

Attractors for 2D-Navier-Stokes models with delays

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Abstract

The existence of an attractor for a 2D-Navier-Stokes system with delay is proved. The theory of pullback attractors is successfully applied to obtain the results since the abstract functional framework considered turns out to be nonautonomous. However, on some occasions, the attractors may attract not only in the pullback sense but in the forward one as well. Also, this formulation allows to treat, in a unified way, terms containing various classes of delay features (constant, variable, distributed delays, etc.). As a consequence, some results for the autonomous model are deduced as particular cases of our general formulation.

Key words: 2D-Navier-Stokes equations, pullback attractor, forward attractor, variable delay, distributed delay

1991 MSC: 35R10, 35B40, 47H20, 58F39, 73K70

1 Introduction

Navier-Stokes equations have received very much attention over the last decades due to their importance in the understanding of fluids motion and turbulence (see [1], [10], [12], [15], [18], [26], amongst others). Very recently, in [7],[8] we started an investigation involving Navier-Stokes models in which the forcing term contains some hereditary features. These situations may appear, for instance, when we want to control the system by applying a force which takes

* Partly supported by Ministerio de Ciencia y Tecnología (Spain) and FEDER (European Community), projects HA2001-0075 and BFM2002-03068.

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into account not only the present state of the system but the history of the solutions.

No doubt at all, the asymptotic behaviour of dynamical systems is an interesting and challenging problem, since it can provide useful information on the future evolution of the system. This will be the main aim of this paper. To this respect, some sufficient conditions ensuring the exponential behaviour of solutions to a 2D-Navier-Stokes delay model were proved in [8]. Roughly speaking, when the viscosity is large, there exists a unique stationary solution to some models and this solution is exponentially stable (which means that the global attractor for these situations becomes the unique stationary solution). However, when the viscosity is small it is expected something similar to what happens in the non-delay framework, i.e., the existence of a compact invariant attracting set (a global attractor for the associated semigroup). But on this occasion, we need to be careful with our analysis since we have to consider the semigroup in a different phase space. In fact, the dynamical system needs to be defined in a phase space of trajectories (for a similar approach for nondelay models see [20]). To be more precise, our intention is to consider an abstract functional model for the delay so that a wide range of hereditary characteristics (constant or variable delay, distributed delay, etc) can be treated in a unified way. Although for some particular cases, the resulting abstract equation becomes autonomous (e.g. for constant delays) and the standard technique for autonomous dynamical systems can be adapted to solve the problem, most cases need of a nonautonomous model to describe the system and, consequently, a nonautonomous technique is necessary to handle the problem. Being possible various options to deal with the problem of attractors for nonautonomous systems (kernel sections [10], skew-product formalism [24], etc.), for our particular situation we have preferred to choose that of pullback attractor (see [9], [16], [17], [23]) which has also proved extremely fruitful, particularly in the case of random dynamical systems (see [13], [14], [23]). The main reason is that, although when one knows the explicit dependence of the delay (e.g. as in the cases of variable or distributed delays) it could be possible to construct the parameters set which is needed to have a skew-product flow (or the symbols set in the theory of kernel sections), it is not known how to construct them when one is trying to develop a general theory concerning abstract delay terms, i.e. under a general functional formulation (see [6] for more details). It is also worth pointing out that, after proving our theory for the nonautonomous delay model, we will obtain similar results for an autonomous version in a straightforward way.

As far as we know, not many papers have been published dealing with the existence of attractors for partial differential equations with delay. We would like to mention that, for instance, a linear partial differential equation containing a nonlinear autonomous term with finite delay is considered in [11], and a class of retarded partial differential equations of second order with respect to

the time variable is analyzed in [3]. However, we do not know any work concerning nonautonomous delay terms. Some results in the finite dimensional context can be found in [6], [5] (see also Mallet-Paret and Sell [21], [22] for some preliminary and interesting results on the structure of the attractors for ordinary differential delay systems).

In Section 2, we will recall some preliminary results on the existence, uniqueness and regularity of solutions of our model as well as some results on the theory of pullback attractors. Section 3 is devoted to prove the existence of the attractor of our nonautonomous delay models. In fact, under suitable uniform assumptions we prove the existence of a pullback attractor. In addition, some applications are exhibited (variable and distributed delays), and we also point out how can be obtained corresponding results for the autonomous framework as a particular case of our general model.

2 Preliminaries

In this section we will include some preliminaries on the existence and uniqueness of solutions to our problem and recall some facts from the theory of pullback attractors.

2.1 Existence and uniqueness of solutions

The general formulation for our model is the following. Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with regular boundary Γ , and consider the following functional $2D$ -Navier-Stokes problem (for further details and notations see Lions [19] and Temam [25]):

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \nu \Delta u + \sum_{i=1}^2 u_i \frac{\partial u}{\partial x_i} = f - \nabla p + g(t, u_t) \quad \text{in } (\tau, +\infty) \times \Omega, \\ \operatorname{div} u = 0 \quad \text{in } (\tau, +\infty) \times \Omega, \\ u = 0 \quad \text{on } (\tau, +\infty) \times \Gamma, \\ u(\tau, x) = u_0(x), \quad x \in \Omega, \\ u(t, x) = \phi(t - \tau, x), \quad t \in (\tau - h, \tau) \quad x \in \Omega, \end{array} \right.$$

where $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p the pressure, $\tau \in \mathbb{R}$ the initial time, u_0 the initial velocity field, f a nondelayed external force field, g another external force with some hereditary characteris-

tics and ϕ the initial datum in the interval of time $(-h, 0)$, where h is a fixed positive number.

To set our problem in the abstract framework, we consider the following usual abstract spaces:

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^2 : \operatorname{div} u = 0 \right\},$$

H = the closure of \mathcal{V} in $(L^2(\Omega))^2$ with norm $|\cdot|$, and inner product (\cdot, \cdot) where for $u, v \in (L^2(\Omega))^2$,

$$(u, v) = \sum_{j=1}^2 \int_{\Omega} u_j(x) v_j(x) dx,$$

V = the closure of \mathcal{V} in $(H_0^1(\Omega))^2$ with norm $\|\cdot\|$, and associated scalar product $((\cdot, \cdot))$, where for $u, v \in (H_0^1(\Omega))^2$,

$$((u, v)) = \sum_{i,j=1}^2 \int_{\Omega} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx.$$

It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and compact.

Finally, we will use $\|\cdot\|_*$ for the norm in V' and $\langle \cdot, \cdot \rangle$ for the duality pairing between V and V' .

Now we define the trilinear form b on $V \times V \times V$ by

$$b(u, v, w) = \sum_{i,j=1}^2 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx \quad \forall u, v, w \in V.$$

Given $T > \tau$ and $u : (\tau - h, T) \rightarrow (L^2(\Omega))^2$, for each $t \in (\tau, T)$ we denote by u_t the function defined on $(-h, 0)$ by the relation $u_t(s) = u(t+s)$, $s \in (-h, 0)$. We also denote $C_H = C^0([-h, 0]; H)$, $C_V = C^0([-h, 0]; V)$, $L_H^2 = L^2(-h, 0; H)$ and $L_V^2 = L^2(-h, 0; V)$.

Now, we establish suitable hypotheses on the term containing the delay. Let $g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2$ satisfy the following assumptions:

- (I) $\forall \xi \in C_H, t \in \mathbb{R} \rightarrow g(t, \xi) \in (L^2(\Omega))^2$ is measurable,
- (II) $\forall t \in \mathbb{R}, g(t, 0) = 0$,
- (III) $\exists L_g > 0$ s.t. $\forall t \in \mathbb{R}, \forall \xi, \eta \in C_H$

$$|g(t, \xi) - g(t, \eta)| \leq L_g \|\xi - \eta\|_{C_H},$$

(IV) $\exists m_0 \geq 0, C_g > 0: \forall m \in [0, m_0], \tau \leq t, u, v \in C^0([\tau - h, t]; H)$

$$\int_{\tau}^t e^{ms} |g(s, u_s) - g(s, v_s)|^2 ds \leq C_g^2 \int_{\tau-h}^t e^{ms} |u(s) - v(s)|^2 ds.$$

Observe that (I)-(III) imply that given $u \in C^0([\tau - h, T]; H)$, the function $g_u : t \in [\tau, T] \rightarrow (L^2(\Omega))^2$ defined by $g_u(t) = g(t, u_t) \forall t \in [\tau, T]$, is measurable (see Bensoussan et al. [2]) and, in fact, belongs to $L^\infty(\tau, T; (L^2(\Omega))^2)$. Then, thanks to (IV), the mapping

$$\mathcal{G} : u \in C^0([\tau - h, T]; H) \rightarrow g_u \in L^2(\tau, T; (L^2(\Omega))^2)$$

has a unique extension to a mapping $\tilde{\mathcal{G}}$ which is uniformly continuous from $L^2(\tau - h, T; H)$ into $L^2(\tau, T; (L^2(\Omega))^2)$. From now on, we will denote $g(t, u_t) = \tilde{\mathcal{G}}(u)(t)$ for each $u \in L^2(\tau - h, T; H)$, and thus, $\forall t \in [\tau, T], \forall u, v \in L^2(\tau - h, T; H)$, we will have

$$\int_{\tau}^t |g(s, u_s) - g(s, v_s)|_{(L^2(\Omega))^2}^2 ds \leq C_g^2 \int_{\tau-h}^t |u(s) - v(s)|^2 ds.$$

