

# Navier-Stokes Equations with Delays

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Some results on the existence and uniqueness of solutions to Navier-Stokes equations when the external force contains some hereditary characteristics are proved.

**Keywords:** Navier-Stokes equations, variable and distributed delays.

## 1. Introduction and statement of the problem

The Navier-Stokes equations govern the motion of usual fluids like water, air, oil, etc. These equations have been the object of numerous works since the first paper of Leray was published in 1933 (see Constantin & Foias 1988; Lions 1969; Temam 1979, and the references therein). However, up to date, we have not found in the literature any work which takes into account the possibility of appearing some kind of delay in these equations. The main aim of this work is to consider several situations in which the external force contains some hereditary features and prove existence of solutions. These situations may appear when we want to control the system (in certain sense) by applying a force which takes into account not only the present state of the system but the history of the solution. It is worth pointing out that a similar analysis was carried out by Artola (1969) for general linear partial differential equations with delays.

Let  $\Omega \subset \mathbb{R}^N$  ( $N = 2$  or  $3$ ) be an open and bounded set with regular boundary  $\Gamma$ ,  $T > 0$  given, and consider the following functional Navier-Stokes problem (for further details and notations see Lions 1969 and Temam 1979):

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \nu \Delta u + \sum_{i=1}^N u_i \frac{\partial u}{\partial x_i} = f - \nabla p + g(t, u_t) \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0 \quad \text{in } (0, T) \times \Omega, \\ u = 0 \quad \text{on } (0, T) \times \Gamma, \\ u(0, x) = u_0(x), \quad x \in \Omega, \\ u(t, x) = \phi(t, x), \quad t \in (-h, 0) \quad x \in \Omega, \end{array} \right.$$

where we assume that  $\nu > 0$  is the kinematic viscosity,  $u$  is the velocity field of the fluid,  $p$  the pressure,  $u_0$  the initial velocity field,  $f$  a nondelayed external force field,  $g$  another external force containing some hereditary characteristic and  $\phi$  the initial datum in the interval of time  $(-h, 0)$ , where  $h$  is a positive fixed number.

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To start, we consider the following usual abstract spaces:

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^N : \operatorname{div} u = 0 \right\},$$

$H$  = the closure of  $\mathcal{V}$  in  $(L^2(\Omega))^N$  with the norm  $|\cdot|$ , and inner product  $(\cdot, \cdot)$  where for  $u, v \in (L^2(\Omega))^N$ ,

$$(u, v) = \sum_{j=1}^N \int_{\Omega} u_j(x) v_j(x) dx,$$

$V$  = the closure of  $\mathcal{V}$  in  $(H_0^1(\Omega))^N$  with the norm  $\|\cdot\|$ , and associated scalar product  $((\cdot, \cdot))$ , where for  $u, v \in (H_0^1(\Omega))^N$ ,

$$((u, v)) = \sum_{i,j=1}^N \int_{\Omega} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx.$$

It follows that  $V \subset H \equiv H' \subset V'$ , where the injections are dense and compact. Now we denote  $a(u, v) = ((u, v))$ , and define the trilinear form  $b$  on  $V \times V \times V$  by

$$b(u, v, w) = \sum_{i,j=1}^N \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx \quad \forall u, v, w \in V.$$

Let  $X$  be a Banach space. Given a function  $u : (-h, T) \rightarrow X$ , for each  $t \in (0, T)$  we denote by  $u_t$  the function defined on  $(-h, 0)$  by the relation  $u_t(s) = u(t+s)$ ,  $s \in (-h, 0)$ .

Finally, we will use  $\|\cdot\|_*$  for the norm in  $V'$  and  $\langle \cdot, \cdot \rangle$  for the duality  $\langle V', V \rangle$ .

In order to state the problem in the correct framework, let us firstly establish the suitable assumptions on the term in which the delay is present.

In a general way, let  $X$  and  $Y$  be two separable Banach spaces, and  $g : [0, T] \times C^0([-h, 0]; X) \rightarrow Y$  such that

(I) for all  $\xi \in C^0([-h, 0]; X)$ , the mapping  $t \in [0, T] \rightarrow g(t, \xi) \in Y$  is measurable,

(II) for each  $t \in [0, T]$ ,  $g(t, 0) = 0$ ,

(III) there exists  $L_g > 0$  such that  $\forall t \in [0, T]$ ,  $\forall \xi, \eta \in C^0([-h, 0]; X)$

$$\|g(t, \xi) - g(t, \eta)\|_Y \leq L_g \|\xi - \eta\|_{C^0([-h, 0]; X)},$$

(IV) there exists  $C_g > 0$  such that  $\forall t \in [0, T]$ ,  $\forall u, v \in C^0([-h, T]; X)$

$$\int_0^t \|g(s, u_s) - g(s, v_s)\|_Y^2 ds \leq C_g \int_{-h}^t \|u(s) - v(s)\|_X^2 ds.$$

