}\int_\tau^t e^{\mu s}((2\nu - \mu\lambda_1^{-1})^{-1}\|f(s)\|_*^2 + 2L_g|u_s|_{C_H}^2)\, ds \quad \forall\, t \geq \tau.$$

Then, by Gronwall lemma we can conclude that (4.2) holds.

Finally, observing that

$$\begin{aligned}\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 &\leq 2\|f(t)\|_*\|u(t)\| + 2L_g|u_t|_{C_H}|u(t)| \\ &\leq \nu\|u(t)\|^2 + \nu^{-1}\|f(t)\|_*^2 + 2L_g|u_t|_{C_H}^2, \quad \text{a.e. } t > \tau,\end{aligned}$$

we conclude (4.3).

From now on we will assume that

$$\text{there exists } 0 < \mu < 2\nu\lambda_1 \text{ such that } 2e^{\mu h}L_g < \mu, \tag{4.4}$$

and

$$\int_{-\infty}^0 e^{(\mu - 2e^{\mu h}L_g)s}\|f(s)\|_*^2\, ds < \infty. \tag{4.5}$$

Remark 4.1 If we assume that $f \in L^2_{loc}(\mathbb{R}; V')$, assumption (4.5) is equivalent to

$$\int_{-\infty}^t e^{(\mu - 2e^{\mu h}L_g)s}\|f(s)\|_*^2\, ds < \infty \quad \forall\, t \in \mathbb{R}.$$

Definition 4.1 *For any $\sigma > 0$, we will denote by $\mathcal{D}_\sigma(C_H)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_H)$ such that*

$$\lim_{\tau\to-\infty}\left(e^{\sigma\tau}\sup_{v\in D(\tau)}|v|_{C_H}^2\right) = 0.$$

Accordingly to the notation introduced in the previous section, $\mathcal{D}_F(C_H)$ will denote the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of C_H.

Remark 4.2 Observe that for any $\sigma > 0$, $\mathcal{D}_F(C_H) \subset \mathcal{D}_\sigma(C_H)$ and that $\mathcal{D}_\sigma(C_H)$ is inclusion-closed.

From now on, for brevity, we will denote

$$\sigma_\mu = \mu - 2e^{\mu h}L_g. \tag{4.6}$$

Corollary 4.1 *Under the assumptions of Proposition 4.1, if moreover conditions (4.4) and (4.5) are satisfied, then the family* $\widehat{D}_{0,\mu} = \{D_{0,\mu}(t) : t \in \mathbb{R}\}$, *with* $D_{0,\mu}(t) = \overline{B}_{C_H}(0, \rho_\mu(t))$, *the closed ball in* C_H *of center zero and radius* $\rho_\mu(t)$, *where*

$$\rho_\mu^2(t) = 1 + e^{\mu h}(2\nu - \mu\lambda_1^{-1})^{-1} \int_{-\infty}^{t} e^{-\sigma_\mu(t-s)} \|f(s)\|_*^2 \, ds,$$

is pullback $\mathcal{D}_{\sigma_\mu}(C_H)$*-absorbing for the process* U *defined by (4.1). Moreover,* $\widehat{D}_{0,\mu} \in \mathcal{D}_{\sigma_\mu}(C_H)$.

Proof. It follows immediately from Lemma 4.1.

Proposition 4.2 *Under the assumptions of Corollary 4.1, the process* U *defined by* (4.1) *is pullback* $\widehat{D}_{0,\mu}$*-asymptotically compact.*

Proof. Let us fix $t_0 \in \mathbb{R}$. Let $\{u^n\}$ with $u^n = u^n(\cdot; \tau_n, \phi^n)$ be a sequence of weak solutions of (1.1), defined in their respective intervals $[\tau_n - h, \infty)$, with initial data $\phi^n \in D_{0,\mu}(\tau_n) = \overline{B}_{C_H}(0, \rho_\mu(\tau_n))$, where $\{\tau_n\} \subset (-\infty, t_0)$ satisfies that $\tau_n \to -\infty$ as $n \to \infty$. We will prove that the sequence $\{u^n_{t_0}\}$ is relatively compact in C_H, i.e., we will see that there exist a subsequence, relabelled $\{u^n_{t_0}\}$, and a function $\psi \in C_H$, such that $u^n_{t_0} \to \psi$ in C_H.

Consider an arbitrary value $T > h$. It follows from (4.2) and (4.5) that there exists $n_0(t_0, T)$ such that $\tau_n < t_0 - T$ for $n \ge n_0(t_0, T)$, and

$$|u^n_t|^2_{C_H} \le R(t_0, T) \quad \forall\, t \in [t_0 - T, t_0],\ n \ge n_0(t_0, T), \tag{4.7}$$

where

$$R(t_0, T) = 1 + e^{\mu h}(2\nu - \mu\lambda_1^{-1})^{-1} e^{-\sigma_\mu(t_0 - T)} \int_{-\infty}^{t_0} e^{\sigma_\mu s} \|f(s)\|_*^2 \, ds,$$

so that, in particular,

$$|u^n(t)|^2 \le R(t_0, T) \quad \forall\, t \in [t_0 - T, t_0],\ n \ge n_0(t_0, T). \tag{4.8}$$

Let us denote $y^n(t) = u^n(t + t_0 - T)$ for all $t \in [0, T]$. In particular, by (4.8), the sequence $\{y^n\}_{n \ge n_0(t_0,T)}$ is bounded in $L^\infty(0, T; H)$.

On the other hand, for each $n \ge n_0(t_0, T)$, the function y^n is a weak solution on $[0, T]$ of a problem similar to (1.1), namely with f and g replaced by

$$\tilde{f}(t) = f(t + t_0 - T) \text{ and } \tilde{g}(t, \cdot) = g(t + t_0 - T, \cdot), \quad t \in (0, T),$$

respectively, and with $y^n_0 = u^n_{t_0 - T}$ and $y^n_T = u^n_{t_0}$. By (4.7), $|y^n_0|^2_{C_H} \le R(t_0, T)$ for all $n \ge n_0(t_0, T)$. From (4.3) we have

$$\|y^n\|^2_{L^2(0,T;V)} \le K(t_0, T) \quad \forall\, n \ge n_0(t_0, T),$$

where

$$K(t_0, T) = \nu^{-1} R(t_0, T) + \nu^{-2} \int_0^T \|\widetilde{f}(s)\|_*^2 \, ds + \nu^{-1} 2 L_g R(t_0, T) T.$$

Hence, the sequence $\{y^n\}_{n \ge n_0(t_0,T)}$ is also bounded in $L^2(0, T; V)$, and the sequence of time derivatives $\{(y^n)'\}_{n \ge n_0(t_0,T)}$ is bounded in $L^2(0, T; V')$. Thus, up to a subsequence (relabelled the same), for some function y we have that

$$\begin{cases} y^n \overset{*}{\rightharpoonup} y & \text{weakly-star in } L^\infty(0, T; H), \\ y^n \rightharpoonup y & \text{weakly in } L^2(0, T; V), \\ (y^n)' \rightharpoonup y' & \text{weakly in } L^2(0, T; V'), \\ y^n \to y & \text{strongly in } L^2(0, T; H), \\ y^n(t) \to y(t) & \text{in } H, \text{ a.e. } t \in (0, T). \end{cases}$$

Observe also that $y \in C([0,T];H)$, and that for every sequence $\{t_n\} \subset [0,T]$ with $t_n \to t^*$, one has

$$y^n(t_n) \rightharpoonup y(t^*) \quad \text{weakly in } H, \tag{4.9}$$

which is a consequence of the boundedness of the sequences $\{y^n\}_{n \geq n_0(t_0,T)}$ and $\{(y^n)'\}_{n \geq n_0(t_0,T)}$ in $L^\infty(0,T;H)$ and $L^2(0,T;V')$ respectively, and the compactness of the injection of H into V' (see the proof of Theorem 2.1 for a similar argument).