Assume now that $u_0 \in H, \phi \in L^2_H, f \in L^2_{loc}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2$ satisfies hypotheses (I)-(IV). For example, when the function g is defined by $g(t, \phi) = G(\phi(-\rho(t)))$ for a suitable differentiable delay function ρ and a Lipschitz continuous mapping $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, the assumptions above hold (see Caraballo & Real [7] for more details and examples). Set $A : V \rightarrow V'$ as $\langle Au, v \rangle = ((u, v))$, $B : V \times V \rightarrow V'$ by $\langle B(u, v), w \rangle = b(u, v, w)$, $\forall u, v, w \in V$, and $B(u) = B(u, u)$. Denoting $D(A) = (H^2(\Omega))^2 \cap V$, then $Au = -P\Delta u, \forall u \in D(A)$, (P the ortho-projector from $(L^2(\Omega))^2$ onto H). For each $\tau \in \mathbb{R}$ we consider the problem:

$$\left\{ \begin{array}{l} \text{To find } u \in L^2(\tau - h, T; H) \cap L^2(\tau, T; V) \cap L^\infty(\tau, T; H) \quad \forall T > \tau, \\ \frac{d}{dt}u(t) + \nu Au(t) + B(u(t)) = f(t) + g(t, u_t) \text{ in } \mathcal{D}'(\tau, +\infty; V'), \\ u(\tau) = u_0, \quad u(t) = \phi(t - \tau), \quad t \in (\tau - h, \tau), \end{array} \right. \quad (1)$$

The following result can be proved as Theorem 2.3 in Caraballo & Real [8].

Theorem 1 *Let us consider $u_0 \in H, \phi \in L^2_H, f \in L^2_{loc}(\mathbb{R}; V')$, and assume that $g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2$ satisfies hypotheses (I)-(IV). Then, for each $\tau \in \mathbb{R}$,*

- a) *There exists a unique solution to (1) which, in addition, belongs to the space $C^0([\tau, +\infty); H)$.*
- b) *If $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$ and $u_0 \in V$, then the solution u to (1) is a strong*

solution, that is,

$$u \in L^2(\tau, T; D(A)) \cap C^0([\tau, T]; V) \text{ and } u' \in L^2(\tau, T; H) \quad \forall T > \tau. \quad (2)$$

In particular, if $\phi \in C_V$ and $u_0 = \phi(0)$, then $u \in C^0([\tau - h, +\infty); V)$.

2.2 Preliminaries on pullback attractors

We now discuss the theory of pullback attractors, as developed in Kloeden and Stonier [16], Kloeden and Schmalzfuss [17], and Crauel *et al.* [14]. As it is well known, in the case of nonautonomous differential equations the initial time is just as important as the final time, and the classical semigroup property of autonomous dynamical systems is no longer available.

Instead of a family of one time-dependent maps $S(t)$ we need to use a two-parameter process $U(t, \tau)$ on the complete metric space X (which in our case will be C_H or $H \times L^2_H$) (cf. Sell [24]); $U(t, \tau)\psi$ uses to denote the value of the solution at time t which was equal to the initial value ψ at time τ .

The semigroup property is replaced by the process composition property

$$U(t, \tau)U(\tau, r) = U(t, r) \quad \text{for all } t \geq \tau \geq r,$$

and, obviously, the initial condition implies $U(\tau, \tau) = \text{Id}$. As with the semigroup composition $S(t)S(\tau) = S(t + \tau)$, this just expresses the uniqueness of solutions.

It is also possible to present the theory within the more general framework of cocycle dynamical systems. In this case the second component of U is viewed as an element of some parameter space J , so that the solution can be written as $U(t, p)\phi$, and a shift map $\theta_t : J \rightarrow J$ is defined so that the process composition becomes the cocycle property,

$$U(t + \tau, p) = U(t, \theta_\tau p)U(\tau, p).$$

However, when one tries to develop a theory which can include several kinds of hereditary characteristics under a unified abstract formulation, what means that we do not know a priori the explicit expression of the delay appearing in the problem, the context of cocycle (or skew-product flows) may not be the most appropriate to deal with the problem, since it is not known how to construct the set J (the same happens with the construction of the symbols set if one wishes to apply the theory of kernel sections as developed by Chepyzhov and Vishik [10]). For this reason, we do not pursue this approach here, but note that it has proved extremely fruitful, particularly in the case of random

dynamical systems. For various examples using this general setting, see Kloeden and Schmalfuss [17], or Sell [24]. For this reason, pullback attractors are often referred to as ‘cocycle attractors’.

As in the standard theory of attractors, we seek an invariant attracting set. However, since the equation is nonautonomous this set also depends on time.

Definition 2 *Let U be a process on a complete metric space X . A family of compact sets $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ is said to be a (global) pullback attractor for U if, for all $\tau \in \mathbb{R}$, it satisfies*

- i) $U(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $t \geq \tau$, and*
- ii) $\lim_{s \rightarrow \infty} \text{dist}(U(t, t-s)D, \mathcal{A}(t)) = 0$, for all bounded subsets D of X .*
The pullback attractor is said to be uniform if the attraction property is uniform in time, i.e.

$$\lim_{s \rightarrow \infty} \sup_{t \in \mathbb{R}} \text{dist}(U(t, t-s)D, \mathcal{A}(t)) = 0, \quad \text{for all bounded subsets } D \subset X.$$

Definition 3 *A family of compact sets $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ is said to be a (global) forward attractor for U if, for all $\tau \in \mathbb{R}$, it satisfies*

- i) $U(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)$ for all $t \geq \tau$, and*
- ii) $\lim_{t \rightarrow \infty} \text{dist}(U(t, \tau)D, \mathcal{A}(t)) = 0$, for all bounded subsets D of X .*
The forward attractor is said to be uniform if the attraction property is uniform in time, i.e.

$$\lim_{t \rightarrow \infty} \sup_{\tau \in \mathbb{R}} \text{dist}(U(t+\tau, \tau)D, \mathcal{A}(t+\tau)) = 0, \quad \text{for all bounded subsets } D \subset X.$$

The reader is referred to Cheban *et al.* [9] for a detailed analysis on the relationship between these concepts. We emphasize that the property of uniform pullback attraction is equivalent to that of uniform forward attraction.

In the definition, $\text{dist}(A, B)$ is the Hausdorff semidistance between A and B , defined as

$$\text{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b), \quad \text{for } A, B \subseteq X.$$

Property i) is a generalization of the invariance property for autonomous dynamical systems. The pullback attracting property ii) considers the state of the system at time t when the initial time $t-s$ goes to $-\infty$ (see also Chepyzhov and Vishik [10])

The notion of an attractor is closely related to that of an absorbing set.

Definition 4 *The family $\{B(t)\}_{t \in \mathbb{R}}$ is said to be (pullback) absorbing with respect to the process U if, for all $t \in \mathbb{R}$ and all $D \subset X$ bounded, there exists*

$T_D(t) > 0$ such that for all $s \geq T_D(t)$

$$U(t, t - s)D \subset B(t).$$

The absorption is said to be uniform if $T_D(t)$ does not depend on the time variable t .

Indeed, just as in the autonomous case, the existence of compact absorbing sets is the crucial property in order to obtain pullback attractors. For the following result see Crauel and Flandoli [13] or Schmalfuss [23].

Theorem 5 *Let $U(t, \tau)$ be a two-parameter process, and suppose $U(t, \tau) : X \rightarrow X$ is continuous for all $t \geq \tau$. If there exists a family of compact (pull-back) absorbing sets $\{B(t)\}_{t \in \mathbb{R}}$, then there exists a pullback attractor $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$, and $\mathcal{A}(t) \subset B(t)$ for all $t \in \mathbb{R}$. Furthermore,*

$$\mathcal{A}(t) = \overline{\bigcup_{\substack{D \subset X \\ \text{bounded}}} \Lambda_D(t)},$$

where

$$\Lambda_D(t) = \bigcap_{n \in \mathbb{N}} \overline{\bigcup_{s \geq n} U(t, t - s)D}.$$

Remark 6 *It is worth mentioning that the uniqueness of the pullback attractor, as defined above, does not hold in general (see Caraballo and Langa [4]). However, the one given in the preceding theorem is minimal with respect to set inclusion (see Crauel and Flandoli [13]). But, if we impose in the definition of pullback attractor that the family $\{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ is uniformly bounded (i.e. there exists a bounded set $B \subset X$ such that $\mathcal{A}(t) \subset B$ for all $t \in \mathbb{R}$) or we are interested in finding uniformly bounded attractors, then the uniqueness of this attractor follows immediately. A sufficient condition ensuring this is that the family of compact absorbing sets in Theorem 5 is also uniformly bounded. Finally, there exists another possibility to ensure the uniqueness of the pullback attractor which is related to the fact that the attractor is asked to belong to a certain class of set valued functions which are attracted by the attractor (see [9]).*

3 Existence of the attractor

We denote by λ_1 the first eigenvalue of the operator A .