Observe that (I)-(III) imply that given  $u \in C^0([-h, T]; X)$ , the function  $g_u : t \in [0, T] \rightarrow Y$  defined by  $g_u(t) = g(t, u_t) \forall t \in [0, T]$ , is measurable (see Bensoussan et al. 1992) and, in fact, belongs to  $L^\infty(0, T; Y)$ . Then, thanks to (IV), the mapping

$$\mathcal{G} : u \in C^0([-h, T]; X) \rightarrow g_u \in L^2(0, T; Y)$$



*Proof.* a) If  $N = 2$  and  $\nu^2 > C_2$ , let  $u, v$  be two solutions to (1) and set  $w = u - v$ . Then, from the energy equality, and the bounds for the trilinear form (see Lions 1969), it follows that for all  $t \in (0, T)$

$$\begin{aligned} |w(t)|^2 + 2\nu \int_0^t \|w(s)\|^2 ds &= -2 \int_0^t b(w(s), u(s), w(s)) ds \\ &\quad + 2 \int_0^t (g_1(s, u_s) - g_1(s, v_s), w(s)) ds \\ &\quad + 2 \int_0^t \langle g_2(s, u_s) - g_2(s, v_s), w(s) \rangle ds \\ &\leq 2k_1 \int_0^t |w(s)| (||w(s)||) ||u(s)|| ds \\ &\quad + 2 \int_0^t |g_1(s, u_s) - g_1(s, v_s)| |w(s)| ds \\ &\quad + 2 \int_0^t \|g_2(s, u_s) - g_2(s, v_s)\|_* ||w(s)|| ds. \end{aligned}$$

Then, from assumption (IV), taking into account that  $w(s) = 0$  for  $s \in (-h, 0)$ , and denoting  $2\varepsilon = \nu - \sqrt{C_2} > 0$ , we have for all  $t \in (0, T)$

$$\begin{aligned} |w(t)|^2 + 2\nu \int_0^t \|w(s)\|^2 ds &\leq \frac{k_1^2}{\varepsilon} \int_0^t |w(s)|^2 |v(s)|^2 ds + \varepsilon \int_0^t \|w(s)\|^2 ds \\ &\quad + \frac{C_1}{\varepsilon} \int_0^t |w(s)|^2 ds + \varepsilon \int_0^t \|w(s)\|^2 ds \\ &\quad + 2\sqrt{C_2} \int_0^t \|w(s)\|^2 ds, \end{aligned}$$

and so,

$$|w(t)|^2 + 2\varepsilon \int_0^t \|w(s)\|^2 ds \leq \frac{k_1^2}{\varepsilon} \int_0^t |w(s)|^2 |v(s)|^2 ds + \frac{C_1}{\varepsilon} \int_0^t |w(s)|^2 ds,$$

from which uniqueness follows thanks to the Gronwall lemma.

b) Now, we assume  $N \in \{2, 3\}$ ,  $\nu^2 > C_2$  and that condition (C) holds. For the proof of existence, we will follow a Galerkin scheme similar to the one in Constantin & Foias (1988), so we only emphasize the details involving the new terms  $g_i$ .

Let us consider  $\{w_j\} \subset V \cap (H^2(\Omega))^N$  the orthonormal basis of  $H$  of all the eigenfunctions of the Stokes problem in  $\Omega$  with homogeneous Dirichlet conditions. The subspace of  $V$  spanned by  $w_1, \dots, w_m$  will be denoted  $V_m$ . Consider the projector  $P_m : H \rightarrow V_m$  given by  $P_m u = \sum_{j=1}^m (u, w_j) w_j$ , and define  $u^m(t) = \sum_{j=1}^m \gamma_{mj}(t) w_j$ , where

$$\begin{cases} u^m \in L^2(-h, T; V_m) \cap C^0([0, T]; V_m) \\ \frac{d}{dt}(u^m(t), w_j) + \nu a(u^m(t), w_j) + b(u^m(t), u^m(t), w_j) = \langle f(t), w_j \rangle + \\ \quad + (g_1(t, u_t^m), w_j) + \langle g_2(t, u_t^m), w_j \rangle \quad \text{in } \mathcal{D}'(0, T), \quad 1 \leq j \leq m, \\ u^m(0) = P_m u_0, \quad u^m(t) = P_m \phi(t), \quad t \in (-h, 0). \end{cases} \quad (2.1)$$

The preceding is a system of ordinary functional differential equations in the unknown  $\gamma^m(t) = (\gamma_{m1}(t), \dots, \gamma_{mm}(t))$ . We can get existence and uniqueness of solution by applying Theorem 3.1 in the Appendix. Observe that, according to Theorem 3.1, we can ensure that problem (2.1) has one solution defined in an interval  $[0, t_*]$  with  $0 < t_* \leq T$ . However, as can be deduced by the a priori estimates below, we can set  $t_* = T$ .

