ALMOST PERIODIC AND ALMOST AUTOMORPHIC SOLUTIONS OF LINEAR DIFFERENTIAL/DIFFERENCE EQUATIONS WITHOUT FAVARD'S SEPARATION CONDITION. I

TOMÁS CARABALLO AND DAVID CHEBAN

Abstract. The well known Favard's Theorem states that the linear differential equation

$$x' = A(t)x + f(t) \tag{1}$$

with Bohr almost periodic coefficients admits at least one Bohr almost periodic solution if it has a bounded solution. The main assumption in this theorem is the separation among bounded solutions of homogeneous equations

$$x' = B(t)x, \tag{2}$$

where $B \in H(A) := \{B \mid B(t) = \lim_{n\to+\infty} A(t+t_n)\}$. If there are bounded solutions which are non-separated, sometimes almost periodic solutions do not exist (R. Johnson, R. Ortega and M. Tarallo, V. Zhikov and B. Levitan).

In this paper we prove that linear differential equation (1) with Levitan almost periodic coefficients has a unique Levitan almost periodic solution, if it has at least one bounded solution, and the bounded solutions of the homogeneous equation

$$x' = A(t)x \tag{3}$$

are homoclinic to zero (i.e. $\lim_{|t|\to+\infty} |\varphi(t)| = 0$ for all bounded solutions φ of (3)). If the coefficients of (1) are Bohr almost periodic and all bounded solutions of all limiting equations (2) are homoclinic to zero, then equation (1) admits a unique almost automorphic solution.

The analogue of this result for difference equations is also given.

We study the problem of existence of Bohr/Levitan almost periodic solutions of equation (1) in the framework of general non-autonomous dynamical systems (cocycles).

Dedicated to Russell Johnson on his 60th birthday

1. Introduction

Recall (see, for example, [16, 20]) that a continuous function φ defined on the real axis $\mathbb{R}$ with values in a Banach space E is called *Bohr almost periodic*, if for all $\varepsilon > 0$ there exists a positive number $l(\varepsilon)$ such that on every interval $[a, a+l]$ $(a \in \mathbb{R})$

Date: April 1, 2008.

1991 *Mathematics Subject Classification.* 34C27, 34K14, 37B20, 37B55, 39A11.

Key words and phrases. almost periodic solution; almost automorphic solutions; non-autonomous dynamical systems; Favard's condition; cocycle.

there exists at least one number τ such that

$$|\varphi(t+\tau)-\varphi(t)|<\varepsilon$$

for all $t \in \mathbb{R}$ (the number τ is called an ε almost period of function φ).

The following result is well known.

Theorem 1.1. *(Bochner's theorem). A continuous function $\varphi : \mathbb{R} \mapsto E$ is Bohr almost periodic if and only if from each sequence $\{t_n\} \subset \mathbb{R}$ there can be extracted a subsequence $\{t_{n_k}\}$ such that the functional sequence $\{\varphi(t+t_{n_k})\}$ converges uniformly on the real axis $\mathbb{R}$.*

A continuous function $\varphi : \mathbb{R} \mapsto E$ is called [20] *Levitan almost periodic*, if there exists a Bohr almost periodic function $\psi : \mathbb{R} \mapsto F$ (F is a Banach space) such that $\mathfrak{N}_\psi \subseteq \mathfrak{N}_\varphi$, where $\mathfrak{N}_\varphi$ is the family of all sequences $\{t_n\} \subset \mathbb{R}$ such that the functional sequence $\{\varphi(t+t_n)\}$ converges to φ uniformly on every compact subset from $\mathbb{R}$.

It is evident that every Bohr almost periodic function is Levitan almost periodic. The inverse statement is not true. For example, the function $\varphi(t) := (2+\sin t + \sin\sqrt{2}t)^{-1}$ is Levitan almost periodic, but not Bohr almost periodic [20].

A continuous function $\varphi : \mathbb{R} \mapsto E$ is called [4] (see also [20, 23, 24]) *almost automorphic* (or *Bohr almost automorphic*) if for every sequence $\{t_n^{'}\}$ there exists a subsequence $\{t_n\}$ for which we have local convergence (i.e. uniform convergence on every compact subset of $\mathbb{R}$)

$$\varphi(t+t_n) \to \tilde{\varphi}(t),$$

the "returning" also holds:

$$\tilde{\varphi}(t-t_n) \to \varphi(t).$$

It is known (see, for example, [20] and also [9]) that every almost automorphic function is Levitan almost periodic. The inverse, generally speaking, is not true because almost periodic functions are bounded, but a Levitan almost periodic function may be unbounded. Recall also that any Bohr almost periodic function is almost automorphic.

This paper is concerned with the study of linear differential (difference) equations with Bohr/Levitan almost periodic and almost automorphic coefficients. This field is called Favard's theory [20, 39], due to the fundamental contributions made by J. Favard [13]. In 1927, J. Favard published his celebrated paper, where he studied the existence of almost periodic solutions of the following equation in $\mathbb{R}^N$:

$$x' = A(t)x + f(t), \tag{4}$$

where the matrix $A(t)$ and the vector-function $f(t)$ are almost periodic in the sense of Bohr (see, for example, [16, 20]).

Along with equation (4), consider the homogeneous equation

$$x' = A(t)x \tag{5}$$

and the corresponding family of *limiting equations*

$$x' = B(t)x, \tag{6}$$

where $B \in H(A)$, and $H(A)$ denotes the hull of almost periodic matrix $A(t)$ which is composed by those functions $B(t)$ obtained as uniform limits on $\mathbb{R}$ of the type $B(t) := \lim_{n\to\infty} A(t+t_n)$, where $\{t_n\}$ is some sequence in $\mathbb{R}$.

From now on, a bounded function on $\mathbb{R}$ will be simply called a bounded function.

Theorem 1.2. *(Favard's theorem* [13]*) The linear differential equation (4) with Bohr almost periodic coefficients admits at least one Bohr almost periodic solution if it has a bounded solution, and each bounded solution $\varphi(t)$ of every limiting equation (6) ($B \in H(A)$) is separated from zero, i.e.*

$$\inf_{t\in\mathbb{R}} |\varphi(t)| > 0.$$

Remark 1.3. *Under the conditions of Favard's theorem, if φ_1 and φ_2 are two different bounded solutions of the same non-homogeneous limiting equation*

$$x' = B(t)x + g(t) \quad ((B,g) \in H(A,g)), \tag{7}$$

then $\inf_{t\in\mathbb{R}} |\varphi_1(t) - \varphi_2(t)| > 0.$

Remark 1.4. *1. Favard's theorem is extended to the case of almost automorphic coefficients in the works of L. Faxing* [15]*, and W. Shen and Y. Yi* [35]*.*

2. For equations with Levitan almost periodic coefficients, Favard's theorem was generalized by B. Levitan [19]*, B. Levitan and V. Zhikov* [20, 39]*, M. Lyubarskii* [21]*, B. Shcherbakov* [33] *and M. Shubin* [36, 37]*.*

Remark 1.5. *In this paper we analyze the existence of Levitan/Bohr almost periodic and almost automorphic solutions of linear differential equations in the framework of a more general problem. Namely, we study the existence of Poisson stable solutions (in particular, periodic, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkgoff, Levitan almost periodic, almost recurrent in the sense of Bebutov, Poisson stable) of linear differential equations with Poisson stable coefficients. The notions of comparability and uniform comparability of motions by the character of recurrence, introduced by B. Shcherbakov* [31]–[34]*, are powerful tools which we will use to study this problem.*

Zhikov and Levitan [20, 39] (see also Johnson [18], Ortega and Tarallo [25], Sell [29]) constructed examples of scalar linear differential equations for which all the solutions are bounded, but none of them is almost periodic. In particular, the following result was established in [25].

Theorem 1.6. *(Ortega and Tarallo* [25]*) Let (5) be a linear differential equation with Bohr almost periodic coefficients, and for some $B \in H(A)$ each nontrivial bounded on $\mathbb{R}$ solution φ of the equation (6) is homoclinic to zero, i.e.*

$$\lim_{|t|\to+\infty} |\varphi(t)| = 0.$$

Then, there exists an almost periodic (in the sense of Bohr) function $g : \mathbb{R} \mapsto \mathbb{R}^n$ such that equation (7) has bounded solutions, but none of them is Bohr almost periodic.

From one of our main result (see Section 4, Theorem 4.1 and Corollary 4.4) it follows that, under the conditions of Theorem 1.6, equation (7) has a unique almost automorphic solution, which is a positive statement in contrast to the one in the previous theorem.

This paper is organized as follows.

In Section 2 we collect some well known facts from the theory of dynamical systems (both autonomous and non-autonomous). Namely, the notions of almost periodic (in the senses of Bohr and Levitan), almost automorphic and recurrent motions; shift dynamical systems and almost periodic and almost automorphic functions; cocycles, skew-product dynamical systems, and general non-autonomous dynamical systems.