Also, by (II), (III), and (4.7), we obtain

$$\int_0^t |\tilde{g}(s, y^n_s)|^2 \, ds \leq Ct,$$

where $C > 0$ does not depend neither on n nor $t \in [0,T]$. Thus, eventually extracting a subsequence, there exists $\xi \in L^2(0,T;(L^2(\Omega))^2)$ such that

$$\tilde{g}(\cdot, y^n_\cdot) \rightharpoonup \xi \quad \text{weakly in } L^2(0,T;(L^2(\Omega))^2),$$

and therefore

$$\begin{aligned} &\int_s^t |\tilde{g}(r, y^n_r)|^2 \, dr \leq C(t-s), \\ &\int_s^t |\xi(r)|^2 \, dr \leq \liminf_{n\to\infty} \int_s^t |\tilde{g}(r, y^n_r)|^2 \, dr \leq C(t-s), \end{aligned} \tag{4.10}$$

for all $0 \leq s \leq t \leq T$. Then, in a standard way, one can prove that $y(\cdot)$ is the unique weak solution to the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \tilde{f}(t) + \xi(t) \quad \text{in } \Omega \times (0,T), \\ \nabla \cdot u \; = 0 \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = y(x,0), \quad x \in \Omega. \end{cases}$$

By the energy equality and (4.10), we obtain that

$$\frac{1}{2}|z(t)|^2 \leq \frac{1}{2}|z(s)|^2 + \int_s^t \langle \tilde{f}(r), z(r) \rangle \, dr + \widetilde{C}(t-s) \quad \forall\, 0 \leq s \leq t \leq T,$$

where $\widetilde{C} = C(4\nu\lambda_1)^{-1}$, and $z = y^n$ or $z = y$. Then, the maps $\tilde{J}_n, \tilde{J} : [0,T] \to \mathbb{R}$ defined by

$$\begin{aligned} \tilde{J}_n(t) &= \frac{1}{2}|y^n(t)|^2 - \int_0^t \langle \tilde{f}(r), y^n(r) \rangle \, dr - Ct, \\ \tilde{J}(t) &= \frac{1}{2}|y(t)|^2 - \int_0^t \langle \tilde{f}(r), y(r) \rangle \, dr - Ct, \end{aligned}$$

are non-increasing and continuous, and satisfy

$$\tilde{J}_n(t) \to \tilde{J}(t) \quad \text{a.e. } t \in (0,T). \tag{4.11}$$

We can use the functionals $\tilde{J}_n$ and $\tilde{J}$ to deduce that $y^n \to y$ in $C([\delta,T];H)$, for any $0 < \delta < T$. If this is not true, then there exist $0 < \delta^* < T$, $\varepsilon^* > 0$, and subsequences $\{y^m\} \subset \{y^n\}_{n \geq n_0(t_0,T)}$ and $\{t_m\} \subset [\delta^*, T]$, with $t_m \to t^*$, such that

$$|y^m(t_m) - y(t^*)| \geq \varepsilon^* \quad \forall\, m. \tag{4.12}$$

Let us fix $\varepsilon > 0$. Observe that $t^* \in [\delta^*, T]$, and therefore, by (4.11) and the continuity and non-increasing character of $\tilde{J}$, there exists $0 < \hat{t}_\varepsilon < t^*$ such that

$$\lim_{m\to\infty} \tilde{J}_m(\hat{t}_\varepsilon) = \tilde{J}(\hat{t}_\varepsilon), \tag{4.13}$$

and

$$0 \le \tilde{J}(\hat{t}_\varepsilon) - \tilde{J}(t^*) \le \varepsilon. \tag{4.14}$$

As $t_m \to t^*$, there exists an m_ε such that $\hat{t}_\varepsilon < t_m$ for all $m \ge m_\varepsilon$. Then, by (4.14),

$$\begin{aligned}\tilde{J}_m(t_m) - \tilde{J}(t^*) &\le \tilde{J}_m(\hat{t}_\varepsilon) - \tilde{J}(t^*)\\ &\le |\tilde{J}_m(\hat{t}_\varepsilon) - \tilde{J}(\hat{t}_\varepsilon)| + |\tilde{J}(\hat{t}_\varepsilon) - \tilde{J}(t^*)|\\ &\le |\tilde{J}_m(\hat{t}_\varepsilon) - \tilde{J}(\hat{t}_\varepsilon)| + \varepsilon\end{aligned}$$

for all $m \ge m_\varepsilon$, and consequently, by (4.13), $\limsup_{m\to\infty} \tilde{J}_m(t_m) \le \tilde{J}(t^*) + \varepsilon$. Thus, as $\varepsilon > 0$ is arbitrary, we deduce that

$$\limsup_{m\to\infty} \tilde{J}_m(t_m) \le \tilde{J}(t^*). \tag{4.15}$$

Taking into account that $t_m \to t^*$, and

$$\int_0^{t_m} \langle \tilde{f}(r), y^m(r)\rangle\, dr \to \int_0^{t^*} \langle \tilde{f}(r), y(r)\rangle\, dr,$$

from (4.15) we deduce that $\limsup_{m\to\infty} |y^m(t_m)| \le |y(t^*)|$. This last inequality and (4.9), imply that $y^m(t_m) \to y(t^*)$ strongly in H, which is in contradiction with (4.12).

We have thus proved that $y^n \to y$ in $C([\delta, T]; H)$, for any $0 < \delta < T$. As $T > h$, we obtain in particular that $u^n_{t_0} \to \psi$ in C_H, where $\psi(s) = y(s+T)$, for $s \in [-h, 0]$.

Joining all the above statements we obtain the existence of minimal pullback attractors for the process U on C_H associated to problem (1.1).

Theorem 4.1 *Assume that $f \in L^2_{loc}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying the assumptions (I)–(III), (4.4) and (4.5), are given. Then, there exist the minimal pullback $\mathcal{D}_F(C_H)$-attractor $\mathcal{A}_{\mathcal{D}_F(C_H)}$, and the minimal pullback $\mathcal{D}_{\sigma_\mu}(C_H)$-attractor $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}$, for the process U defined by (4.1). The family $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}$ belongs to $\mathcal{D}_{\sigma_\mu}(C_H)$, and the following relation holds:*

$$\mathcal{A}_{\mathcal{D}_F(C_H)}(t) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t) \subset \overline{B}_{C_H}(0, \rho_\mu(t)) \quad \forall\, t \in \mathbb{R}.$$

Proof. The result is a direct consequence of Theorem 3.1, Remark 3.2, Corollary 3.1, Proposition 4.1, Corollary 4.1, and Proposition 4.2.

