AN ESTIMATE ON THE FRACTAL DIMENSION OF ATTRACTORS OF GRADIENT-LIKE DYNAMICAL SYSTEMS

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Abstract. This paper is dedicated to estimate the fractal dimension of exponential global attractors of some generalized gradient-like semigroups in a general Banach space in terms of the maximum of the *dimension* of the local unstable manifolds of the isolated invariant sets, Lipschitz properties of the semigroup and rate of exponential attraction. We also generalize this result for some special evolution processes, introducing a concept of Morse decomposition with pullback attractivity. Under suitable assumptions, if (A, A^*) is an attractor-repeller pair for the attractor $\mathcal{A}$ of a semigroup $\{T(t) : t \geq 0\}$, then the fractal dimension of $\mathcal{A}$ can be estimated in terms of the fractal dimension of the local unstable manifold of A^*, the fractal dimension of A, the Lipschitz properties of the semigroup and the rate of the exponential attraction. The ingredients of the proof are the notion of generalized gradient-like semigroups and their regular attractors, Morse decomposition and a fine analysis of the structure of the attractors. As we said previously, we generalize this result for some evolution processes using the same basic ideas.

1. Introduction

Over the last forty years, the study of qualitative properties of semigroups in Banach spaces has received very much attention (see, for instance, [3],[7],[13], [21] and [36]). In particular, the study of global attractors has created a deep area of research and greatly improved the understanding of qualitative properties of solutions for these infinite dimensional dynamical systems.

A particular aspect that has called the attention of many researchers, and for which a very nice theory has been developed, is the fractal dimension of attractors. Starting with the pioneering works [25] and [28], the theory has grown considerably and new strategies to find bounds for the fractal dimension have been proposed (see for example [36, 14, 21, 7] and references therein).

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Before we proceed, let us briefly recall the definitions of *topological*, *Hausdorff* and *fractal dimension*.

If K is a topological space, we say that K has finite *topological* dimension if there exists a natural number d such that, for every open covering $\mathcal{U}$ of K, there is another covering $\mathcal{U}'$ of K refining $\mathcal{U}$ with the property that each point of K belongs to at most $d+1$ sets in $\mathcal{U}'$. In this case, the topological dimension $\dim_T(K)$ of K is the minimum d with this property. With this notion, a subset of $\mathbb{R}^n$ with non-empty interior has topological dimension n and, if K is a compact metric space with topological dimension $\dim_T(K) < \infty$, then it is homeomorphic to a subset of $\mathbb{R}^n$ with $n = 2\dim_T(K) + 1$ (see [27], [33]).

Next we introduce the notion of Hausdorff dimension. For a given metric space (X, ρ), $\alpha > 0$, $\epsilon > 0$ and $A \subset X$ let

$$\mu_\epsilon^{(\alpha)}(A) = \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam}(B_i))^\alpha : \; A \subset \cup_{i=1}^\infty B_i, \; \operatorname{diam}(B_i) < \epsilon \right\},$$

with the convention $\inf \varnothing = \infty$. Since $\mu_\epsilon^{(\alpha)}(A)$ increases as ϵ decreases, we define

$$\mu^{(\alpha)}(A) = \lim_{\epsilon \to 0} \mu_\epsilon^{(\alpha)}(A).$$

Definition 1.1. *For $A \subset X$, the* Hausdorff dimension *of A is defined by*

$$\inf\{\alpha \geqslant 0 : \mu^{(\alpha)}(A) = 0\} = \sup\{\alpha \geqslant 0 : \mu^{(\alpha)}(A) = \infty\}.$$

It is known (see [33]) that $\dim_T(K) \leqslant \dim_H(K)$.

Now we turn our attention to the attractors of gradient semigroups in Banach spaces. Let X be a Banach space with norm $\|\cdot\|_X$ and recall that a *semigroup* on X is a family $\{T(t) : t \geq 0\}$ of continuous maps from X into itself such that

i) $T(0) = I$,
ii) $T(t+s) = T(t) \circ T(s)$, for all $t, s \geq 0$ and
iii) $\mathbb{R}^+ \times X \ni (t, x) \mapsto T(t)x \in X$ is continuous.

A *global solution* for $\{T(t) : t \geq 0\}$ through $x \in X$ is a continuous function $\phi : \mathbb{R} \to X$ such that $T(t)\phi(s) = \phi(t+s)$ for all $t \geq 0$, $s \in \mathbb{R}$ and $\phi(0) = x$.
A subset $\mathcal{A}$ of X is said *invariant under the action of the semigroup* $\{T(t) : t \geq 0\}$ if $T(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and we say that $\mathcal{A}$ attracts B under the action of $\{T(t) : t \geq 0\}$ if

$$\operatorname{dist}_H(T(t)B, A) := \sup_{b \in b} \inf_{a \in A} \|T(t)b - a\|_X \xrightarrow{t \to \infty} 0.$$

A subset of X is the *global attractor* for $\{T(t) : t \geq 0\}$ if it is compact, invariant and attracts bounded subsets of X under the action of $\{T(t) : t \geq 0\}$.

A semigroup is said to be *gradient* if there is a continuous function $V : X \to \mathbb{R}$ such that $R^+ \ni t \mapsto V(T(t)x) \in \mathbb{R}$ is non-increasing for each $x \in X$ and $V(T(t)x) = V(x)$ for all $t \geq 0$ if and only if $T(t)x = x$ for all $t \geq 0$; that is, x is a *stationary solution* for $\{T(t) : t \geq 0\}$. Denote by $\mathcal{E}$ the set of stationary solutions for $\{T(t) : t \geq 0\}$.

If a gradient semigroup $\{T(t) : t \geq 0\}$ has a global attractor $\mathcal{A}$ and its set of stationary solutions $\mathcal{E}$ is finite, then

$$\mathcal{A} = \bigcup_{e \in \mathcal{E}} W^u(e),$$

where

$$W^u(e) = \{x \in X : \text{ there is a global solution } \phi : \mathbb{R} \to X \text{ through } x \text{ such that } \phi(t) \xrightarrow{t \to -\infty} e\}.$$

For gradient semigroups we define $W^u_{\mathrm{loc}}(e)$ as the intersection of $W^u(e)$ with a neighborhood of e. Assume that $W^u_{\mathrm{loc}}(e)$ is the graph of a Lipschitz map with domain in a finite dimensional afine linear manifold $e + Q_e(X)$ where Q_e is a projection with finite dimensional rank.

We know (following [6]) that

$$\begin{aligned} &\dim_H(W^u_{\mathrm{loc}}(e)) = \mathrm{rank}(Q_e) < \infty, \text{ for each } e \in \mathcal{E}, \\ &\dim_H(T(n)W^u_{\mathrm{loc}}(e)) \leqslant \dim_H(W^u_{\mathrm{loc}}(e)), \ \ n \geq 0. \end{aligned}$$

It is not difficult to see that $W^u(e) = \bigcup_{n=0}^{\infty} T(n)W^u_{\mathrm{loc}}(e)$ and, from the $\sigma-$sub-additivity property of the Hausdorff measure it follows that

$$\begin{aligned} \mathrm{rank}(Q_e) &= \dim_H(W^u_{\mathrm{loc}}(e)) \\ &\leqslant \dim_H(W^u(e)) \\ &= \dim_H\left(\bigcup_{n=0}^{\infty} T(n)W^u_{\mathrm{loc}}(e)\right) \\ &\leqslant \sup_{n\in\mathbb{N}} \dim_H(T(n)W^u_{\mathrm{loc}}(e)) \\ &\leqslant \dim_H(W^u_{\mathrm{loc}}(e)) \\ &= \mathrm{rank}(Q_e), \end{aligned}$$

and therefore $\dim_H(W^u(e)) = \mathrm{rank}(Q_e)$, for all $e \in \mathcal{E}$. Hence, since $\mathcal{A} = \bigcup_{e\in\mathcal{E}} W^u(e)$, we have that

$$\dim_H(\mathcal{A}) = \max_{e\in\mathcal{E}} \mathrm{rank}(Q_e). \tag{1.1}$$

In particular $\mathcal{A}$ is homeomorphic to a subset of $\mathbb{R}^N$ where $N = 2 \max\limits_{e\in\mathcal{E}} \mathrm{rank}(Q_e) + 1$.

Finally we introduce the notion of fractal dimension. If K is a compact metric space let $N(r, K)$ be the least number of balls of radius r necessary to cover K. The *fractal dimension* (or also known as *capacity* or *box-counting dimension*) $c(K)$ of K is defined by:

$$c(K) = \limsup_{r\to 0} \frac{\log N(r,K)}{\log(1/r)}.$$

Alternatively, $c = c(K)$ is the least real number such that, for all $\epsilon > 0$ there exists $\delta > 0$ with

$$N(r,K) \leqslant \left(\frac{1}{r}\right)^{c+\epsilon}, \quad 0 < r < \delta.$$

From this, it is easy to see that

$$\dim_H(K) \leqslant c(K). \tag{1.2}$$

The fractal and Hausdorff dimension may differ significantly. One can easily see that the set $\{\frac{1}{n} : n \in \mathbb{N}^*\} \cup \{0\}$ is a compact subset of $\mathbb{R}$ with zero Hausdorff dimension and fractal dimension equal to $\frac{1}{2}$. It may even happen that the Hausdorff dimension is zero with the fractal dimension being infinite (see [28] for such an example).

One particular result that makes the fractal dimension a very interesting object of research is the following result (see [28])

Theorem 1.2. *Given a Banach space X, a compact subset K of X with fractal dimension $c(K) < \infty$ and a finite dimensional subspace Y with* $\dim Y > 2c(K)+1$*, if $\mathcal{P}(X,Y)$ is the subspace of $\mathcal{L}(X,Y)$ of the projections with range Y, the set $\{P \in \mathcal{P}(X,Y) : P|_K$ is injective $\}$ is of second category in $\mathcal{P}(X,Y)$.*

The inverse of the projection restricted to K is continuous. In fact, in some situations, this inverse is Hölder continuous (see [33]).

It would be very nice to be able to prove a similar result to (1.1) for the fractal dimension, and this will be indeed our main objective in this paper. Nonetheless, such result would not be expected since the manner in which the unstable manifold of a given equilibria accumulates on other equilibria may be at a slow rate causing the dimension to increase (like it happens with the set $\{\frac{1}{n} : n \in \mathbb{N}^*\} \cup \{0\}$). However, if we take the sequence $\{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0\}$, it is not difficult to see that the Hausdorff and fractal dimension coincide. Inspired by this, we seek a bound for the fractal dimension of regular attractors with exponentially attracting local unstable manifolds.

The result will be proved for generalized gradient-like semigroups and will make use of the Morse decomposition of a generalized-gradient like semigroup (as in [1]). In Section 2 we introduce the basic concepts and results needed to prove the main result. Section 3 is dedicated to obtain an estimate on the fractal dimension of global attractors for generalized gradient-like Lipschitz semigroups for which the local unstable set of an isolated invariant set is the graph of a Lipschitz map over a finite dimensional subspace of the phase space.

2. Generalized gradient-like semigroups and attractor-repeller pairs

In this section we present the notion of a generalized gradient-like semigroup and some basic results, some other results concerning to attractor-repeller pairs can be found in [1].

To introduce the notion of generalized gradient-like semigroups (see [5]) we first need the definition of isolated invariant set:

Definition 2.1. *Let $\{T(t) : t \geqslant 0\}$ be a semigroup. We say that an invariant set $\Xi \subset X$ for the semigroup $\{T(t) : t \geq 0\}$ is an isolated invariant set if there is an $\epsilon > 0$ such that Ξ is the maximal invariant subset of $\mathcal{O}_\epsilon(\Xi)$.*

A disjoint family of isolated invariant sets is a family $\{\Xi_1, \cdots, \Xi_n\}$ of isolated invariant sets with the property that, for some $\epsilon > 0$,

$$\mathcal{O}_\epsilon(\Xi_i) \cap \mathcal{O}_\epsilon(\Xi_j) = \varnothing,\ 1 \leq i < j \leq n.$$

Definition 2.2. *Let $\{T(t) : t \geq 0\}$ be a semigroup which has a disjoint family of isolated invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. A* homoclinic structure *associated to $\boldsymbol{\Xi}$ is a subset $\{\Xi_{k_1}, \cdots, \Xi_{k_p}\}$ of $\boldsymbol{\Xi}$ ($p \leq n$) together with a set of global solutions $\{\phi_1, \cdots, \phi_p\}$ such that*

$$\Xi_{k_j} \overset{t\to-\infty}{\longleftarrow} \phi_j(t) \overset{t\to\infty}{\longrightarrow} \Xi_{k_{j+1}},\ 1 \leq j \leq p,$$

where $\Xi_{k_{p+1}} := \Xi_{k_1}$.

We are now ready to define generalized gradient-like semigroups.

Definition 2.3. *Let $\{T(t) : t \geq 0\}$ be a semigroup with a global attractor $\mathcal{A}$ and a disjoint family of isolated invariant sets $\boldsymbol{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$. We say that $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup relative to $\boldsymbol{\Xi}$ if*

(i) For any global solution $\xi : \mathbb{R} \to \mathcal{A}$ there are $1 \leq i, j \leq n$ such that

$$\Xi_i \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to\infty}{\longrightarrow} \Xi_j.$$

(ii) There is no homoclinic structure associated to $\boldsymbol{\Xi}$.

Now we will introduce the notion attractor-repeller pairs in a global attractor $\mathcal{A}$.

Definition 2.4. *Let $\{T(t) : t \geq 0\}$ be a semigroup with a global attractor $\mathcal{A}$. We say that a non-empty subset Ξ of $\mathcal{A}$ is a local attractor if there is an $\epsilon > 0$ such that $\omega(\mathcal{O}_\epsilon(\Xi)) = \Xi$. The repeller Ξ^* associated to a local attractor Ξ is the set defined by*

$$\Xi^* = \{x \in \mathcal{A} : \omega(x) \cap \Xi = \varnothing\}.$$

The pair (Ξ, Ξ^*) is called attractor-repeller pair *for $\{T(t) : t \geq 0\}$.*

Note that if Ξ is a local attractor, then Ξ^* is closed and invariant.