Section 3 is devoted to the existence of motions comparable (respectively, uniformly comparable) by the character of recurrence in the sense of Shcherbakov [31]-[34]. The main results of this section are Theorems 3.6, 3.12 and Corollaries 3.9–3.10, 3.13 (which are, in fact, the main abstract results of the paper) which provide sufficient conditions for the existence of a unique motion comparable (respectively, uniformly comparable) by the character of recurrence for two-sided non-autonomous dynamical systems.

In Section 4 we analyze the compatible (respectively, uniformly compatible) solutions of ordinary differential and difference equations in a Banach space. Here we present a test for the existence of Bohr (respectively, Levitan) almost periodic and almost automorphic solutions of non-homogeneous linear differential/difference equations with Bohr (respectively, Levitan) almost periodic and almost automorphic coefficients.

2. Almost Periodic and Almost Automorphic Motions of Dynamical Systems

Let us collect in this section some well known concepts and results from the theory of dynamical systems which will be necessary for our analysis in this paper.

2.1. **Recurrent, Almost Periodic and Almost Automorphic Motions.**

Let X be a complete metric space, $\mathbb{R}$ ($\mathbb{Z}$) be the group of real (integer) numbers. By $\mathbb{T}$ we will denote either $\mathbb{R}$ or $\mathbb{Z}$.

Let $(X,\mathbb{T},\pi)$ be *a dynamical system* on X, i.e. let $\pi : \mathbb{T}\times X \to X$ be a continuous function such that $\pi(0,x) = x$ for all $x \in X$, and $\pi(t_1 + t_2, x) = \pi(t_2, \pi(t_1, x))$, for all $x \in X$, and $t_1, t_2 \in \mathbb{T}$.

Given $\varepsilon > 0$, a number $\tau \in \mathbb{T}$ is called an $\varepsilon-$shift (respectively, an $\varepsilon-$almost period) of x, if $\rho(\pi(\tau,x),x) < \varepsilon$ (respectively, $\rho(\pi(\tau+t,x),\pi(t,x)) < \varepsilon$ for all $t \in \mathbb{T}$).

A point $x \in X$ is called *almost recurrent* (respectively, *Bohr almost periodic*), if for any $\varepsilon > 0$ there exists a positive number l such that in any segment of length l there is an $\varepsilon-$shift (respectively, an $\varepsilon-$almost period) of the point $x \in X$.

If the point $x \in X$ is almost recurrent, and the set $H(x) := \overline{\{\pi(t,x) \mid t \in \mathbb{T}\}}$ is compact, then x is called *recurrent*, where the bar denotes the closure in X.

Denote by $\mathfrak{N}_x := \{\{t_n\} \subset \mathbb{T} \ : \ \text{such that } \{\pi(t_n,x)\} \to x \text{ and } \{t_n\} \to \infty\}$.

A point $x \in X$ is said to be *Levitan almost periodic* (see [20]) for the dynamical system $(X,\mathbb{T},\pi)$ if there exists a dynamical system $(Y,\mathbb{T},\lambda)$, and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Remark 2.1. *Let $x_i \in X_i$ $(i = 1,2,\dots,m)$ be a Levitan almost periodic point of the dynamical system $(X_i,\mathbb{T},\pi_i)$. Then the point $x := (x_1,x_2,\dots,x_m)) \in X := X_1 \times X_2 \times \ldots \times X_m$ is also Levitan almost periodic for the product dynamical system $(X,\mathbb{T},\pi)$, where $\pi : \mathbb{T} \times X \to X$ is defined by the equality $\pi(t,x) := (\pi_1(t,x_1),\pi_2(t,x_2),\ \ldots,\ \pi_m(t,x_m))$ for all $t \in \mathbb{T}$ and $x := (x_1,x_2,\dots,x_m) \in X$.*

A point $x \in X$ is called *stable in the sense of Lagrange* (*st*.L), if its trajectory $\{\pi(t,x) \ : \ t \in \mathbb{T}\}$ is relatively compact.

A point $x \in X$ is called *almost automorphic* [20, 35] for the dynamical system $(X,\mathbb{T},\pi)$, if the following conditions hold:

(i) x is st.L;
(ii) there exists a dynamical system $(Y,\mathbb{T},\lambda)$, a homomorphism h from $(X,\mathbb{T},\pi)$ onto $(Y,\mathbb{T},\lambda)$ and an almost periodic (in the sense of Bohr) point $y \in Y$ such that $h^{-1}(y) = \{x\}$.

Remark 2.2. *The following facts hold true.*
1. Every almost automorphic point $x \in X$ is also Levitan almost periodic.

2. A Levitan almost periodic point x with relatively compact trajectory $\{\pi(t,x)\ \ t \in \mathbb{T}\}$ is also almost automorphic (see [2]–[5], [10],[22] *and* [35]*). In other words, a Levitan almost periodic point x is almost automorphic, if and only if its trajectory $\{\pi(t,x)\ \ t \in \mathbb{T}\}$ is relatively compact.*

3. Let $(X,\mathbb{T},\pi)$ and $(Y,\mathbb{T},\lambda)$ be two dynamical systems, $x \in X$, and assume that the following conditions are fulfilled:

(i) *there exists a point $y \in Y$ which is Levitan almost periodic;*
(ii) $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Then, the point x is also Levitan almost periodic.

4. Let $x \in X$ be a st.L point, $y \in Y$ an almost automorphic point, and $\mathfrak{N}_y \subseteq \mathfrak{N}_x$. Then, the point x is almost automorphic.

2.2. **Shift Dynamical Systems, Almost Periodic and Almost Automorphic Functions.** We show below a general method for the construction of dynamical systems on the space of continuous functions. In this way we will obtain many well known dynamical systems on functional spaces (see, for example, [5, 31]).

Let $(X,\mathbb{T},\pi)$ be a dynamical system on X, Y be a complete pseudo metric space, and P be a family of pseudo metrics on Y. We denote by $C(X,Y)$ the family of all continuous functions $f : X \to Y$ equipped with the compact-open topology. This topology is given by the following family of pseudo metrics $\{d_K^p\}$ ($p \in P$, $K \in \mathcal{K}(X)$), where

$$d_K^p(f,g) := \sup_{x\in K} p(f(x),g(x)),$$

and $\mathcal{K}(X)$ denotes the family of all compact subsets of X. For every $\tau \in \mathbb{T}$ we define a mapping $\sigma_\tau : C(X,Y) \to C(X,Y)$ by the following equality: $(\sigma_\tau f)(x) := f(\pi(\tau,x))$, $x \in X$. We note that the family of mappings $\{\sigma_\tau : \tau \in \mathbb{T}\}$ possesses the next properties:

a. $\sigma_0 = id_{C(X,Y)}$;
b. $\sigma_{\tau_1} \circ \sigma_{\tau_2} = \sigma_{\tau_1+\tau_2} \quad \forall \tau_1, \tau_2 \in \mathbb{T}$,
c. σ_τ is continuous, $\quad \forall \tau \in \mathbb{T}$.

Furthermore, the next lemma ensures that $(C(X,Y),\mathbb{T},\sigma)$ is a dynamical system.

Lemma 2.3. [8] *The mapping $\sigma : \mathbb{T} \times C(X,Y) \to C(X,Y)$, defined by the equality $\sigma(\tau,f) := \sigma_\tau f \quad (f \in C(X,Y),\ \tau \in \mathbb{T})$ is continuous, and, consequently, the triple $(C(X,Y),\mathbb{T},\sigma)$ is a dynamical system on $C(X,Y)$.*

Consider now some examples of dynamical systems given by the form $(C(X,Y),\mathbb{T},\sigma)$ which are useful in applications.

Example 2.4. *Let $X = \mathbb{T}$, and denote by $(X,\mathbb{T},\pi)$ a dynamical system on $\mathbb{T}$, where $\pi(t,x) := x+t$. The dynamical system $(C(\mathbb{T},Y),\mathbb{T},\sigma)$ is called* Bebutov's dynamical system [31] *(a dynamical system of translations, or shifts dynamical system). For example, the equality*

$$d(f,g) := \sup_{L>0} \max\{d_L(f,g), L^{-1}\},$$

*where $d_L(f,g) := \max_{|t|\le L} \rho(f(t),g(t))$, defines a complete metric (*Bebutov's metric*) on the space $C(\mathbb{T},Y)$ which is compatible with the compact-open topology on $C(\mathbb{T},Y)$.*

Remark 2.5. *It is known* [31, 34] *that $d(f,g) < \varepsilon$ (respectively, $d(f,g) > \varepsilon$ or $d(f,g) = \varepsilon$) is equivalent to the inequality $d_{\frac{1}{\varepsilon}}(f,g) < \varepsilon$ (respectively, $d_{\frac{1}{\varepsilon}}(f,g) > \varepsilon$ or $d_{\frac{1}{\varepsilon}}(f,g) = \varepsilon$).*

It is said that the function $\varphi \in C(\mathbb{T},Y)$ *possesses a property* (A), if the motion $\sigma(\cdot,\varphi) : \mathbb{T} \to C(\mathbb{T},Y)$ possesses this property in the Bebutov's dynamical system $(C(\mathbb{T},Y),\mathbb{T},\sigma)$, generated by the function φ. As property (A) we can consider periodicity, quasi-periodicity, almost periodicity, almost automorphy, recurrence etc.