3.1 Construction of the associated process

Now we will apply the theory in the previous section to prove the existence of an attractor for our nonautonomous Navier-Stokes model with delay. To this end, we consider $g : \mathbb{R} \times C_H \rightarrow (L^2(\Omega))^2$ satisfying (I)-(IV) and assume that $u_0 \in H$, $\phi \in L^2_H$ and $f \in L^2_{loc}(\mathbb{R}; V')$. Then, for each initial time $\tau \in \mathbb{R}$, Theorem 1 ensures that problem (1) possesses a unique solution $u(\cdot; \tau, (u_0, \phi))$ which belongs to the space $L^2(\tau, T; V) \cap L^2(\tau - h, \tau; H) \cap C^0([\tau, T]; H)$ for all $T > \tau$. We can now proceed in two different forms to construct the evolution process which can help us in the analysis of the long-time behaviour of our model. On the one hand, we can define a process in the phase space C_H as the family of mappings $U(t, \tau) : C_H \rightarrow C_H$ given by

$$U(t, \tau)\phi = u_t(\cdot; \tau, (\phi(0), \phi)), \quad \text{for any } \phi \in C_H, \text{ and any } \tau \leq t. \quad (3)$$

However, it may seem that the product space $M_H^2 = H \times L^2_H$ can be more convenient since this is the usual space where the initial data are taken. This space is a Hilbert space with associated norm

$$\|(u_0, \phi)\|_{M_H^2}^2 = |u_0|^2 + \int_{-h}^0 |\phi(s)|^2 ds, \quad \text{for } (u_0, \phi) \in M_H^2.$$

In this way, we can define the corresponding process as

$$S(t, \tau)(u_0, \phi) = (u(t; \tau, (u_0, \phi)), u_t(\cdot; \tau, (u_0, \phi))), \quad \text{for } (u_0, \phi) \in M_H^2, \tau \leq t. \quad (4)$$

Although, due to the continuity of trajectories, it seems sensible to consider only the first case, with a little more of additional work we will be able to handle both situations at the same time. Of course, it is sensible to expect that the attractors for both situations should be related. We will prove that this is indeed the case.

Remark 7 *Associated to the processes $U(\cdot, \cdot)$ and $S(\cdot, \cdot)$ we will consider the family of mappings $\tilde{U}(\cdot, \cdot) : M_H^2 \rightarrow L^2_H$ defined as*

$$\tilde{U}(t, \tau)(u_0, \phi) = u_t(\cdot; \tau, (u_0, \phi)), \quad \text{for } (u_0, \phi) \in M_H^2, \text{ and } \tau \leq t. \quad (5)$$

Observe that

$$U(t, \tau)\phi = \tilde{U}(t, \tau)(\phi(0), \phi) \quad \text{for any } t \geq \tau, \text{ and any } \phi \in C_H. \quad (6)$$

In this way, we then have that the process $S(t, \tau)$ can be rewritten as

$$S(t, \tau)(u_0, \phi) = (u(t; \tau, (u_0, \phi)), \tilde{U}(t, \tau)(u_0, \phi)). \quad (7)$$

These facts will allow us to prove the estimates for the processes U and S in a straightforward way by using the previously obtained ones for the process \tilde{U} .

To be more precise, let us consider the linear mapping

$$j : \phi \in C_H \mapsto j(\phi) = (\phi(0), \phi) \in H \times C_H.$$

This map is obviously continuous from C_H into $H \times C_H$ and into M_H^2 . Noticing that for all $(u_0, \phi) \in M_H^2$ it holds that $\tilde{U}(t, \tau)(u_0, \phi) \in C_H$ provided that $t \geq \tau + h$, we then can write

$$S(t, \tau)(u_0, \phi) = j(\tilde{U}(t, \tau)(u_0, \phi)), \quad \text{for } (u_0, \phi) \in M_H^2, t \geq \tau + h.$$

Before proving that $S(\cdot, \cdot)$ and $U(\cdot, \cdot)$ are continuous processes, we need the following result.

Lemma 8 *Let $(u_0, \phi), (v_0, \psi) \in M_H^2$ be two couples of initial data for our problem (1), and let $\tau \in \mathbb{R}$ be an initial time. Denote by $u(\cdot) = u(\cdot; \tau, (u_0, \phi))$ and $v(\cdot) = u(\cdot; \tau, (v_0, \psi))$ the corresponding solutions to (1). Then, there exists a constant $c_0' > 0$ which does not depend on the initial data and time, such that*

$$\begin{aligned} |u(t) - v(t)|^2 &\leq \left(C_g^2 \|\phi - \psi\|_{L_H^2}^2 + |u_0 - v_0|^2 \right) \times \\ &\quad \times \exp \left(\int_{\tau}^t (C_g^2 + c_0'' \|u(s)\|^2 + 1) ds \right), \quad \forall t \geq \tau. \end{aligned} \quad (8)$$

Therefore, it also holds

$$\begin{aligned} \|u_t - v_t\|_{C_H}^2 &\leq \left(C_g^2 \|\phi - \psi\|_{L_H^2}^2 + |u_0 - v_0|^2 \right) \times \\ &\quad \times \exp \left(\int_{\tau}^t (C_g^2 + c_0'' \|u(s)\|^2 + 1) ds \right), \quad \forall t \geq \tau + h. \end{aligned} \quad (9)$$

PROOF. It follows from (1) that

$$\frac{d}{dt}(u - v) + \nu A(u - v) + B(u) - B(v) = g(t, u_t) - g(t, v_t).$$

If we set $w = u - v$, we deduce

$$\frac{1}{2} \frac{d}{dt} |w|^2 + \nu ((w, w)) + (B(u) - B(v), w) = (g(t, u_t) - g(t, v_t), w)$$

and

$$\frac{1}{2} \frac{d}{dt} |w|^2 + \nu \|w\|^2 - b(w, u, w) = (g(t, u_t) - g(t, v_t), w),$$

and therefore

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |w|^2 + \nu \|w\|^2 &\leq c_0 |w| \|u\| \|w\| + |g(t, u_t) - g(t, v_t)| |w| \\ &\leq c_0' |w|^2 \|u\|^2 + \nu \|w\|^2 + \frac{1}{2} |g(t, u_t) - g(t, v_t)|^2 + \frac{1}{2} |w|^2, \end{aligned}$$

whence

$$\frac{d}{dt} |w|^2 \leq (c_0'' \|u\|^2 + 1) |w|^2 + |g(t, u_t) - g(t, v_t)|^2 \quad (10)$$

and

$$\begin{aligned} |w(t)|^2 - |w(\tau)|^2 &\leq \int_{\tau}^t |g(s, u_s) - g(s, v_s)|^2 ds + \int_{\tau}^t (c_0'' \|u(s)\|^2 + 1) |w(s)|^2 ds \\ &\leq C_g^2 \int_{\tau-h}^t |u(s) - v(s)|^2 ds + \int_{\tau}^t (c_0'' \|u(s)\|^2 + 1) |w(s)|^2 ds \\ &\leq C_g^2 \|\phi - \psi\|_{L_H^2}^2 + \int_{\tau}^t (C_g^2 + c_0'' \|u(s)\|^2 + 1) |w(s)|^2 ds. \end{aligned}$$

Consequently,

$$|w(t)|^2 \leq C_g^2 \|\phi - \psi\|_{L_H^2}^2 + |u_0 - v_0|^2 + \int_{\tau}^t (C_g^2 + c_0'' \|u(s)\|^2 + 1) |w(s)|^2 ds, \forall t \geq \tau.$$

The Gronwall lemma implies now for any $t \geq \tau$,

$$|w(t)|^2 \leq (C_g^2 \|\phi - \psi\|_{L_H^2}^2 + |u_0 - v_0|^2) \exp \left(\int_0^t (C_g^2 + c_0'' \|u(s)\|^2 + 1) ds \right).$$

We have therefore proved (8).

Assume now that $t \geq \tau + h$. Then $t + \theta \geq \tau$ for any $\theta \in [-h, 0]$ and it holds

$$\begin{aligned} |w(t + \theta)|^2 &\leq (C_g^2 \|\phi - \psi\|_{L_H^2}^2 + |u_0 - v_0|^2) \exp \left(\int_{\tau}^{t+\theta} (C_g^2 + c_0'' \|u(s)\|^2 + 1) ds \right) \\ &\leq (C_g^2 \|\phi - \psi\|_{L_H^2}^2 + |u_0 - v_0|^2) \exp \left(\int_{\tau}^t (C_g^2 + c_0'' \|u(s)\|^2 + 1) ds \right), \end{aligned}$$

and thus

$$\|w_t\|_{C_H}^2 \leq (C_g^2 \|\phi - \psi\|_{L_H^2}^2 + |u_0 - v_0|^2) \exp \left(\int_{\tau}^t (C_g^2 + c_0'' \|u(s)\|^2 + 1) ds \right).$$

The proof is now complete.

Theorem 9 *Under the previous assumptions, the mappings $U(\cdot, \cdot)$ defined in (3) and $S(\cdot, \cdot)$ defined in (4) are processes. Moreover, $U(t, \tau) : C_H \rightarrow C_H$ and $S(t, \tau) : M_H^2 \rightarrow M_H^2$ are continuous for any $\tau \leq t$.*

PROOF. The uniqueness of solutions obviously implies that $U(\cdot, \cdot)$ and $S(\cdot, \cdot)$ are processes. The continuity of both families of mappings follows from (8) and (9). Indeed, assume that $\phi, \psi \in C_H$, and consider the solutions $u(\cdot), v(\cdot)$ to (1) corresponding to the initial data $(\phi(0), \phi), (\psi(0), \psi)$, we deduce from (8) that

$$|u(t) - v(t)|^2 \leq (C_g^2 h + 1) \|\phi - \psi\|_{C_H}^2 \exp \left(\int_{\tau}^t (C_g^2 + c_0'' \|u(s)\|^2 + 1) ds \right), \quad \forall t \geq \tau.$$

As, on the other hand,

$$u(t) - v(t) = \phi(t - \tau) - \psi(t - \tau), \quad \text{for } \tau - h \leq t \leq \tau,$$

it then holds that

$$|u(t) - v(t)|^2 \leq (C_g^2 h + 1) \|\phi - \psi\|_{C_H}^2 \exp\left(\int_{\tau-h}^t (C_g^2 + c_0'' \|u(s)\|^2 + 1) ds\right), \quad \forall t \geq \tau - h,$$

whence

$$\|u_t - v_t\|_{C_H}^2 \leq (C_g^2 h + 1) \|\phi - \psi\|_{C_H}^2 \exp\left(\int_{\tau-h}^t (C_g^2 + c_0'' \|u(s)\|^2 + 1) ds\right), \quad \forall t \geq \tau,$$

what implies the continuity of $U(t, \tau)$.