3. An estimate on the fractal dimension of attractors for gradient-like semigroups

Recall that, from the definition, if $K \subset G$ are both compact subspaces of X, then $c(K) \leqslant c(G)$.

Now assume that X, Y are Banach spaces, $K \subset X$, $G \subset Y$ compact subsets and $f : K \to G$ a Lipschitz function with Lipschitz constant $L_f > 0$. Then $c(f(K)) \leqslant c(K)$. In fact, since $N(\epsilon, f(K)) \leqslant N(\epsilon/L_f, K)$ we have that

$$\begin{aligned} c(f(K)) &= \limsup_{\epsilon \to 0^+} \frac{\ln N(\epsilon, f(K))}{\ln(1/\epsilon)} \leqslant \limsup_{\epsilon \to 0^+} \frac{\ln N(\epsilon/L_f, K)}{\ln(1/\epsilon)} \\ &= \limsup_{\epsilon \to 0^+} \frac{\ln N(\epsilon/L_f, K)}{\ln(L_f/L_f\epsilon)} = \limsup_{\epsilon \to 0^+} \frac{\ln N(\epsilon/L_f, K)}{\ln(L_f/\epsilon) - \ln(L_f)} \\ &= \limsup_{\epsilon \to 0^+} \frac{1}{1 - \frac{\ln(L_f)}{\ln(L_f/\epsilon)}} \frac{\ln N(\epsilon/L_f, K)}{\ln(L_f/\epsilon)} \leqslant c(K). \end{aligned}$$

As a consequence of this result, if we assume the above hypotheses and in addition $X = Y$ and $K \subset f(K)$, then $c(K) = c(f(K))$.

Throughout this section we are interested in the calculation of the fractal dimension of the attractor, in terms of the fractal dimensions of the unstable manifolds associated to the isolated invariant sets. First we need to start with some results concerning the isolated invariant sets for a given gradient-like semigroup $\{T(t) : \ t \geqslant 0\}$.

Definition 3.1. *Let $\{T(t) : \ t \geqslant 0\}$ be a generalized gradient-like semigroup with global attractor $\mathcal{A}$, and $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ a family of associated isolated invariant sets. We say that an isolated invariant set Ξ_i is a* source, *if $W^s_{loc}(\Xi_i) \cap \mathcal{A} = \Xi_i$; and a* sink *if $W^u(\Xi_i) = \Xi_i$. Otherwise, we say that Ξ_i is a* saddle.

Theorem 3.2. *Let $\{T(t) : \ t \geqslant 0\}$ be a generalized gradient-like semigroup with global attractor $\mathcal{A}$ and $\mathbf{\Xi} = \{\Xi_1, \cdots, \Xi_n\}$ the associated isolated invariant sets. Then, there is at least one source and at least one sink.*

Proof. Assume there are no sources. Then given Ξ_i, there exists a Ξ_j $(j \neq i)$ and a global solution ξ such that

$$\Xi_i \overset{t \to -\infty}{\longleftarrow} \xi(t) \overset{t \to \infty}{\longrightarrow} \Xi_j.$$

Inductively, we can construct a homoclinic structure since there is a finite number of isolated invariant sets, which leads us to a contradiction. A similar argument proves the existence of a sink. $\square$

Remark 3.3. *Assume that $\{T(t): t \geqslant 0\}$ is a generalized gradient-like semigroup with global attractor $\mathcal{A}$ and $\Xi = \{\Xi_1, \cdots, \Xi_n\}$ the associated isolated invariant sets. We can easily show that the attractor $\mathcal{A}$ of $\{T(t): t \geqslant 0\}$ coincides with the attractor $\mathcal{A}'$ of the discrete generalized gradient-like semigroup $\{S^n : n \in \mathbb{N}\}$, where $S = T(1)$. In fact, it is clear that $\mathcal{A} \subset \mathcal{A}'$. Conversely, the attractor $\mathcal{A}'$ is given as the union of unstable manifolds of the isolated invariant sets, and given a point $z \in \mathcal{A}'$, there exists an isolated invariant set Ξ_i and a global solution ξ such that $\xi(0) = z$ and $\xi(-n) \stackrel{n\to\infty}{\to} \Xi_i$. Now, we can define $\phi(-t)$ for all $t \geqslant 0$ as follows: given $n \in \mathbb{N}$, define*

$$\phi(-t) = T(n-t)\xi(-n), \text{ for all } 0 \leqslant t \leqslant n.$$

This obviously gives us a global solution ϕ of $\{T(t): t \geqslant 0\}$ such that $\phi(0) = z$ and $\xi(-t) \stackrel{t\to\infty}{\to} \Xi_i$, which proves that $\mathcal{A} = \mathcal{A}'$.

Due to this remark, we can now consider only the case of discrete generalized gradient-like semigroups and we begin stating our first result on fractal dimension.

Proposition 3.4. *Let $\{T^n : n \in \mathbb{N}\}$ be a discrete semigroup with global attractor $\mathcal{A}$. Let $S = T_{|\mathcal{A}}$ and assume that S is Lipschitz continuous with Lipschitz constant $L > 1$. Let (A, A^*) be an attractor-repeller pair in $\mathcal{A}$, and assume that there exist constants $M \geqslant 1$ and $\omega > 0$ such that, for all K compact subset of $\mathcal{A}$ with $K \cap A^* = \varnothing$, we have $\mathrm{dist_H}(S^n K, A) \leqslant M e^{-\omega n}$, for all $n \in \mathbb{N}$. Assume also that there is a neighbourhood B of A^* in $\mathcal{A}$ such that $\overline{B} \cap A = \varnothing$.*

Then

$$c(B) \leqslant c(\mathcal{A}) \leqslant \max\left\{\frac{\omega + \ln(L)}{\omega} c(B), c(A)\right\}.$$

Proof. Clearly, since $B \subset \mathcal{A}$, $c(B) \leqslant c(\mathcal{A})$. We only have to prove the right inequality. For this, we divide the proof into four steps:

Step 1: Define $\Omega_n = S^n(\mathcal{A} \setminus B) \setminus S^{n+1}(\mathcal{A} \setminus B)$, for all $n \in \mathbb{N}$. Note that $\Omega_0 = (\mathcal{A} \setminus B) \setminus S(\mathcal{A} \setminus B) \subset S(B) \setminus B \subset S(B)$ and therefore $c(\Omega_0) \leqslant c(S(B)) = c(B)$, because $B \subset S(B)$ and S is a Lipschitz continuous function.

Now we obtain an estimate on the minimum number of r-balls $N(r, \Omega_k)$ necessary to cover Ω_k in terms of the numbers of balls necessary to cover Ω_0. Let $n_0^{r,k} = N(r/L^k, \Omega_0)$ and $\{x_1, \ldots, x_{n_0^{r,k}}\}$ a finite sequence of points in Ω_0 such that

$$\Omega_0 \subset \bigcup_{i=1}^{n_0^{r,k}} B(x_i, r/L^k).$$

Set, for each $i = 1, \ldots, n_0^{r,k}$, $\xi_i = S^k(x_i) \in \Omega_k$. Then, for each $y \in \Omega_k$ there exists $z \in \Omega_0$ such that $y = S^k(z)$, $z \in B(x_i, r/L^k)$ for some $i = 1, \ldots, n_0^{r,k}$ and we have

$$\|y-\xi_i\| = \|S^k(z)-S^k(x_i)\| \leqslant L^k\|z-x_i\| < r, \text{ for all } y\in\Omega_k.$$

So, we just proved $\Omega_k \subset \cup_{i=1}^{n_0^{r,k}} B(\xi_i,r)$, which gives $N(r,\Omega_k)\leqslant n_0^{r,k}$.

Step 2: Given $r>0$, since $\mathrm{dist}_{\mathrm{H}}(S^n(\mathcal{A}\setminus B),A)\leqslant Me^{-\omega n}$ for all $n\geqslant 0$, there exists $n_0(r)=\lceil \frac{1}{\omega}\ln(\frac{M}{r})\rceil$ such that

$$G(r) := \left(\bigcup_{j\geqslant n_0(r)} \Omega_j\right)\cup A \subset \mathcal{O}_r(A),$$

where $\mathcal{O}_r(A)$ denotes the r-neighborhood of A. So, if $A\subset \cup_{i=1}^{N(r,A)} B(x_i,r)$ with $x_i\in A$ for all $i=1,\ldots,N(r,A)$, then $\mathcal{O}_r(A)\subset \cup_{i=1}^{N(r,A)}B(x_i,2r)$ therefore $N(2r,\mathcal{O}_r(A))\leqslant N(r,A)$. We conclude that $N\left(r,G(\frac{r}{2})\right)\leqslant N(\frac{r}{2},A)$.

Step 3: From Step 1, if $H(r):=\bigcup_{j=0}^{n_0(r)}\Omega_j$ we have

$$N(r,H(r))\leqslant n_0(r)\max_{k=0,\ldots,n_0(r)} N(r/L^k,\Omega_0) = n_0 N(r/L^{n_0(r)},\Omega_0),$$

since $L>1$.

Step 4: First, note that for each $r>0$, we have that $\mathcal{A}=B\cup G(\frac{r}{2})\cup H(\frac{r}{2})$ and therefore

$$\begin{aligned} N(r,\mathcal{A}) &\leqslant 3\max\{N(r,B);\ N(r,H(r/2));\ N(r,G(r/2))\}\\ &\leqslant 3\max\{N(r,B);\ N\left(r/2,H(r/2)\right);\ N(r/2,A)\}\\ &\leqslant 3\max\{N(r,B);\ n_0(r/2)N(r/L^{n_0(r/2)},\Omega_0);\ N(r/2,A)\}.\end{aligned}$$

As the logarithm function is increasing, we obtain

$$\ln N(r,\mathcal{A})\leqslant \ln 3+\max\{\ln N(r,B);\ \ln n_0(r/2)+\ln N(r/L^{n_0(r/2)},\Omega_0);\ \ln N(r/2,A)\}.$$

Hence

$$\frac{\ln N(r,\mathcal{A})}{\ln(1/r)}\leqslant \frac{\ln 3}{\ln(1/r)}+\max\left\{\frac{\ln N(r,B)}{\ln(1/r)};\ \frac{\ln n_0(r/2)}{\ln(1/r)}+\frac{\ln N(r/L^{n_0(r/2)},\Omega_0)}{\ln(1/r)};\frac{\ln N(r/2,A)}{\ln(1/r)}\right\}.$$

Obviously, $\limsup\limits_{r\to 0^+}\dfrac{\ln 3}{\ln(1/r)}=0$. Now, we compute the other terms:

(a)

$$\limsup_{r\to 0^+}\frac{\ln n_0(r/2)}{\ln(1/r)} = \limsup_{r\to 0^+}\frac{\ln 1/\omega}{\ln(1/r)}+\limsup_{r\to 0^+}\frac{\ln(\ln(2M/r))}{\ln(1/r)}=0;$$

(b)

$$\begin{aligned}\limsup_{r\to 0^+}\frac{\ln N(r/L^{n_0(r/2)},\Omega_0)}{\ln(1/r)} &= \limsup_{r\to 0^+}\frac{\ln N(r/L^{n_0(r/2)},\Omega_0)}{\ln(L^{n_0(r/2)}/rL^{n_0})}\\ &= \limsup_{r\to 0^+}\frac{1}{1-\frac{n_0(r/2)\ln L}{\ln(L^{n_0(r/2)}/r)}}\frac{\ln N(r/L^{n_0(r/2)},\Omega_0)}{\ln(L^{n_0(r/2)}/r)},\end{aligned}$$

but

$$\limsup_{r\to 0^+}\frac{1}{1-\frac{n_0(r/2)\ln L}{\ln(L^{n_0(r/2)}/r)}} = \limsup_{r\to 0^+}\left(\frac{n_0(r/2)\ln(L)}{\ln(1/r)}+1\right),$$

and since $\frac{1}{\omega}\ln(\frac{2M}{r})\leqslant n_0\leqslant\frac{1}{\omega}\ln(\frac{2M}{r})+1$,

$$\limsup_{r\to 0^+}\left(\frac{n_0(r/2)\ln(L)}{\ln(1/r)}+1\right)=\frac{\omega+\ln(L)}{\omega},$$

which shows that

$$\limsup_{r\to 0^+}\frac{\ln N(r/L^{n_0(r/2)},\Omega_0)}{\ln(1/r)}\leqslant\frac{\omega+\ln(L)}{\omega}c(\Omega_0).$$

(c)

$$\begin{aligned}\limsup_{r\to 0^+}\frac{\ln N(r/2,A)}{\ln(1/r)} &= \limsup_{r\to 0^+}\frac{\ln N(r/2,A)}{\ln(2/2r)}\\ &\limsup_{r\to 0^+}\frac{1}{1+\frac{\ln(1/2)}{\ln(1/r)}}\frac{\ln N(r/2,A)}{\ln(2/r)}\leqslant c(A).\end{aligned}$$

Joining (a), (b) and (c), we obtain

$$c(\mathcal{A})\leqslant\max\left\{c(B),\frac{\omega+\ln(L)}{\omega}c(\Omega_0),c(A)\right\}\leqslant\max\left\{\frac{\omega+\ln(L)}{\omega}c(B),c(A)\right\},$$

using the fact $c(\Omega_0)\leqslant c(B)$. The proof is now complete. □

Now, using this proposition we can estimate the fractal dimension of a global attractor of a discrete generalized gradient-like semigroup $\{T^n : n\in\mathbb{N}\}$ in terms of the fractal dimensions of the local unstable manifolds of the isolated invariant sets.