Example 2.6. *Let $X := \mathbb{T} \times W$, where W is a metric space, and let $(X,\mathbb{T},\pi)$ denote a dynamical system on X defined in the following way: $\pi(t,(s,w)) := (s+t,w)$. Using the general method proposed above we can define a dynamical system of translations $(C(\mathbb{T}\times W,Y),\mathbb{T},\sigma)$ on $C(\mathbb{T}\times W,Y)$.*

The function $f \in C(\mathbb{T}\times W,Y)$ is called almost periodic (recurrent, almost automorphic, etc.) with respect to $t \in \mathbb{T}$, uniformly with respect to $w \in W$ on every compact from W, if the motion $\sigma(\cdot,f)$ is almost periodic (recurrent, almost automorphic, etc.) in the dynamical system $(C(\mathbb{T}\times W,Y),\mathbb{T},\sigma)$.

Remark 2.7. *Let W be a compact metric space, then the topology on $C(W,Y)$ is metrizable. For example, the equality*

$$d(f,g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(f,g)}{1+d_k(f,g)}, \quad \text{where} \quad d_k(f,g) := \max_{|t|\le k,\ x\in W} \rho(f(t,x), g(t,x)),$$

defines a complete metric on the space $C(W,X)$ which is compatible with the compact-open topology on $C(W,X)$. The space $C(\mathbb{T}\times W,Y)$ is topologically isomorphic to $C(\mathbb{T},C(W,Y))$ [31]*, and also the shifts dynamical systems $(C(\mathbb{T}\times W,Y),\mathbb{T},\sigma)$ and $(C(\mathbb{T},C(W,Y)),\mathbb{T},\sigma)$ are dynamically isomorphic.*

2.3. Cocycles, Skew-Product Dynamical Systems and Non-Autonomous Dynamical Systems.

Consider now two dynamical systems $(X,\mathbb{T},\pi)$ and $(Y,\mathbb{T},\lambda)$. A triplet $\langle (X,\mathbb{T},\pi),\ (Y,\mathbb{T},\lambda),\ h\rangle$ is called *a non-autonomous dynamical system* if h is a homomorphism from $(X,\mathbb{T},\pi)$ onto $(Y,\mathbb{T},\lambda)$.

Let $(Y,\mathbb{T},\lambda)$ be a dynamical system on Y, W a complete metric space, and φ a continuous mapping from $\mathbb{T}\times W\times Y$ in W possessing the following properties:

a. $\varphi(0,u,y) = u$ $(u\in W, y\in Y)$;
b. $\varphi(t+\tau,u,y) = \varphi(\tau,\varphi(t,u,y),\lambda(t,y))$ $(t,\tau\in\mathbb{T},\ u\in W, y\in Y)$.

Then, the triplet $\langle W,\varphi,(Y,\mathbb{T},\lambda)\rangle$ (or shortly φ) is called [28] *a cocycle* on $(Y,\mathbb{T},\lambda)$ with the fiber W.

Given a cocycle $\langle W,\varphi,(Y,\mathbb{T},\lambda)\rangle$, let us set $X := W\times Y$, and define a mapping $\pi:\mathbb{T}\times X\to X$ as follows: $\pi(t,(u,y)) := (\varphi(t,u,y),\lambda(t,y))$ (i.e. $\pi=(\varphi,\lambda)$). Then, $(X,\mathbb{T},\pi)$ is a dynamical system on X, which is called *a skew-product* dynamical system [28], and $h = pr_2 : X\to Y$ is a homomorphism from $(X,\mathbb{T},\pi)$ onto $(Y,\mathbb{T},\lambda)$ and, hence, $\langle (X,\mathbb{T},\pi),\ (Y,\mathbb{T},\lambda), h\rangle$ is a non-autonomous dynamical system.

Thus, if $\langle W,\varphi,(Y,\mathbb{T},\lambda)\rangle$ is a cocycle on the dynamical system $(Y,\mathbb{T},\lambda)$ with the fiber W, then it generates a non-autonomous dynamical system $\langle (X,\mathbb{T},\pi),\ (Y,\mathbb{T},\lambda), h\rangle$ $(X := W\times Y)$, called a non-autonomous dynamical system generated by the cocycle $\langle W,\varphi,(Y,\mathbb{T},\lambda)\rangle$ on $(Y,\mathbb{T},\lambda)$.

Non-autonomous dynamical systems (cocycles) play a very important role in the study of non-autonomous evolutionary differential equations. Under appropriate assumptions, every non-autonomous differential equation generates a cocycle (a non-autonomous dynamical system).

3. Comparability and Uniform Comparability of Motions by the Character of Recurrence in the Sense of Shcherbakov

We will prove now the main abstract results in this paper. First, we start with the following definitions.

Let $(\Omega,\mathbb{T},\lambda)$ be a dynamical system. A point $\omega\in\Omega$ is said to be (see, for example, [34] and [38]) *positively (respectively, negatively) stable in the sense of Poisson*, if there exists a sequence $t_n\to+\infty$ (respectively, $t_n\to-\infty$) such that $\lambda(t_n,\omega)\to\omega$ If the point ω is Poisson stable in both directions, it is called *Poisson stable.*

Denote by $\mathfrak{N}_\omega = \{\{t_n\} \subset \mathbb{T} \mid \lambda(t_n, \omega) \to \omega, \quad \text{as} \quad n \to +\infty\}$.

Let (X, h, Ω) be a fiber space, i.e. X and Ω are two metric spaces and $h : X \to \Omega$ is a homomorphism from X onto Ω. The subset $M \subseteq X$ is said to be *conditionally relatively compact* [7, 8] if the pre-image $h^{-1}(\Omega') \bigcap M$ of every relatively compact subset $\Omega' \subseteq \Omega$ is a relatively compact subset of X, in particular $M_\omega := h^{-1}(\omega) \bigcap M$ is relatively compact for every ω. The set M is called *conditionally compact* if it is closed and conditionally relatively compact.

Example 3.1. *Let K be a compact space, $X := K \times \Omega$, and consider $h = pr_2 : X \to \Omega$. Then, the triplet (X, h, Ω) is a fiber space, the space X is conditionally compact, but it is not compact.*

The following result characterizes when a closed set is conditionally compact.

Lemma 3.2. *Let M be a closed subset of X. Then, M is conditionally compact with respect to (X, h, Ω) if and only if the following conditions hold:*

(i) *the set $M_\omega := h^{-1}(\omega) = \{x \in M \mid h(x) = \omega\}$ is compact for all $\omega \in \Omega$;*
(ii) *the mapping $\omega \mapsto M_\omega$ is upper semi-continuous.*

Proof. Necessity. If the closed set M is conditionally compact, then the set M_ω is evidently compact for all ω. Assume now that $\omega_n \to \omega$ as $n \to +\infty$, and take $x_n \in M_{\omega_n}$. Since the set M is conditionally compact, we can assume, without loss of generality, that the sequence $\{x_n\}$ is convergent. Denote by x its limit, then $h(x) = \lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} \omega_n = \omega$ and, consequently, $x \in M_\omega$, i.e. the mapping $\omega \mapsto M_\omega$ is upper semi-continuous.

Sufficiency. Let $\Omega^{'} \subseteq \Omega$ be an arbitrary compact set. Since the mapping $\omega \mapsto M_\omega$ is upper semi-continuous, then the set $M \cap h^{-1}(\Omega^{'}) = \cup\{M_\omega \mid \omega \in \Omega^{'}\}$ is compact. □

The next lemma provides some examples of conditionally compact sets for a non-autonomous dynamical system.

Lemma 3.3. *Let $\langle W, \varphi, (\Omega, \mathbb{T}, \lambda)\rangle$ be a cocycle and $\langle (X, \mathbb{T}, \pi), (\Omega, \mathbb{T}, \lambda), h\rangle$ be a non-autonomous dynamical system associated to the cocycle φ. Suppose that $x_0 := (u_0, \omega_0) \in X := W \times \Omega$, and that the set $Q_{(u_0,\omega_0)} := \overline{\{\varphi(t, u_0, \omega_0) \mid t \in \mathbb{T}\}}$ (respectively, $Q^+_{(u_0,\omega_0)} := \overline{\{\varphi(t, u_0, \omega_0) \mid t \in \mathbb{T}_+\}}$, where $\mathbb{T}_+ := \{t \in \mathbb{T} \mid t \geq 0\}$) is compact.*

Then, the set $H(x_0) := \overline{\{\pi(t, x_0) \mid t \in \mathbb{T}\}}$ (respectively, $\overline{\{\pi(t, x_0) \mid t \in \mathbb{T}_+\}} := H^+(x_0)$) is conditionally compact.