As for the continuity of $S(t, \tau)$, we consider the initial data $(u_0, \phi), (v_0, \psi) \in M_H^2$ and their corresponding solutions $u(\cdot), v(\cdot)$ as described in the previous lemma. We first notice that if $t \geq \tau + h$, we obtain from (9)

$$\begin{aligned} \|u_t - v_t\|_{L_H^2}^2 &= \int_{-h}^0 |u(t + \theta) - v(t + \theta)|^2 d\theta \\ &\leq \int_{-h}^0 \sup_{s \in [-h, 0]} |u(t + s) - v(t + s)|^2 d\theta \\ &\leq h \left(C_g^2 \|\phi - \psi\|_{L_H^2}^2 + |u_0 - v_0|^2 \right) \exp\left(\int_{\tau}^t (C_g^2 + c_0'' \|u(s)\|^2 + 1) ds\right). \end{aligned}$$

On the other hand, if $\tau \leq t < \tau + h$ we immediately deduce

$$\begin{aligned} \|u_t - v_t\|_{L_H^2}^2 &= \int_{-h}^0 |u(t + \theta) - v(t + \theta)|^2 d\theta \\ &\leq \left((C_g^2 h + 1) \|\phi - \psi\|_{L_H^2}^2 + h |u_0 - v_0|^2 \right) \times \\ &\quad \times \exp\left(\int_{\tau}^t (C_g^2 + c_0'' \|u(s)\|^2 + 1) ds\right). \end{aligned}$$

Thus, we have for all $t \geq \tau$

$$\begin{aligned} \|u_t - v_t\|_{L_H^2}^2 &\leq \left((C_g^2 h + 1) \|\phi - \psi\|_{L_H^2}^2 + h |u_0 - v_0|^2 \right) \times \\ &\quad \times \exp\left(\int_{\tau}^t (C_g^2 + c_0'' \|u(s)\|^2 + 1) ds\right), \end{aligned}$$

and the continuity of $S(t, \tau)$ follows immediately from (8) and this inequality.

Now, we will prove that, under suitable assumptions, there exists a family of compact absorbing sets for the processes $U(t, \tau)$ and $S(\cdot, \cdot)$. Although we could carry out our programme for more general nonautonomous terms f and g , for the sake of clarity we prefer to consider an autonomous force f to develop

our theory. Later on, we will comment on how our results can be adapted to deal with more general nonautonomous terms. It is remarkable that, in this particular case (i.e. autonomous forcing term f) the absorbing family we will construct does not depend on the time variable.

We will proceed in the following way. First, we will establish existence of (uniformly bounded) absorbing families of sets in different phase spaces for the family of mappings $\tilde{U}(\cdot, \cdot)$. Then, taking into account the relationship between $\tilde{U}(\cdot, \cdot)$, $U(\cdot, \cdot)$ and $S(\cdot, \cdot)$ and the results in Lemma 11, we will be able to construct appropriate families of compact pullback absorbing sets for the processes $U(\cdot, \cdot)$ and $S(\cdot, \cdot)$.

3.2 Existence of absorbing families of sets in C_H and M_H^2

We first need a technical lemma which will be very helpful in our analysis. It relates the absorption and attraction properties for the mapping $\tilde{U}(\cdot, \cdot)$ with those of $U(\cdot, \cdot)$ and $S(\cdot, \cdot)$ in such a way that, proving those for \tilde{U} yields to similar properties for U and S .

Definition 10 *The family of bounded sets $\{B(t)\}_{t \in \mathbb{R}}$ in C_H is said to be (pullback) absorbing¹ for $\tilde{U}(\cdot, \cdot)$ in M_H^2 if for any given bounded set $\tilde{D} \subset M_H^2$ and any $t \in \mathbb{R}$, there exists $\tilde{T}_{\tilde{D}}(t) > 0$ such that for all $s \geq \tilde{T}_{\tilde{D}}(t)$ it holds*

$$\tilde{U}(t, t-s)\tilde{D} \subset B(t).$$

In the same sense, the family of bounded sets $\{B(t)\}_{t \in \mathbb{R}}$ in C_H is said to be (pullback) attracting for $\tilde{U}(\cdot, \cdot)$ in M_H^2 if for any given bounded set $\tilde{D} \subset M_H^2$ and any $t \in \mathbb{R}$ it holds

$$\lim_{s \rightarrow +\infty} \text{dist}_{C_H}(\tilde{U}(t, t-s)\tilde{D}, B(t)) = 0.$$

Lemma 11 *Assume that the family of bounded sets $\{B(t)\}_{t \in \mathbb{R}}$ in C_H is absorbing (resp. attracting) for $\tilde{U}(\cdot, \cdot)$. Then,*

- (a) *The family $\{B(t)\}_{t \in \mathbb{R}}$ is absorbing (resp. attracting) for the process $U(\cdot, \cdot)$.*
- (b) *The family of bounded sets $\{j(B(t))\}_{t \in \mathbb{R}}$ in $H \times C_H$ is absorbing (resp. attracting) for the process $S(\cdot, \cdot)$.*

¹ Notice that the word absorbing used here should be interpreted in a generalized sense, since \tilde{U} is not a process.

PROOF. (a) Let $D \subset C_H$ be a bounded set, i.e. there exists $d > 0$ such that $\|\phi\|_{C_H} \leq d$ for all $\phi \in D$. Let us consider the set $\tilde{D} = j(D) = \{(\phi(0), \phi) : \phi \in D\}$. This set is bounded in M_H^2 , namely, it holds that

$$\|(u_0, \phi)\|_{M_H^2}^2 \leq \tilde{d}^2 \quad \text{for any } (u_0, \phi) \in \tilde{D},$$

where $\tilde{d}^2 = (1+h)d^2$. Recalling now that $U(t, t-s)\phi = \tilde{U}(t, t-s)(\phi(0), \phi)$ for any $\phi \in D$, and taking into account the absorbing property of \tilde{U} , it follows immediately the absorption property for the process U . In fact, the absorption time $T_D(t)$ is the corresponding $\tilde{T}_{\tilde{D}}(t)$ given in Definition 10 for the absorbing property for the set \tilde{D} and the map \tilde{U} . The proof of the respective statement on the attraction follows in the same way.

(b) Let us consider now a bounded set $\tilde{D} \subset M_H^2$. Then, there exists $\tilde{T}_{\tilde{D}}(t) \geq h$ such that

$$\tilde{U}(t, t-s)(\tilde{D}) \subset B(t) \quad \text{for all } s \geq \tilde{T}_{\tilde{D}}(t).$$

Taking into account that

$$S(t, t-s)(\tilde{D}) = j(\tilde{U}(t, t-s)(\tilde{D})) \quad \text{for } s \geq \tilde{T}_{\tilde{D}}(t),$$

it follows that

$$S(t, t-s)(\tilde{D}) = j(\tilde{U}(t, t-s)(\tilde{D})) \subset j(B(t)) \quad \text{for } s \geq \tilde{T}_{\tilde{D}}(t).$$

The attraction result can be proved analogously.

Theorem 12 *Assume that (I)-(IV) hold for any $\tau \leq t$ with $m_0 > 0$, and $f \in (L^2(\Omega))^2$. Then, if $\nu\lambda_1 > C_g$, there exists a family $\{B(t)\}_{t \in \mathbb{R}}$ of bounded absorbing sets in C_H for the family of mappings $\{\tilde{U}(t, \tau) : t \geq \tau\}$. Moreover, this family is given by $B(t) = B_1$ for all $t \in \mathbb{R}$, where $B_1 \subset C_H$ is bounded.*

PROOF. As $\tilde{D} \subset M_H^2$ is bounded, then there exists $\tilde{d} > 0$ such that

$$|u_0|^2 + \|\phi\|_{L_H^2}^2 \leq \tilde{d}^2, \quad \text{for all } (u_0, \phi) \in \tilde{D}.$$

Let us take $(u_0, \phi) \in \tilde{D}$ and $\tau \in \mathbb{R}$, and denote as usual $u(\cdot) = u(\cdot; \tau, (u_0, \phi))$. As $\nu\lambda_1 > C_g$, we can choose $\sigma > 0$ small enough such that $2\nu\lambda_1 > 2C_g + \sigma$. Then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u|^2 + \nu \|u\|^2 &= (f, u) + (g(t, u_t), u) \\ &\leq \frac{|f|^2}{2\sigma} + \frac{\sigma}{2} |u|^2 + \frac{1}{2C_g} |g(t, u_t)|^2 + \frac{C_g}{2} |u|^2, \end{aligned}$$

and,

$$\frac{d}{dt} |u|^2 + 2\nu \|u\|^2 \leq \frac{|f|^2}{\sigma} + (\sigma + C_g) |u|^2 + \frac{1}{C_g} |g(t, u_t)|^2, \quad (11)$$

what implies

$$\frac{d}{dt} |u|^2 \leq \frac{|f|^2}{\sigma} + \frac{1}{C_g} |g(t, u_t)|^2 - (2\nu\lambda_1 - (\sigma + C_g)) |u|^2. \quad (12)$$