Theorem 3.5. *Let $\{T^n : n\in\mathbb{N}\}$ be a discrete generalized gradient-like semigroup with global attractor $\mathcal{A}$ and $\boldsymbol{\Xi}=\{\Xi_1,\ldots,\Xi_p\}$ the associated isolated invariant sets. Assume that the restriction $T_{|\mathcal{A}}$ to $\mathcal{A}$ of the operator T is a Lipschitz continuous function with Lipschitz constant $L>1$ and assume also that there exist constants $M>1$ and $\omega>0$ such that for every attractor-repeller pair (A,A^*) in $\mathcal{A}$ and every compact subset $K\subset\mathcal{A}$ with $K\cap A^*=\varnothing$ we have*

$$\mathrm{dist}_{\mathrm{H}}(T^n(K),A)\leqslant Me^{-\omega n},\ \textit{for all } n\geqslant 0.$$

Finally, assume that the local unstable manifolds $\{W^u_{loc}(\Xi_i), i, \ldots, p\}$ *are given as graphs of Lipschitz functions. Under these conditions*

$$\max_{i=1,\ldots,p} c(W^u_{loc}(\Xi_i)) \leqslant c(\mathcal{A}) \leqslant \frac{\omega + \ln(L)}{\omega} \max_{i=1,\ldots,p} c(W^u_{loc}(\Xi_i)).$$

Proof. Since $\{T^n : \quad n \in \mathbb{N}\}$ is a discrete gradient-like semigroup, there exists at least one source. Let Ξ_i be one of these sources and B_i a neighbourhood of Ξ_i in $\mathcal{A}$ such that $B_i \subset W^u_{loc}(\Xi_i)$ and $T(B_i) \subset W^u_{loc}(\Xi_i)$, so that $c(B_i) = c(T(B_i)) = c(W^u_{loc}(\Xi_i))$. Now, it is easy to see that $\Xi_i = A_i^*$, where $A_i = \cup_{j\neq i} W^u_{loc}(\Xi_j)$. By Proposition 3.4,

$$c(B_i) \leqslant c(\mathcal{A}) \leqslant \max\left\{\frac{\omega + \ln(L)}{\omega} c(B_i), c(A_i)\right\},$$

that is

$$c(W^u_{loc}(\Xi_i)) \leqslant c(\mathcal{A}) \leqslant \max\left\{\frac{\omega + \ln(L)}{\omega} c(W^u_{loc}(\Xi_i)), c(A_i)\right\}.$$

Now, restrict the operator T to the attractor A_i. Thus, we have a discrete generalized gradient-like semigroup with attractor A and $\mathbf{\Xi^1} = \mathbf{\Xi} \setminus \{\Xi_i\}$, which has at least one source Ξ_k, with $k \neq i$. We can use the same argument above to prove that

$$c(W^u_{loc}(\Xi_k)) \leqslant c(A_i) \leqslant \max\left\{\frac{\omega + \ln(L)}{\omega} c(W^u_{loc}(\Xi_k)), c(A_k)\right\}.$$

And joining these two results, we obtain

$$\max_{j=i,k} c(W^u_{loc}(\Xi_j)) \leqslant c(\mathcal{A}) \leqslant \max\left\{\frac{\omega + \ln(L)}{\omega} c(W^u_{loc}(\Xi_i)), \frac{\omega + \ln(L)}{\omega} c(W^u_{loc}(\Xi_k)), c(A_k)\right\}.$$

This process must stop, since there are just a finite number of isolated invariant sets, and proceeding inductively we obtain the desired result. □

Remark 3.6. *The proof of this theorem suggests a certain order in the family of isolated invariant sets and, after a possible index rearrangement, we can assume that* $\mathbf{\Xi} = \{\Xi_1, \ldots, \Xi_p\}$ *and in the proof, the first source in* $\mathcal{A}$ *to be chosen is* Ξ_p*, the second is* Ξ_{p-1} *and so on. Such an ordering can be used to form a new family* $\mathcal{N} = \{\mathcal{N}_1, \ldots, \mathcal{N}_m\}$ *with* $m \leqslant p$ *called* energy level decomposition for the attractor $\mathcal{A}$, *which is a* Morse decomposition *for* $\mathcal{A}$. *For more details see Section 5 of* [1]. *Using this decomposition we can see that the fractal dimension of the sets* $W^u_{loc}(\Xi_i)$ *is a non-increasing function of the index* i*, and we have that*

$$c(W^u_{loc}(\Xi_1)) \leqslant c(\mathcal{A}) \leqslant \frac{\omega + \ln(L)}{\omega} c(W^u_{loc}(\Xi_1)).$$

Our next result is an immediate corollary of the preceding theorem, once we recall some basic facts concerning discrete gradient-like semigroups $\{T^n : n \in \mathbb{N}\}$ with an attractor $\mathcal{A}$ and a finite set $\mathcal{E} = \{e_1, \ldots, e_p\}$ of fixed hyperbolic points. First, the reader can check (see [15] for a proof) that the local unstable (stable) manifold $W^u_{loc}(e_i)$ (W^s_{loc}) is given by a graph of a Lipschitz function. Now, in these conditions it is easy to see that there are only a finite number of attractor-repeller pairs (A, A^*), namely, the pairs (A, A^*), with

$$A = \bigcup_{\substack{i\in I \\ I\subset\{1,\dots,p\}}} W^u(e_i).$$

Using this fact and the exponential attraction of each fixed point, we can prove that there exist constants $M \geqslant 1$ and $\omega > 0$ such that for every attractor-repeller pair (A, A^*) and every compact subset K of $\mathcal{A}$ with $K \cap A^* = \varnothing$, we have

$$\mathrm{dist}_H(T^n(K), A) \leqslant Me^{-\omega n}, \text{ for all } n \geqslant 0.$$

From these two facts it follows the next result:

Corollary 3.7. *Let $\{T^n : n \in \mathbb{N}\}$ be a discrete gradient-like semigroup with an attractor $\mathcal{A}$ and a finite set $\mathcal{E} = \{e_1, \dots, e_p\}$ of fixed hyperbolic points. Assume that the restriction of T to $\mathcal{A}$ is a Lipschitz function with Lipschitz constant $L > 1$. Let $M \geqslant 1$ and $\omega > 0$ be two constants such that for every attractor-repeller pair (A_i, A_i^*), with $A_i = \cup_{j\neq i} W^u_{loc}(e_j)$ and every compact subset K of $\mathcal{A}$ with $K \cap A^* = \varnothing$ we have*

$$\mathrm{dist}_H(T^n(K), A) \leqslant Me^{-\omega n}, \text{ for all } n \geqslant 0.$$

Then

$$\max_{i=1,\dots,p} c(W^u_{loc}(e_i)) \leqslant c(\mathcal{A}) \leqslant \frac{\omega + \ln(L)}{\omega} \max_{i=1,\dots,p} c(W^u_{loc}(e_i)).$$

Under similar, although appropriately modified, hypotheses it is possible to show an analogous result to Proposition 3.4, but using now local stable manifolds.

Proposition 3.8. *Let $\{T^n : n \in \mathbb{N}\}$ be a discrete semigroup with global attractor $\mathcal{A}$. Let $S = T_{|\mathcal{A}}$ and assume that S is invertible with inverse S^{-1} a Lipschitz continuous map, with Lipschitz constant $L > 1$. Let (A, A^*) be an attractor-repeller pair in $\mathcal{A}$, and assume that there exist constants $M \geqslant 1$ and $\omega > 0$ such that, for all K compact subset of $\mathcal{A}$ with $K \cap A = \varnothing$, we have $\mathrm{dist}_{\mathrm{H}}(S^{-n}K, A^*) \leqslant Me^{-\omega n}$, for all $n \in \mathbb{N}$. Assume also that there is a neighbourhood B of A in $\mathcal{A}$ such that $\overline{B} \cap A^* = \varnothing$.*

Then

$$c(B) \leqslant c(\mathcal{A}) \leqslant \max\left\{\frac{\omega + \ln(L)}{\omega} c(B), c(A^*)\right\}.$$

Additionally we can also establish the next result.

Theorem 3.9. *Let $\{T^n : n \in \mathbb{N}\}$ be a discrete generalized gradient-like semigroup with global attractor $\mathcal{A}$ and $\Xi = \{\Xi_1, \dots, \Xi_p\}$ the associated isolated invariant sets. Assume that the restriction $T_{|\mathcal{A}}$ to $\mathcal{A}$ of the operator T is invertible with its inverse T^{-1} a Lipschitz continuous function with Lipschitz constant $L > 1$ and assume also that there exist constants $M > 1$*

and $\omega > 0$ such that for every attractor-repeller pair (A, A^*) in $\mathcal{A}$ and every compact subset $K \subset \mathcal{A}$ with $K \cap A = \varnothing$ we have

$$\mathrm{dist}_{\mathrm{H}}(T^{-n}(K), A^*) \leqslant M e^{-\omega n}, \text{ for all } n \geqslant 0.$$

Finally, assume that the intersection of the local stable manifolds $\{W^s_{loc}(\Xi_i), i, \dots, p\}$ with the global attractor $\mathcal{A}$ are given as graphs of Lipschitz functions. Under these conditions

$$\max_{i=1,\dots,p} c(W^s_{loc}(\Xi_i) \cap \mathcal{A}) \leqslant c(\mathcal{A}) \leqslant \frac{\omega + \ln(L)}{\omega} \max_{i=1,\dots,p} c(W^s_{loc}(\Xi_i) \cap \mathcal{A}).$$

Remark 3.10. *If the hypotheses of Corollary 3.7 are satisfied, S is invertible and the hypotheses of exponential attraction for the inverse to the local repellers are also satisfied then*

$$c(\mathcal{A}) \leqslant \min\left\{ \frac{\omega + \ln L}{\omega} \max_{i=1,\cdots,p} c(W^u_{loc}(e_i)), \max_{i=1,\cdots,p} c(W^s_{loc}(e_i) \cap \mathcal{A}) \right\}.$$

This can be easily seen if we return to the proof of Proposition 3.4. If S is Lipschitz continuous with Lipschitz constant $L > 1$, then S^{-1} is also Lipschitz with Lipschitz constant $1/L < 1$, and the proof in this case is modified. More precisely, in Step 3,

$$N(r, H(r)) \leqslant n_0 N(r, \Omega_0).$$

Also, with the reversed hypotheses

$$c(\mathcal{A}) \leqslant \min\left\{ \max_{i=1,\cdots,p} c(W^u_{loc}(e_i)), \frac{\omega + \ln L}{\omega} \max_{i=1,\cdots,p} c(W^s_{loc}(e_i) \cap \mathcal{A}) \right\}.$$

Remark 3.11. *Consider the autonomous equation*

$$u_t = u_{xx} + \lambda(u - \beta u^3) \tag{3.1}$$

for $x \in [0, \pi]$ with Dirichlet boundary conditions. We consider the family of attractors $\{\mathcal{A}_\lambda :\ \lambda > 0\}$, varying with the parameter λ. Note that our argument implies that if we approach a bifurcation point $\lambda = n^2$, $n \in \mathbb{N}$, our estimate on the fractal dimension of the attractor $\mathcal{A}_\lambda$ explodes, since the rate of exponential attraction ω approaches to zero (see [20] *where it is proved that this attraction is in fact polynomial). However, we know that the fractal dimension of the above Chafee-Infante equation is finite and of order $\sqrt{\lambda}$ for all values of $\lambda \geq \lambda_1$ (the first eigenvalue of the Laplacian operator) (see, for instance,* [36, 33]*). Despite of this fact, if we choose λ' near λ ($\lambda' > \lambda$), then the estimate is finite, as we have the exponential attraction to hyperbolic equilibria. Moreover, for any sequence $\{\lambda_n :\ n \in \mathbb{N}\}$ such that λ_n is away from the endpoints of the interval $(n^2, (n+1)^2)$, uniformly for $n \in \mathbb{N}$, our estimate is of order $\sqrt{\lambda_n}$.*

4. Exponential attraction of some generalized gradient-like semigroups

In this section, we give a result concerning the exponential attraction for a global attractor of a given discrete generalized gradient-like semigroup $\{T^n : n \in \mathbb{N}\}$.

Definition 4.1. *We say that a discrete semigroup $\{T^n : n \in \mathbb{N}\}$ has a pointwise exponentially attracting local unstable set of an invariant set $\Xi \in \boldsymbol{\Xi}$, if there are positive constants C_0, ϱ_0 and δ_0 such that*

$$\mathrm{dist}_H(T^n u_0, W^u_{\mathrm{loc}}(\Xi)) \leqslant C_0 e^{-\varrho_0 n}, \tag{4.1}$$

whenever $u_0 \in \mathcal{O}_{\delta_0}(\Xi)$, $n \in \mathbb{N}$ and $\{T^k u_0 : 0 \le k \le n\} \subset \mathcal{O}_{\delta_0}(\Xi)$.

Lemma 4.2. *Let $\{T^n : n \in \mathbb{N}\}$ be a generalized gradient-like semigroup with a set of disjoint compact invariant sets $\boldsymbol{\Xi}$.*

Then, given $\Xi \in \boldsymbol{\Xi}$ and $\epsilon > 0$, there is $\delta > 0$ such that, if $v \in \mathcal{O}_\delta(\Xi)$ and for some $n_1 > 0$, $T^{n_1} v \notin \mathcal{O}_\epsilon(\Xi)$ then $T^n v \notin \mathcal{O}_\delta(\Xi)$ for all $n \ge n_1$.

Proof: Assume that there are $\epsilon > 0$, a sequence $\{v_k\}$ in V with $v_k \overset{n\to\infty}{\longrightarrow} \Xi$, sequences $\{n_k\}$ and $\{m_k\}$ in $\mathbb{N}$ with $m_k > n_k$, $\mathrm{dist}(T^{n_k} v_k, \Xi) \ge \epsilon$ and $\mathrm{dist}(T^{m_k} v_k, \Xi) \overset{k\to\infty}{\longrightarrow} 0$. Then, Ξ is chain recurrent relative to $\boldsymbol{\Xi}$, which is a contradiction. ■

Lemma 4.3. *Let $\{T^n : n \in \mathbb{N}\}$ be a generalized gradient-like semigroup with a set of disjoint compact invariant sets $\boldsymbol{\Xi}$. If V is a bounded positively invariant subset of X and $B = \cup_{\Xi \in \boldsymbol{\Xi}} \Xi$, given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\{T^n v : 0 \le n \le n_0\} \cap \mathcal{O}_\epsilon(B) \neq \emptyset, \quad \forall v \in V.$$

Proof: This lemma is proved by contradiction. Assume that there are sequences $\{x_k\} \subset V$, $n_k \overset{k\to\infty}{\longrightarrow} \infty$ such that $\{T^j x_k : 0 \le j \le n_k\} \cap \mathcal{O}_\epsilon(B) = \emptyset$.