Remark 4.3 *(i) If, additionally, we assume that*

$$\sup_{r\le 0} \int_{-\infty}^{r} e^{-\sigma_\mu(r-s)} \|f(s)\|_*^2\, ds < \infty,$$

where σ_μ is given by (4.6)*, then, taking into account Remark 3.3, we deduce that*

$$\mathcal{A}_{\mathcal{D}_F(C_H)}(t) = \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t) \quad \forall\, t \in \mathbb{R}.$$

(ii) Observe that a natural question concerning the existence of more families of pullback attractors is to strengthen the conditions on the parameter μ that satisfies (32) and (33). More exactly, if

$\sigma < \sigma'$, then $\mathcal{D}_\sigma(C_H) \subset \mathcal{D}_{\sigma'}(C_H)$. Therefore, in order to obtain attractors for bigger universes, we would wonder if there exists $\mu' \in (0, 2\nu\lambda_1)$ such that $\sigma_{\mu'} > \sigma_\mu$. In such a case, conditions (32) and (33) would be satisfied automatically. The key point for having $\sigma_{\mu'} > \sigma_\mu$ is to analyze the growth behaviour of the map $\mu \mapsto \sigma_\mu$. Namely, if the map $\mu \mapsto \sigma_\mu$ is non-decreasing, we look for $\mu < \mu' < 2\nu\lambda_1$ (this may involve a smallness condition on the delay); otherwise, we seek for $0 < \mu' < \mu$. Under any of these conditions, we would obtain new families of pullback attractors and new relations among them (see [9, Remark 4.15] or [1, Remark 5] for similar results in a simpler context).

5 Regularity of pullback attractors and V attraction for the process associated to (1.1)

Now, we strengthen the regularity of solutions and a second energy equality for them, in order to obtain additional attraction, namely, in the H^1 norm instead of L^2 as in Section 4. For any $\tilde{h} \in [0, h]$, let us denote

$$C_H^{\tilde{h},V} = \{\varphi \in C_H : \varphi|_{[-\tilde{h},0]} \in B([-\tilde{h}, 0]; V)\},$$

where $B([-\tilde{h}, 0]; V)$ is the space of bounded functions from $[-\tilde{h}, 0]$ into V. The space $C_H^{\tilde{h},V}$ is a Banach space with the norm

$$\|\varphi\|_{\tilde{h},V} = |\varphi|_{C_H} + \sup_{\theta\in[-\tilde{h},0]} \|\varphi(\theta)\|.$$

Observe that the space $C_V = C([-h, 0]; V)$ is a Banach subspace of $C_H^{h,V}$.

Proposition 5.1 *Assume that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$, satisfying the assumptions (I)–(III), are given. Then, for any bounded set $B \subset C_H$, one has:*

(i) The set of weak solutions of (1.1) $\{u(\cdot; \tau, \phi) : \phi \in B\}$ is bounded in $L^\infty(\tau + \varepsilon, T; V)$, for any $\varepsilon > 0$ and any $T > \tau + \varepsilon$.

(ii) Moreover, if $\{\phi(0) : \phi \in B\}$ is bounded in V, then $\{u(\cdot; \tau, \phi) : \phi \in B\}$ is bounded in $L^\infty(\tau, T; V)$, for all $T > \tau$.

Proof. By (2.5), the regularity property (a) in Theorem 2.1, Cauchy-Schwartz and Young inequalities, we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{d\theta}\|u(\theta)\|^2 + \nu|Au(\theta)|^2 + b(u(\theta), u(\theta), Au(\theta)) \\ \leq\ & \frac{2}{\nu}(|f(\theta)|^2 + |g(\theta, u_\theta)|^2) + \frac{\nu}{4}|Au(\theta)|^2, \quad \text{a.e. } \theta > \tau. \end{aligned}$$

Since the trilinear term b can be estimated using (2.2) as

$$\begin{aligned} |b(u(\theta), u(\theta), Au(\theta))| &\leq C_1|u(\theta)|^{1/2}\|u(\theta)\||Au(\theta)|^{3/2} \\ &\leq \frac{\nu}{4}|Au(\theta)|^2 + C^{(\nu)}|u(\theta)|^2\|u(\theta)\|^4, \end{aligned}$$

where

$$C^{(\nu)} = 27C_1^4(4\nu^3)^{-1}, \tag{5.1}$$

this, combined with the above and the properties of g, gives

$$\frac{d}{d\theta}\|u(\theta)\|^2 + \nu|Au(\theta)|^2 \leq \frac{4}{\nu}|f(\theta)|^2 + 2C^{(\nu)}|u(\theta)|^2\|u(\theta)\|^4 + \frac{4L_g^2}{\nu}|u_\theta|_{C_H}^2, \quad \text{a.e. } \theta > \tau. \tag{5.2}$$

Integrating, in particular we deduce that for all $\tau < s \leq r$

$$\|u(r)\|^2 \leq \|u(s)\|^2 + \frac{4}{\nu}\int_s^r |f(\theta)|^2\, d\theta + 2C^{(\nu)}\int_s^r |u(\theta)|^2\|u(\theta)\|^4\, d\theta + \frac{4L_g^2}{\nu}\int_s^r |u_\theta|_{C_H}^2\, d\theta.$$

By Gronwall lemma we obtain again that for all $\tau < s \leq r$

$$\begin{aligned}\|u(r)\|^2 \quad &\leq \quad \Big(\|u(s)\|^2 + \frac{4}{\nu}\int_s^r |f(\theta)|^2\, d\theta + \frac{4L_g^2}{\nu}\int_s^r |u_\theta|_{C_H}^2\, d\theta\Big) \\ &\quad \times \exp\Big(2C^{(\nu)}\int_s^r |u(\theta)|^2\|u(\theta)\|^2\, d\theta\Big).\end{aligned} \tag{5.3}$$

Integrating once more with respect to $s \in (\tau, r)$, it yields

$$\begin{aligned}(r-\tau)\|u(r)\|^2 \quad &\leq \quad \Big(\int_\tau^T \|u(s)\|^2\, ds + \frac{4(T-\tau)}{\nu}\int_\tau^T |f(\theta)|^2\, d\theta + \frac{4L_g^2(T-\tau)}{\nu}\int_\tau^T |u_\theta|_{C_H}^2\, d\theta\Big) \\ &\quad \times \exp\Big(2C^{(\nu)}\int_\tau^T |u(\theta)|^2\|u(\theta)\|^2\, d\theta\Big) \quad \forall\, \tau < r \leq T.\end{aligned}$$

In particular, for $\tau + \varepsilon \leq r \leq T$, we have

$$\begin{aligned}\|u(r)\|^2 \quad &\leq \quad \frac{1}{\varepsilon}\Big(\int_\tau^T \|u(s)\|^2\, ds + \frac{4(T-\tau)}{\nu}\int_\tau^T |f(\theta)|^2\, d\theta + \frac{4L_g^2(T-\tau)}{\nu}\int_\tau^T |u_\theta|_{C_H}^2\, d\theta\Big) \\ &\quad \times \exp\Big(2C^{(\nu)}\int_\tau^T |u(\theta)|^2\|u(\theta)\|^2\, d\theta\Big).\end{aligned}$$

Taking into account (4.2) and (4.3), the claim (i) is proved.

The proof of claim (ii) is simpler. If $\phi(0)$ belongs to V, then from (5.2) one deduces that for all $\tau \leq r \leq T$,

$$\|u(r)\|^2 \leq \|u(\tau)\|^2 + \frac{4}{\nu}\int_\tau^r |f(\theta)|^2\, d\theta + 2C^{(\nu)}\int_\tau^r |u(\theta)|^2\|u(\theta)\|^4\, d\theta + \frac{4L_g^2}{\nu}\int_\tau^r |u_\theta|_{C_H}^2\, d\theta.$$

Hence, one may apply directly Gronwall lemma and proceed analogously as before to conclude (ii).