Proof. Let Ω' be an arbitrary compact subset of Ω, and $\{x_n\}$ an arbitrary sequence in $H(x_0)$ (respectively, in $H^+(x_0)$). Then, for any $n \in \mathbb{N}$ there exists $t_n \in \mathbb{T}$ (respectively, $t_n \in \mathbb{T}_+$) such that $\rho(\pi(t_n, x_0), x_n) \leq 1/n$ (or equivalently, $\rho_2(\lambda(t_n, \omega_0), \omega_n) \leq 1/n$ and $\rho_1(\varphi(t_n, u_0, \omega_0), u_n) \leq 1/n$, where $x_n := (u_n, \omega_n)$ and ρ_1 (respectively, ρ_2) denotes the distance on the space W (respectively, Ω). Since the sets Ω' and $Q_{(u_0,\omega_0)}$ are compact, the sequences $\{u_n\}$ and $\{\omega_n\}$ are relatively compact and, consequently, so is the sequence $\{x_n\}$. □

Let $\langle (X,\mathbb{T},\pi),(\Omega,\mathbb{T},\lambda),h\rangle$ be a two-sided (a group) non-autonomous dynamical system, and $\omega\in\Omega$ be a positively Poisson stable point. Denote by

$$\mathcal{E}_\omega := \{\xi|\quad \exists\{t_n\}\in\mathfrak{N}_\omega \quad \text{such that} \quad \pi(t_n,\cdot)|_{X_\omega}\to\xi\},$$

where $X_\omega := \{x\in X| \quad h(x)=\omega\}$ and $\to$ means the pointwise convergence.

Lemma 3.4. [7, 8] *Let $\omega\in\Omega$ be a Poisson stable point, $\langle (X,\mathbb{T},\pi),\ (\Omega,\mathbb{T},\lambda),h\rangle$ a non-autonomous dynamical system, and X a conditionally compact space. Then, $\mathcal{E}_\omega$ is a nonempty compact sub-semigroup of the semigroup $X_\omega^{X_\omega}$ (w.r.t. the composition of mappings).*

Recall that if X is a compact metric space, then X^X denotes the collection of all maps from X to itself, provided with the product topology, or, in other words, the topology of pointwise convergence. By Tychonoff's theorem, X^X is compact.

X^X possesses a semi-group structure defined by the composition of maps.

Let $\mathcal{E}$ be a semi-group. *A right ideal* in $\mathcal{E}$ is a non-empty subset I such that $I\mathcal{E}\subset I$, where $I\mathcal{E} := \{\xi\circ\eta\ :\xi\in I,\ \eta\in\mathcal{E}\}$, and $\xi\circ\eta$ is the composition of ξ and η defined in the following way:

$$(\xi\circ\eta)(x) := \eta(\xi(x)), \quad \text{for} \quad x\in X.$$

It is worth noticing that we are using the original notation for the composition which was used in the works [1], [5] and [11]. Needless to say that this notation can be misunderstanding, and it would be possible to use the standard definition, but then we should change all the terminology about right and left ideals, and the results already proved in the literature concerning these sets. For this reason, we prefer to keep the original notation and recommend the reader to be careful with this notation.

A right ideal is said to be *a minimal right ideal* if it does not contain any proper right ideal.

An idempotent in a semigroup $\mathcal{E}$ is an element $u\in\mathcal{E}$ such that $u^2=u$.

Remark 3.5. *1. Every compact semigroup admits at least one minimal right ideal* [1, 5, 11].

2. Every compact semigroup contains at least one idempotent element [1, 5, 11].

Now, we can prove our first main abstract result.

Theorem 3.6. *Let X be a conditionally compact metric space, and $\langle (X,\mathbb{T},\pi)\ (\Omega,\mathbb{T},\lambda),h\rangle$ a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *There exists a Poisson stable point $\omega\in\Omega$;*
(ii) $\lim\limits_{|t|\to+\infty}\rho(\pi(t,x_1),\pi(t,x_2))=0$ *for all* $x_1,x_2\in X_\omega := h^{-1}(\omega)=\{x\in X\ :\ h(x)=\omega\}$.

Then there exists a unique point $x_\omega\in X_\omega$ such that $\xi(x_\omega)=x_\omega$ for all $\xi\in\mathcal{E}_\omega$.

Proof. Let $I \subseteq \mathcal{E}_\omega$ be a minimal right ideal of the compact semigroup $\mathcal{E}_\omega$ and $u \in I$ be an arbitrary idempotent element of I. Then $u^2 = u$. Since $\mathcal{E}_\omega \circ u = I$ (see, for instance, Chapter 1 in Bronsteyn [5]), we have $\mathcal{E}_\omega \circ u(x) = I(x)$ for all $x \in X_\omega$. Under the conditions of the theorem, for every $\xi \in \mathcal{E}_\omega$ there exists a unique $x_\xi \in X_\omega$ such that $\xi(x) = x_\xi$ for all $x \in X_\omega$, i.e. the set $\xi(X_\omega)$ consists of a single point. Denote by $M_\omega := u(X_\omega)$, then $u(x) = x$ for all $x \in M_\omega$ because $u^2 = u$. On the other hand, the set M_ω consists of a single point x_ω. Notice that $\mathcal{E}_\omega \circ u(x) = I(x) = M_\omega$ for all $x \in M_\omega = \{x_\omega\}$. Thus we have $\xi(x_\omega) = x_\omega$ for all $\xi \in \mathcal{E}_\omega$.

Finally, we will prove that the semigroup $\mathcal{E}_\omega$ admits a unique fixed point. If we suppose that it is not true, then there exist $x_1, x_2 \in X_\omega$ ($x_1 \neq x_2$) such that $\xi(x_i) = x_i$ ($i = 1, 2$) for all $\xi \in \mathcal{E}_\omega$. In particular, there exists a sequence $\{|t_n|\} \to +\infty$ ($t_n \in \mathbb{T}$) such that $\{\pi(t_n, x_i)\} \to x_i$ ($i = 1, 2$). On the other hand, we have

$$\rho(x_1, x_2) = \lim_{n \to +\infty} \rho(\pi(t_n, x_1), \pi(t_n, x_2)) = 0,$$

i.e., $x_1 = x_2$, and this contradiction proves our statement. □

Corollary 3.7. *Let $\langle (X, \mathbb{T}, \pi),\ (\Omega, \mathbb{T}, \lambda), h \rangle$ be a non-autonomous dynamical system, and $x_0 \in X$. Suppose that the following conditions are fulfilled:*

(i) *the set $H(x_0) := \overline{\{\pi(t, x_0) \mid t \in \mathbb{T}\}}$ is conditionally compact;*
(ii) *the point $\omega := h(x_0) \in \Omega$ is Poisson stable;*
(iii) $\lim\limits_{|t| \to +\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0$ *for all $x_1, x_2 \in H(x_0) \cap X_\omega$, where $X_\omega := h^{-1}(\omega) = \{x \in X \ :\ h(x) = \omega\}$.*

Then, there exists a unique point $x_\omega \in H(x_0) \cap X_\omega$ such that $\xi(x_\omega) = x_\omega$ for all $\xi \in \mathcal{E}_\omega$.

Proof. To prove this statement it is sufficient to apply Theorem 3.6 to the non-autonomous dynamical system $\langle (H(x_0), \mathbb{T}, \pi),\ (H(\omega), \mathbb{T}, \lambda), h \rangle$. □

A point $x \in X$ is said to be *comparable with $\omega \in \Omega$ by the character of recurrence* (see [31]–[34]) if $\mathfrak{N}_\omega \subseteq \mathfrak{N}_x$.

Remark 3.8. *If a point $x \in X$ is comparable with $\omega \in \Omega$ by the character of recurrence, and ω is stationary (respectively, τ-periodic, recurrent, Poisson stable), then so is the point x* [34].

Corollary 3.9. *Let X be a conditionally compact metric space, and $\langle (X, \mathbb{T}, \pi),$ $(\Omega, \mathbb{T}, \lambda), h \rangle$ be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *there exists a Poisson stable point $\omega \in \Omega$;*
(ii) $\lim\limits_{|t| \to +\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = 0$ *for all $x_1, x_2 \in X_\omega := h^{-1}(\omega) = \{x \in X \ :\ h(x) = \omega\}$.*

Then, there exists a unique point $x_\omega \in X_\omega$ which is comparable with $\omega \in \Omega$ by the character of recurrence, such that

$$\lim_{|t| \to +\infty} \rho(\pi(t, x), \pi(t, x_\omega)) = 0 \tag{8}$$

for all $x \in X_\omega$.