Choose $m_k :=$ the largest integer smaller than $\frac{n_k}{2}$. Then, there is a subsequence of $\{T^{m_k} x_k\}$ (which we denote the same) convergent to a certain $x_0 \in V$. It is easy to see that $\{T^n x_0 : n \in \mathbb{N}\} \cap \mathcal{O}_\epsilon(B) = \varnothing$ and this is in contradiction with the fact that B attracts points. ■

Lemma 4.4. *Let $\{T^n : n \in \mathbb{N}\}$ be a discrete generalized gradient-like semigroup with global attractor $\mathcal{A}$ and $\boldsymbol{\Xi} = \{\Xi_1, \ldots, \Xi_p\}$ the associated isolated invariant sets. Let V be a bounded and positively invariant closed neighborhood of $\mathcal{A}$, and assume that the restriction $T_{|_V}$ to V of the operator T is a Lipschitz continuous function with Lipschitz constant $e^L > 1$. Assume that each set $\Xi \in \boldsymbol{\Xi}$ has pointwise exponentially attracting local unstable sets.*

Then, there are constants $\tilde{\gamma} > 0$, $\tilde{c} > 0$ such that for any $v \in V$

$$\mathrm{dist}(T^n v, \mathcal{A}) \le \tilde{c} e^{-\tilde{\gamma} n}, \ n \in \mathbb{N}. \tag{4.2}$$

Proof: As e^L is the Lipschitz constant of T in V, then

$$\mathrm{dist}(T^n w_1, T^n w_2) \leq e^{nL}\mathrm{dist}(w_1, w_2). \tag{4.3}$$

Choose $\delta, \gamma > 0$ and $c > 0$, such that

$$\mathrm{dist}(T^n w, W^u_{loc}(\Xi_j)) \leq ce^{-\gamma n} \text{ for all } j = 1, \ldots, k, \tag{4.4}$$

whenever $w \in \mathcal{O}_\delta(\Xi_j)$ and $n \in \mathbb{N}$ is such that $\{T^k w : 0 \leq k \leq n\} \subset \mathcal{O}_\delta(\Xi_j)$.

From Lemma 4.2, choose $\delta' < \delta$ such that, if $v \in \mathcal{O}_{\delta'}(\Xi_j)$, and for some $n_1 > 0$

$$T^{n_1} v \notin \mathcal{O}_\delta(\Xi_j),$$

then

$$T^n v \notin \mathcal{O}_{\delta'}(\Xi_j), \text{ for all } n \geqslant n_1.$$

Now, from Lemma 4.3, there exists $n_0 \in \mathbb{N}$ such that, for all $v \in V$

$$\{T^n v : 0 \leq n \leq n_0\} \cap \bigcup_{j=1}^{k} \mathcal{O}_{\delta'}(\Xi_j) \neq \varnothing.$$

Thus, given $v \in V$, there are sequences $\{n^-_{i(j)}\}_{j=0}^m$, $\{n^+_{i(j)}\}_{j=0}^m$ and $\{\Xi_{i(j)}\}_{j=0}^m$ such that $i(j) \in \{1, \ldots, p\}$, $1 \leq j \leq m \leqslant p)$,

$$n^-_{i(1)} \leqslant n_0, \qquad 0 < n^-_{i(j)} - n^+_{i(j-1)} \leqslant n_0, \quad 1 \leqslant j \leqslant m, \qquad n^+_{i(m)} = +\infty,$$

for which $T^n v \in \mathcal{O}_\delta(\Xi_{i(j)})$ for all $n^-_{i(j)} \leqslant n < n^+_{i(j)}$, $T^{n^+_{i(j)}} v \in \overline{\mathcal{O}_\delta(\Xi_{i(j)})}$ and $j \in \{0, 1, \ldots, m\}$.

Choose any $v \in V$. Then, the positive orbit through v visits neighborhoods of some of the compact invariant sets that belong to $\boldsymbol{\Xi}$. We simply enumerate such sets as $\Xi_1, \ldots, \Xi_m$, $m \leq p$, using the order in which their δ'-neighborhood is visited by the orbit of v.

We now choose a point $y_1 \in \Xi$ and, for each $n^-_j \leqslant n \leqslant n^+_j$, choose ψ_n such that

$$\mathrm{dist}(T^n v, \overline{W^{\mathrm{u}}_{\mathrm{loc}}(\Xi_j)}) = \mathrm{dist}(T^n v, \psi_n), \quad 1 \leqslant j \leqslant m.$$

Define

$$\tilde{u}_n = \begin{cases} y_1, \ 0 \leqslant n < n^-_1, \\ \psi_n, \ n^-_1 \leqslant n \leqslant n^+_1. \end{cases}$$

By assumption we have

$$\mathrm{dist}(T^n v, \tilde{u}_n) \leq \sup_{v \in V} \mathrm{dist}(v, y_1) e^{\gamma n_0} e^{-\gamma n} =: \tilde{c}_1 e^{-\gamma n}, \ 0 \leq n < n^-_1,$$

$$\mathrm{dist}(T^n v, \tilde{u}_n) \leq ce^{-\gamma(n - n^-_1)} \leq ce^{\gamma n_0} e^{-\gamma n} =: \hat{c}_1 e^{-\gamma n}, \ n^-_1 \leq n \leq n^+_1,$$

and we next denote

$$c_1 = \max\{\tilde{c}_1, \hat{c}_1\}, \quad n^+_0 := 0, \quad n^1_0 := n^+_1, \quad \gamma_1 := \gamma.$$

Having this done, we define step by step

$$\gamma_j = \frac{\gamma_{j-1}^2}{L+2\gamma_{j-1}}, \quad \kappa_j^0 = \min\left\{\frac{L+2\gamma_{j-1}}{L+\gamma_{j-1}} n_j^-, n_j^+\right\}, \ j=2,\dots,m,$$

and we extend $\tilde{u}$ onto the whole $\mathbb{N}$ letting

$$\tilde{u}_n = \begin{cases} T^{n-n_{j-1}^+}\tilde{u}_{n_{j-1}^+}, \ n_{j-1}^+ \le n < n_j^-, \\ T^{n-n_j^-}T^{n_j^- - n_{j-1}^+}\tilde{u}_{n_{j-1}^+}, \ n_j^- \le n \le \kappa_j^0, \\ \psi(n), \ \kappa_j^0 < n \le n_j^+. \end{cases}$$

Note that κ_j^0 may not be an integer, and is this case n will not achieve its value.

We will show that, for each $j=2,\dots,m$, the following implication holds:

if (i) $\operatorname{dist}(T^n v, \tilde{u}_n) \leqslant c_{j-1}e^{-\gamma_{j-1}n}$, $n_{j-2}^+ \leqslant n < n_{j-1}^+$ with some $c_{j-1}>0$,

then (ii) $\operatorname{dist}(T^n v, \tilde{u}_n) \leqslant c_j e^{-\gamma_j n}$, $n_{j-1}^+ \leqslant n < n_j^+$ with some $c_j > 0$,.

First note that, by assumption, if $n_{j-1}^+ \leqslant n \leqslant n_j^-$,

$$\begin{aligned} \operatorname{dist}(T^n v, \tilde{u}_n) &\leqslant ce^{L(n-n_{j-1}^+)}\operatorname{dist}(T^{n_{j-1}^+}v, \tilde{u}_{n_{j-1}^+}) \\ &\overset{\text{(i)}}{\leqslant} cc_{j-1}e^{L(n-n_{j-1}^+)-\gamma_{j-1}n_{j-1}^+}. \end{aligned} \tag{4.5}$$

Before we proceed with further estimates note that, by assumption and due to the above construction, if $n_j^- < \kappa_j^0 \leqslant n_j^+$ then, for $n_j^- \leqslant n \leqslant \kappa_j^0$,

$$\begin{aligned} \operatorname{dist}(T^n v, \tilde{u}_n) &\leqslant ce^{L(n-n_j^-)}\operatorname{dist}(T^{n_j^-}v, T^{n_j^- - n_{j-1}^+}\tilde{u}_{n_{j-1}^+}) \\ &\overset{(4.5)}{\leqslant} cc_{j-1}e^{Ln_0}e^{L(n-n_j^-)-\gamma_{j-1}n_{j-1}^+} \\ &\leqslant cc_{j-1}e^{(L+\gamma)n_0}e^{L(n-n_j^-)-\gamma_{j-1}n_j^-} \end{aligned} \tag{4.6}$$

and for $\kappa_j^0 \leqslant n \leqslant n_j^+$,

$$\operatorname{dist}(T^n v, \psi^n_{j,n_j^-,n_j^+}) \leqslant ce^{-\gamma(t-n_j^-)} \leqslant ce^{-\gamma_{j-1}(t-n_j^-)}. \tag{4.7}$$

Taking a closer look at (4.6)-(4.7) it can be noticed that, whenever $n_j^- < \kappa_j^0 < n_j^+$, we have

$$L(\kappa_j^0 - n_j^-) - \gamma_{j-1}n_j^- = -\gamma_{j-1}(\kappa_j^0 - n_j^-).$$

In fact, we infer that

$$L(t-n_j^-) - \gamma_{j-1}n_j^- \leqslant -\gamma_j n, \quad n_j^- \leqslant n \leqslant \kappa_j^0. \tag{4.8}$$

$$-\gamma_{j-1}(t-n_j^-) \leqslant -\gamma_j n, \quad \kappa_j^0 < n \leqslant n_j^+. \tag{4.9}$$

Now, we are ready to complete the estimate. From (4.6) and (4.8) we obtain that, for $n_j^- \leqslant n \leqslant \kappa_j^0$

$$\operatorname{dist}(T^n v, \tilde{u}_n) \leqslant c c_{j-1} e^{(L+\gamma)n_0} e^{-\gamma_j n},$$

whereas (4.7) and (4.9) ensures that, $\kappa_j^0 < n \leqslant n_j^+$,

$$\operatorname{dist}(T^n v, \tilde{u}_n) \leqslant c e^{-\gamma_{j-1}(t-n_j^-)} \leqslant c e^{-\gamma_j t}.$$

From (4.5), for $n_{j-1}^+ \leqslant n \leqslant n_j^-$,

$$\operatorname{dist}(T^n v, \tilde{u}_n) \leqslant c c_{j-1} e^{L(n-n_{j-1}^+)-\gamma_{j-1}(n_{j-1}^+-n+n)} \leqslant c c_{j-1} e^{(L+\gamma)n_0} e^{-\gamma_j n}, \tag{4.10}$$

condition (ii) thus holds with

$$c_j = \max\{c, c c_{j-1} e^{(L+\gamma)n_0}\}$$

and the proof is completed. ■

5. Non-autonomous dynamical systems and attractor-repeller pairs

In this section we are interested in obtaining an estimate for the fractal dimension of a pullback attractor for a gradient-like evolution process. As usual, X is a Banach space and we define a nonlinear evolution process as a two-parameter family $\{T(t,s) : \ t \geqslant s \in \mathbb{R}\}$ of continuous operators from X into itself such that

(1) $T(t,t) = I$,
(2) $T(t,\sigma)T(\sigma,s) = T(t,s)$, for each $t \geqslant \sigma \geqslant s$, and
(3) $(t,s) \mapsto T(t,s)x_0$ is continuous for $t \geqslant s$, $x_0 \in X$.

A continuous function $\xi : \mathbb{R} \to X$ is called a global solution for the evolution process $\{T(t,s) : t \geqslant s\}$ if it satisfies

$$T(t,s)\xi(s) = \xi(t), \text{ for all } t \geqslant s \in \mathbb{R}.$$

A non-linear semigroup (or autonomous evolution process) is a family $\{T(t) : \ t \geqslant 0\}$ with the property that $\{T(t,s) = T(t-s) : \ t \geqslant s \in \mathbb{R}\}$ is an evolution process. We recall that, for a semigroup $\{T(t) : \ t \geqslant 0\}$ a set $\mathcal{A}$ is said to be invariant if $T(t)\mathcal{A} = \mathcal{A}$ for all $t \geqslant 0$. We now define *invariance* in this context as follows

Definition 5.1. *A family* $\{\mathcal{A}(t) \subset X : \ t \in [\sigma, \infty)\}$ *is* invariant *under* $T(\cdot,\cdot)$ *if* $T(t,s)\mathcal{A}(s) = \mathcal{A}(t)$ *for all* $t \geqslant s \geqslant \sigma$.

We have already seen that in the autonomous case, the attractor, when it exists, is exactly the union of all its global bounded orbits,

$$\mathcal{A} = \{x \in X : \text{ there is a bounded global solution through } x\}. \tag{5.1}$$

In the non-autonomous case, the 'attractor' which coincides with the union of all globally-defined bounded solutions; that is,

$$\{\mathcal{A}(t):\ t\in\mathbb{R}\}=\{\xi(t)|\ \xi(\cdot):\mathbb{R}\to X \text{ is bounded and } T(t,s)\xi(s)=\xi(t)\} \tag{5.2}$$

is the pullback attractor (see [7, 10, 18, 35, 12]).

We now will define attractor-repeller pairs and extract some of their properties. For this purpose we follow the ideas in [19], and some demonstrations are omitted since they can be found in this reference.