In fact, multiplying in (2.1) by  $\gamma_{m,j}(t)$  and summing in  $j$ , we get for all  $t \in [0, t_*]$

$$\begin{aligned} |u^m(t)|^2 + 2\nu \int_0^t \|u^m(s)\|^2 ds &\leq |u_0|^2 + 2 \int_0^t \langle f(s), u^m(s) \rangle \\ &\quad + 2 \int_0^t (g_1(s, u_s^m), u^m(s)) ds \\ &\quad + 2 \int_0^t \langle g_2(s, u_s^m), u^m(s) \rangle ds, \end{aligned}$$

and arguing in a similar manner as we did in the proof of uniqueness in the 2-dimensional case, we easily get two constants (depending on  $\phi, \nu, f, g_1, g_2, h, T$ , but not on  $m$  nor  $t_*$ )  $K_1$  and  $K_2$  such that

$$\sup_{t \in [0, t_*]} |u^m(t)|^2 \leq K_1, \quad \int_0^{t_*} \|u^m(s)\|^2 ds \leq K_2.$$

So we can take  $t_* = T$ , and obtain that  $\{u^m\}$  is bounded in  $L^2(0, T; V) \cap L^\infty(0, T; H)$ . Moreover, observe that  $u^m = P_m \phi$  in  $(-h, 0)$  and, by the choice of the basis  $\{w_j\}$ , the sequence  $u^m$  converges to  $\phi$  in  $L^2(-h, 0; V)$ , and, in particular,  $g_1(\cdot, u^m) + g_2(\cdot, u^m)$  is bounded in  $L^2(0, T; V')$ . Now, it is a standard matter to bound the nonlinear term  $b(u^m, u^m, \cdot)$ , and using the same reasoning that in Constantin & Foias (1988) (see page 67), one can obtain that  $\{\frac{du^m}{dt}\}$  is bounded in  $L^{4/3}(0, T; V')$  (in fact, if  $N = 2$ ,  $\{\frac{du^m}{dt}\}$  is bounded in  $L^2(0, T; V')$ ). Using the compactness of the injection of the space  $W = \{u \in L^2(0, T; V) : \frac{du}{dt} \in L^{4/3}(0, T; V')\}$  into  $L^2(0, T; H)$ , from the preceding analysis and the assumptions on  $g_1$  and  $g_2$ , we can deduce that there exist a subsequence (denoted again  $u^m$ ) and  $u \in L^2(-h, T; V)$  such that:

$$\begin{aligned} u^m &\rightharpoonup u \text{ weakly in } L^2(-h, T; V), \\ u^m &\rightharpoonup u \text{ weakly star in } L^\infty(0, T; H), \\ u^m &\rightarrow u \text{ in } L^2(-h, T; H), \\ g_i(\cdot, u^m) &\rightarrow g_i(\cdot, u) \text{ weakly in } L^2(0, T; V'), \quad i = 1, 2. \end{aligned}$$

Now, as in the non-delay case, we can take limits in (2.1) after integrating over the interval  $(0, t)$  (for  $t \in (0, T)$ ), getting that  $u$  is a solution to our problem (1.1) (see once again, e.g., Constantin & Foias 1988 for the complete details).  $\square$

**Remark 2.2.** Observe that if  $g_1$  satisfies (I)-(IV) with  $X = H$  and  $Y = (L^2(\Omega))^N$ , then, as a direct consequence of (IV),  $g_1$  satisfies assumption (C).

### 3. Some general situations

In this section, we are going to show some situations where our theory can be applied. The cases considered include situations such as distributed delay, variable delay, and delay in gradient or second order derivatives terms.

#### (a) Case 1

Let  $G : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  be a measurable function satisfying  $G(t, 0) = 0$  for all  $t \in [0, T]$ , and assume that there exists  $L_1 > 0$  such that

$$|G(t, u) - G(t, v)|_{\mathbb{R}^N} \leq L_1 |u - v|_{\mathbb{R}^N}, \quad \forall u, v \in \mathbb{R}^N.$$

Consider a function  $\omega(t)$ , which is going to play the role of the delay function. We suppose that  $\omega \in C^1([0, T])$ ,  $\omega(t) \geq 0$  for all  $t \in [0, T]$ ,  $h = \max_{t \in [0, T]} \omega(t) > 0$  and  $\omega_* = \max_{t \in [0, T]} \omega'(t) < 1$ . Then, we define  $g_1(t, \xi)(x) = G(t, \xi(-\omega(t)))(x)$  for each  $\xi \in C^0([0, T]; H)$ ,  $x \in \Omega$  and  $t \in [0, T]$ . Notice that, in this case, the delayed term  $g_1$  in our problem turns to  $g_1(t, u_t) = G(t, u(t - \omega(t)))$ . Then,  $g_1$  satisfies the hypotheses in Theorem 2.1 with  $X = H$  and  $Y = L^2(\Omega)^N$ .