Corollary 5.1 *Under the assumptions of Proposition 5.1, the process U defined by (4.1) satisfies that $U(t,\tau)$ maps bounded sets of C_H into bounded sets of C_H, for all $t \geq \tau$. Moreover, for any $\tilde{h} \in [0,h]$, the family of mappings $U(t,\tau)|_{C_H^{\tilde{h},V}}$, with $t \geq \tau$, is also a well defined process on $C_H^{\tilde{h},V}$, and maps bounded sets of $C_H^{\tilde{h},V}$ into bounded sets of $C_H^{\tilde{h},V}$.*

Proposition 5.2 *Assume that $f \in L^2_{loc}(\mathbb{R};(L^2(\Omega))^2)$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfying the assumptions (I)–(III), are given. Let us denote $u = u(\cdot;\tau,\phi)$ and $v = v(\cdot;\tau,\psi)$ the solutions of (1.1) corresponding to initial data ϕ and $\psi \in C_H^{0,V}$. Then, the following estimate holds:*

$$\begin{aligned}\|u(s)-v(s)\|^2 \quad &\leq \quad \Big(\|\phi(0)-\psi(0)\|^2 + \frac{L_g^2}{\nu}\int_\tau^t |u_\theta - v_\theta|_{C_H}^2\, d\theta\Big) \\ &\quad \times \exp\Big[\int_\tau^t \Big(2C^{(\nu)}\lambda_1^{-1}\|u(\theta)\|^4 + \frac{2C_1^2}{\nu}|v(\theta)\|Av(\theta)|\Big)d\theta\Big]\end{aligned} \tag{5.4}$$

for all $\tau \le s \le t$, where $C^{(\nu)}$ is given in (5.1).

As a consequence, for all $\tilde{h} \in [0,h]$ and any $\tau \le t$, the mapping $U(t,\tau) : C_H^{\tilde{h},V} \to C_H^{\tilde{h},V}$ given by (4.1), is continuous.

Proof. In order to prove the statement, we only have to check (5.4) and combine it with Proposition 2.1, and claim (ii) in Proposition 5.1.

Let us denote $w = u - v$. If we apply the energy equality to w, we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|w(t)\|^2 + \nu|Aw(t)|^2 + b(u(t),u(t),Aw(t)) - b(v(t),v(t),Aw(t)) \\ = \; & (g(t,u_t) - g(t,v_t), Aw(t)) \\ \le \; & \frac{L_g^2}{2\nu}|w_t|_{C_H}^2 + \frac{\nu}{2}|Aw(t)|^2, \quad \text{a.e. } t > \tau, \end{aligned}$$

where we have used Young inequality and the property (III) of g.

The trilinear terms can be estimated, using (2.2), as follows:

$$\begin{aligned} & |b(u(t),u(t),Aw(t)) - b(v(t),v(t),Aw(t))| \\ = \; & |b(w(t),u(t),Aw(t)) + b(v(t),w(t),Aw(t))| \\ \le \; & C_1|w(t)|^{1/2}\|u(t)\||Aw(t)|^{3/2} + C_1|v(t)|^{1/2}|Av(t)|^{1/2}\|w(t)\||Aw(t)| \\ \le \; & C^{(\nu)}|w(t)|^2\|u(t)\|^4 + \frac{C_1^2}{\nu}|v(t)||Av(t)|\|w(t)\|^2 + \frac{\nu}{2}|Aw(t)|^2. \end{aligned}$$

Therefore, from above we obtain that

$$\frac{d}{dt}\|w(t)\|^2 \le 2C^{(\nu)}|w(t)|^2\|u(t)\|^4 + \frac{2C_1^2}{\nu}|v(t)||Av(t)|\|w(t)\|^2 + \frac{L_g^2}{\nu}|w_t|_{C_H}^2, \quad \text{a.e. } t > \tau.$$

Integrating, it yield,s for all $\tau \le s \le t$,

$$\|w(s)\|^2 \le \|w(\tau)\|^2 + \frac{L_g^2}{\nu}\int_\tau^s |w_\theta|_{C_H}^2\, d\theta + \int_\tau^s \|w(\theta)\|^2\Big(2C^{(\nu)}\lambda_1^{-1}\|u(\theta)\|^4 + \frac{2C_1^2}{\nu}|v(\theta)||Av(\theta)|\Big)\, d\theta.$$

From this inequality, using Gronwall lemma, we deduce (5.4).

Definition 5.1 *For any $\sigma > 0$ and $\tilde{h} \in [0,h]$, we will denote by $\mathcal{D}_\sigma^{\tilde{h},V}(C_H)$ the class of families $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\sigma(C_H)$ such that for any $t \in \mathbb{R}$ and for any $\varphi \in D(t)$, it follows that $\varphi|_{[-\tilde{h},0]} \in B([-\tilde{h},0];V)$.*

Analogously, we will denote by $\mathcal{D}_F^{\tilde{h},V}(C_H)$ the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of C_H such that for any $\varphi \in D$, it holds that $\varphi|_{[-\tilde{h},0]} \in B([-\tilde{h},0];V)$.

Finally, we will denote by $\mathcal{D}_F(C_H^{\tilde{h},V})$ the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of $C_H^{\tilde{h},V}$.

Remark 5.1 The chain of inclusions for the universes in the above definition and the universes introduced in Section 3, is the following:

$$\mathcal{D}_F(C_H^{\tilde{h},V}) \subset \mathcal{D}_F^{\tilde{h},V}(C_H) \subset \mathcal{D}_\sigma^{\tilde{h},V}(C_H) \subset \mathcal{D}_\sigma(C_H),$$

and

$$\mathcal{D}_F(C_H^{\tilde{h},V}) \subset \mathcal{D}_F^{\tilde{h},V}(C_H) \subset \mathcal{D}_F(C_H) \subset \mathcal{D}_\sigma(C_H),$$

for all $\sigma > 0$ and any $\tilde{h} \in [0, h]$. It must also be pointed out that $\mathcal{D}_\sigma^{\tilde{h},V}(C_H)$ is also inclusion-closed, which will be important (cf. Remark 3.2). Finally, it is clear that if $0 \le \tilde{h}_1 < \tilde{h}_2 \le h$, then

$$\mathcal{D}_F(C_H^{\tilde{h}_2,V}) \subset \mathcal{D}_F(C_H^{\tilde{h}_1,V}), \quad \mathcal{D}_F^{\tilde{h}_2,V}(C_H) \subset \mathcal{D}_F^{\tilde{h}_1,V}(C_H), \quad \mathcal{D}_\sigma^{\tilde{h}_2,V}(C_H) \subset \mathcal{D}_\sigma^{\tilde{h}_1,V}(C_H).$$

We establish now some results on absorbing properties of $U : \mathbb{R}^2_d \times C_H^{\tilde{h},V} \to C_H^{\tilde{h},V}$.