Definition 5.2 (Attraction universe)**.** *An* attraction universe $\mathcal{D}$ *for a nonlinear evolution process* $\{T(t,s):\ t\geqslant s\}$ *is a collection of bounded families* $D=\{D(t):\ t\in\mathbb{R}\}$*, i.e.,* $D(t)\subset X$ *bounded, for all* $t\in\mathbb{R}$*, such that if* $\varnothing\subsetneq D'\subseteq D$ *for some* $D\in\mathcal{D}$*, then* $D'\in\mathcal{D}$*; where* $D'\subseteq D$ *means* $D'(t)\subseteq D(t)$ *for all* $t\in\mathbb{R}$*.*

Definition 5.3 (Pullback attractor with respect to an attraction universe)**.** *Let* $\{T(t,s):\ t\geqslant s\}$ *be an evolution process in a Banach space* X*. A nonempty, compact, invariant family* $A=\{A(t):\ t\in\mathbb{R}\}\in\mathcal{D}$ *is called a* pullback attractor with respect to an attraction universe $\mathcal{D}$ *if*

$$\lim_{s\to-\infty}\operatorname{dist}_H(T(t,s)D(s),A(t))=0,$$

for all $t\in\mathbb{R}$ *and all family* $D\in\mathcal{D}$*.*

Proposition 5.4. *Given an attraction universe* $\mathcal{D}$*, the pullback attractor with respect to* $\mathcal{D}$ *is unique.*

Proof. Let A and A' be two pullback attractors with respect to the attraction universe $\mathcal{D}$. Since $A'\in\mathcal{D}$, we have for every $t\in\mathbb{R}$ that

$$\operatorname{dist}_H(A'(t),A(t))=\lim_{s\to-\infty}\operatorname{dist}_H(T(t,s)A'(s),A(t))=0,$$

and since $A'(t)$ and $A(t)$ are both compact, it follows that $A'(t)\subseteq A(t)$. Analogously we show that $A(t)\subseteq A'(t)$ which concludes the result. □

Definition 5.5 (Pullback absorbing set with respect to an attraction universe)**.** *Let* $\mathcal{D}$ *be an attraction universe of a nonlinear evolution process* $\{T(t,s):\ t\geqslant s\}$ *in a Banach space* X*. A nonempty family* $B=\{B(t):\ t\in\mathbb{R}\}\in\mathcal{D}$ *is called* pullback absorbing with respect to $\mathcal{D}$ *if for each* $D\in\mathcal{D}$ *and* $t\in\mathbb{R}$ *there exists* $s_0\leqslant t$ *such that*

$$T(t,s)D(s)\subset B(t),\ \textit{for all } s\leqslant s_0.$$

Theorem 5.6 (Existence of a pullback attractor with respect to an attraction universe)**.** *Let $\{T(t,s):\ t\geqslant s\}$ be a nonlinear evolution process in a Banach space X. Assume that $B=\{B(t):\ t\in\mathbb{R}\}$ is a compact pullback absorbing family with respect to an attraction universe $\mathcal{D}$. Then there exists a pullback attractor $\mathcal{A}=\{\mathcal{A}(t):t\in\mathbb{R}\}$ with respect to $\mathcal{D}$, where for each $t\in\mathbb{R}$, the fibers $A(t)$ are defined by*

$$\mathcal{A}(t)=\omega_p(B(t),t)=\bigcap_{\sigma\leqslant t}\overline{\bigcup_{s\leqslant\sigma}T(t,s)B(s)}. \tag{5.3}$$

Proof. Let $\mathcal{A}$ be defined by (5.3). Firstly, we will show that for every $t\in\mathbb{R}$

$$\lim_{s\to-\infty}\operatorname{dist}_H(T(t,s)B(s),\mathcal{A}(t))=0.$$

Assume to contrary that there exist $t\in\mathbb{R}$, a sequence $\{s_n\}_{n\geqslant 0}\subseteq(-\infty,t]$ with $s_n\to-\infty$ as $n\to\infty$, a sequence $\{x_n\}_{n\geqslant 0}$ with $x_n\in B(s_n)$ and an $\epsilon>0$ such that

$$\operatorname{dist}_H(T(t,s_n)x_n,\mathcal{A}(t))\geqslant\epsilon,\ \text{for every } n\geqslant 0.$$

Since B is an absorbing family with respect to $\mathcal{D}$, $B\in\mathcal{D}$ and $s_n\to-\infty$ as $n\to\infty$, we can assume that $y_n\doteq T(t,s_n)x_n\in B(t)$ for every $n\geqslant 0$. By the compactness of $B(t)$, we can also assume that there exists $y\in B(t)$ such that the sequence $\{y_n\}_{n\geqslant 0}$ satisfies $y_n\to y$ as $n\to\infty$.

But $\operatorname{dist}_H(y_n,A(t))\geqslant\epsilon$ for every $n\geqslant 0$ and so $\operatorname{dist}_H(y,A(t))\geqslant\epsilon$, which is a contradiction since $y\in\omega_p(B(t),t)=A(t)$.

Now let $D=\{D(t):t\in\mathbb{R}\}\in\mathcal{D}$. The above calculation gives us that, given $t\in\mathbb{R}$ and $\epsilon>0$, there exists $s_0\leqslant t$ such that

$$\operatorname{dist}_H(T(t,s)B(s),A(t))<\epsilon,\ \text{ for all } s\leqslant s_0.$$

Now, the family B is pullback absorbing with respect to $\mathcal{D}$ and so, for the $s_0\leqslant t$ given above, there exists $s_1\leqslant 0$ such that $T(s_0,s)D(s)\subset B(s_0)$ for every $s\leqslant s_1$.

Thus, for $s\leqslant s_1$ we have

$$\begin{aligned}\operatorname{dist}_H(T(t,s)D(s),A(t))&=\operatorname{dist}_H(T(t,s_0)T(s_0,s)D(s),A(t))\\&\leqslant\operatorname{dist}_H(T(t,s_0)B(s_0),A(t))<\epsilon,\end{aligned}$$

which proves that A pullback attracts every family $D\in\mathcal{D}$.

The compactness of $A(t)$ follows since $A(t)\subset B(t)$ and $A(t)$ is closed, for every $t\in\mathbb{R}$.

It remains to show the invariance of the family $A=\{A(t):\ t\in\mathbb{R}\}$. Let $x\in A(s)$ and $t\geqslant s$. Then there are sequences $\{s_n\}_{n\geqslant 0}\subset(-\infty,s]$ and $\{x_n\}_{n\geqslant 0}$ such that $s_n\to-\infty$ as $n\to\infty$, $x_n\in B(s_n)$ for every $n\geqslant 0$ and $T(s,s_n)s_n\to x$ as $n\to\infty$. Using the continuity of $T(t,s)$, we have that

$$T(t,s_n)x_n=T(t,s)T(s,s_n)x_n\to T(t,s)x,\ \text{ as } n\to\infty,$$

which proves that $T(t,s)x \in A(t)$.

Now if $x \in A(t)$ and $s \leqslant t$, there exist sequences $\{s_n\}_{n\geqslant 0} \subset (\infty, t]$ and $\{x_n\}_{n\geqslant 0}$ such that $s_n \to -\infty$ as $n \to \infty$, $x_n \in B(s_n)$ for every $n \geqslant 0$ and $T(t,s_n)x_n \to x$ as $n \to \infty$. Since $s_n \to -\infty$ as $n \to \infty$, we can assume that $s_n \leqslant s$ for all $n \geqslant 0$.

We have then $T(t,s_n)x_n = T(t,s)T(s,s_n)x_n$ and since B is absorbing, we can also assume that the sequence $\{T(s,s_n)x_n\}_{n\geqslant 0}$ is contained in $B(s)$. But $B(s)$ is compact, and we can assume that there exists $y \in B(s)$ such that $T(s,s_n)x_n \to y$. Thus $y \in A(s)$ and, by the continuity of $T(t,s)$, we have $T(t,s)y = x$, which concludes the invariance of A and also the theorem. $\square$

We now introduce the concepts of local attractivity and repulsion, following [32] (see also [19]).

Definition 5.7 (Local attractivity)**.** *Let $\{T(t,s): \ t \geqslant s\}$ be a nonlinear evolution process in a Banach space X with a pullback attractor $\{\mathcal{A}(t): \ t \in \mathbb{R}\}$. A compact invariant family $\{A(t): \ t \in \mathbb{R}\}$ with $A(t) \subseteq \mathcal{A}(t)$ for every $t \in \mathbb{R}$ is called a* local pullback attractor *if there exists an $\eta > 0$ such that*

$$\lim_{s\to-\infty} \operatorname{dist}_H(T(t,s)\mathcal{O}_\eta(A(s)), A(t)) = 0, \ \textit{for all } t \in \mathbb{R},$$

where $\mathcal{O}_\eta(A(s)) \doteq \{x \in \mathcal{A}(s): \ \operatorname{dist}_H(x, A(s)) < \eta\}$, for $s \in \mathbb{R}$. The supremum of all $\eta > 0$ for which the above relation holds is called local pullback radius of attraction *of A.*

Remark 5.8. *We see that a local pullback attractor is a pullback attractor with respect to the attraction universe $\mathcal{D}$ defined by all the families $\{\mathcal{O}_\zeta(A(t)): \ t \in \mathbb{R}\}$ where $\zeta \in (0, \eta]$.*

In order to introduce the concept of local repeller, an injectivity condition of the evolution process over its pullback attractor will be necessary. Assume then that we have a nonlinear evolution process $\{T(t,s) : t \geqslant s\}$ in a Banach space X with pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$. Assume also that $T(t,s)|_{\mathcal{A}(s)} : \mathcal{A}(s) \to \mathcal{A}(t)$ is injective for every $t \geqslant s$. By the compactness of $\mathcal{A}(s)$, $T(t,s)|_{\mathcal{A}(s)} : \mathcal{A}(s) \to \mathcal{A}(t)$ is an homeomorphism for every $t \geqslant s$, its inverse is defined and is continuous. In this case, we write $T(s,t) = (T(t,s)|_{\mathcal{A}(s)})^{-1}$ for $s \leqslant t$ and we say that the evolution process is *invertible*.

Definition 5.9 (Local repulsion)**.** *Let $\{T(t,s): \ t \geqslant s\}$ be an invertible nonlinear evolution process in a Banach space X with a pullback attractor $\{\mathcal{A}(t): \ t \in \mathbb{R}\}$. A compact invariant family $A^* = \{A^*(t): \ t \in \mathbb{R}\}$ with $A^*(t) \subset \mathcal{A}(t)$ for every $t \in \mathbb{R}$ is called a* local repeller *if there exists $\eta > 0$ such that*

$$\lim_{s\to-\infty} \operatorname{dist}_H(T(s,t)\mathcal{O}_\eta(A^*(t)), A^*(s)) = 0, \ \textit{for all } t \in \mathbb{R}.$$

The supremum of all $\eta > 0$ such that the above relation holds is called local radius of repulsion *of A^*.*

Theorem 5.10 (Existence of attractor-repeller pairs)**.** *Let $\{T(t,s) : t \geqslant s\}$ be an invertible nonlinear evolution process with a pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ and $A^* = \{A^*(t) : t \in \mathbb{R}\}$ a local repeller. Then, there exists a uniquely determined local pullback attractor $A = \{A(t) : t \in \mathbb{R}\}$, which is the maximal local pullback attractor outside A^* in the sense that $A(t) \cap A^*(t) = \varnothing$ for all $t \in \mathbb{R}$ and any local pullback attractor $A' = \{A'(t) : t \in \mathbb{R}\}$ with $A' \supsetneq A$ has nonempty intersection with A^*; i.e., there exists $t \in \mathbb{R}$ such that $A'(t) \cap A^*(t) \neq \varnothing$. The pair (A, A^*) is called an* attractor-repeller pair.

Proof. Since A^* is a local repeller, if $\eta > 0$ is the local radius of repulsion of A^*, we have that

$$\lim_{s \to -\infty} \operatorname{dist}_H(T(s,t)\mathcal{O}_\eta(A^*(t)), A^*(s)) = 0, \text{ for all } t \in \mathbb{R}. \tag{5.4}$$

Consider the universe of attraction containing all the families D^ζ for $\zeta \in (0, \eta]$ which are defined by

$$D^\zeta(t) \doteq \mathcal{A}(t) \setminus \mathcal{O}_\zeta(A^*(t)), \text{ for all } t \in \mathbb{R}.$$

Now we will show that the family D^η is pullback absorbing with respect to $\mathcal{D}$ (note that D^η is a compact family). Choose $\zeta \in (0, \eta]$ and $t \in \mathbb{R}$ arbitrarily. Equation (5.4) gives us a $s_0 \leqslant t$ such that

$$\operatorname{dist}_H(T(s,t)\mathcal{O}_\eta(A^*(t)), A^*(s)) < \frac{\zeta}{2} \text{ for all } s \leqslant s_0,$$

which means that $T(s,t)\mathcal{O}_\eta(A^*(t)) \subseteq \mathcal{O}_{\zeta/2}(A^*(s))$ for all $s \leqslant s_0$. Thus, we obtain

$$\begin{aligned} T(s,t)D^\eta(t) &= T(s,t)(\mathcal{A}(t) \setminus \mathcal{O}_\eta(A^*(t))) \\ &= \mathcal{A}(s) \setminus T(s,t)\mathcal{O}_\eta(A^*(t)) \\ &\supseteq D^\zeta(s), \text{ for all } s \leqslant s_0. \end{aligned}$$

Applying $T(t,s)$ in both sides we obtain the relation $T(t,s)D^\zeta(s) \subseteq D^\eta(t)$ for all $s \leqslant s_0$ which proves that the family D^η is pullback absorbing with respect to $\mathcal{D}$.

Theorem 5.6 guarantees the existence of a pullback attractor $A = \{A(t) : t \in \mathbb{R}\}$ with respect to $\mathcal{D}$ with $A \subset D^\eta$. Now, since $A(t) \subseteq D^{\eta/2}$ for all $t \in \mathbb{R}$ we have that $B_{\eta/2}(A(t)) \subseteq D^\eta(t)$ for all $t \in \mathbb{R}$. But $D^\eta \in \mathcal{D}$ and since A pullback attracts D^η, A pullback attracts $\{B_{\eta/2}(A(t)) : t \in \mathbb{R}\}$, which shows that A is a local pullback attractor.