Indeed, (I)-(III) follow immediately. On the other hand, if  $u, v \in L^2(-h, T; H)$ , using the change of variable  $\tau = s - \omega(s)$  it is easy to see that

$$\int_0^t |g_1(s, u_s) - g_1(s, v_s)|^2 ds \leq \int_{-h}^t |u(\tau) - v(\tau)|^2 d\tau \quad \forall t \in [0, T],$$

and, consequently, (IV) and (C) are fulfilled.

(b) *Case 2*

Let now  $G : [0, T] \times [-h, 0] \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  be a measurable function satisfying  $G(t, s, 0) = 0$  for all  $(t, s) \in [0, T] \times [-h, 0]$  and such that there exists a function  $\gamma \in L^2(-h, 0)$  such that

$$|G(t, s, u) - G(t, s, v)|_{\mathbb{R}^N} \leq \gamma(s)|u - v|_{\mathbb{R}^N}, \quad \forall u, v \in \mathbb{R}^N \quad \forall (t, s) \in [0, T] \times [-h, 0].$$

Then, we define  $g_1(t, \xi)(x) = \int_{-h}^0 G(t, s, \xi(s)(x)) ds$  for each  $\xi \in C^0([0, T]; H)$ ,  $x \in \Omega$  and  $t \in [0, T]$ . In this case, the delayed term  $g_1$  in our problem becomes

$$g_1(t, u_t) = \int_{-h}^0 G(t, s, u(t+s)) ds.$$

As in Case 1,  $g_1$  satisfies the hypotheses in Theorem 2.1 with  $X = H$  and  $Y = (L^2(\Omega))^N$ .

Indeed, (I) and (II) can be deduced immediately. On the other hand, if  $\xi, \eta \in C^0([0, T]; H)$ , for each  $t \in [0, T]$  we obtain

$$\begin{aligned} |g_1(t, \xi) - g_1(t, \eta)|^2 &\leq \int_{\Omega} \left( \int_{-h}^0 |G(t, s, \xi(s)(x)) - G(t, s, \eta(s)(x))|_{\mathbb{R}^N} ds \right)^2 dx \\ &\leq \int_{\Omega} \left( \int_{-h}^0 \gamma(s) |\xi(s)(x) - \eta(s)(x)|_{\mathbb{R}^N} ds \right)^2 dx \\ &\leq \int_{\Omega} \|\gamma\|_{L^2(-h, 0)}^2 \left( \int_{-h}^0 |\xi(s)(x) - \eta(s)(x)|_{\mathbb{R}^N}^2 ds \right) dx \\ &\leq h \|\gamma\|_{L^2(-h, 0)}^2 \|\xi - \eta\|_{C^0([0, T]; H)}^2. \end{aligned}$$

Finally, if  $u, v \in L^2(-h, T; H)$  then, for each  $t \in [0, T]$  it follows

$$\int_0^t |g_1(\tau, u_\tau) - g_1(\tau, v_\tau)|^2 d\tau \leq h \|\gamma\|_{L^2(-h, 0)}^2 \int_0^t \left( \int_{-h}^0 |u(s+\tau) - v(s+\tau)|^2 ds \right) d\tau,$$

and, with the change  $r = s + \tau$ ,

$$\begin{aligned} \int_0^t |g_1(\tau, u_\tau) - g_1(\tau, v_\tau)|^2 d\tau &\leq h \|\gamma\|_{L^2(-h, 0)}^2 \int_0^t \left( \int_{\tau-h}^{\tau} |u(r) - v(r)|^2 dr \right) d\tau \\ &\leq hT \|\gamma\|_{L^2(-h, 0)}^2 \int_{-h}^t |u(r) - v(r)|^2 dr. \end{aligned}$$

## (c) Case 3

Now, we shall exhibit a situation where certain delay can appear in terms containing partial derivatives with respect to the spatial variables.

Let  $B(\cdot) \in L^\infty(0, T; \mathcal{L}(V; (L^2(\Omega))^N))$  and  $\omega \in C^1([0, T])$ , such that  $\omega(t) \geq 0$  for all  $t \in [0, T]$ ,  $h = \max_{t \in [0, T]} \omega(t) > 0$  and  $\omega_* = \max_{t \in [0, T]} \omega'(t) < 1$ . We now define  $g_1(t, \xi) = B(t)\xi(-\omega(t))$  for each  $\xi \in C^0([0, T]; V)$ , and  $t \in [0, T]$ . Thus, in this case the delayed term  $g_1$  in problem (1.1) turns to  $g_1(t, u_t) = B(t)u(t - \omega(t))$ . It is easy to see that  $g_1$  satisfies the hypotheses in Theorem 2.1 with  $X = V$  and  $Y = (L^2(\Omega))^N$ .