Now, we can also choose $m \in (0, m_0)$ such that $2\nu\lambda_1 > 2C_g + \sigma + m$. Then

$$\begin{aligned} \frac{d}{dt} (e^{mt} |u(t)|^2) &= m e^{mt} |u(t)|^2 + e^{mt} \frac{d}{dt} |u(t)|^2 \\ &\leq \frac{e^{mt} |f|^2}{\sigma} + e^{mt} \left(m - (2\nu\lambda_1 - (\sigma + C_g)) |u(t)|^2 + \frac{1}{C_g} |g(t, u_t)|^2 \right), \end{aligned} \quad (13)$$

and, integrating between τ and $t (\geq \tau)$,

$$\begin{aligned} e^{mt} |u(t)|^2 - e^{m\tau} |u_0|^2 &\leq \int_{\tau}^t \frac{e^{ms} |f|^2}{\sigma} ds + \int_{\tau}^t \frac{e^{ms}}{C_g} |g(s, u_s)|^2 ds \\ &\quad + \int_{\tau}^t e^{ms} (m - (2\nu\lambda_1 - (\sigma + C_g))) |u(s)|^2 ds \\ &\leq \frac{e^{mt} |f|^2}{m\sigma} + C_g \int_{\tau-h}^{\tau} e^{ms} |\phi(s - \tau)|^2 ds \\ &\quad + \int_{\tau}^t e^{ms} (m + C_g - (2\nu\lambda_1 - (\sigma + C_g))) |u(s)|^2 ds \\ &\leq \frac{e^{mt} |f|^2}{m\sigma} + C_g e^{m\tau} \int_{-h}^0 |\phi(\theta)|^2 d\theta. \end{aligned}$$

Thus,

$$e^{mt} |u(t)|^2 \leq \frac{e^{mt} |f|^2}{m\sigma} + e^{m\tau} \tilde{d}^2 (1 + C_g),$$

and

$$|u(t)|^2 \leq \frac{|f|^2}{m\sigma} + \tilde{d}^2 (1 + C_g) e^{-mt} e^{m\tau}, \quad \text{for all } t \geq \tau. \quad (14)$$

Now, if we take $t \geq \tau + h$, we have for $\theta \in [-h, 0]$

$$\begin{aligned} |u(t + \theta)|^2 &\leq \frac{|f|^2}{m\sigma} + \tilde{d}^2 (1 + C_g) e^{-m(t+\theta)} e^{m\tau} \\ &\leq \frac{|f|^2}{m\sigma} + \tilde{d}^2 e^{mh} (1 + C_g) e^{-mt} e^{m\tau}, \end{aligned}$$

whence

$$\|u_t\|_{C_H}^2 \leq \frac{|f|^2}{m\sigma} + \tilde{d}^2 e^{mh} (1 + C_g) e^{-mt} e^{m\tau} \text{ for all } t \geq \tau + h.$$

If we now consider the time $t-s$ instead of τ (i.e. $u(\cdot)$ denotes now $u(\cdot; t-s, (u_0, \phi))$), so that we can use more easily the definition of absorbing sets) we have

$$\|\tilde{U}(t, t-s)(u_0, \phi)\|_{C_H} = \|u_t\|_{C_H}^2 \leq \frac{|f|^2}{m\sigma} + \tilde{d}^2 e^{mh} (1 + C_g) e^{-ms} \text{ for all } t, \text{ and } s \geq h.$$

and denoting by $\tilde{\rho}^2 = \frac{|f|^2}{m\sigma}$ and $\tilde{\rho}_H^2 = 2\tilde{\rho}^2$, it easily follows that there exists $\tilde{T}_{\tilde{D}}(t) (= \tilde{T}_{\tilde{D}}) \geq h$ such that for all $s \geq \tilde{T}_{\tilde{D}}(t)$ and all $(u_0, \phi) \in M_H^2$, it holds $\|\tilde{U}(t, t-s)(u_0, \phi)\|_{C_H} \leq \tilde{\rho}_H$, which means that the balls $B(t) = B_{C_H}(0, \tilde{\rho}_H)$ form an absorbing family of bounded sets for the mappings $\tilde{U}(t, \tau)$.

Corollary 13 *Under the assumptions in Theorem 12, there exists a family $\{B(t)\}_{t \in \mathbb{R}}$ of bounded absorbing sets in C_H for the process U , which is given by $B(t) = B_1 = B_{C_H}(0, \tilde{\rho}_H)$ for all $t \in \mathbb{R}$. Moreover, the family $\{\mathbf{B}(t)\}_{t \in \mathbb{R}}$ given by $\mathbf{B}(t) = B_H(0, \tilde{\rho}_H) \times B_{L_H^2}(0, h^{1/2}\tilde{\rho}_H) \subset M_H^2$ for all $t \in \mathbb{R}$ is absorbing for the process S .*

PROOF. The first part follows from the previous Theorem 12 and Lemma 11. As for the second, observe that $\{j(B(t))\}_{t \in \mathbb{R}}$ is a family of bounded absorbing sets for $S(\cdot, \cdot)$. On the other hand, as $\|\phi\|_{L_H^2}^2 \leq h \|\phi\|_{C_H}^2$ and

$$j(B(t)) = \{(\phi(0), \phi) : \phi \in B_{C_H}(0, \tilde{\rho}_H)\},$$

it follows that

$$j(B(t)) \subset B_H(0, \tilde{\rho}_H) \times B_{L_H^2}(0, h^{1/2}\tilde{\rho}_H) = \mathbf{B}(t),$$

what implies that the family $\{\mathbf{B}(t)\}_{t \in \mathbb{R}}$ is absorbing for the process $S(\cdot, \cdot)$.

Remark 14 *If we assume that $f \in V'$, the previous results also hold true by modifying slightly the proofs and substituting $|f|$ by $\|f\|_*$.*

3.3 Existence of an absorbing family of sets in C_V

We now prove the existence of an absorbing family of sets in C_V and a necessary bound on the term $\int_{t+\theta_1}^{t+\theta_2} |Au(s)|^2 ds$. We proceed in a similar way as we have already done in the previous subsection.

Theorem 15 *Under the assumptions in Theorem 12, there exist positive constants $\tilde{\rho}_V, \tilde{\beta}_1, \tilde{\beta}_2$ such that for any bounded set $\tilde{D} \subset M_H^2$ and for $\tilde{T}_{\tilde{D}}$ the ab-*

sorbing time corresponding to the set B_1 in Theorem 12, it follows

$$\begin{aligned} \left\| \tilde{U}(t, t-s)(u_0, \phi) \right\|_{C_V}^2 &= \max_{\theta \in [-h, 0]} \|u(t+\theta; t-s, (u_0, \phi))\|^2 \leq \tilde{\rho}_V^2, \\ \int_{t+\theta_1}^{t+\theta_2} |Au(\sigma; t-s, (u_0, \phi))|^2 d\sigma &\leq \tilde{\beta}_1 |\theta_2 - \theta_1| + \tilde{\beta}_2, \end{aligned}$$

for all $s \geq \tilde{T}_{\tilde{D}} + 1 + h$, $t \in \mathbb{R}$, $(u_0, \phi) \in \tilde{D}$, and $\theta_1, \theta_2 \in [-h, 0]$.

PROOF. As in the proof of Theorem 12, let $\tilde{D} \subset M_H^2$ be a bounded set, i.e. there exists $\tilde{d} > 0$ such that $\|(u_0, \phi)\|_{M_H^2} \leq \tilde{d}$ for all $(u_0, \phi) \in \tilde{D}$. Denote $u(\cdot) = u(\cdot; t_0 - s, (u_0, \phi))$ for $(u_0, \phi) \in \tilde{D}$, where $t_0 \in \mathbb{R}$ is a fixed number, and let us take $s \geq \tilde{T}_{\tilde{D}}$, where we have chosen the same σ and m than in that proof. We can then integrate in (11) between t and $t+1$ for $t \geq t_0$ and $s \geq \tilde{T}_{\tilde{D}}$. We obtain

$$\begin{aligned} |u(t+1)|^2 - |u(t)|^2 + (2\nu - (\sigma + C_g) \lambda_1^{-1}) \int_t^{t+1} \|u(r)\|^2 dr \\ \leq \frac{|f|^2}{\sigma} + \frac{1}{C_g} \int_t^{t+1} |g(r, u_r)|^2 dr \\ \leq \frac{|f|^2}{\sigma} + \frac{1}{C_g} \left[C_g^2 \int_{t-h}^{t+1} |u(r)|^2 dr \right] \\ \leq \frac{|f|^2}{\sigma} + C_g \int_{t-h}^t |u(r)|^2 dr + C_g \int_t^{t+1} |u(r)|^2 dr \\ \leq \frac{|f|^2}{\sigma} + C_g \int_{t-h}^t |u(r)|^2 dr + C_g \lambda_1^{-1} \int_t^{t+1} \|u(r)\|^2 dr, \end{aligned}$$

and

$$\begin{aligned} (2\nu - (\sigma + 2C_g) \lambda_1^{-1}) \int_t^{t+1} \|u(r)\|^2 dr &\leq \frac{|f|^2}{\sigma} + C_g \int_{t-h}^t |u(r)|^2 dr + |u(t)|^2 \\ &\leq \frac{|f|^2}{\sigma} + C_g \int_{t-h}^t \|u_r\|_{C_H}^2 dr + \tilde{\rho}_H^2 \\ &\leq \frac{|f|^2}{\sigma} + (1 + hC_g) \tilde{\rho}_H^2. \end{aligned}$$