If $A' = \{A'(t) : t \in \mathbb{R}\}$ is another pullback attractor with $A' \supsetneq A$, there exists a $t_0 \in \mathbb{R}$ such that $A'(t_0) \supsetneq A(t_0)$. Let $x \in A'(t_0) \setminus A(t_0)$. Since A' is a local pullback attractor and $x_0 \in A'(t_0)$ (then $T(s,t_0)x \in A(t_0)$ for all $s \in \mathbb{R}$), there exists $\tilde{\eta} > 0$ such that

$$\lim_{s \to -\infty} \operatorname{dist}_H(T(t_0,s)\mathcal{O}_{\tilde{\eta}}(T(s,t_0)x), A'(t_0)) = 0.$$

But, by Theorem 5.11 (ii), $\lim\limits_{s\to-\infty} \mathrm{dist}_H(T(s,t_0)x, A^*(s)) = 0$, and so for each $n \geqslant 0$ we can find $s_n < -n$ and $x_n \in A^*(s_n)$ such that $\lim_{n\to\infty} \mathrm{dist}_H(T(s_n,t_0)x, x_n) < \tilde{\eta}$. In this way, $x_n \in \mathcal{O}_{\tilde{\eta}}(T(s_n,t_0)x)$ and

$$\lim_{n\to\infty} \mathrm{dist}_H(T(t_0,s_n)x_n, A'(t_0)) = 0.$$

Since $x_n \in A^*(s_n)$ and A^* is invariant, the sequence $\{T(t_0,s_n)\}_{n\geqslant 0}$ is in $A^*(t_0)$. By the compactness of $A^*(t_0)$ we can assume that there exists $z \in A^*(t_0)$ such that $T(t_0,s_n)x_n \to z$ as $n \to \infty$ and then it is clear that $\mathrm{dist}_H(z, A'(t_0)) = 0$, and by the compactness of $A'(t_0)$, we have that $z \in A'(t_0)$, which proves that $A'(t_0) \cap A^*(t_0) \neq \varnothing$ and completes the proof. $\square$

Theorem 5.11 (Dynamics of attractor-repeller pairs)**.** *Let $\{T(t,s) : t \geqslant s\}$ be an invertible nonlinear evolution process in a Banach space X with pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$. Let (A, A^*) be an attractor-repeller pair. The following statements hold:*

- **(i)** *There exists a $\beta > 0$ such that*
$$\mathcal{O}_\beta(A(t)) \cap \mathcal{O}_\beta(A^*(t)) = \varnothing, \text{ for all } t \in \mathbb{R}.$$
- **(ii)** *Let $t_0 \in \mathbb{R}$ be a fixed real number and $C \subseteq \mathcal{A}(t_0) \setminus A(t_0)$ a compact set. Then*
$$\lim_{s\to-\infty} \mathrm{dist}_H(T(s,t_0)C, A^*(s)) = 0.$$
- **(iii)** *Let $\{K(t) : t \in \mathbb{R}\}$ be a family of compact sets with $K(t) \subseteq \mathcal{A}(t)$ for all $t \in \mathbb{R}$ and $\liminf_{s\to-\infty} \mathrm{dist}_H(K(t), A^*(t)) > 0$. Then*
$$\lim_{s\to-\infty} \mathrm{dist}_H(T(t,s)K(s), A(t)) = 0, \text{ for all } t \in \mathbb{R}.$$

Proof. Let $\eta > 0$ and $\mathcal{D}$ as in the proof of Theorem 5.10; i.e., η is the local radius of repulsion of A^* and $\mathcal{D}$ is the attraction universe containing all the families D^ζ, with $\zeta \in (0, \eta]$.

- **(i)** It was shown in the proof of Theorem 5.10 that $A \subseteq D^\eta$. This assertion then follows by taking $\beta \doteq \frac{\eta}{2}$.
- **(ii)** Let $t_0 \in \mathbb{R}$ be a fixed real number, $C \subseteq \mathcal{A}(t_0) \setminus A(t_0)$ be a compact set and $\epsilon \in (0, \eta)$. Since A is a pullback attractor with respect to $\mathcal{D}$ and $D^\epsilon \in \mathcal{D}$, there exists $s_0 \leqslant t_0$ such that
$$\mathrm{dist}_H(T(t_0,s)D^\epsilon(s), A(t_0)) \leq \frac{\mathrm{dist}_H(C, A(t_0))}{2}, \text{ for all } s \leqslant s_0,$$
which implies that $C \cap T(t_0,s)D^\epsilon(s) = \varnothing$ for all $s \leqslant s_0$. Hence, $D^\epsilon(s) \cap T(s,t_0)C = \varnothing$ for all $s \leqslant s_0$, which means that $\mathrm{dist}_H(T(s,t_0)C, A^*(s)) < \epsilon$ for all $s \leqslant s_0$.
- **(iii)** Choose $\zeta < \min\{\eta, \liminf_{s\to-\infty} \mathrm{dist}_H(K(s), A^*(s))\}$. Since $\zeta < \liminf\limits_{s\to-\infty} \mathrm{dist}_H(K(s), A^*(s))$, there exists $s_0 \in \mathbb{R}$ such that
$$\mathrm{dist}_H(K(s), A^*(s)) \geqslant \zeta, \text{ for all } s \leqslant s_0,$$

which implies that $K(s) \subset D^{\zeta}(s)$ for all $s \leqslant s_0$. This finishes the proof since $D^{\zeta} \in \mathcal{D}$ and A is a pullback attractor with respect to $\mathcal{D}$.

□

Proposition 5.12 (Nonuniqueness of attractor-repeller pairs)**.** *Let $\{T(t,s): \ t \geqslant s\}$ be an invertible nonlinear evolution process with a pullback attractor $\{\mathcal{A}(t): \ t \in \mathbb{R}\}$. Let A^* and R^* be two local repellers such that their corresponding attractors, A and R respectively, are equal; i.e., $A = R$. Then,*

$$\lim_{s\to-\infty} \operatorname{dist}_H(A^*(s), R^*(s)) = 0.$$

Proof. Arguing by contradiction, assume that there are sequences $\{s_n\}_{n\geqslant 0} \subseteq \mathbb{R}$, $\{x_n\}_{n\geqslant 0}$ and $\epsilon > 0$ such that $s_n \to -\infty$ as $n \to \infty$, $x_n \in A^*(s_n)$ and

$$\operatorname{dist}_H(x_n, R^*(s_n)) \geqslant \epsilon, \text{ for all } n \geqslant 0.$$

Applying Theorem 5.11 (iii) for the attractor-repeller pair (R, R^*), since

$$\liminf_{n\to\infty} \operatorname{dist}_H(x_n, R^*(s_n)) \geqslant \epsilon > 0,$$

we have that

$$\lim_{n\to\infty} \operatorname{dist}_H(T(0,s_n)x_n, R(0)) = 0.$$

Since $T(0,s_n)x_n \in R^*(0)$ for all $n \geqslant 0$ and both $A^*(0)$ and $A(0) = R(0)$ are compact sets, it follows that $R(0) \cap R^*(0) \neq \varnothing$, which is a contradiction and proves the result. □

6. Morse decomposition for nonlinear evolution processes

The definition of a Morse decomposition via finite attractor-repeller pair sequence is basically the same as in the autonomous case.

Definition 6.1. *Let $\{T(t,s): \ t \geqslant s\}$ be an invertible nonlinear evolution process in a Banach space X with pullback attractor $\{\mathcal{A}(t): \ t \in \mathbb{R}\}$. Assume that there exists a sequence of attractor-repeller pairs $(A_i(t), A_i^*(t))$, for $i = 0, \cdots, n$, satisfying*

$$\varnothing = A_n^*(t) \subsetneq A_{n-1}^*(t) \subsetneq \cdots \subsetneq A_0^*(t) = \mathcal{A}(t),$$

for all $t \in \mathbb{R}$, and also

$$\varnothing = A_0(t) \subsetneq A_1(t) \subsetneq \cdots \subsetneq A_n(t) = \mathcal{A}(t),$$

for all $t \in \mathbb{R}$. The collection $\{M_1, M_2, \cdots, M_n\}$ defined by

$$M_i(t) \doteq A_i(t) \cap A_{i-1}^*(t), \text{ for all } t \in \mathbb{R} \text{ and } i \in \{1, \cdots, n\}$$

is called a Morse decomposition*. Each family $\{M_i(t): \ t \in \mathbb{R}\}$ is called a* Morse set*.*

Note here that, unlike the autonomous case, we need to impose the condition $\varnothing = A_0(t) \subsetneq A_1(t) \subsetneq \cdots \subsetneq A_n(t) = \mathcal{A}(t)$ on the local pullback attractors, since Proposition 5.12 indicates that local pullback attractors of the attractor sequence may coincide.

The definition of a Morse decomposition is a generalization of an attractor-repeller pair in the sense that, if (A, A^*) is an attractor-repeller pair such that $\varnothing \subsetneq A \subsetneq \mathcal{A}$, then $\{A, A^*\}$ is a Morse decomposition.

We now present a proposition that summarizes the general properties of a Morse decomposition.

Proposition 6.2. *Let $\{T(t,s) : \ t \geqslant s\}$ be an invertible nonlinear evolution process in a Banach space X with a pullback attractor $\{\mathcal{A}(t) : \ t \in \mathbb{R}\}$. The Morse sets of a Morse decomposition $\{M_1, \cdots, M_n\}$ are nonempty, invariant and isolated; i.e., there exists a $\beta > 0$ such that, for $i \neq j$*

$$\mathcal{O}_\beta(M_i(t)) \cap \mathcal{O}_\beta(M_j(t)) = \varnothing, \text{ for all } t \in \mathbb{R}, \text{ and } i \neq j.$$

Proof. Firstly, choose an arbitrary Morse set $M_i = A_i \cap A^*_{i-1}$. Since $A_{i-1} \subsetneq A_i$ there exist $t_0 \in \mathbb{R}$ and a point $x \in A_i(t_0) \setminus A_{i-1}(t_0)$. But $x \in A_i(t_0)$ and, by the invariance, $T(s,t_0)x \in A_i(s)$ for all $s \leqslant t_0$ and since A_i is a local pullback attractor, for $\eta > 0$ being the local radius of attraction, we have that

$$\lim_{s \to -\infty} \operatorname{dist}_H(T(t_0,s)\mathcal{O}_\eta(T(s,t_0)x), A_i(t_0)) = 0.$$

Theorem 5.11 (ii) gives that $\lim\limits_{s \to -\infty} \operatorname{dist}_H(T(s,t_0)x, A^*_{i-1}(s)) = 0$, and we can construct a sequence $\{y_n\}_{n \geqslant 0}$ in $A^*_{i-1}(t_0)$ with

$$\lim_{n \to \infty} \operatorname{dist}_H(y_n, A^*_i(t_0)) = 0.$$

Since $A^*_{i-1}(t_0)$ and $A_i(t_0)$ are both compact, we have that $M_i(t_0) = A_i(t_0) \cap A^*_{i-1}(t_0) \neq \varnothing$.

Furthermore, $T(t,s)M_i(s) = T(t,s)A_i(s) \cap T(t,s)A^*_{i-1}(s) = A_i(t) \cap A^*_{i-1}(t) = M_i(t)$ for all $t \in \mathbb{R}$ and $i \in \{1, \cdots, n\}$ since $T(t,s)$ is injective in $\mathcal{A}(s)$.

Choose now another Morse set M_j. We can assume without loss of generality that $j > i$ and then

$$\begin{aligned} M_i \cap M_j &= A_i \cap A^*_{i-1} \cap A_j \cap A^*_{j-1} \\ &= A_i \cap A_j \cap A^*_{i-1} \cap A^*_{j-1} \\ &= A_i \cap A^*_{j-1} \subseteq A_{j-1} \cap A^*_{j-1} \\ &= \varnothing. \end{aligned}$$

Finally the isolation property is a straightforward consequence of Theorem 5.11 (i). $\square$

The Morse decompositions are not uniquely defined, as in the autonomous case.

Definition 6.3. *A Morse decomposition* $\{M_1, \cdots, M_n\}$ *is said to be* finer *than the Morse decomposition* $\{N_1, \cdots, N_m\}$ *if*

$$\lim_{s\to-\infty} \mathrm{dist}_H(\cup_{i=1}^n M_i(s), \cup_{i=1}^m N_i(s)) = 0.$$

Remark 6.4. *Let* $\{M_1, \cdots, M_n\}$ *be a Morse decomposition given by the local repellers*

$$\varnothing = A_n^* \subsetneq A_{n-1}^* \subsetneq \cdots \subsetneq A_0^* = \mathcal{A},$$

and its corresponding local pullback attractors

$$\varnothing = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = \mathcal{A}.$$

Assume we have a new local repeller B^* *and its corresponding local attractor* B *satisfying*

$$\varnothing = A_n^* \subsetneq A_{n-1}^* \subsetneq \cdots \subsetneq A_i^* \subsetneq B^* \subsetneq A_{i-1}^* \subsetneq \cdots \subsetneq A_0^* = \mathcal{A},$$

and

$$\varnothing = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_{i-1} \subsetneq B \subsetneq A_i \subsetneq \cdots \subsetneq A_n = \mathcal{A},$$

Then, the Morse decomposition $\{M_1, \cdots, M_n\}$ *is finer than the Morse decomposition defined by the new sequence, and this is seen simply noting that* $M_i = A_i \cap A_{i-1}^* \subset B \cap A_{i-1}^*$.

The next result shows the importance of the Morse sets for the asymptotic behaviour of a nonautonomous dynamical system.