Proof. By Theorem 3.6 there exists a unique fixed point $x_\omega \in X_\omega$ of the semigroup $\mathcal{E}_\omega$. To prove this statement it is sufficient to show that the point x_ω is as required. Let $\{t_n\} \in \mathfrak{N}_\omega$, then $\{t_n\} \in \mathfrak{N}_{x_\omega}$. We argue now by contradiction. If we suppose that it is not true, then there are two subsequences $\{t_{n_k^i}\} \subset \{t_n\}$ $(i = 1,2)$ such that $\lim\limits_{k\to+\infty} \pi(t_{n_k^i}, x_\omega) = x_i$ (i=1,2) and $x_1 \neq x_2$. Without loss of generality, we can assume that the sequences $\{\pi(t_{n_k^i}, \cdot)\}$ are convergent in X^X. Denoting by $\xi_i := \lim\limits_{k\to+\infty} \pi(t_{n_k^i}, \cdot)$, then $\xi_i \in \mathcal{E}_\omega$ and we have $x_1 = \xi_1(x) = \xi_2(x) = x_2$, which is a contradiction, and the proof is therefore complete. □

Corollary 3.10. *Let $\omega \in \Omega$ be a stationary (respectively, τ-periodic, almost automorphic, recurrent, Levitan almost periodic, Poisson stable) point. Then, under the conditions of Corollary 3.9, there exists a unique stationary (respectively, τ-periodic, almost automorphic, recurrent, Levitan almost periodic, Poisson stable) point $x_\omega \in X_\omega$ such that the equality (8) holds for all $x \in X_\omega$.*

Proof. This statement directly follows from Corollary 3.9 and Remark 3.8. □

Denote by $\mathfrak{M}_\omega := \{\{t_n\} \subset \mathbb{T} \mid \text{the sequence } \{\lambda(t_n, \omega)\} \text{ is convergent }\}$.

A point $x \in X$ is said to be *uniformly comparable with $\omega \in \Omega$ by the character of recurrence* (see [31]–[34]) if $\mathfrak{M}_\omega \subseteq \mathfrak{M}_x$.

Remark 3.11. *1. If a point $x \in X$ is uniformly comparable with $\omega \in \Omega$ by the character of recurrence, and ω is stationary (respectively, τ-periodic, almost periodic, almost automorphic, recurrent, Poisson stable), then so is the point x* [31]–[34].

2. Every almost periodic point is recurrent.

Theorem 3.12. *Let X be a compact metric space and $\langle (X,\mathbb{T},\pi),\ (\Omega,\mathbb{T},\lambda), h\rangle$ be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

(i) *The point $\omega \in \Omega$ is recurrent;*
(ii) $\lim\limits_{|t|\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = 0$ *for all $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$.*

Then, there exists a unique point $x_\omega \in X_\omega$ which is uniformly comparable with $\omega \in \Omega$ by the character of recurrence, and such that (8) holds for all $x \in X_\omega$.

Proof. By Theorem 3.6, there exists a unique fixed point $x_\omega \in X_\omega$ of the semigroup $\mathcal{E}_\omega$. By Corollary 3.10, the point x_ω is recurrent. It is now sufficient to show that the point x_ω is as required. Let $M := \overline{\{\pi(t,x_\omega)\ : t \in \mathbb{T}\}}$. Then, M is a compact minimal set because the point x_ω is recurrent. We will show that $M_q := M \cap X_q$ (for all $q \in H(\omega) := \overline{\{\sigma(t,\omega)\ : t \in \mathbb{T}\}}$) consists of a single point. If we suppose that it is not true, then there exist $q_0 \in H(\omega)$ and $x_1, x_2 \in M_{q_0}$ such that $x_1 \neq x_2$. By Corollary 3.9, there exists a unique point $x_{q_0} \in M_{q_0}$ which is comparable with q_0 by the character of recurrence. Without loss of generality, we can suppose that

$x_{q_0} = x_1$. Since the set M is minimal, then there exists a sequence $\{t_n\} \in \mathfrak{N}_{q_0}$ such that $\{\pi(t_n, x_1)\} \to x_2$. On the other hand, in view of the inclusion $\mathfrak{N}_{q_0} \subseteq \mathfrak{N}_{x_1}$, we have $\{\pi(t_n, x_1)\} \to x_1$ and, consequently, $x_1 = x_2$. This contradiction proves our statement.

Now we will prove that $\mathfrak{M}_\omega \subseteq \mathfrak{M}_{x_\omega}$. Let $\{t_n\} \in \mathfrak{M}_\omega$, then $\{t_n\} \in \mathfrak{M}_{x_\omega}$. Arguing once more by contraction, if we suppose that it is not true, then there are two subsequences $\{t_{n_k^i}\}$ $(i = 1, 2)$ such that $\lim\limits_{k\to+\infty} \pi(t_{n_k^i}, x_\omega) = x_i$ (i=1,2) and $x_1 \neq x_2$. Denoting $q_0 := \lim\limits_{n\to+\infty} \sigma(t_n, \omega)$, then $q_0 \in H(\omega)$ and $x_1, x_2 \in M_{q_0}$. But this is a contradiction since we proved above that M_q consisted of a single point for all $q \in H(\omega)$. The proof is therefore finished. □

Corollary 3.13. *Let $\omega \in \Omega$ be a stationary (respectively, τ-periodic, Bohr almost periodic, recurrent, Poisson stable) point. Then, under the conditions of Theorem 3.12, there exists a unique stationary (respectively, τ-periodic, Bohr almost periodic, recurrent, Poisson stable) point $x_\omega \in X_\omega$ such that (8) is fulfilled for all $x \in X_\omega$.*

Proof. This statement follows directly from Theorem 3.12 and Remark 3.11. □

Remark 3.14. *1. Note that the algebraic approach using ideal and idempotent was originally proposed in the works of R. Ellis* [11].

2. Application of the Ellis semigroup theory to non–autonomous systems (non–autonomous ordinary differential equations, functional differential equations, partial differential equations) with compact base (driving system) has already been made in many works including those due to I. Bronsteyn [5]*, D. Cheban* [8]*, R. Ellis and R. Johnson* [12]*, R. Johnson* [17]*, B. Levitan and V. Zhikov* [39]*, R. Sacker and G. Sell* [26, 27]*, G. Sell, W. Shen and Y. Yi* [30]*, W. Shen and Y. Yi* [35]*. As for the non–autonomous systems with noncompact base (driving system), the Ellis semigroup theory was applied in the works of D. Cheban* [7, 8, 9]*.*

4. Compatible and Uniformly Compatible Solutions of Linear Differential/Difference Equations

In this final section we will apply our abstract theory, previously developed in Section 3, to analyze two important applications: non-homogeneous linear differential equations, and non-homogeneous linear difference equations.

4.1. **Linear Differential Equations.** Let E denote a Banach space with norm $|\cdot|$. Let $[E]$ be the Banach space of all bounded linear operators that act on a Banach space E equipped with the operator norm. Let $C(\mathbb{R}, [E])$ be the space of all continuous operator-valued functions $A : \mathbb{R} \to [E]$ equipped with the compact-open topology and let $(C(\mathbb{R}, [E]), \mathbb{R}, \sigma)$ be the dynamical system of shifts on $C(\mathbb{R}, [E])$.

Consider the differential equation

$$u' = A(t)u + f(t) \tag{9}$$

and the corresponding homogeneous equation

$$u' = A(t)u, \tag{10}$$

where $(A,f) \in C(\mathbb{R},[E]) \times C(\mathbb{R},E)$. Along with equations (9) and (10) we also consider the *H-class* of the equation (9) (respectively, (10)), which is the family of equations

$$v' = B(t)v + g(t), \tag{11}$$

(respectively,

$$v' = B(t)v \quad) \tag{12}$$

with $(B,g) \in H(A,f) := \overline{\{(A_\tau, f_\tau) \mid \tau \in \mathbb{R}\}}$ (respectively, $B \in H(A)$), where $A_\tau(t) = A(t+\tau)$, $f_\tau(t) := f(t+\tau)$ and $t \in \mathbb{R}$, and the bar denotes closure in $C(\mathbb{R},[E]) \times C(\mathbb{R},E)$ (respectively, $C(\mathbb{R},[E])$). Let $\varphi(t,v,(B,g))$ (respectively, $\varphi(t,v,B)$) be the solution of equation (11) (respectively, (12)) satisfying the condition $\varphi(0,\, v,\, (B,g)) = v$ (respectively, $\varphi(0,v,B) = v$).

We set $Y := H(A,f)$ and denote the dynamical system of shifts on $H(A,f)$ by $(Y,\mathbb{R},\sigma)$. We put $X := E \times Y$ and define a dynamical system on X by setting $\pi(t,(v,B)) := (\varphi(t,v,(B,g)), B_t, g_t)$ for all $(v,(B,g)) \in E \times Y$ and $t \in \mathbb{R}$. Then $\langle (X,\mathbb{R},\pi), (Y,\mathbb{R},\sigma), h\rangle$ is a group (two-sided) non-autonomous dynamical system, where $h := pr_2 : X \to Y$.