Therefore,

$$\int_t^{t+1} \|u(r)\|^2 dr \leq \tilde{I}_V, \forall t \geq t_0, \quad (15)$$

where

$$\tilde{I}_V = \frac{1}{2\nu - (\sigma + 2C_g) \lambda_1^{-1}} \left(\frac{|f|^2}{\sigma} + (1 + hC_g) \tilde{\rho}_H^2 \right).$$

On the other hand, we take the inner product with Au and obtain for $r \geq t_0$

$$\frac{1}{2} \frac{d}{dr} \|u\|^2 + \nu |Au|^2 + b(u, u, Au) \leq (f, Au) + (g(r, u_r), Au). \quad (16)$$

Now we evaluate the terms. First, notice that

$$\begin{aligned} |(f, Au)| + |(g(r, u_r), Au)| &\leq |Au| (|f| + |g(r, u_r)|) \\ &\leq \frac{\nu}{4} |Au|^2 + \frac{2}{\nu} (|f|^2 + |g(r, u_r)|^2). \end{aligned} \quad (17)$$

Next,

$$\begin{aligned} |b(u, u, Au)| &\leq c_1 |u|^{1/2} \|u\| |Au|^{3/2} \\ &\leq \frac{\nu}{4} |Au|^2 + \frac{c'_1}{\nu^3} |u|^2 \|u\|^4. \end{aligned} \quad (18)$$

Thanks to (17)-(18), and the fact that $\|\varphi\| \leq \lambda_1^{-1} |A\varphi|$ for $\varphi \in D(A)$, we can deduce from Eq. (16)

$$\frac{d}{dr} \|u\|^2 + \nu |Au|^2 \leq \frac{\nu}{4} (|f|^2 + L_g^2 \|u_r\|_{C_H}^2) + \frac{2c'_1}{\nu^3} |u|^2 \|u\|^4, \quad (19)$$

and

$$\begin{aligned} \frac{d}{dr} \|u\|^2 + \nu \lambda_1 \|u\|^2 &\leq \frac{\nu}{4} (|f|^2 + L_g^2 \|u_r\|_{C_H}^2) + \frac{2c'_1}{\nu^3} |u|^2 \|u\|^4 \\ &\leq \frac{\nu}{4} (|f|^2 + L_g^2 \tilde{\rho}_H^2) + \frac{2c'_1}{\nu^3} |u|^2 \|u\|^4. \end{aligned}$$

Now, we can apply the uniform Gronwall lemma for $s \geq \tilde{T}_{\tilde{D}}$ (see Temam [26]). Then,

$$\|u(r)\|^2 \leq (a_3 + a_2) e^{a_1}, \quad \text{for all } r \geq t_0 + 1, \text{ provided } s \geq \tilde{T}_{\tilde{D}},$$

where

$$\begin{aligned} a_3 &= \tilde{I}_V \\ a_2 &= \frac{\nu}{4} (|f|^2 + L_g^2 \tilde{\rho}_H^2) \\ a_1 &= \frac{2c'_1}{\nu^3} \tilde{\rho}_H^2 \tilde{I}_V, \end{aligned}$$

and, consequently, if we take $s \geq \tilde{T}_{\tilde{D}} + 1 + h$,

$$\sup_{\theta \in [-h, 0]} \|u(t_0 + \theta)\|^2 \leq (a_3 + a_2) e^{a_1} = \tilde{\rho}_V^2, \quad (20)$$

where the constants appearing in (20) are independent of the fixed time $t_0 \in \mathbb{R}$. So, (20) holds true for all $t_0 \in \mathbb{R}$. Denoting from now on

$$u(\cdot) = u(\cdot; t - s, (u_0, \phi)),$$

and, taking into account that part b) in Theorem 1 ensures that $u_t(\cdot) \in C_V$ for $s > h$, we indeed have

$$\|u_t\|_{C_V} \leq \tilde{\rho}_V, \quad \text{for all } t \in \mathbb{R}, \text{ provided } s \geq \tilde{T}_{\tilde{D}} + 1 + h.$$

Finally we will obtain the bound on the term $\int_{t+\theta_1}^{t+\theta_2} |Au(r)|^2 dr$. Indeed, from (19) it follows

$$|Au|^2 \leq \alpha_1 + \alpha_2 |u|^2 \|u\|^4 - \frac{1}{\nu} \frac{d}{dr} \|u\|^2.$$

If we choose $s \geq \tilde{T}_D + 1 + h$ and $\theta_1, \theta_2 \in [-h, 0]$ with e.g. $\theta_2 > \theta_1$, we have

$$\begin{aligned} \int_{t+\theta_1}^{t+\theta_2} |Au(r)|^2 dr &\leq \alpha_1 |\theta_2 - \theta_1| + \alpha_2 \int_{t+\theta_1}^{t+\theta_2} |u(r)|^2 \|u(r)\|^4 dr \\ &\quad - \frac{1}{\nu} \|u(t+\theta_2)\|^2 + \frac{1}{\nu} \|u(t+\theta_1)\|^2 \\ &\leq \left(\alpha_1 + \alpha_2 \tilde{\rho}_H^2 \tilde{\rho}_V^4 \right) |\theta_2 - \theta_1| + \frac{1}{\nu} \tilde{\rho}_V^2, \end{aligned}$$

as desired.

Corollary 16 *Under the assumptions in Theorem 12, there exist positive constants ρ_V, β_1, β_2 such that for any bounded set $D \subset C_H$ and for $T_D = \tilde{T}_{j(D)}$ with $\tilde{T}_{j(D)}$ the absorbing time corresponding to the set B_1 in Theorem 12, it follows*

$$\|U(t, t-s)\phi\|_{C_V}^2 = \|u_t(\cdot; t-s, j(\phi))\|_{C_V}^2 = \max_{\theta \in [-h, 0]} \|u(t+\theta; t-s, j(\phi))\|^2 \leq \rho_V^2,$$

$$\int_{t+\theta_1}^{t+\theta_2} |Au(\sigma; t-s, j(\phi))|^2 d\sigma \leq \beta_1 |\theta_2 - \theta_1| + \beta_2,$$

for all $s \geq T_D + 1 + h, t \in \mathbb{R}, \phi \in D$, and $\theta_1, \theta_2 \in [-h, 0]$. In particular, the family $\{B_2(t)\}_{t \in \mathbb{R}}$, where $B_2(t) = B_2 = B_{C_V}(0, \rho_V)$, is absorbing for the process $U(\cdot, \cdot)$.

Moreover, the family $\{B_S(t)\}_{t \in \mathbb{R}}$, where $B_S(t) = B_{C_V}(0, \rho_V) \times B_{L_V^2}(0, h^{1/2} \rho_V)$, is absorbing for $S(\cdot, \cdot)$.

PROOF. The proof follows the same lines as those of Corollary 13.

3.4 Existence of the pullback attractors

Now we can prove the following result.

Theorem 17 *Under the assumptions in Theorem 12, there exist a unique uniformly bounded pullback attractor $\{\mathcal{A}_{C_H}(t)\}_{t \in \mathbb{R}}$ for the process $U(\cdot, \cdot)$ in C_H , and a unique uniformly bounded pullback attractor $\{\mathcal{A}_{M_H^2}(t)\}_{t \in \mathbb{R}}$ for $S(\cdot, \cdot)$ in M_H^2 . Furthermore, $\mathcal{A}_{M_H^2}(t) \subset H \times C_H$ for all $t \in \mathbb{R}$ and both attractors are related by means of*

$$\mathcal{A}_{M_H^2}(t) = j(\mathcal{A}_{C_H}(t)), \quad \text{for all } t \in \mathbb{R}.$$

PROOF. Let us consider the family $\{B_2(t)\}_{t \in \mathbb{R}}$, where $B_2(t) = B_2 = B_{C_V}(0; \rho_V)$ for all $t \in \mathbb{R}$. This is a family of bounded sets in C_V , which is also (uniformly) absorbing for $\tilde{U}(\cdot, \cdot)$. Take now $\tilde{B}_2 = j(B_2)$. Then, using the previous notation, there exists $\tilde{T}'_{\tilde{B}_2} = T_{B_2} + 1 + h > 0$ such that

$$\tilde{U}(t, t-s)\tilde{B}_2 \subset B_2, \quad \text{for all } t \in \mathbb{R}, \text{ and all } s \geq \tilde{T}'_{\tilde{B}_2}.$$

Now, for each $t \in \mathbb{R}$, consider the set

$$B_3(t) = \bigcup_{s \geq \tilde{T}'_{\tilde{B}_2}} \tilde{U}(t, t-s)\tilde{B}_2 \subset B_2 \subset C_V.$$

Thus, $\{B_3(t)\}_{t \in \mathbb{R}}$ is a family of uniformly bounded sets in C_V which is (uniformly) absorbing for $\tilde{U}(\cdot, \cdot)$.

If we prove that each $B_3(t)$ is relatively compact in C_H , then $\{\overline{B_3(t)}\}_{t \in \mathbb{R}}$ (where the closure is taken in C_H) is a family of compact absorbing set in C_H for $\tilde{U}(\cdot, \cdot)$. Consequently, it is also a family of compact (uniform) absorbing sets for the process $U(\cdot, \cdot)$ in C_H , and $\{j(\overline{B_3(t)})\}_{t \in \mathbb{R}}$ is another family of compact (uniform) absorbing sets for $S(\cdot, \cdot)$ in M_H^2 , what ensures the existence of the pullback attractors for the processes. The uniqueness of these attractors holds since they are uniformly bounded (see Remark 6).