Theorem 6.5. *Let* $\{T(t,s) : \ t \geqslant s\}$ *be a nonlinear evolution process in a Banach space* X *with a global attractor* $\{\mathcal{A}(t) : \ t \in \mathbb{R}\}$*, and* $\{M_1, \cdots, M_n\}$ *be a Morse decomposition obtained by the sequence of local repellers* $\mathcal{A} = A_0^* \supsetneq \cdots \supsetneq A_n^* = \varnothing$*. Then, all families* $\{\gamma(t) : t \in \mathbb{R}\}$ *of points for which* $\gamma(t) \in \mathcal{A}(t)$ *for all* $t \in \mathbb{R}$ *and* $\liminf_{s\to-\infty} \mathrm{dist}_H(\gamma(s), \cup_{j=1}^n \partial A_i^*(s)) > 0$ *satisfy*

$$\lim_{s\to-\infty} \mathrm{dist}_H(T(t,s)\gamma(s), \cup_{i=1}^n M_j(t)) = 0, \ \textit{for all } t \in \mathbb{R}.$$

Proof. Fix $t \in \mathbb{R}$. By the above remark, without loss of generality, we can assume that there exists an $i \in \{1, \cdots, n\}$ with

$$\gamma(s) \in A_{i-1}^*(s) \text{ and } \gamma(s) \notin A_i^*(s), \text{ for all } s \in \mathbb{R}.$$

Then $\lim_{s\to-\infty} \mathrm{dist}_H(\gamma(s), \partial A_i^*(s)) > 0$ implies that $\lim_{s\to-\infty} \mathrm{dist}_H(\gamma(s), A_i^*(s)) > 0$. Theorem 5.11 (iii) implies that

$$\lim_{s\to-\infty} \mathrm{dist}_H(T(t,s)\gamma(s), A_i(t)) = 0. \tag{6.1}$$

We now show that, in fact, $\lim_{s\to-\infty} \mathrm{dist}_H(T(t,s)\gamma(s), M_i(t)) = 0$. Assume that there are $\epsilon > 0$ and a sequence $\{s_n\}_{n\geqslant 0} \subset (-\infty, t]$ with $s_n \to -\infty$ as $n \to \infty$, and

$$\mathrm{dist}_H(T(t,s_n)\gamma(s_n), M_i(t)) \geqslant \epsilon, \text{ for all } n \geqslant 0.$$

Since $A^*_{i-1}(t)$ is compact, we can assume that the sequence $\{T(t,s_n)\gamma(s_n)\}_{n\geqslant 0} \subset A^*_{i-1}(t)$ converges to a point $x \in A^*_{i-1}(t)$. Moreover, by (6.1), $x \in A_i(t)$. Thus, $x \in M_i(t)$ and

$$0 = \mathrm{dist}_H(x, M_i(t)) = \lim_{n\to\infty} \mathrm{dist}_H(T(t,s_n)\gamma(s_n), M_i(t)) \geqslant \epsilon,$$

which is a contradiction and proves the result. □

To finish this section we show a result of uniqueness of the local pullback attractors in a Morse decomposition under stronger convergence hypotheses.

Proposition 6.6. *Let $\{T(t,s) : t \geqslant s\}$ be an invertible nonlinear evolution process with pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ and $\{M_1, \cdots, M_n\}$ be a Morse decomposition obtained by the finite sequence of local repellers*

$$\mathcal{A} = A^*_0 \supsetneq A^*_1 \supsetneq \cdots \supsetneq A^*_n = \varnothing.$$

Moreover, assume that for all $t \in \mathbb{R}$ and $x \in \mathcal{A}(t)$ there is an $i \in \{1, \cdots, n\}$ with

$$\lim_{s\to-\infty} \mathrm{dist}_H(T(s,t)x, M_i(s)) = 0.$$

Then, the representation

$$A_i(t) = \{x \in \mathcal{A}(t) : \lim_{s\to-\infty} \mathrm{dist}_H(T(s,t)x, \cup_{j=1}^i M_j(s)) = 0\}$$

holds for all $i \in \{1, \cdots, n\}$; i.e., the local pullback attractors of the Morse decomposition are uniquely defined.

Proof. ($\subseteq$) Let $t \in \mathbb{R}$ be a fixed real number and $x \in A_i(t)$. By the hypotheses, choose $j \in \{1, \cdots, n\}$ such that

$$\lim_{s\to-\infty} \mathrm{dist}_H(T(s,t)x, M_j(t)) = 0.$$

Then,

$$\lim_{s\to-\infty} \mathrm{dist}_H(T(s,t)x, A^*_{j-1}(t)) \leqslant \lim_{s\to-\infty} \mathrm{dist}_H(T(s,t)x, M_j(t)) = 0.$$

Now, if $j > i$, then $\lim_{s\to-\infty} \mathrm{dist}_H(T(s,t)x, A^*_i(t)) = 0$ since $A^*_{j-1}(t) \subset A^*_i(t)$, which contradicts Theorem 5.11 (i) since $T(s,t)x \in A_i(s)$ for all $s \in \mathbb{R}$.

($\supseteq$) Fix $t \in \mathbb{R}$ and let $x \in \mathcal{A}(t) \setminus A_i(t)$. Then, Theorem 5.11 (ii) implies

$$\lim_{s\to-\infty} \mathrm{dist}_H(T(s,t)x, A^*_i(s)) = 0. \tag{6.2}$$

If $\lim_{s\to-\infty} \mathrm{dist}_H(T(s,t)x, \cup_{j=1}^i M_j(s)) = 0$, then

$$\lim_{s\to-\infty} \mathrm{dist}_H(T(s,t)x, A_i(s)) = 0,$$

since $M_j \subset A_i$ for $j \in \{1, \cdots, i\}$, which contradicts (6.2) in view of Theorem 5.11 (i). □

Example 6.7. *Consider the differential equation*

$$x'(t) = x(t) - b(t)x^3, \quad x(s) = x_0,$$

with $0 < b_0 \leq b(t) \leq B_0$*, for all* $t \in \mathbb{R}$*. This equation can be solved explicitly, so that we get*

$$x^2(t,s;x_0) = \frac{e^{2t}}{e^{2s}x_0^{-2} + \int_s^t 2e^{2r}b(r)dr}.$$

Thus, the pullback attractor is given by $\mathcal{A}(t) = [-\xi(t), \xi(t)]$*, with*

$$\xi^2(t) = \frac{e^{2t}}{\int_{-\infty}^t 2e^{2r}b(r)dr}.$$

Note that 0*,* $\xi(t) \in [1/\sqrt{B_0}, 1/\sqrt{b_0}]$ *and* $\xi(t) \in [-1/\sqrt{b_0}, -1/\sqrt{B_0}]$ *are global solutions with special stability properties. In particular,* 0 *is a local repeller with* $\eta = \inf_{t\in\mathbb{R}} \xi(t) \geq 1/\sqrt{B_0}$ *and, by Theorem 5.10,* $D^\eta(t) = \mathcal{A}(t)\mathcal{O}_\eta(0)$ *is a pullback absorbing set with associated pullback attractor* $\mathcal{A}(t) = \{-\xi(t), \xi(t)\}$*. As a consequence, we observe that*

$$A_0^*(t) = \mathcal{A}(t),\ A_1^*(t) = \{0\},\ A_2^*(t) = \emptyset,$$

$$A_0(t) = \emptyset,\ A_1(t) = \{-\xi(t), \xi(t)\},\ A_2(t) = \mathcal{A}(t),$$

are the corresponding family of repeller and local attractors, so that we get

$$M_1(t) = \{-\xi(t), \xi(t)\},\ M_2(t) = \{0\}$$

as the associated Morse decomposition for this example.

7. An estimate on the fractal dimension of pullback attractors

Let us begin this section by stating an abstract result concerning the fractal dimension of a pullback attractor of a nonlinear evolution process.

Proposition 7.1. *Let* $\{T(t,s) : t \geqslant s\}$ *be an invertible nonlinear evolution process with a pullback attractor* $\{\mathcal{A}(t) : t \in \mathbb{R}\}$*. Let also* $\{A^*(t) : t \in \mathbb{R}\}$ *be a local repeller in* $\mathcal{A}$ *and* $\{A(t) : t \in \mathbb{R}\}$ *its corresponding local pullback attractor. Assume that the following conditions are satisfied*

- **(a)** *There is a constant* $L > 1$ *such that, for all* $t \in \mathbb{R}$*,* $T(t,t-1)|_{\mathcal{A}(t-1)}$ *is a Lipschitz mapping with Lipschitz constant* L*;*
- **(b)** *There are a family* $\{B(t) : t \in \mathbb{R}\}$ *and constants* $c_1, c_2 \geqslant 0$ *such that* $B(t)$ *is a neighborhood of* $A^*(t)$ *in* $\mathcal{A}(t)$ *for all* $t \in \mathbb{R}$*,* $\overline{B(t)} \cap A(t) = \varnothing$ *and*

$$c_1 \leqslant c(B(t)) \leqslant c_2, \ \textit{for all}\ t \in \mathbb{R};$$

(c) *There are constants $M,\omega>0$ such that if $\{K(t):\ t\in\mathbb{R}\}$ is a family of compact sets with $K(t)\subset\mathcal{A}(t)$ and $K(t)\cap A^*(t)=\varnothing$, for all $t\in\mathbb{R}$ then*

$$\mathrm{dist}_H(T(t,s)K(s),A(t))\leqslant Me^{\omega(s-t)},\ \textit{for all } s\leqslant t.$$

Then, for all $t\in\mathbb{R}$, we have that

$$c_1\leqslant c(\mathcal{A}(t))\leqslant \max\left\{\frac{\omega+\ln L}{\omega}c_2,c(A(t))\right\}.$$

Proof. Let us fix $t\in\mathbb{R}$ and, for $n\in\mathbb{N}$, we define the compact sets

$$K_n\doteq\mathcal{A}(t-n)\setminus B(t-n)$$

and also we define subsets $\tilde{K}_n\subset K_n$ by

$$\tilde{K}_n\doteq K_n\setminus T(t-n,t-n-1)K_{n+1},\ \text{for } n\in\mathbb{N}.$$

Clearly we have that $K_n\subset\mathcal{A}(t-n)$ and $K_n\cap A^*(t-n)=\varnothing$ for all $n\in\mathbb{N}$.

We also note that if $z\in\tilde{K}_n$ then $z\in K_n$, but $z\notin T(t-n,t-n-1)K_{n+1}$. However $z\in\mathcal{A}(t-n)=T(t-n,t-n-1)\mathcal{A}(t-n-1)$, and $\mathcal{A}(t-n-1)=(\mathcal{A}(t-n-1)\setminus B(t-n-1))\cup B(t-n-1)$ and hence $z\in T(t-n,t-n-1)B(t-n-1)$. Thus, $\tilde{K}_n\subset T(t-n,t-n-1)B(t-n-1)$.

By the precedent estimates we have that

$$c(\tilde{K}_n)\leqslant c(T(t-n,t-n-1)B(t-n-1))\leqslant c(B(t-n-1)),$$

because $T(t-n,t-n-1)|_{\mathcal{A}(t-n-1)}$ is a Lipschitz mapping for every $n\in\mathbb{N}$, and, in this way we obtain $c(\tilde{K}_n)\leqslant c_2$, for all $n\in\mathbb{N}$.

Now, let us define Ω_n by

$$\Omega_n\doteq T(t,t-n)\tilde{K}_n,\ \text{for all } n\in\mathbb{N}.$$

Also, since $\tilde{K}_n\subset K_n$, by the hypotheses **(c)** we have that

$$\mathrm{dist}_H(\Omega_n,A(t))\leqslant Me^{-\omega n},\ \text{for all } n\in\mathbb{N}.$$

Claim: It holds that $\mathcal{A}(t)\setminus B(t)\subset(\cup_{n\geqslant 0}\Omega_n)\cup A(t)$.

Indeed, let $x\in\mathcal{A}(t)\setminus B(t)$. We have two possibilities for x:

(i) $x\notin T(t,t-1)K_1$ and, in this case, $x\in\Omega_0=(\mathcal{A}(t)\setminus B(t))\setminus T(t,t-1)K_1$;

(ii) $x\in T(t,t-1)K_1$ and, in this case, there is $y_1\in K_1$ such that $x=T(t,t-1)y_1$.

For y_1 we also have two possibilities

(iii) $y_1\notin T(t-1,t-2)K_2$ and, in this case, $y_1\in\tilde{K}_1$ and $x=T(t,t-1)y_1\in T(t,t-1)\tilde{K}_1=\Omega_1$;

(iv) $y_1 \in T(t-1,t-2)K_2$ and, in this case, there is $y_2 \in K_2$ such that $y_1 = T(t-1,t-2)y_2$ and so $x = T(t,t-2)y_2$.

Now, applying this reasoning inductively, we obtain two possibilities for x: either $x \in \Omega_n$ for some $n \in \mathbb{N}$ or there is a sequence $\{y_n\}_{n\in\mathbb{N}}$ satisfying $y_n \in K_n$, $x = y_0$ and $y_n = T(t-n,t-n-1)y_{n+1}$ for all $n \in \mathbb{N}$ (and so $x = T(t,t-n)y_n$ for all $n \in \mathbb{N}$).

In the first possibility, clearly we have $x \in \cup_{n\geqslant 0}\Omega_n$. Now, if the second possibility happens, using the hypothesis **(c)**, we have for all $n \in \mathbb{N}$:

$$\mathrm{dist}_H(x, A(t)) = \mathrm{dist}_H(T(t,t-n)y_n, A(t)) \leqslant \mathrm{dist}_H(T(t,t-n)K_n, A(t)) \leqslant Me^{-\omega n},$$

and it follows that $\mathrm{dist}_H(x, A(t)) = 0$ and, since $A(t)$ is compact, $x \in A(t)$, which concludes the proof of our claim.

We define now, for every $r > 0$ and $k \geqslant 0$,

$$N_{r,k} \doteq N\left(r/c^k, \tilde{K}_k\right);$$

i.e., there are $x_1^k, \cdots, x_{N_{r,k}^k} \in \tilde{K}_k$ such that $\tilde{K}_k \subset \cup_{i=1}^{N_{r,k}} B(x_i^k, r/L^k)$.

In this way, if $z \in \Omega_k$, then there is $x \in \tilde{K}_k$ such that $z = T(t,t-k)x$, and there is $i \in \{1,\cdots,N_{r,k}\}$ such that $\|x - x_i^k\| < r/L^k$. Now, if we define $\xi_i^k \doteq T(t,t-k)x_i^k$ for all $i = 1,\cdots,N_{r,k}$, we have

$$\|z - \xi_i^k\| = \|T(t,t-k)x - T(t,t-k)x_i^k\| \leqslant L^k\|x - x_i^k\| < r,$$

thus $\Omega_k \subset \cup_{i=1}^{N_{r,k}} B(\xi_i^k, r)$ and so $N(r,\Omega_k) \leqslant N_{r,k}$.

With the same arguments used in the autonomous case, namely in Proposition 3.4, we know that, from hypothesis **(c)**, if $n(r) \doteq \lceil \frac{1}{\omega}\ln\left(\frac{M}{r}\right)\rceil$ and $G(r) \doteq \left(\cup_{i\geqslant n(r)+1}\Omega_i\right) \cup A(t)$, we have that $G(r) \subset \mathcal{O}_r(A(t))$ and hence $N(2r, G(r)) \leqslant N(2r, A(t)) \leqslant N(r, A(t))$.