Indeed, (I)-(IV) obviously hold. On the other hand, if  $v^m$  converges to zero weakly in  $L^2(-h, T; V)$  and  $\psi \in L^2(0, T; V)$  is given, we have

$$\int_0^T \langle g_1(t, v_t^m), \psi(t) \rangle dt = \int_0^T \langle B^*(t)\psi(t), v^m(t - \omega(t)) \rangle dt,$$

with  $B^*(\cdot) \in L^\infty(0, T; \mathcal{L}(L^2(\Omega)^N; V')) \subset L^\infty(0, T; \mathcal{L}(V; V'))$  the adjoint of  $B(\cdot)$ . Using the change of variables  $\tau = t - \omega(t) = \rho(t)$ , we obtain

$$\begin{aligned} \int_0^T \langle g_1(t, v_t^m), \psi(t) \rangle dt &= \int_{\rho(0)}^{\rho(T)} \langle B^*(\rho^{-1}(\tau))\psi(\rho^{-1}(\tau)), v^m(\tau) \rangle \frac{1}{\rho'(\rho^{-1}(\tau))} d\tau \\ &= \int_{-h}^T \langle \Psi(\tau), v^m(\tau) \rangle d\tau, \end{aligned}$$

with

$$\Psi(\tau) = \begin{cases} \frac{1}{\rho'(\rho^{-1}(\tau))} B^*(\rho^{-1}(\tau))\psi(\rho^{-1}(\tau)) & \text{if } \tau \in [\rho(0), \rho(T)], \\ 0 & \text{if } \tau \in [-h, T] \setminus [\rho(0), \rho(T)]. \end{cases}$$

For this function  $\Psi$  it follows

$$\int_{-h}^T \|\Psi(\tau)\|_*^2 d\tau = \int_{\rho(0)}^{\rho(T)} \frac{1}{(\rho'(\rho^{-1}(\tau)))^2} \|B^*(\rho^{-1}(\tau))\psi(\rho^{-1}(\tau))\|_*^2 d\tau,$$

and thus, by means of the change  $\tau = \rho(t) = t - \omega(t)$ ,

$$\int_{-h}^T \|\Psi(\tau)\|_*^2 d\tau = \int_0^T \frac{1}{1 - \omega'(t)} \|B^*(t)\psi(t)\|_*^2 dt \leq \frac{b_0^2}{1 - \omega_*} \int_0^T \|\psi(t)\|^2 dt,$$

where  $b_0 = \|B^*(\cdot)\|_{L^\infty(0, T; \mathcal{L}(V; V'))}$ . Consequently,  $\Psi \in L^2(-h, T; V')$  and

$$\lim_{m \rightarrow \infty} \int_0^T \langle g_1(t, v_t^m), \psi(t) \rangle dt = \lim_{m \rightarrow \infty} \int_{-h}^T \langle \Psi(\tau), v^m(\tau) \rangle d\tau = 0.$$

Therefore, hypothesis (C) is satisfied and once again we can apply our theory to this situation.

## (d) Case 4

Let  $K \in L^\infty(-h, T; \mathcal{L}(V; V'))$  and consider in problem (1.1) a term of the form  $g_2(t, u_t) = \int_{-h}^0 K(t+s)u(t+s) ds$ , defined for all  $u \in L^2(-h, T; V)$ . This term corresponds to the situation  $g_2(t, \xi) = \int_{-h}^0 K(t+s)\xi(s) ds$  for each  $t \in [0, T]$  and  $\xi \in C^0([0, T]; V)$ . In this case, it is easy to see that  $g_2$  is well defined and satisfies (I)-(IV) with  $X = V$  and  $Y = V'$ . In particular, if we denote  $k = \|K(\cdot)\|_{L^\infty(-h, T; \mathcal{L}(V; V'))}$ , we can see that, for each  $t \in [0, T]$  and each  $u \in L^2(-h, T; V)$ , we have

$$\int_0^t \|g_2(s, u_s)\|_*^2 ds \leq k^2 h \min(h, T) \int_{-h}^t \|u(s)\|^2 ds,$$

and thus, (IV) holds by setting  $C_2 = k^2 h \min(h, T)$ .