Proposition 5.3 *Let g satisfying (I)-(III) be given. Assume that* $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$ *is such that there exists* $0 < \mu < 2\nu\lambda_1$ *such that* $\mu > 2e^{\mu h} L_g$*, and*

$$\int_{-\infty}^0 e^{\sigma_\mu s} |f(s)|^2 \, ds < \infty, \tag{5.5}$$

where σ_μ *is given by (4.6). Then, for any* $\tilde{h} \in [0, h]$*, the family* $\widehat{D}_{0,\mu,\tilde{h}} = \{D_{0,\mu,\tilde{h}}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_H^{\tilde{h},V})$*, with*

$$D_{0,\mu,\tilde{h}}(t) = D_{0,\mu}(t) \cap C_H^{\tilde{h},V},$$

where $D_{0,\mu}(t)$ *defined in Corollary 4.1, is a family of closed sets of* $C_H^{\tilde{h},V}$*, which is pullback* $\mathcal{D}_{\sigma_\mu}^{\tilde{h},V}(C_H)$*-absorbing for the process* $U : \mathbb{R}^2_d \times C_H^{\tilde{h},V} \to C_H^{\tilde{h},V}$ *given by (4.1). Moreover,* $\widehat{D}_{0,\mu,\tilde{h}}$ *belongs to* $\mathcal{D}_{\sigma_\mu}^{\tilde{h},V}(C_H)$*.*

Proof. It is a consequence of Corollary 4.1.

Lemma 5.1 *Under the assumptions of Proposition 5.3, for any* $\widehat{D} \in \mathcal{D}_{\sigma_\mu}(C_H)$ *and any* $r > h$*, the family* $\widehat{D}^{(r)} = \{D^{(r)}(\tau) : \tau \in \mathbb{R}\}$*, where* $D^{(r)}(\tau) = U(\tau + r, \tau) D(\tau)$*, for any* $\tau \in \mathbb{R}$*, belongs to* $\mathcal{D}_{\sigma_\mu}^{h,V}(C_H)$*.*

Proof. From (4.2), we deduce

$$\sup_{\psi \in D^{(r)}(\tau)} \left(e^{\sigma_\mu \tau} |\psi|^2_{C_H} \right) \le e^{\mu h - \sigma_\mu r} \sup_{\phi \in D(\tau)} \left(e^{\sigma_\mu \tau} |\phi|^2_{C_H} \right) + (2\nu\lambda_1 - \mu)^{-1} e^{\mu h - \sigma_\mu r} \int_\tau^{\tau + r} e^{\sigma_\mu s} |f(s)|^2 \, ds.$$

From this inequality, property (a) in Theorem 2.1, and assumption (5.5), we deduce the result.

Now, we establish several estimates in finite intervals of time when the initial time is sufficiently shifted in a pullback sense (cf. [8, 9] for similar results in a context without delays).

Lemma 5.2 *Under the assumptions of Proposition 5.3, for any* $t \in \mathbb{R}$ *and* $\widehat{D} \in \mathcal{D}_{\sigma_\mu}(C_H)$*, there exist* $\tau_1(\widehat{D}, t, h) < t - 2h - 2$ *and functions* $\{\rho_i\}_{i=1}^4$ *depending on t and h, such that for any* $\tau \le \tau_1(\widehat{D}, t, h)$ *and any* $\phi^\tau \in D(\tau)$*, we have*

$$\begin{cases} |u(r; \tau, \phi^\tau)|^2 \le \rho_1(t) & \forall\, r \in [t - 2h - 2, t], \\ \|u(r; \tau, \phi^\tau)\|^2 \le \rho_2(t) & \forall\, r \in [t - h - 1, t], \\ \nu \displaystyle\int_{r-1}^r |Au(\theta; \tau, \phi^\tau)|^2 \, d\theta \le \rho_3(t) & \forall\, r \in [t - h, t], \\ \displaystyle\int_{r-1}^r |u'(\theta; \tau, \phi^\tau)|^2 \, d\theta \le \rho_4(t) & \forall\, r \in [t - h, t], \end{cases} \tag{5.6}$$

where

$$
\begin{aligned}
\rho_1(t) &= 1 + e^{\mu h}(2\nu\lambda_1 - \mu)^{-1} e^{-\sigma_\mu(t-2h-2)} \int_{-\infty}^{t} e^{\sigma_\mu s} |f(s)|^2 \, ds,\\
\rho_2(t) &= \left(\nu^{-1}\left(1 + 2\nu^{-1}\lambda_1^{-1}L_g^2 + 4L_g^2\right)\rho_1(t) + \nu^{-1}\left(4 + 2\nu^{-1}\lambda_1^{-1}\right)\int_{t-h-2}^{t} |f(\theta)|^2 \, d\theta\right)\\
&\quad \times \exp\left\{2\nu^{-1}C^{(\nu)}\rho_1(t)\left[\left(1 + 2\nu^{-1}\lambda_1^{-1}L_g^2\right)\rho_1(t) + 2\nu^{-1}\lambda_1^{-1}\int_{t-h-2}^{t} |f(\theta)|^2 \, d\theta\right]\right\},\\
\rho_3(t) &= \rho_2(t) + 2C^{(\nu)}\rho_1(t)\rho_2^2(t) + 4L_g^2\nu^{-1}\rho_1(t) + 4\nu^{-1}\int_{t-h-1}^{t} |f(\theta)|^2 \, d\theta,\\
\rho_4(t) &= \nu\rho_2(t) + 4L_g^2\rho_1(t) + 2C_1^2\nu^{-1}\rho_2(t)\rho_3(t) + 4\int_{t-h-1}^{t} |f(\theta)|^2 \, d\theta,
\end{aligned}
$$

and $C^{(\nu)}$ is given in (5.1).

Proof. Let $\tau_1(\widehat{D}, t, h) < t - 2h - 2$ be such that

$$e^{\mu h} e^{-\sigma_\mu(t-2h-2)} e^{\sigma_\mu \tau} |\phi^\tau|_{C_H}^2 \le 1 \quad \forall\, \tau \le \tau_1(\widehat{D}, t, h),\ \phi^\tau \in D(\tau).$$

Consider fixed $\tau \le \tau_1(\widehat{D}, t, h)$ and $\phi^\tau \in D(\tau)$. First estimate in (5.6) follows directly from (4.2), using the increasing character of the exponential.

Now, for the rest of the estimates, let us consider again the Galerkin approximations already used in Theorem 2.1, and denote for short $u^m(r) = u^m(r; \tau, \phi^\tau)$. Multiplying in (2.7) by $\alpha_{m,j}(t)$, and summing from $j = 1$ to m, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|u^m(t)|^2 + \nu\|u^m(t)\|^2 &= (f(t) + g(t, u_t^m), u^m(t))\\
&\le \frac{1}{\nu\lambda_1}(|f(t)|^2 + |g(t, u_t^m)|^2) + \frac{\nu}{2}\lambda_1|u^m(t)|^2, \quad \text{a.e. } t > \tau,
\end{aligned}
$$

where we have used Young inequality. Now, by the assumptions (II) and (III) on g, we obtain

$$\frac{d}{dt}|u^m(t)|^2 + \nu\|u^m(t)\|^2 \le \frac{2}{\nu\lambda_1}(|f(t)|^2 + L_g^2|u_t^m|_{C_H}^2), \quad \text{a.e. } t > \tau.$$

Integrating, we deduce that

$$\nu \int_{r-1}^{r} \|u^m(\theta)\|^2 \, d\theta \le |u^m(r-1)|^2 + \frac{2}{\nu\lambda_1}\int_{r-1}^{r} (|f(\theta)|^2 + L_g^2|u_\theta^m|_{C_H}^2) \, d\theta \quad \forall\, \tau \le r - 1. \tag{5.7}$$