If we define now $H(r) \doteq \cup_{i=0}^{n(r)}\Omega_i$ it follows that

$$N(r, H(r)) \leqslant n(r). \max_{i=0,\cdots,n(r)} N(r\Omega_i) \leqslant n(r). \max_{i=0,\cdot,n(r)} N_{r,i},$$

where $N_{r,i} = N(r/L^i, \tilde{K}_i)$.

From the previous claim we see that $\mathcal{A}(t) = B(t) \cup G(r/2) \cup H(r/2)$ for every $r > 0$, and therefore

$$\begin{aligned} N(r, \mathcal{A}(t)) \leqslant& 3\max\{N(r, B(t)); N(r, G(r/2)); N(r, H(r/2))\} \\ \leqslant& 3\max\{N(r, B(t)); N(r, G(r/2)); N(r/2, H(r/2))\} \\ \leqslant& \max\left\{N(r, B(t)); N(r/2, A(t)); n(r/2). \max_{i=0,\cdots,n(r/2)} N_{r/2,i}\right\}. \end{aligned}$$

Since the logarithm function is increasing we have, choosing $r > 0$ small so that $\ln(1/r) > 0$,

$$\begin{aligned}\frac{\ln N(r, \mathcal{A}(t))}{\ln(1/r)} \leqslant \frac{\ln 3}{\ln(1/r)} + \max \Bigg\{ & \frac{\ln N(r, B(t))}{\ln(1/r)}; \frac{\ln N(r/2, A(t))}{\ln(1/r)}; \\ & \frac{\ln n(r/2)}{\ln(1/r)} + \max_{i=0,\cdots,n(r/2)} \frac{\ln N_{r/2,i}}{\ln(1/r)} \Bigg\}.\end{aligned} \tag{7.1}$$

We now compute the last term on the right hand side of (7.1):

$$\begin{aligned}\frac{\ln N_{r/2,i}}{\ln(1/r)} &= \frac{\ln N(r/2L^i, \tilde{K}_i)}{\ln(1/r)} \\ &= \frac{\ln N(r/2L^i, \tilde{K}_i)}{\ln(2L^i/r)}. \left(\frac{i \ln L + \ln 2 + \ln(1/r)}{\ln(1/r)} \right) \\ &\leqslant \frac{\ln N(r/2L^i, \tilde{K}_i)}{\ln(2L^i/r)}. \left(\frac{n(r/2)}{\ln(1/r)} \ln L + \frac{\ln 2}{\ln(1/r)} + 1 \right),\end{aligned}$$

and using the calculation from Proposition 3.4, taking lim sup for $r \to 0^+$ in both sides of (7.1), we have that

$$c(\mathcal{A}(t)) \leqslant \max \left\{ c(B(t)), c(A(t)), \frac{\omega + \ln L}{\omega}. \sup_{n \geqslant 0} c(\tilde{K}_n) \right\}$$

and thus

$$c(\mathcal{A}(t)) \leqslant \max \left\{ c(A(t)), \frac{\omega + \ln L}{\omega}.c_2 \right\}.$$

The first inequality is straightforward and we conclude the proof of this proposition. □

Corollary 7.2. *Let $\{T(t,s) : t \geqslant s\}$ be an invertible nonlinear evolution process in a Banach space X with pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$. Assume that the process possesses a Morse decomposition $\{M_1, \cdots, M_n\}$ given by the finite sequence of local repellers $\mathcal{A} = A_0^* \supsetneq A_1^* \supsetneq \cdots \supsetneq A_n^* = \varnothing$, and its associated local pullback attractors $\varnothing = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = \mathcal{A}$. Assume that the following conditions hold:*

- **(a)** *There is a constant $L > 1$ such that, for all $t \in \mathbb{R}$, $T(t, t-1)|_{\mathcal{A}(t-1)}$ is a Lipschitz mapping with Lipschitz constant L;*
- **(b)** *For each $i \in \{1, \cdots, n\}$ there is a family $\{B_i(t) : t \in \mathbb{R}\}$ such that $B_i(t)$ is a neighbourhood of $M_i(t)$ in $A_i(t)$ for all $t \in \mathbb{R}$, $\overline{B_i(t)} \cap A_{i-1}(t) = \varnothing$ and assume also that there exist constants c_1, c_2, independent of i, such that*

$$c_1 \leqslant c(B_i(t)) \leqslant c_2, \text{ for all } t \in \mathbb{R} \text{ and } i \in \{1, \cdots, n\};$$

where we set $M_{n+1} = \mathcal{A}$.

(c) *There are constants $M, \omega > 0$ such that if $\{K(t) : t \in \mathbb{R}\}$ is a family of compact sets with $K(t) \subset A_i(t)$ and $K(t) \cap M_i(t) = \varnothing$, for all $t \in \mathbb{R}$ then*

$$\operatorname{dist}_H(T(t,s)K(s), A_{i-1}(t)) \leqslant Me^{\omega(s-t)}, \text{ for all } s \leqslant t \text{ and } i \in \{1, \cdots, n\}.$$

Then, for all $t \in \mathbb{R}$, we have that

$$c_1 \leqslant c(\mathcal{A}(t)) \leqslant \frac{\omega + \ln L}{\omega} c_2.$$

Proof. Firstly, on account of hypothesis **(b)** $i = n$, we have that there is a family $\{B_n(t) : t \in \mathbb{R}\}$ such that $B_n(t)$ is a neighborhood of $M_n(t) = A_n(t) \cap A^*_{n-1}(t) = \mathcal{A}(t) \cap A^*_{n-1}(t) = A^*_{n-1}(t)$ in $A_n(t) = \mathcal{A}(t)$ for all $t \in \mathbb{R}$, $\overline{B_n(t)} \cap A_{n-1}(t) = \varnothing$, and

$$c_1 \leqslant c(B_n(t)) \leqslant c_2, \text{ for all } t \in \mathbb{R}.$$

Hypothesis **(c)**, for $i = n$, implies that

$$\operatorname{dist}_H(T(t,s)K(s), A_{n-1}(t)) \leqslant Me^{\omega(s-t)}, \text{ for all } s \leqslant t,$$

for every family $\{K(t) : t \in \mathbb{R}\}$ of compact sets satisfying $K(t) \subset A_n(t) = \mathcal{A}(t)$ and $K(t) \cap M_n(t) = K(t) \cap A^*_{n-1}(t) = \varnothing$ for all $t \in \mathbb{R}$. Then, we can apply Proposition 7.1 to obtain

$$c_1 \leqslant c(\mathcal{A}(t)) \leqslant \max\left\{c(A_{n-1}(t)), \frac{\omega + \ln L}{\omega}.c_2\right\}, \text{ for all } t \in \mathbb{R}.$$

Now define $S(t,s) \doteq T(t,s)|_{A_{n-1}(s)}$ for all $s \leqslant t$. Note that the important fact in Proposition 7.1 is that the process is defined on a compact invariant family $\{\mathcal{A}(t) : t \in \mathbb{R}\}$, and it does not matter if this family is a pullback attractor or not. Then, we can apply this proposition to the invertible evolution process $\{S(t,s) : t \geqslant s\}$ as long as we can verify the hypotheses.

To check the hypotheses we take $i = n-1$. We have the following:

(i) The pair (A_{n-2}, M_{n-1}) is an attractor-repeller pair of the evolution process $\{S(t,s) : t \geqslant s\}$, since $A_{n-2}(t) \subset A_{n-1}(t)$ and $M_{n-1}(t) = A_{n-1}(t) \cap A^*_{n-2}(t)$ for all $t \in \mathbb{R}$;

(ii) $S(t, t-1)$ is a Lipschitz map with constant $L > 1$ for all $t \in \mathbb{R}$;

(iii) There is a family $\{B_{n-1}(t) : t \in \mathbb{R}\}$ such that $B_{n-1}(t)$ is a neighborhood of $M_{n-1}(t)$ in $A_{n-1}(t)$ for all $t \in \mathbb{R}$, $\overline{B_{n-1}(t)} \cap A_{n-2}(t) = \varnothing$ and

$$c_1 \leqslant c(B_{n-1}(t)) \leqslant c_2, \text{ for all } t \in \mathbb{R};$$

(iv) Hypothesis **(c)**, for $i = n-1$, implies that

$$\operatorname{dist}_H(T(t,s)K(s), A_{n-2}(t)) \leqslant Me^{\omega(s-t)}, \text{ for all } s \leqslant t,$$

for every family $\{K(t) : \ t \in \mathbb{R}\}$ of compact sets satisfying $K(t) \subset A_{n-1}(t) = \mathcal{A}(t)$ and $K(t) \cap M_{n-1}(t) = \varnothing$ for all $t \in \mathbb{R}$.

Hence, we can apply Proposition 7.1 to the process $\{T(t,s) : \ t \geqslant s\}$ defined in the compact invariant family $\{A_{n-1}(t) : \ t \in \mathbb{R}\}$ and the attractor-repeller pair (A_{n-2}, M_{n-1}) to deduce

$$c_1 \leqslant c(A_{n-1}(t)) \leqslant \max\left\{c(A_{n-2}(t)), \frac{\omega + \ln L}{\omega}.c_2\right\}, \text{ for all } t \in \mathbb{R}.$$

Joining these two results we obtain

$$c_1 \leqslant c(\mathcal{A}(t)) \leqslant \max\left\{c(A_{n-2}(t)), \frac{\omega + \ln L}{\omega}.c_2\right\}, \text{ for all } t \in \mathbb{R}.$$

Arguing now inductively, since $A_0(t) = \varnothing$ for all $t \in \mathbb{R}$, we finally arrive at

$$c_1 \leqslant c(\mathcal{A}(t)) \leqslant \frac{\omega + \ln L}{\omega}.c_2, \text{ for all } t \in \mathbb{R}.$$

□

8. Example

To illustrate our results, consider the following non-autonomous logistic equation

$$u_t = u_{xx} + \lambda u - \beta(t)u^3, \tag{8.1}$$

for $x \in [0, \pi]$ with Dirichlet boundary conditions. We assume that there are positive constants β_1, β_2 such that $0 < \beta_1 \leq \beta(t) \leq \beta_2$ for all $t \in \mathbb{R}$. The existence of global pullback attractors for this equation is already known (see, for instance, [22]).

We consider the positive cone within $H_0^1(0, \pi)$,

$$\mathcal{V}^+ = \{u \in H_0^1(0, \pi) : \ u(x) \geq 0 \text{ for a.e. } x \in \Omega\}.$$

For (8.1), we can define an order with respect to $\mathcal{V}^+$. That is, $u_0 \leq v_0$ if $v_0 - u_0 \in \mathcal{V}^+$.

In order to investigate further the behaviour of positive solutions the following definition ([23]) is crucial.

Definition 8.1. *A positive function with values in $C(\overline{\Omega})$ is non–degenerate at ∞ (respectively $-\infty$) if there exists $t_0 \in \mathbb{R}$ such that u is defined in $[t_0, \infty)$ (respectively $(-\infty, t_0]$) and there exists a $C^1(\overline{\Omega})$ function $\varphi_0(x) > 0$ in Ω, (vanishing on $\partial\Omega$ in case of Dirichlet boundary conditions) and satisfying $\frac{\partial \varphi_0}{\partial n} < 0$, such that*

$$u(t, x) \geq \varphi_0(x) \qquad \text{for all} \quad x \in \Omega \text{ and all } t \geq t_0$$

(respectively for all $t \leq t_0$).

From [23], we know that there exist two extremal (minimal and maximal) bounded global solutions, $\xi_m(\cdot)$ and $\xi_M(\cdot)$ for (8.1), i.e. if $\psi(\cdot)$ is any bounded global solution for $S(t,s)$ then

$$\xi_m(t) \leq \psi(t) \leq \xi_M(t), \text{ for all } t \in \mathbb{R}.$$

Moreover, (8.1) has a pullback attractor $A(t)$ with

$$A(t) \subset [\xi_m(t), \xi_M(t)], \text{ for all } t \in \mathbb{R},$$

with $\xi_m(t), \xi_M(t) \in A(t)$ for all $t \in \mathbb{R}$.

As a direct application of the results in [23], [24] we obtain the following description of the pullback attractor within the positive cone.

Theorem 8.2.

a) *If $\lambda < \lambda_1$ then $\xi_M(t) \equiv 0$ for all $t \in \mathbb{R}$.*
b) *If $\lambda > \lambda_1$ then $\xi_M(t)$ is strictly positive and is the unique non-degenerate global solution at $-\infty$ and $+\infty$.*
c) *The pullback attractor for* (8.1) *in the positive cone satisfies $\mathcal{A}^+(t) \subset [0, \xi_M(t)]$. In particular, any global solution in $\mathcal{A}^+(t)$ is no non-degenerate at $-\infty$.*
d) *$\xi_M(t)$ pullback attracts exponentially fast every bounded set $B \subset int\mathcal{V}^+$.*

As the linearization around the zero solution of (8.1) coincides with that of the autonomous case $\beta(t) = 1$, if we suppose that $\lambda \in (\lambda_1, \lambda_2)$ we know from Henry [15] that 0 is an unstable equilibria with associated unstable manifold included in the positive cone of codimension 1. From our point of view on Morse decomposition, we can conclude that 0 is a local repeller in $\mathcal{A}^+(t)$. Now, by Theorem 8.2, item d), we obtain $\xi_M(t)$ as the associated local attractor in the positive cone.

Thus, a direct application of Corollary 7.2 yields

$$c(\mathcal{A}^+(t)) \leqslant \frac{\omega + \ln L}{\omega},$$

with ω the exponential rate of attraction to $\xi_M(t)$ (see [23, 24] for estimation of this parameter) and L the Lipschitz constant for $T(t,s;u_0) = u(t,s;u_0)$ with respect to the initial data u_0.

Remark 8.3. *Observe that, since the nonlinear term is odd, if u is a solution of* (8.1) *then so is $v = -u$. As a consequence, the behaviour of solutions in the positive and negative cones are symmetric and thus, if we denote by $\xi_M(t)$ the maximal bounded solution in the positive cone, the minimal bounded solution in the negative cone is just $-\xi_M(t)$, so that we get in this infinite dimensional dynamical system a similar behaviour as in Example 6.7.*

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