On the other hand, let  $v^m$  be weakly converging to zero in  $L^2(-h, T; V)$ , and fix  $\psi \in L^2(0, T; V)$ . Then

$$\int_0^T \langle g_2(t, v_t^m), \psi(t) \rangle dt = \int_0^T \left\langle \int_{t-h}^t K(\tau) v^m(\tau) d\tau, \psi(t) \right\rangle dt,$$

and, by Fubini's theorem, it is easy to see that

$$\int_0^T \langle g_2(t, v_t^m), \psi(t) \rangle dt = \int_{-h}^T \langle \Sigma(\tau), v^m(\tau) \rangle d\tau,$$

with  $\Sigma(\tau) = K^*(\tau)\Psi(\tau)$  and

$$\Psi(\tau) = \begin{cases} \int_0^{\tau+h} \psi(t) dt & \text{if } -h \leq \tau < 0, \\ \int_\tau^{\tau+h} \psi(t) dt & \text{if } 0 \leq \tau < T-h, \\ \int_\tau^T \psi(t) dt & \text{if } T-h \leq \tau \leq T, \end{cases}$$

in the case  $h \leq T$ , and

$$\Psi(\tau) = \begin{cases} \int_0^{\tau+h} \psi(t) dt & \text{if } -h \leq \tau < T-h, \\ \int_0^T \psi(t) dt & \text{if } T-h \leq \tau < 0, \\ \int_\tau^T \psi(t) dt & \text{if } 0 \leq \tau \leq T, \end{cases}$$

in the case  $h > T$ . In both cases  $\Psi \in C^0([0, T]; V)$ , and in particular  $\Sigma \in L^2(0, T; V')$ . Consequently, if  $v^m$  converges weakly to zero in  $L^2(-h, T; V)$ , then  $g_2(\cdot, v^m)$  converges weakly to zero in  $L^2(-h, T; V')$  and thus,  $g_2$  satisfies hypothesis (C).

## Appendix

In this section we will prove a theorem on the existence of solutions for a finite-dimensional problem. This result has been used in the proof of Theorem 2.1, and is a variant of Theorem 3.2 Chapter 4 (p. 213) in Bensoussan et al. (1992). We present the proof for the sake of completeness. However, it is worth mentioning that a similar result is proved in Hale & Lunel (1995) in the case of continuous initial datum, although this result cannot be applied to our situation.



**Theorem 3.1.** Let  $u_0 \in \mathbb{R}^m$ ,  $\phi \in L^2(-h, 0; \mathbb{R}^m)$ ,  $k \in L^2(0, T; \mathbb{R}^m)$ ,  $g : [0, T] \times C^0([-h, 0]; \mathbb{R}^m) \rightarrow \mathbb{R}^m$  satisfying hypotheses (I)-(IV) with  $X = Y = \mathbb{R}^m$ , and  $f : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  a continuous function such that  $f(t, 0) = 0$  and for all  $n > 0$  there exists  $L_n > 0$  such that

$$|f(t, u) - f(t, v)|_{\mathbb{R}^m} \leq L_n |u - v|_{\mathbb{R}^m}, \quad \forall |u|_{\mathbb{R}^m} \leq n, |v|_{\mathbb{R}^m} \leq n, \quad \forall t \in [0, T].$$

Then:

a) For each  $t_* \in (0, T]$  there exists at most one solution to the problem

$$\begin{cases} \text{To find } u \in L^2(-h, t_*; \mathbb{R}^m) \cap C^0([0, t_*]; \mathbb{R}^m) \text{ such that} \\ u(t) = \phi(t), \quad t \in (-h, 0), \\ u(t) = u_0 + \int_0^t f(s, u(s)) ds + \int_0^t g(s, u_s) ds + \int_0^t k(s) ds \quad \forall t \in [0, t_*]. \end{cases} \quad (3.1)$$

b) There exists  $t_* \in (0, T]$  such that there exists one (and only one) solution to the problem (3.1).

c) Suppose that there exists a constant  $C > 0$  such that if  $t_* \in (0, T]$  is such that there is a solution  $u$  of (3.1), then  $\max_{t \in [0, t_*]} |u(t)|_{\mathbb{R}^m} \leq C$ . Then, under this additional assumption, there exists a solution to problem (3) with  $t_* = T$ .

*Proof.* a) If  $u$  and  $v$  are two solutions of (3.1) then, denoting  $w = u - v$ , we obtain  $w = 0$  in  $(-h, 0)$ , and for all  $t \in [0, t_*]$

$$|w(t)|_{\mathbb{R}^m} \leq L_n \int_0^t |w(s)|_{\mathbb{R}^m} ds + \left( C_g t_* \int_0^t |w(s)|_{\mathbb{R}^m}^2 ds \right)^{1/2},$$

with  $n = \max(\max_{t \in [0, t_*]} |u(t)|_{\mathbb{R}^m}, \max_{t \in [0, t_*]} |v(t)|_{\mathbb{R}^m})$ .