Let us now prove this compactness property. To this end, we will use the Ascoli-Arzelà theorem, in other words, we have to check

- (A) The set $\bigcup_{s \geq \tilde{T}'_{\tilde{B}_2}} \tilde{U}(t, t-s)\tilde{B}_2$ is equicontinuous (i.e. $\forall \varepsilon > 0, \exists \delta > 0$ such that if $|\theta_1 - \theta_2| \leq \delta$, then $|\tilde{U}(t, t-s)(j(\phi))(\theta_1) - \tilde{U}(t, t-s)(j(\phi))(\theta_2)| \leq \varepsilon, \forall t \in \mathbb{R}, s \geq \tilde{T}'_{\tilde{B}_2}, \forall \phi \in B_2$.)
- (B) For each $\theta \in [-h, 0]$,

$$\overline{\bigcup_{s \geq \tilde{T}'_{\tilde{B}_2} \bigcup_{\phi \in B_2} \tilde{U}(t, t-s)(j(\phi))(\theta)}} \text{ is a compact set in } H.$$

To prove (B) we need to check that, for any fixed $\theta \in [-h, 0]$ and $t \in \mathbb{R}$, the set

$$\{u(t + \theta; t-s, j(\phi)) : s \geq \tilde{T}'_{\tilde{B}_2}, \phi \in B_2\}$$

is relatively compact. But this holds since this set is bounded in V (see Theorem 15) and the injection $V \subset H$ is compact.

Finally, in order to prove (A) we proceed by estimating

$$\begin{aligned} & \left| \tilde{U}(t, t-s)(j(\phi))(\theta_1) - \tilde{U}(t, t-s)(j(\phi))(\theta_2) \right| \\ &= |u(t+\theta_1; t-s, j(\phi)) - u(t+\theta_2; t-s, j(\phi))| \end{aligned}$$

for $t \in \mathbb{R}$, $\theta_1, \theta_2 \in [-h, 0]$, $s \geq \tilde{T}'_{\tilde{B}_2}$ and $\phi \in B_2$. Then we obtain (denoting for simplicity $u(\cdot; t-s, j(\phi))$ by $u(\cdot)$ and assuming $\theta_2 > \theta_1$)

$$\begin{aligned} |u(t+\theta_1) - u(t+\theta_2)| &= \left| \int_{t+\theta_1}^{t+\theta_2} u'(r) dr \right| \\ &\leq \int_{t+\theta_1}^{t+\theta_2} |u'(r)| dr \\ &\leq \int_{t+\theta_1}^{t+\theta_2} (\nu |Au(r)| + |B(u(r))| + |f| + |g(r, u_r)|) dr \\ &\leq |f| |\theta_1 - \theta_2| \\ &\quad + \int_{t+\theta_1}^{t+\theta_2} (\nu |Au(r)| + c_1 |Au(r)| \|u(r)\| + L_g \|u_r\|_{C_H}) dr \\ &\leq |f| |\theta_1 - \theta_2| \\ &\quad + \int_{t+\theta_1}^{t+\theta_2} ((\nu + c_1 \|u(r)\|) |Au(r)| + L_g \|u_r\|_{C_H}) dr, \end{aligned} \tag{21}$$

and, consequently, for $t \in \mathbb{R}$, $s \geq \tilde{T}'_{\tilde{B}_2}$

$$\begin{aligned} |u(t+\theta_1) - u(t+\theta_2)| &\leq |f| |\theta_1 - \theta_2| \\ &\quad + \int_{t+\theta_1}^{t+\theta_2} ((\nu + c_1 \|u(r)\|) |Au(r)| + L_g \|u_r\|_{C_H}) dr \\ &\leq (|f| + \rho_H L_g) |\theta_1 - \theta_2| + \int_{t+\theta_1}^{t+\theta_2} (\nu + c_1 \rho_V) |Au(r)| dr \\ &\leq (|f| + \rho_H L_g) |\theta_1 - \theta_2| \\ &\quad + (\nu + c_1 \rho_V) |\theta_1 - \theta_2|^{1/2} \int_{t+\theta_1}^{t+\theta_2} |Au(r)|^2 dr \\ &\leq (|f| + \rho_H L_g) |\theta_1 - \theta_2| \\ &\quad + (\nu + c_1 \rho_V) (\beta_1 |\theta_1 - \theta_2| + \beta_2) |\theta_1 - \theta_2|^{1/2}, \end{aligned}$$

which implies the needed equicontinuity.

Finally, we will prove the interesting relationship that there exists between the attractors $\{\mathcal{A}_{C_H}(t)\}_{t \in \mathbb{R}}$ and $\{\mathcal{A}_{M_H^2}(t)\}_{t \in \mathbb{R}}$. Observe that from the properties of the mapping $j(\cdot)$, the results in Lemma 11 and Corollary 16, it is straightforward to check that $\{j(\mathcal{A}_{C_H}(t))\}_{t \in \mathbb{R}}$ is a uniformly bounded family of compact sets in M_H^2 which is pullback attracting for the process $S(\cdot, \cdot)$, and it is also invariant. Taking into account the uniqueness of the uniformly

bounded attractors, it follows immediately that

$$\mathcal{A}_{M_H^2}(t) = j(\mathcal{A}_{C_H}(t)) \quad \text{for all } t \in \mathbb{R}.$$

The proof is now complete.

Remark 18 *As we have already mentioned, our analysis can be extended to deal with more general nonautonomous f and g . The technique we have used in the previous subsections can be performed to treat this case in a straightforward way, although with additional difficulties in the computations. For instance, if we assume that $f \in L_{loc}^2(\mathbb{R}; L^2(\Omega)^2)$, and satisfies*

$$\int_{-\infty}^t e^{ms} |f(s)|^2 ds < +\infty, \quad \text{for all } t \in \mathbb{R}, \quad \text{and } m > 0,$$

then, under assumptions (I)-(IV) with $m_0 > 0$, and $\nu\lambda_1 > C_g$, it is not difficult to check that there exists a family $\{B(t)\}_{t \in \mathbb{R}}$ of bounded absorbing sets in C_H for $\tilde{U}(\cdot, \cdot)$. To be more precise, $B(t) = B_{C_H}(0, \rho_H(t))$ where $\rho_H^2(t) = 2e^{mh}e^{-mt} \int_{-\infty}^t e^{ms} |f(s)|^2 ds$, for a positive but small enough m . Under these assumptions, we can then prove similarly the existence of the nonautonomous absorbing family in C_V , and conclude with the existence of the pullback attractor. We leave the details to the reader.

3.5 An application: a forcing term with variable delay

Consider that operator g is given by

$$g(t, u_t) = G(u(t - \rho(t))),$$

with $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a function satisfying $G(0) = 0$ and such that there exists $L_1 > 0$ for which

$$|G(u) - G(v)|_{\mathbb{R}^2} \leq L_1 |u - v|_{\mathbb{R}^2}, \quad \forall u, v \in \mathbb{R}^2,$$

and $\rho \in C^1(\mathbb{R})$, $\rho(t) \geq 0$ for all $t \in \mathbb{R}$, $h = \sup_{t \in \mathbb{R}} \rho(t) \in (0, +\infty)$ and $\rho_* = \sup_{t \in \mathbb{R}} \rho'(t) < 1$. This situation is within our framework and satisfies our assumptions (Conditions (I)-(IV)) ensuring the existence and uniqueness of solutions (see Caraballo & Real [7]). Moreover, (IV) is fulfilled by setting

$C_g^2 = L_1^2 e^{m_0 h} / (1 - \rho_*)$ for any $m_0 > 0$. Indeed, it follows for $t \geq \tau$

$$\begin{aligned} \int_{\tau}^t e^{ms} |g(s, u_s) - g(s, v_s)|^2 ds &= \int_{\tau}^t e^{ms} |G(u(s - \rho(s))) - G(v(s - \rho(s)))|^2 ds \\ &\leq L_1^2 \int_{\tau}^t e^{ms} |u(s - \rho(s)) - v(s - \rho(s))|^2 ds \\ &\leq \frac{L_1^2 e^{mh}}{1 - \rho_*} \int_{\tau - \rho(\tau)}^{t - \rho(t)} e^{m\sigma} |u(\sigma) - v(\sigma)|^2 d\sigma \\ &\leq \frac{L_1^2 e^{m_0 h}}{1 - \rho_*} \int_{\tau - h}^t e^{ms} |u(s) - v(s)|^2 ds, \quad m \in [0, m_0]. \end{aligned}$$