Now, observe that the first estimate in (5.6) and the estimates obtained in the proof of Proposition 5.1 also hold for the u^m. From (5.3), integrating with respect to $s \in (r-1, r)$, and using the first estimate in (5.6), we obtain

$$
\begin{aligned}
\|u^m(r)\|^2 &\le \left(\int_{r-1}^{r} \|u^m(s)\|^2 \, ds + 4\nu^{-1}\int_{r-1}^{r} |f(\theta)|^2 \, d\theta + 4L_g^2\nu^{-1}\rho_1(t)\right)\\
&\quad \times \exp\left(2C^{(\nu)}\rho_1(t)\int_{r-1}^{r} \|u^m(\theta)\|^2 \, d\theta\right) \quad \forall\, r \in [t-h-1, t].
\end{aligned}
$$

From this, jointly with (5.7) and the first estimate in (5.6) for u^m, one deduces

$$\|u^m(r; \tau, \phi^\tau)\|^2 \le \rho_2(t) \quad \forall\, r \in [t-h-1, t]. \tag{5.8}$$

From this inequality and Remark 2.4, we deduce that

$$u^m \overset{*}{\rightharpoonup} u(\cdot;\tau,\phi^\tau) \quad \text{weakly-star in } L^\infty(t-h-1,t;V).$$

So, taking inferior limit when m goes to infinity in (5.8), and using the fact that $u(\cdot;\tau,\phi^\tau) \in C([t-h-1,t];V)$, we obtain the second estimate in (5.6).

On other hand, from (5.2) we also have

$$\begin{aligned}\nu\int_{r-1}^{r}|Au^m(\theta)|^2\,d\theta \quad &\leq \quad \|u^m(r-1)\|^2+4\nu^{-1}\int_{r-1}^{r}|f(\theta)|^2\,d\theta+2C^{(\nu)}\int_{r-1}^{r}|u^m(\theta)|^2\|u^m(\theta)\|^4\,d\theta\\ &\quad +4L_g^2\nu^{-1}\int_{r-1}^{r}|u_\theta^m|_{C_H}^2\,d\theta \quad \forall\,\tau\leq r-1.\end{aligned}$$

Therefore,

$$\nu\int_{r-1}^{r}|Au^m(\theta;\tau,\phi^\tau)|^2\,d\theta \leq \rho_3(t) \quad \forall\,r\in[t-h,t]. \tag{5.9}$$

From Remark 2.4 and (5.9), we deduce that

$$u^m \rightharpoonup u(\cdot;\tau,\phi^\tau) \quad \text{weakly in } L^2(r-1,r;D(A))\;\forall\,r\in[t-h,t].$$

Thus, taking inferior limit when m goes to infinity in (5.9), we obtain the third inequality in (5.6).

Finally, multiplying in (2.7) by $\alpha'_{m,j}(t)$, and summing from $j=1$ till m, we obtain

$$\begin{aligned}&|(u^m)'(\theta)|^2+\frac{\nu}{2}\frac{d}{d\theta}\|u^m(\theta)\|^2+b(u^m(\theta),u^m(\theta),(u^m)'(\theta))\\ =\quad &(f(\theta),(u^m)'(\theta))+(g(\theta,u_\theta^m),(u^m)'(\theta)), \quad \text{a.e. } \theta>\tau.\end{aligned}$$

Observing that by Young inequality and (2.3),

$$\begin{aligned}|(f(\theta),(u^m)'(\theta))| \quad &\leq \quad \frac{1}{8}|(u^m)'(\theta)|^2+2|f(\theta)|^2,\\ |(g(\theta,u_\theta^m),(u^m)'(\theta))| \quad &\leq \quad \frac{1}{8}|(u^m)'(\theta)|^2+2|g(\theta,u_\theta^m)|^2,\\ |b(u^m(\theta),u^m(\theta),(u^m)'(\theta))| \quad &\leq \quad C_1|Au^m(\theta)|\|u^m(\theta)\||(u^m)'(\theta)|\\ &\leq \quad \frac{1}{4}|(u^m)'(\theta)|^2+C_1^2|Au^m(\theta)|^2\|u^m(\theta)\|^2,\end{aligned}$$

we obtain that

$$|(u^m)'(\theta)|^2+\nu\frac{d}{d\theta}\|u^m(\theta)\|^2 \leq 4|f(\theta)|^2+4|g(\theta,u_\theta^m)|^2+2C_1^2|Au^m(\theta)|^2\|u^m(\theta)\|^2, \quad \text{a.e. } \theta>\tau.$$

From the properties of g, and integrating above, we conclude

$$\begin{aligned}\int_{r-1}^{r}|(u^m)'(\theta)|^2\,d\theta \quad &\leq \quad \nu\|u^m(r-1)\|^2+4\int_{r-1}^{r}|f(\theta)|^2\,d\theta+2C_1^2\int_{r-1}^{r}|Au^m(\theta)|^2\|u^m(\theta)\|^2\,d\theta\\ &\quad +4L_g^2\int_{r-1}^{r}|u_\theta^m|_{C_H}^2\,d\theta \quad \forall\,\tau\leq r-1.\end{aligned}$$

From the first estimate in (5.6) for u^m, (5.8) and (5.9), we deduce that

$$\int_{r-1}^{r}|(u^m)'(\theta;\tau,\phi^\tau)|^2\,d\theta \leq \rho_4(t) \quad \forall\,r\in[t-h,t]. \tag{5.10}$$

From Remark 2.4 and (5.10), we deduce that

$$(u^m)' \rightharpoonup u'(\cdot;\tau,\phi^\tau) \quad \text{weakly in } L^2(r-1,r;H)\,\forall\, r\in[t-h,t].$$

Thus, taking inferior limit when m goes to infinity in (5.10), we obtain the fourth inequality in (5.6).

Remark 5.2 Under the assumptions of Lemma 5.2, $\lim_{t\to-\infty} e^{\sigma_\mu t}\rho_1(t)=0$.

Now, we can prove the $\mathcal{D}^{\tilde h,V}_{\sigma_\mu}(C_H)$-asymptotic compactness of the process U restricted to the space $C_H^{\tilde h,V}$. The proof relies on an energy method with continuous functions, which is similar to that used in the proof of Proposition 4.2, but starting with the energy equality (2.5), as in [9, Lem.4.13]; we reproduce it here just for the sake of completeness.

Lemma 5.3 *Under the assumptions of Proposition 5.3, and for any $\tilde h\in[0,h]$, the process $U:\mathbb{R}^2_d\times C_H^{\tilde h,V}\to C_H^{\tilde h,V}$ is pullback $\mathcal{D}^{\tilde h,V}_{\sigma_\mu}(C_H)$-asymptotically compact.*

Proof. Let $\tilde h\in[0,h]$ be fixed. Since, taking into account Remark 3.1, the asymptotic compactness in the norm of C_H was already established in Proposition 4.2, we only must care about the sup norm in $B([-\tilde h,0];V)$. So, let us fix $t\in\mathbb{R}$, a family $\widehat{D}=\{D(t):t\in\mathbb{R}\}\in\mathcal{D}^{\tilde h,V}_{\sigma_\mu}(C_H)$, a sequence $\{\tau_n\}\subset(-\infty,t]$ with $\tau_n\to-\infty$, and a sequence $\{\phi^{\tau_n}\}\subset C_H^{\tilde h,V}$, with $\phi^{\tau_n}\in D(\tau_n)$ for all n.