Consequently,

$$|w(t)|_{\mathbb{R}^m}^2 \leq (L_n + C_g^{1/2})^2 t_* \int_0^t |w(s)|_{\mathbb{R}^m}^2 ds, \quad \forall t \in [0, t_*],$$

and thus,  $w = 0$  on  $[0, t_*]$ .

b) Take any  $C > 0$  such that  $|u_0|_{\mathbb{R}^m} \leq C$ . Denote

$$M = 1 + C + C_g^{1/2} \|\phi\|_{L^2(-h, 0; \mathbb{R}^m)} + C(C_g h)^{1/2},$$

and fix  $t_* \in (0, T]$  such that

$$2t_* \leq \left( 1 + M(L_M + C_g^{1/2}) + \|k\|_{L^2(0, T; \mathbb{R}^m)} \right)^{-2},$$

and  $\frac{T}{t_*}$  being an integer.

Let

$$X = \{u \in L^2(-h, t_*; \mathbb{R}^m) \cap C^0([0, t_*]; \mathbb{R}^m); \quad u = \phi \text{ in } (-h, 0), \quad |u(t)|_{\mathbb{R}^m} \leq M \quad \forall t \in [0, t_*]\},$$

with the metric  $d(\cdot, \cdot)$  given by

$$d(u, v) = \max_{t \in [0, t_*]} |u(t) - v(t)|_{\mathbb{R}^m}.$$

It is evident that  $(X, d)$  is a complete metric space and  $X \neq \emptyset$ .

For each  $u \in X$ , denote

$$\mathcal{T}u(t) = \begin{cases} \phi(t), & \text{if } t \in (-h, 0), \\ u_0 + \int_0^t f(s, u(s)) ds + \int_0^t g(s, u_s) ds + \int_0^t k(s) ds, & \text{if } t \in [0, t_*]. \end{cases}$$

It is obvious that  $\mathcal{T}u \in L^2(-h, t_*; \mathbb{R}^m) \cap C^0([0, t_*]; \mathbb{R}^m)$  and, by definition,  $\mathcal{T}u(t) = \phi(t)$  if  $t \in (-h, 0)$ . Moreover, if  $t \in [0, t_*]$ ,

$$|\mathcal{T}u(t)|_{\mathbb{R}^m} \leq |u_0|_{\mathbb{R}^m} + L_M M t_* + (C_g t_*)^{1/2} \left( \int_{-h}^t |u(s)|_{\mathbb{R}^m}^2 ds \right)^{1/2} + (t_*)^{1/2} \|k\|_{L^2(0, T; \mathbb{R}^m)},$$

and consequently, thanks to the choice of  $t_*$ ,  $|\mathcal{T}u(t)|_{\mathbb{R}^m} \leq M$  for all  $t \in [0, t_*]$ . Thus,  $\mathcal{T}$  is a mapping from  $X$  into  $X$ . Finally, it is not difficult to see that  $\sqrt{2}d(\mathcal{T}u, \mathcal{T}v) \leq d(u, v)$  for all  $u, v \in X$ . Consequently,  $\mathcal{T}$  has a fixed point in  $X$ .

c) If there exists a constant  $C > 0$  such that if  $t_* \in (0, T]$  is such that there is a solution  $u$  of (3.1), then  $\max_{t \in [0, t_*]} |u(t)|_{\mathbb{R}^m} \leq C$ , with this  $C$  fixed, take  $M$  and  $t_*$  as in b), that is,

$$M = 1 + C + C_g^{1/2} \|\phi\|_{L^2(-h, 0; \mathbb{R}^m)} + C(C_g h)^{1/2},$$

and  $t_* \in (0, T]$  such that

$$2t_* \leq \left( 1 + M(L_M + C_g^{1/2}) + \|k\|_{L^2(0, T; \mathbb{R}^m)} \right)^{-2},$$

with  $\frac{T}{t_*}$  being an integer.

We know that then, there exists one and only one function  $u \in L^2(-h, t_*; \mathbb{R}^m) \cap C^0([0, t_*]; \mathbb{R}^m)$  such that

$$\begin{cases} u(t) = \phi(t), & t \in (-h, 0), \\ u(t) = u_0 + \int_0^t f(s, u(s)) ds + \int_0^t g(s, u_s) ds + \int_0^t k(s) ds, & \forall t \in [0, t_*]. \end{cases}$$

If  $t_* < T$ , then denote  $\tilde{\phi}(s) = u(t_* + s)$  for each  $s \in (-h, 0)$ , and consider the problem

$$\begin{cases} v \in L^2(-h, t_*; \mathbb{R}^m) \cap C^0([0, t_*]; \mathbb{R}^m), \\ v(t) = \tilde{\phi}(t), & t \in (-h, 0), \\ v(t) = u(t_*) + \int_0^t f(s + t_*, v(s)) ds + \int_0^t g(s + t_*, v_s) ds \\ \quad + \int_0^t k(s + t_*) ds, & \forall t \in [0, t_*]. \end{cases} \quad (3.2)$$