Observe that if $\nu \lambda_1 > L_1 / (1 - \rho_*)^{1/2}$, our result ensures the existence of a pullback attractor $\mathcal{A}_{C_H}(t) \subset C_H$ for the process $U(\cdot, \cdot)$ (and also another pullback attractor $\mathcal{A}_{M_H^2}(t)$ for $S(\cdot, \cdot)$). Indeed, we only need to check that $\nu \lambda_1 > C_g = L_1 e^{m_0 h/2} / (1 - \rho_*)^{1/2}$. But, if $\nu \lambda_1 > L_1 / (1 - \rho_*)^{1/2}$, then for a sufficiently small but positive m_0 , we have that $\nu \lambda_1 > L_1 e^{m_0 h/2} / (1 - \rho_*)^{1/2}$. Notice that the analysis done in Caraballo and Real [8] ensures that if the viscosity ν is larger, i.e., if for instance, for certain positive constants k_1 and k_2 (depending only on Ω), it holds that

$$2\nu \lambda_1 > \frac{(2 - \rho_*) L_1}{(1 - \rho_*)} + \frac{k_1 |f|}{\nu - \lambda_1^{-1} L_1} + \frac{k_2 |f|^3}{\nu^2 (\nu - \lambda_1^{-1} L_1)^3},$$

then, there exists a unique stationary solution $u_{\infty} \in V$ to our problem and every solution approaches this stationary solution exponentially fast. In other words, $\mathcal{A}_{C_H}(t)$ consists of this unique stationary solution. Notice that in the particular case $\rho_* = 0$ (which means that the delay function ρ is not increasing) we obtain an attractor for our model if $\nu \lambda_1 > L_1$, and this attractor becomes a unique point if

$$2\nu \lambda_1 > 2L_1 + \frac{k_1 |f|}{\nu - \lambda_1^{-1} L_1} + \frac{k_2 |f|^3}{\nu^2 (\nu - \lambda_1^{-1} L_1)^3}.$$

3.6 Remarks on the autonomous case

We are now interested in the following autonomous version of our problem

$$\left\{ \begin{array}{l} \text{To find } u \in L^2(0, T; H) \cap L^2(-h, T; V) \cap L^{\infty}(0, T; H) \quad \forall T > 0, \text{ s.t.} \\ \frac{d}{dt} u(t) + \nu A u(t) + B(u(t)) = f + g(u_t) \text{ in } \mathcal{D}'(0, +\infty; V'), \\ u(0) = u_0, \quad u(t) = \phi(t), \quad t \in (-h, 0), \end{array} \right. \quad (22)$$

where $f \in (L^2(\Omega))^2$, and $g : C_H \rightarrow (L^2(\Omega))^2$ satisfies (II), (III) and (IV) in Section 2. Owing to the fact that g does not explicitly depend on the time

variable t , these conditions can be rewritten as follows:

(g1) $g(0) = 0$

(g2) there exists $L_g > 0$ such that $\forall \xi, \eta \in C_H$

$$|g(\xi) - g(\eta)| \leq L_g \|\xi - \eta\|_{C_H},$$

(g3) $\exists m_0 \geq 0, C_g > 0: \forall m \in [0, m_0], 0 \leq t, u, v \in C^0([-h, t]; H)$

$$\int_0^t e^{ms} |g(u_s) - g(v_s)|^2 ds \leq C_g^2 \int_{-h}^t e^{ms} |u(s) - v(s)|^2 ds.$$

For each initial function $\phi \in C_H$ and taking as initial value $u_0 = \phi(0)$, there exists a unique solution $u(\cdot; \phi)$ to problem (22) such that $u \in C^0([-h, +\infty); H)$. Then, for any $t \geq 0$ we can define an operator $U_0(t) : C_H \rightarrow C_H$ as

$$U_0(t)\phi = u_t(\cdot; \phi).$$

Bearing in mind the analysis done in the previous section, we can proceed only on the phase space C_H since the existence of an attractor in C_H enables us to obtain another one in M_H^2 . In this sense, it is not difficult to prove, in a similar fashion as we have done in the preceding subsections, that this dynamical system $U_0(\cdot)$ possesses a global attractor in C_H . But, we note that, considering this problem as a nonautonomous one and setting $U(t, \tau)$ for its associated process, it holds that

$$U(t, \tau) = U(t - \tau, 0), \quad \text{for all } t \geq \tau,$$

and, consequently,

$$U_0(t) = U(t, 0), \quad \text{for all } t \geq 0,$$

is a semigroup of nonlinear continuous operators.

Now, our previously developed theory allows the reader to prove as an easy exercise the following result.

Theorem 19 (*Existence of global attractor*) Assume that (g1), (g2) and (g3) hold with $m_0 > 0$. If, in addition, $\nu\lambda_1 > C_g$, then there exists the global attractor $\mathcal{A}_{C_H} \subset C_H$ for the semigroup $U_0(t)$.

Remark 20 *Needless to say that a similar result can also be proved if we consider the semigroup $S_0(t) = S(t, 0)$.*

As an application, we will now consider an example in which the forcing term contains a distributed delay.

Let $G : [-h, 0] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a measurable function satisfying $G(s, 0) = 0$ for all $s \in [-h, 0]$ and assume that there exists a function $\gamma \in L^2(-h, 0)$ such

that

$$|G(s, u) - G(s, v)|_{\mathbb{R}^N} \leq \gamma(s)|u - v|_{\mathbb{R}^N}, \forall u, v \in \mathbb{R}^N \quad \forall s \in [-h, 0].$$

Then, we define $g(\xi)(x) = \int_{-h}^0 G(s, \xi(s)(x)) ds$ for each $\xi \in C^0([0, T]; H)$ and $x \in \Omega$. In this case, the delayed term g in our problem becomes

$$g(u_t) = \int_{-h}^0 G(s, u(t+s)) ds.$$

It holds that g satisfies the hypotheses in Theorem 19.

Indeed, (g1) is evident. As for (g2), notice that, if $\xi, \eta \in C_H$, we obtain

$$\begin{aligned} |g(\xi) - g(\eta)|^2 &\leq \int_{\Omega} \left(\int_{-h}^0 |G(s, \xi(s)(x)) - G(s, \eta(s)(x))|_{\mathbb{R}^N} ds \right)^2 dx \\ &\leq \int_{\Omega} \left(\int_{-h}^0 \gamma(s) |\xi(s)(x) - \eta(s)(x)|_{\mathbb{R}^N} ds \right)^2 dx \\ &\leq \int_{\Omega} \|\gamma\|_{L^2(-h,0)}^2 \left(\int_{-h}^0 |\xi(s)(x) - \eta(s)(x)|_{\mathbb{R}^N}^2 ds \right) dx \\ &\leq h \|\gamma\|_{L^2(-h,0)}^2 \|\xi - \eta\|_{C_H}^2. \end{aligned}$$

Finally, if $u, v \in C^0([-h, T]; H)$ then, for each $t > 0$, $m_0 > 0$ and all $m \in [0, m_0]$, it follows

$$\begin{aligned} \int_0^t e^{m\tau} |g(u_\tau) - g(v_\tau)|^2 d\tau &\leq \|\gamma\|_{L^2(-h,0)}^2 \int_0^t e^{m\tau} \left(\int_{-h}^0 |u(s+\tau) - v(s+\tau)|^2 ds \right) d\tau \\ &\leq \|\gamma\|_{L^2(-h,0)}^2 \int_{-h}^0 \left(\int_0^t e^{m\tau} |u(s+\tau) - v(s+\tau)|^2 d\tau \right) ds \\ &\leq \|\gamma\|_{L^2(-h,0)}^2 \int_{-h}^0 \left(\int_s^{t+s} e^{m(r-s)} |u(r) - v(r)|^2 dr \right) ds \\ &\leq \|\gamma\|_{L^2(-h,0)}^2 \int_{-h}^0 e^{-ms} \left(\int_{-h}^t e^{mr} |u(r) - v(r)|^2 dr \right) ds \\ &\leq \|\gamma\|_{L^2(-h,0)}^2 h e^{m_0 h} \int_{-h}^t e^{mr} |u(r) - v(r)|^2 dr. \end{aligned}$$

Consequently, Theorem 19 ensures the existence of the global attractor in C_H provided $\nu\lambda_1 > \|\gamma\|_{L^2(-h,0)} h^{1/2} e^{m_0 h/2}$. But, we note that if $\nu\lambda_1 > \|\gamma\|_{L^2(-h,0)} h^{1/2}$, we can choose m_0 small enough such that $\nu\lambda_1 > \|\gamma\|_{L^2(-h,0)} h^{1/2} e^{m_0 h/2}$. It is also remarkable that when $h \rightarrow 0$ the sufficient condition ensuring the existence of the global attractor becomes $\nu\lambda_1 > 0$, which is the usual one in the case without delays (and trivially fulfilled).

Conclusions and final comments

We have proved some results concerning the asymptotic behaviour of solutions to a two dimensional Navier-Stokes model containing delay forcing terms. In fact, we have proved the existence of attractors in the pullback sense by rewriting our problem in a nonautonomous context, so that the theory of pullback attractor can be successfully applied. Observe that, what we actually have proved in Theorem 17, is that the set $\mathbf{B} = \overline{\cup_{t \in \mathbb{R}} B_3(t)}$ is compact and uniformly pullback, and so forward, attracting. Then, by means of the theory developed by Chepyzhov and Vishik (see [10]), there exists a uniform forward attractor $\mathbb{A}_f \subset C_H$ which contains the sets $\mathcal{A}_{C_H}(t)$, for all $t \in \mathbb{R}$. Finally, the autonomous case has been deduced from our general results.

However, there is still much work to be done in this field. For instance, it could be very helpful to obtain some results on the finite dimensionality of the pullback attractor. Also, we could consider a different framework for our process in the product space $H \times L^2(-h, 0; V)$, if we allow that the term g can contain first order derivatives with respect to the spatial variables. Some results on the regularity of the attractor will be also welcome. Needless to mention the interesting and important three-dimensional case. Of course, these topics will be analyzed in some forthcoming papers.

Acknowledgements. This work was started while the first author was visiting the Mathematics Intitute (University of Warwick) during June-August 2002, within the Programa de Movilidad de Profesores de Universidad y de Escuelas Universitarias (Secretaría de Estado de Universidades e Investigación, Spain). T.C. thanks James Robinson for his continual and very appreciated hospitality and the Royal Society of London for their generosity.

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