For short, let us denote $u^n(\cdot)=u(\cdot;\tau_n,\phi^{\tau_n})$. It is enough to prove that the sequence $\{u^n(t+\cdot)\}$ is relatively compact in C_V. By the asymptotic compactness in the norm of C_H and using a recursive argument in a finite number of steps, we may assume without loss of generality that there exists $\xi\in C([-2h-1,0];H)$ such that

$$u^n(t+\cdot)\to\xi(\cdot) \quad \text{strongly in } C([-2h-1,0];H). \tag{5.11}$$

From Lemma 5.2 we know that there exists a value $\tau_1(\widehat{D},t,h)<t-2h-2$ such that the subsequence $\{u^n:\tau_n\le\tau_1(\widehat{D},t,h)\}$ is bounded in $L^\infty(t-h-1,t;V)\cap L^2(t-h-1,t;D(A))$ with $\{(u^n)'\}$ bounded in $L^2(t-h-1,t;H)$. Moreover, using the Aubin-Lions compactness lemma (e.g., cf. [13]), and taking into account (5.11), we may ensure that if we denote $u(t+r)=\xi(r)$ for all $r\in[-2h-1,0]$, then $u\in L^\infty(t-h-1,t;V)\cap L^2(t-h-1,t;D(A))$ with $u'\in L^2(t-h-1,t;H)$, and for a subsequence (relabelled the same) the following convergences hold:

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(t-h-1,t;V),\\ u^n\rightharpoonup u & \text{weakly in } L^2(t-h-1,t;D(A)),\\ (u^n)'\rightharpoonup u' & \text{weakly in } L^2(t-h-1,t;H),\\ u^n\to u & \text{strongly in } L^2(t-h-1,t;V),\\ u^n(s)\to u(s) & \text{strongly in } V, \text{ a.e. } s\in(t-h-1,t). \end{cases} \tag{5.12}$$

Indeed, $u\in C([t-h-1,t];V)$ satisfies, thanks to (5.11) and (5.12), the equation (2.4) in $(t-h-1,t)$.

From the boundedness of $\{u^n\}$ in $C([t-h-1,t];V)$, we have that for any sequence $\{s_n\}\subset[t-h-1,t]$ with $s_n\to s_*$, it holds that

$$u^n(s_n)\rightharpoonup u(s_*) \quad \text{weakly in } V, \tag{5.13}$$

where we have used (5.11) to identify the weak limit. We will prove that

$$u^n\to u \quad \text{strongly in } C([t-h,t];V), \tag{5.14}$$

using an energy method for continuous functions analogous to that employed in the proof of Proposition 4.2, but starting with the energy equality (2.5) as in [9].

Indeed, if (5.14) is false, there exist $\varepsilon > 0$, a value $t_* \in [t-h, t]$, and subsequences (which we relabel the same) $\{u^n\}$ and $\{t_n\} \subset [t-h, t]$, with $\lim_{n\to\infty} t_n = t_*$, such that

$$\|u^n(t_n) - u(t_*)\| \geq \varepsilon \quad \forall\, n \geq 1. \tag{5.15}$$

Recall that by (5.13) we have that

$$\|u(t_*)\| \leq \liminf_{n\to\infty} \|u^n(t_n)\|. \tag{5.16}$$

On the other hand, using the energy equality (2.5) for u and all u^n, and reasoning as for the obtention of (5.2), we have that for all $t-h-1 \leq s_1 \leq s_2 \leq t$,

$$\begin{aligned} &\|u^n(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au^n(r)|^2\,dr \\ \leq\ & \|u^n(s_1)\|^2 + 2C^{(\nu)} \int_{s_1}^{s_2} |u^n(r)|^2 \|u^n(r)\|^4\,dr + \frac{4}{\nu}\int_{s_1}^{s_2} |f(r)|^2\,dr + \frac{4L_g^2}{\nu}\int_{s_1}^{s_2} |u^n_r|^2_{C_H}\,dr, \end{aligned}$$

and

$$\begin{aligned} &\|u(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au(r)|^2\,dr \\ \leq\ & \|u(s_1)\|^2 + 2C^{(\nu)} \int_{s_1}^{s_2} |u(r)|^2 \|u(r)\|^4\,dr + \frac{4}{\nu}\int_{s_1}^{s_2} |f(r)|^2\,dr + \frac{4L_g^2}{\nu}\int_{s_1}^{s_2} |u_r|^2_{C_H}\,dr. \end{aligned}$$

In particular, we can define the functions

$$\bar{J}_n(s) = \|u^n(s)\|^2 - 2C^{(\nu)} \int_{t-h-1}^{s} |u^n(r)|^2 \|u^n(r)\|^4\,dr - \frac{4}{\nu}\int_{t-h-1}^{s} |f(r)|^2\,dr - \frac{4L_g^2}{\nu}\int_{t-h-1}^{s} |u^n_r|^2_{C_H}\,dr,$$

$$\bar{J}(s) = \|u(s)\|^2 - 2C^{(\nu)} \int_{t-h-1}^{s} |u(r)|^2 \|u(r)\|^4\,dr - \frac{4}{\nu}\int_{t-h-1}^{s} |f(r)|^2\,dr - \frac{4L_g^2}{\nu}\int_{t-h-1}^{s} |u_r|^2_{C_H}\,dr.$$

These are continuous functions on $[t-h-1, t]$, and from the above inequalities, both $\bar{J}_n$ and $\bar{J}$ are non-increasing. Now, reasoning analogously as in the proofs of Theorem 2.1 (Step 2) and of Proposition 4.2, we may conclude that $\limsup_{n\to\infty} \bar{J}_n(t_n) \leq \bar{J}(t_*)$, and therefore, by (5.11) and (5.12), $\limsup_{n\to\infty} \|u^n(t_n)\| \leq \|u(t_*)\|$, which joined to (5.16) and (5.13) implies that $u^n(t_n) \to u(t_*)$ strongly in V, in contradiction with (5.15). Thus, (5.14) is proved as desired.

Now, we can establish our main result.

Theorem 5.1 *Let g satisfying (I)-(III) be given. Assume that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$ satisfies (5.5) for some $0 < \mu < 2\nu\lambda_1$ such that $\mu > 2e^{\mu h} L_g$. Then, for any $\tilde{h} \in [0, h]$, the process U on $C_H^{\tilde{h},V}$ possesses a minimal pullback $\mathcal{D}^{\tilde{h},V}_{\sigma_\mu}(C_H)$-attractor $\mathcal{A}_{\mathcal{D}^{\tilde{h},V}_{\sigma_\mu}(C_H)}$, a minimal pullback $\mathcal{D}^{\tilde{h},V}_F(C_H)$-attractor $\mathcal{A}_{\mathcal{D}^{\tilde{h},V}_F(C_H)}$, and a minimal pullback $\mathcal{D}_F(C_H^{\tilde{h},V})$-attractor $\mathcal{A}_{\mathcal{D}_F(C_H^{\tilde{h},V})}$. Besides, the following relations hold:*

$$\begin{aligned} \mathcal{A}_{\mathcal{D}_F(C_H^{\tilde{h},V})}(t) &\subset \mathcal{A}_{\mathcal{D}^{\tilde{h},V}_F(C_H)}(t) \\ &\subset \mathcal{A}_{\mathcal{D}_F(C_H)}(t) \\ &\subset \mathcal{A}_{\mathcal{D}^{\tilde{h},V}_{\sigma_\mu}(C_H)}(t) = \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t) \\ &\subset C_V \quad \forall\, t \in \mathbb{R}, \end{aligned} \tag{5.17}$$

and for any family $\widehat{D} \in \mathcal{D}_{\sigma_\mu}(C_H)$,

$$\lim_{\tau \to -\infty} \mathrm{dist}_{C_V}(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t)) = 0 \quad \forall\, t \in \mathbb{R}. \tag{5.18}$$