A solution $\varphi \in C(\mathbb{R},E)$ of equation (9) is called [31],[34] *compatible by the character of recurrence* (or simply, *compatible*) if $\mathfrak{N}_{(A,f)} \subseteq \mathfrak{N}_\varphi$, where $\mathfrak{N}_{(A,f)} := \{\{t_n\} \subset \mathbb{R} \mid (A_{t_n}, f_{t_n}) \to (A,f)\}$, and, respectively, $\mathfrak{N}_\varphi := \{\{t_n\} \subset \mathbb{R} \mid \varphi_{t_n} \to \varphi\}$.

Applying the results from Sections 2–3 to this system, we obtain the following statements.

Theorem 4.1. *Let $(A,f) \in C(\mathbb{R},[E]) \times C(\mathbb{R},E)$ be Poisson stable. Suppose that the following conditions hold:*

- (i) *equation (9) admits a relatively compact solution $\varphi(t,u_0,(A,f))$, i.e. there exists $u_0 \in E$ such that $Q_{(u_0,(A,f))} := \overline{\varphi(\mathbb{R},u_0,(A,f))}$ is a compact subset of E;*
- (ii) *all the relatively compact solutions on $\mathbb{R}$ of equation (10) tend to zero as the time tends to ∞, i.e. $\lim\limits_{|t|\to+\infty} |\varphi(t,u,A)| = 0$ if $\varphi(t,u,A)$ is a relatively compact solution (this means that the set $\varphi(\mathbb{R},u,A)$ is relatively compact in E).*

Then, equation (9) possesses a unique compatible solution $\varphi(t,\bar{u},f)$ with values in the compact subset $Q_{(u_0,(A,f))}$.

Proof. Denote by $\langle (X,\mathbb{R},\pi), (Y,\mathbb{R},\sigma), h\rangle$ the group non-autonomous dynamical system generated by equation (9) (see the construction above). By Lemma 3.3, the invariant set $H(x_0) \subset X$ (where $x_0 := (u_0,(A,f)) \in X$ and $\overline{\{\pi(t,x_0) \mid t \in \mathbb{R}\}} := H(x_0)$) is conditionally compact. Let now $x_1, x_2 \in H(x_0) \cap X_{(A,f)}$, where $X_{(A,f)} := E \times \{(A,f)\}$ (i.e. $x_i = (u_i,(A,f))$ and $u_i \in E$ (i=1,2)), then

$$\lim_{|t|\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = \lim_{|t|\to+\infty} |\varphi(t,u_1,(A,f)) - \varphi(y,u_2,(A,f))| = 0.$$

To finish the proof it is sufficient to refer to Theorem 3.6, Corollary 3.7 and Corollary 3.9. $\square$

Corollary 4.2. *Under the assumptions in Theorem 4.1, if* $(A, f) \in C(\mathbb{R}, [E]) \times C(\mathbb{R}, E)$ *is* τ*-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable), then equation (9) admits a unique* τ*-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable) solution.*

Proof. This statement follows from Theorem 4.1 and Corollary 3.10. □

Corollary 4.3. *Under the conditions of Theorem 4.1, if* $(A, f) \in C(\mathbb{R}, [E]) \times C(\mathbb{R}, E)$ *is almost automorphic, then equation (9) admits a unique almost automorphic solution.*

Proof. Since the function $\varphi(t, \bar{u}, (A, f))$ is relatively compact and the functions $A \in C(\mathbb{R}, [E])$ and $f \in C(\mathbb{R}, E)$ are bounded on $\mathbb{R}$, then $\varphi(t, \bar{u}, (A, f))$ is uniformly continuous on $\mathbb{R}$. Thus $\bar{\varphi} := \varphi(\cdot, \bar{u}, (A, f)) \in C(\mathbb{R}, E)$ is a Lagrange stable point of the dynamical system $(C(\mathbb{R}, E), \mathbb{R}, \sigma)$. On the other hand, by Corollary 4.2 the function $\bar{\varphi}$ is Levitan almost periodic and, consequently, it is almost automorphic. □

Corollary 4.4. *Under the conditions of Theorem 4.1 if* $(A, f) \in C(\mathbb{R}, [E]) \times C(\mathbb{R}, E)$ *is Bohr almost periodic, then equation (9) admits a unique almost automorphic solution.*

Proof. This statement follows from Corollary 4.3 because every Bohr almost periodic function is almost automorphic. □

Remark 4.5. *Under the conditions of Corollary 4.4, the unique almost automorphic solution, generally speaking, is not Bohr almost periodic. In* [25]*, one can find an example of a finite-dimensional differential equation of type (9) with Bohr almost periodic coefficients* $A \in C(\mathbb{R}, [E])$*, and* $f \in C(\mathbb{R}, E)$ *such that all bounded solutions* $\varphi(t, u, A)$ *of equation (10) tend to* 0 *as* $|t| \to +\infty$*, but equation (9) does not admit Bohr almost periodic solutions.*

A solution $\varphi \in C(\mathbb{R}, E)$ of equation (9) is called [31],[34] *uniformly compatible by the character of recurrence* if $\mathfrak{M}_{(A,f)} \subseteq \mathfrak{M}_{\varphi}$, where $\mathfrak{M}_{(A,f)} := \{\{t_n\} \subset \mathbb{R} \mid \text{ such that the sequence } \{(A_{t_n}, f_{t_n})\} \text{ is convergent}\}$, and, respectively, $\mathfrak{M}_{\varphi} := \{\{t_n\} \subseteq \mathbb{R} \mid \text{ such that the sequence } \{\varphi_{t_n}\} \text{ is convergent}\}$.

Theorem 4.6. *Let* $(A, f) \in C(\mathbb{R}, [E]) \times C(\mathbb{R}, E)$ *be recurrent. Suppose that the following conditions hold:*

- (i) *equation (9) admits a relatively compact solution* $\varphi(t, u_0, (A, f))$*;*
- (ii) *for all* $B \in H(A)$ *the relatively compact solutions of equation (12) tend to zero as the time tends to* ∞*, i.e.* $\lim\limits_{|t| \to +\infty} |\varphi(t, u, B)| = 0$ *if* $\varphi(t, u, B)$ *is relatively compact on* $\mathbb{R}$ *.*

Then, equation (9) possesses a unique uniformly compatible solution $\varphi(t, \bar{u}, f)$ *with values in the compact subset* $Q_{(u_0,(A,f))}$*.*

Proof. Denote by $\langle (X, \mathbb{R}, \pi), (Y, \mathbb{R}, \sigma), h \rangle$ the group non-autonomous dynamical system, generated by equation (9). Under the conditions of the theorem the invariant set $H(x_0) \subset X$ (where $x_0 := (u_0, (A, f)) \in X$ and $H(x_0) := \overline{\{\pi(t, x_0) \mid t \in \mathbb{R}\}}$) is compact. Let now $x_1, x_2 \in H(x_0) \cap X_{(B,g)}$, where $(B, g) \in H(A, f)$ and $X_{(B,g)} := E \times \{(B, g)\}$ (i.e. $x_i = (u_i, (B, g))$ and $u_i \in E$ (i=1,2)), then

$$\lim_{|t|\to+\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = \lim_{|t|\to+\infty} |\varphi(t, u_1, (B, g)) - \varphi(y, u_2, (B, g))| = 0.$$

To finish the proof it is sufficient to apply now Theorem 3.12 and Corollary 3.13. □

Corollary 4.7. *Under the conditions of Theorem 4.6 if* $(A, f) \in C(\mathbb{R}, [E]) \times C(\mathbb{R}, E)$ *is* τ*-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent), then equation (9) admits a unique* τ*-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent) solution.*

To conclude this subsection we consider particular examples that illustrate the above results.

Example 4.8. *Let* $a \in C(\mathbb{R}, \mathbb{R})$ *be the Bohr almost periodic function defined by*

$$a(t) := \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{3/2}} \sin \frac{t}{2k+1}, \tag{13}$$

and let

$$h(t) := \int_0^t a(s)ds = \sum_{k=0}^{\infty} \frac{2}{(2k+1)^{1/2}} \sin^2 \frac{t}{2(2k+1)}.$$

Note that $a(t + t_n) \to -a(t)$ *uniformly on* $\mathbb{R}$*, where* $t_n := (2n + 1)!!$*. Therefore,* $-a \in H(a) := \overline{\{a_\tau \mid \tau \in \mathbb{R}\}}$*. Using the inequality* $|\sin t| \geq \frac{1}{2}|t|$ *with* $|t| \leq 1$*, we obtain that*

$$h(t) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{1/2}} \sin^2 \frac{t}{2(2k+1)} \geq \sum_{k \geq \frac{1}{2}(\frac{|t|}{2}-1)} \frac{t^2}{8} \frac{1}{(2k+1)^{5/2}}$$

$$\geq \frac{t^2}{8} \int_{|s| \geq \frac{1}{2}(\frac{|t|}{2}-1)} \frac{ds}{(2s+1)^{5/2}} = \frac{t^2 2^{3/2}}{24|t|^{3/2}} = \frac{1}{6\sqrt{2}}|t|^{1/2} \to +\infty$$

as $|t| \to +\infty$*. This implies that the module of all non-zero solutions of the equation*

$$x' = a(t)x \tag{14}$$

tends to $+\infty$ *as* $|t| \to +\infty$*, whereas those of the equation*

$$y' = b(t)y, \tag{15}$$

with $b := -a \in H(a)$ *tend to zero.*

Thus, if $f \in C(\mathbb{R}, \mathbb{R})$ *is a Bohr almost periodic function and the equation*

$$y' = b(t)y + f(t) \tag{16}$$

admits a bounded solution, then according to Theorem 4.1 and Corollary 4.4 it has a unique almost automorphic solution.