The problem (3.2) has the same structure than (3.1), with  $|u(t_*)|_{\mathbb{R}^m} \leq C$  and

$$\int_{-h}^0 |\tilde{\phi}(s)|_{\mathbb{R}^m}^2 ds = \int_{t_*-h}^{t_*} |u(\tau)|_{\mathbb{R}^m}^2 d\tau \leq C^2 h + \|\phi\|_{L^2(-h, 0; \mathbb{R}^m)}^2.$$

Moreover, if we denote

$$\tilde{f}(t, w) = f(t + t_*, w) \quad \forall t \in [0, t_*] \quad \forall w \in \mathbb{R}^m,$$

and

$$\tilde{g}(t, \xi) = f(t + t_*, \xi) \quad \forall t \in [0, t_*] \quad \forall \xi \in C^0([-h, 0]; \mathbb{R}^m),$$

then it is easy to see that  $\tilde{f}$  and  $\tilde{g}$  satisfy the hypotheses in the theorem, with  $T$  substituted by  $t_*$ , and the same constants  $L_n$ ,  $C_g$  and  $L_g$ .

For example, if  $u, v \in C^0([-h, t_*]; \mathbb{R}^m)$  and  $t \in [0, t_*]$ , then

$$\begin{aligned} \int_0^t |\tilde{g}(s, u_s) - \tilde{g}(s, v_s)|_{\mathbb{R}^m}^2 ds &= \int_0^t |g(s + t_*, u_s) - g(s + t_*, v_s)|_{\mathbb{R}^m}^2 ds \\ &= \int_{t_*}^{t_*+t} |g(\tau, u_{\tau-t_*}) - g(\tau, v_{\tau-t_*})|_{\mathbb{R}^m}^2 d\tau. \end{aligned}$$

If we denote

$$\hat{u}(s) = \begin{cases} u(s - t_*) & \text{if } s \in [t_* - h, 2t_*], \\ 0 & \text{if } s \in [-h, t_* - h), \end{cases} \quad \hat{v}(s) = \begin{cases} v(s - t_*) & \text{if } s \in [t_* - h, 2t_*], \\ 0 & \text{if } s \in [-h, t_* - h), \end{cases}$$

then  $\hat{u}$  and  $\hat{v}$  are in  $L^2(-h, 2t_*; \mathbb{R}^m)$ , and  $\forall \tau \in [t_*, t_* + t]$ ,  $\forall \theta \in [-h, 0]$ ,

$$u_{\tau-t_*}(\theta) = \hat{u}_\tau(\theta), \quad v_{\tau-t_*}(\theta) = \hat{v}_\tau(\theta),$$

thus we obtain

$$\begin{aligned} \int_0^t |\tilde{g}(s, u_s) - \tilde{g}(s, v_s)|_{\mathbb{R}^m}^2 ds &= \int_{t_*}^{t_*+t} |g(\tau, \hat{u}_\tau) - g(\tau, \hat{v}_\tau)|_{\mathbb{R}^m}^2 d\tau \\ &\leq \int_0^{t_*+t} |g(\tau, \hat{u}_\tau) - g(\tau, \hat{v}_\tau)|_{\mathbb{R}^m}^2 d\tau \\ &\leq C_g \int_{-h}^{t_*+t} |\hat{u}(\tau) - \hat{v}(\tau)|_{\mathbb{R}^m}^2 d\tau \\ &= C_g \int_{t_*-h}^{t_*+t} |\hat{u}(\tau) - \hat{v}(\tau)|_{\mathbb{R}^m}^2 d\tau \\ &= C_g \int_{t_*-h}^{t_*+t} |u(\tau - t_*) - v(\tau - t_*)|_{\mathbb{R}^m}^2 d\tau, \end{aligned}$$

and, consequently,

$$\int_0^t |\tilde{g}(s, u_s) - \tilde{g}(s, v_s)|_{\mathbb{R}^m}^2 ds \leq C_g \int_{-h}^t |u(s) - v(s)|_{\mathbb{R}^m}^2 ds.$$

By the considerations above, we can ensure that there exists one and only one solution to (3.2). Now, it is easy to see that the function

$$w(t) = \begin{cases} u(t) & \text{if } t \in (-h, t_*], \\ v(t) & \text{if } t \in (t_*, 2t_*], \end{cases}$$

with  $u$  the solution to (3.1) in  $(-h, t_*]$ , and  $v$  the solution to (3.2), is solution of (3.1) in the interval  $(-h, 2t_*]$ . If  $2t_* = T$ , we have finished, but if  $2t_* < T$ , we then can use a similar construction to the preceding one, and obtain a solution to (3.1) in the interval  $(-h, 3t_*]$ . After a finite number of steps, we can construct the solution of (3.1) in the interval  $(-h, T]$ .  $\square$

Partly supported by DGICYT (Spain) Proyecto PB98-1134.

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