Finally, if moreover f *satisfies*

$$\sup_{s \le 0} \left(e^{-\sigma_\mu s} \int_{-\infty}^{s} e^{\sigma_\mu \theta} |f(\theta)|^2 \, d\theta \right) < \infty, \tag{5.19}$$

then all attractors in (5.17) coincide, and this family is tempered in C_V*, in the sense that*

$$\lim_{t \to -\infty} \left(e^{\sigma_\mu t} \sup_{v \in \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t)} \|v\|_{C_V}^2 \right) = 0, \tag{5.20}$$

where for $v \in C_V$*,* $\|v\|_{C_V} = \max_{s \in [-h,0]} \|v(s)\|$.

Proof. Let us fix $\tilde{h} \in [0,h]$. The existence of $\mathcal{A}_{\mathcal{D}^{\tilde{h},V}_{\sigma_\mu}(C_H)}$ is a consequence of Theorem 3.1, Proposition 5.2, Proposition 5.3, and Lemma 5.3.

The existence of $\mathcal{A}_{\mathcal{D}^{\tilde{h},V}_{F}(C_H)}$ and $\mathcal{A}_{\mathcal{D}_F(C_H^{\tilde{h},V})}$ follows from the above facts, and the inclusions $\mathcal{D}_F(C_H^{\tilde{h},V}) \subset \mathcal{D}^{\tilde{h},V}_F(C_H) \subset \mathcal{D}^{\tilde{h},V}_{\sigma_\mu}(C_H)$.

In (5.17), the chain of inclusions follows from Corollary 3.1, Theorem 3.2, and Remark 5.1. The equality is a consequence of Theorem 3.2 and Remark 3.4, by using Theorem 2.1, Remark 5.1, Lemma 5.2, Remark 5.2, and Lemma 5.3. The last inclusion is a consequence of the regularity result (a) in Theorem 2.1.

Property (5.18) is a consequence of Lemma 5.1, and the fact that by the regularity result (a) in Theorem 2.1, for any $\widehat{D} \in \mathcal{D}_{\sigma_\mu}(C_H)$ and any $\tau < t - h - 1$,

$$\begin{aligned}
& \mathrm{dist}_{C_V}(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t)) \\
\le\ & \mathrm{dist}_{C_H^{h,V}}(U(t,\tau+h+1)(U(\tau+h+1,\tau)D(\tau)), \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t)) \\
=\ & \mathrm{dist}_{C_H^{h,V}}(U(t,\tau+h+1)D^{(h+1)}(\tau), \mathcal{A}_{\mathcal{D}^{h,V}_{\sigma_\mu}(C_H)}(t)).
\end{aligned}$$

The coincidence of all attractors in (5.17) under the additional assumption (5.19) holds by applying once more Theorem 3.2 and Remark 3.4, Theorem 2.1, Proposition 5.1 (i), Remark 5.1, and Lemma 5.3, since (5.19) is equivalent to

$$\sup_{s \le 0} \int_{s-1}^{s} |f(\theta)|^2 \, d\theta < \infty. \tag{5.21}$$

The tempered condition (5.20) comes from (5.19) (and therefore (5.21)) and the expression of $\rho_2(t)$ given in Lemma 5.2.

Remark 5.3 (i) Observe that, under the assumptions of Theorem 5.1, one has $\mathcal{A}_{\mathcal{D}^{\tilde{h},V}_{\sigma_\mu}(C_H)} \equiv \mathcal{A}_{\mathcal{D}^{h,V}_{\sigma_\mu}(C_H)}$ for any $\tilde{h} \in [0,h]$, i.e., the pullback attractor $\mathcal{A}_{\mathcal{D}^{\tilde{h},V}_{\sigma_\mu}(C_H)}$ is independent of $\tilde{h}$. Actually, if f also satisfies (5.19), then $\mathcal{A}_{\mathcal{D}^{\tilde{h},V}_{F}(C_H)} \equiv \mathcal{A}_{\mathcal{D}^{h,V}_{F}(C_H)}$, and $\mathcal{A}_{\mathcal{D}_F(C_H^{\tilde{h},V})} \equiv \mathcal{A}_{\mathcal{D}_F(C_H^{h,V})}$.

(ii) Observe that since $\widehat{D}_{0,\mu,h} \in \mathcal{D}^{h,V}_{\sigma_\mu}(C_H)$, and that for each $t \in \mathbb{R}$, $D_{0,\mu,h}(t)$ is closed in $C_H^{h,V}$, from Remark 3.2 and Remark 5.1, we deduce that $\mathcal{A}_{\mathcal{D}^{h,V}_{\sigma_\mu}(C_H)} \in \mathcal{D}^{h,V}_{\sigma_\mu}(C_H)$.

Remark 5.4 We can also consider, for each $0 \leq \tilde{h} \leq h$, the class $\mathcal{D}_{\sigma_\mu}(C_H^{\tilde{h},V})$ of all families $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_H^{\tilde{h},V})$ such that

$$\lim_{\tau \to -\infty} \left(e^{\sigma_\mu \tau} \sup_{v \in D(\tau)} \|v\|^2_{\tilde{h},V} \right) = 0.$$

For this universe we have the chain of inclusions

$$\mathcal{D}_F(C_H^{\tilde{h},V}) \subset \mathcal{D}_{\sigma_\mu}(C_H^{\tilde{h},V}) \subset \mathcal{D}^{\tilde{h},V}_{\sigma_\mu}(C_H) \subset \mathcal{D}_{\sigma_\mu}(C_H).$$

Under the assumptions of Theorem 5.1, we deduce the existence of the minimal pullback $\mathcal{D}_{\sigma_\mu}(C_H^{\tilde{h},V})$-attractor $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H^{\tilde{h},V})}$. Moreover, this pullback attractor satisfies

$$\mathcal{A}_{\mathcal{D}_F(C_H^{\tilde{h},V})}(t) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H^{\tilde{h},V})}(t) \subset \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}(t) \quad \forall\, t \in \mathbb{R}.$$

In fact, if assumption (5.19) is satisfied, then $\mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H^{\tilde{h},V})} \equiv \mathcal{A}_{\mathcal{D}_{\sigma_\mu}(C_H)}$.

Acknowledgments

This work is dedicated to the memory of Prof. José Real, with our deepest admiration, gratitude, and love. He was Ph.D. advisor of the first two authors of this paper - a clever mathematician with a sharp view on problems, generous and wonderful colleague, and better friend. He passed away too soon, being only 60 years old. We miss him deeply, but he will stay forever in our hearts.

The authors thank one referee's comment (see Remark 4.3 (ii)), that improved the previous version of this paper.

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