The above example is a slight modification of the well-known example of Favard (see [13]-[14]). Our case differs from Favard's example in that the solutions of equation (15) are not only bounded on $\mathbb{R}$, but they tend to zero as $|t| \to +\infty$. Thus, a non-zero solution of equation (15) is asymptotically stable, but the zero solution of equation (14) is not, even though $a \in H(b)$.

Example 4.9. *Consider the following two-dimensional system of linear differential equations*

$$x' = A(t)x,$$

where

$$A(t) = \begin{pmatrix} -a(t) & -b(t) \\ b(t) & -a(t) \end{pmatrix}, \tag{17}$$

the function $a(t)$ *is defined by equality (13) and* $b(t) := (2 + \sin t + \sin \sqrt{2}t)^{-1}$ *for all* $t \in \mathbb{R}$. *It is easy to check that the matrix* $A(t)$ *is Levitan almost periodic, but not almost automorphic because it is unbounded on* $\mathbb{R}$.

Let $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ *be a solution of the system (17), then we have*

$$\frac{d}{dt}(x_1^2(t) + x_2^2(t)) = -2a(t)(x_1^2(t) + x_2^2(t)) \tag{18}$$

for all $t \in \mathbb{R}$. *It follows from (18) that*

$$|\varphi(t, x, A)| = e^{-h(t)}|x| \tag{19}$$

for all $x \in \mathbb{R}^2$ *and* $t \in \mathbb{R}$, *where* $\varphi(t, x, A)$ *is a solution of equation (17) passing through the point* $x \in \mathbb{R}^2$ *at the initial moment, and* $h(t) := \int_0^t a(s)ds$. *Taking into account the results in Example 4.8 we conclude that* $\lim\limits_{|t| \to +\infty} |\varphi(t, x, A)| = 0$.

Thus, if $f \in C(\mathbb{R}, \mathbb{R}^2)$ *is a Levitan almost periodic function and the equation*

$$y' = A(t)y + f(t)$$

admits a bounded solution then, according to Theorem 4.1 and Corollary 4.2, it has a unique Levitan almost periodic solution.

4.2. **Linear Difference Equations.** As our second class of applications, consider the following difference equation

$$u(t + 1) = A(t)u(t) + f(t) \tag{20}$$

and its corresponding homogeneous equation

$$u(t + 1) = A(t)u(t), \tag{21}$$

where $(A, f) \in C(\mathbb{Z}, [E]) \times C(\mathbb{Z}, E)$. Along with equations (20) and (21), we also consider the H-class of equation (20) (respectively, (21)), that is the family of equations

$$v(t + 1) = B(t)v(t) + g(t), \tag{22}$$

(respectively,

$$v(t + 1) = B(t)v(t) \quad) \tag{23}$$

with $(B, g) \in H(A, f) := \overline{\{(A_\tau, f_\tau) \mid \tau \in \mathbb{Z}\}}$ (respectively, $B \in H(A)$), $A_\tau(t) = A(t + \tau)$, $f_\tau(t) := f(t + \tau)$ and $t \in \mathbb{Z}$, where the bar denotes closure in $C(\mathbb{Z}, [E]) \times C(\mathbb{Z}, E)$ (respectively, $C(\mathbb{Z}, [E])$). Let $\varphi(t, v, (B, g))$ (respectively, $\varphi(t, v, B)$) be the solution of equation (22) (respectively, (23)) that satisfies the condition $\varphi(0, v, (B, g)) = v$ (respectively, $\varphi(0, v, B) = v$).

Now, in order to have a two-sided non-autonomous dynamical system, we need to impose the following condition on the rest of the paper:

Example 4.10. ***Condition (C)****: the operator $B(n)$ is invertible for all $B \in H(A)$ and $n \in \mathbb{Z}$.*

Remark 4.11. *Assuming that condition (C) is fulfilled from now on, we can ensure that the solution $\varphi(n, v, (B, g))$ of equation (22) is defined on the whole $\mathbb{Z}$.*

We set now $Y := H(A, f)$, and denote the dynamical system of shifts on $H(A, f)$ by $(Y, \mathbb{Z}, \sigma)$. Consider $X := E \times Y$, and define a dynamical system on X by setting $\pi(\tau, (v, B, g)) := (\varphi(\tau, v, (B, g)), B_\tau, g_\tau)$ for all $(v, (B, g)) \in E \times Y$ and $\tau \in \mathbb{Z}$. Then $\langle (X, \mathbb{Z}, \pi), (Y, \mathbb{Z}, \sigma), h \rangle$ is a group non-autonomous dynamical system, where $h := pr_2 : X \to Y$.

Our aim now is to apply the results of Sections 2–3 to this system, and obtain some results concerning the difference equation (22).

As before, a solution $\varphi \in C(\mathbb{Z}, E)$ of equation (20) is called [34] compatible by the character of recurrence if $\mathfrak{N}_{(A,f)} \subseteq \mathfrak{N}_\varphi$, where $\mathfrak{N}_{(A,f)} := \{\{t_n\} \subset \mathbb{Z} \mid (A_{t_n}, f_{t_n}) \to (A, f)\}$, and, respectively, $\mathfrak{N}_\varphi := \{\{t_n\} \subset \mathbb{Z} \mid \varphi_{t_n} \to \varphi\}$.

Following a scheme similar to the one used in the first application (and which is motivated by the structure of the general theory developed in Section 3) we can prove now similar results for the discrete non-autonomous dynamical system generated by equation (20). Although the proofs may seem a repetition of the previous ones, they are necessary to justify every statement, and this is the reason why we prefer not to omit them.

Theorem 4.12. *Let $(A, f) \in C(\mathbb{Z}, [E]) \times C(\mathbb{Z}, E)$ be Poisson stable. Suppose that the following conditions hold:*

- (i) *equation (20) admits a relatively compact solution $\varphi(t, u_0, (A, f))$, i.e. the set $Q_{(u_0,(A,f))} := \overline{\varphi(\mathbb{Z}, u_0, (A, f))}$ is compact in E;*
- (ii) *all the relatively compact solutions of equation (21) tend to zero as the time t tends to ∞, i.e. $\lim\limits_{|t|\to+\infty} |\varphi(t, u, A)| = 0$ if $\varphi(n, u, A)$ is relatively compact (this means that the set $\varphi(\mathbb{Z}, u, A))$ is relatively compact in E).*

Then, equation (20) has a unique compatible solution $\varphi(n, \bar{u}, f)$ with values from the compact $Q_{(u_0,(A,f))}$.

Proof. Denote by $\langle (X, \mathbb{Z}, \pi), (Y, \mathbb{Z}, \sigma), h \rangle$ the group non-autonomous dynamical system generated by equation (20) (see construction above). By Lemma 3.3, under the conditions of the theorem, the invariant set $H(x_0) \subset X$ (where $x_0 := (u_0, (A, f)) \in X$ and $H(x_0) := \overline{\{\pi(t, x_0) \mid t \in \mathbb{R}\}}$) is conditionally compact. Let now $x_1, x_2 \in H(x_0) \cap X_{(A,f)}$, where $X_{(A,f)} := E \times \{(A, f)\}$ (i.e. $x_i = (u_i, (A, f))$ and $u_i \in E$ (i=1,2)), then

$$\lim_{|t|\to+\infty} \rho(\pi(t, x_1), \pi(t, x_2)) = \lim_{|t|\to+\infty} |\varphi(t, u_1, (A, f)) - \varphi(t, u_2, (A, f))| = 0.$$

Now to finish the proof it is sufficient to refer to Theorem 3.6, Corollary 3.7 and Corollary 3.9. □

Corollary 4.13. *Under the conditions of Theorem 4.12 if $(A,f) \in C(\mathbb{Z},[E]) \times C(\mathbb{Z},E)$ is τ-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable), then equation (20) admits a unique τ-periodic (respectively, Levitan almost periodic, almost recurrent, Poisson stable) solution.*

Proof. This statement follows from Theorem 4.12 and Corollary 3.10. □

Corollary 4.14. *Under the conditions of Theorem 4.12 if $(A,f) \in C(\mathbb{Z},[E]) \times C(\mathbb{Z},E)$ is almost automorphic, then equation (9) admits a unique almost automorphic solution.*

Proof. Since the function $\varphi(t,\bar{u},(A,f))$ is relatively compact, it easily follows that $\bar{\varphi} := \varphi(\cdot,\bar{u},(A,f)) \in C(\mathbb{Z},E)$ is a Lagrange stable point of the dynamical system $(C(\mathbb{Z},E),\mathbb{Z},\sigma)$. On the other hand, by Corollary 4.13 the function $\bar{\varphi}$ is Levitan almost periodic and, consequently, almost automorphic. □

Corollary 4.15. *Under the conditions of Theorem 4.12 if $(A,f) \in C(\mathbb{Z},[E]) \times C(\mathbb{Z},E)$ is Bohr almost periodic, then equation (20) admits a unique almost automorphic solution.*

Proof. This statement follows from Corollary 4.14 because every Bohr almost periodic function is almost automorphic. □

A solution $\varphi \in C(\mathbb{Z},E)$ of equation (20) is called (see [31],[34]) *uniformly compatible by the character of recurrence*, if $\mathfrak{M}_{(A,f)} \subseteq \mathfrak{M}_{\varphi}$, where $\mathfrak{M}_{(A,f)} := \{\{t_n\} \subset \mathbb{Z} \mid$ such that the sequence $\{(A_{t_n},f_{t_n})\}$ is convergent $\}$ (respectively, $\mathfrak{M}_{\varphi} := \{\{t_n\} \subset \mathbb{Z} \mid$ such that the sequence $\{\varphi_{t_n}\}$ is convergent $\}$).

Theorem 4.16. *Let $(A,f) \in C(\mathbb{Z},[E]) \times C(\mathbb{Z},E)$ be recurrent. Suppose that the following conditions hold:*

(i) *the equation (20) admits a relatively compact on $\mathbb{Z}$ solution $\varphi(t,u_0,(A,f))$;*
(ii) *for all $B \in H(A)$ the relatively compact on $\mathbb{Z}$ solutions of equation (23) tend to zero as the time tends to ∞, i.e. $\lim_{|t|\to+\infty} |\varphi(t,u,B)| = 0$, if $\varphi(t,u,B)$ is relatively compact on $\mathbb{Z}$.*

Then, equation (20) has a unique uniformly compatible solution $\varphi(t,\bar{u},f)$ with values from the compact $Q_{(u_0,(A,f))}$.

Proof. Denote by $\langle (X,\mathbb{Z},\pi),(Y,\mathbb{Z},\sigma),h\rangle$ the group non-autonomous dynamical system generated by equation (20). Under the conditions of the theorem, the invariant set $H(x_0) \subset X$ (where $x_0 := (u_0,(A,f)) \in X$ and $H(x_0) := \overline{\{\pi(t,x_0) \mid t \in \mathbb{Z}\}}$) is compact. Let now $x_1,x_2 \in H(x_0) \cap X_{(B,g)}$, where $(B,g) \in H(A,f)$ and $X_{(B,g)} := E \times \{(B,g)\}$ (i.e. $x_i = (u_i,(B,g))$ and $u_i \in E$ (i=1,2)). Then

$$\lim_{|t|\to+\infty} \rho(\pi(t,x_1),\pi(t,x_2)) = \lim_{|t|\to+\infty} |\varphi(t,u_1,(B,g)) - \varphi(y,u_2,(B,g))| = 0.$$

Now to finish the proof it is sufficient to refer to Theorem 3.12. □

Corollary 4.17. *Under the conditions of Theorem 4.16, if* $(A, f) \in C(\mathbb{Z}, [E]) \times C(\mathbb{Z}, E)$ *is* τ*-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent), then equation (20) admits a unique* τ*-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent) solution.*

Proof. This statement follows from Theorem 4.16 and Corollary 3.13. □

Remark 4.18. *Note that the results of this subsection can also hold even without assuming condition (C), but the proofs in this case may need of some new ideas and abstract results. Briefly, the main difference is as follows: as we have seen, equation (22) with condition (C) generates a group non-autonomous dynamical system, but without condition (C) the non-autonomous dynamical system generated by (22) is only one-sided (i.e. a semi-group system). We plan to develop this situation in our next paper* [6].

Acknowledgments

1. The authors would like to thank the anonymous referee for his/her comments and suggestions on a preliminary version of this article.

2. This paper was written while the second author was visiting the Department of Differential Equations and Numerical Analysis, University of Seville (Spain), in the winter of 2007/2008. He would like to thank people of this university for their very kind hospitality. He also gratefully acknowledges the financial support of the Consejería de Innovación, Ciencia y Empresa de la Junta de Andalucía, and the Ministerio de Educación y Ciencia (Spain) under the project MTM2005-01412. The first author is also partially supported by Proyecto de Excelencia de la Junta de Andalucía P07-FQM-02468.

References

[1] J. Auslander, *Minimal Flows and Theirs Extensions,* North Holland-Amsterdam/New York/Oxford/Tokyo, 1988.

[2] B.R. Basit, A connection between the almost periodic functions of Levitan and almost automorphic functions, *Vestnik Moskov. Univ. Ser. I Mat. Meh.* 26 (1971), no. 4, 11–15.

[3] B. Basit, Les fonctions abstraites presques automorphiques et presque périodiques au sens de Levitan, et leurs différence, *Bull. Sci. Math. (2),* 101 (1977), no. 2, 131–148.

[4] S. Bochner, A new approach to almost periodicity, *Proc. Nat. Acad. Sci. U.S.A.,* 48 (1962), 2039–2043.

[5] I.U. Bronsteyn, *Extensions of Minimal Transformation Group,* Noordhoff, 1979.

[6] T. Caraballo, D.N. Cheban, Almost periodic and almost automorphic solutions of linear differential equations without Favard's separation condition. II, in preparation.

[7] D.N. Cheban, Global pullback attractors of $\mathbb{C}$-analytic non-autonomous dynamical systems, *Stochastics and Dynamics* 1 (2001), no. 4, 511-536.

[8] D.N. Cheban, *Global Attractors of Non-Autonomous Dissipative Dynamical Systems*, Interdisciplinary Mathematical Sciences 1, River Edge, NJ: World Scientific, 2004.

[9] D. N. Cheban, Levitan almost periodic and almost automorphic solutions of V-monotone differential equations, *Journal of Dynamics and Differential Equations,* DOI:10.1007/s10884-008-9101-x (2008).

[10] J. Egawa, A characterization of almost automorphic functions, *Proc. Japan Acad.* Ser. A Math. Sci. 61 (1985), no. 7, 203–206.

[11] R. Ellis, *Lectures on Topological Dynamics,* W. A. Benjamin, Inc. New York, 1969.

[12] R. Ellis, R.A. Johnson, Topological dynamics and linear differential systems. *J. Diff. Eqns.* 44 (1982), no. 1, 21–39.

[13] J. Favard, Sur les equations differentielles a coefficients presque-periodiques, *Acta Math.* 51 (1927), 31–81.

[14] J. Favard, *Leçons sur Fonction Presque-Periodiques,* Gauthier-Villars, Paris, 1933.

[15] L. Faxing, The existence of almost–automorphic solutions of almost–automorphic systems, *Ann. of Diff. Eqs.* 3 (1987), no. 3, 329–349.

[16] A.M. Fink, *Almost Periodic Differential Equations,* Lecture notes in Mathematics, Vol. 377, Springer-Verlag, 1974.

[17] R. A. Johnson, Ergodic theory and linear differential equations, *J. Diff. Eqns.* 28 (1978), 23–34.

[18] R.A. Johnson, A linear almost periodic equation with almost automorphic solution, *Proc. Amer. Math. Soc.* 82 (1981), 199–205.

[19] B. M. Levitan, *Almost Periodic Functions,* Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953 (in Russian).

[20] B.M. Levitan, V.V. Zhikov, *Almost Periodic Functions and Differential Equations,* Cambridge Univ. Press, London, 1982.

[21] M. G. Lyubarskii, An extension of Favard's theory to the case of a system of linear differential equations with unbounded Levitan almost periodic coefficients (Russian), *Dokl. Akad. Nauk SSSR* 206 (1972), 808–810.

[22] P. Milnes, Almost automorphic functions and totally bounded groups, *Rocky Mountain J. Math.* 7 (1977), no. 2, 231–250.

[23] G.M. N'Guerekata, *Almost automorphic and almost periodic functions in abstract spaces,* Kluwer Academic/Plenum Publishers, New York, 2001.

[24] G.M. N'Guerekata, *Topics in Almost Automorphy,* Springer-Verlag, New York, 2005.

[25] R. Ortega, M. Tarallo, Almost periodic linear differential equations with non-separated solutions, *J. Funct. Anal.,* 237 (2006), no. 2, 402–426.

[26] R. Sacker, G. R. Sell, Finite extensions of minimal transformation groups, *Trans. Amer. Math. Soc.,* 190 (1974), 325–334.

[27] R. Sacker R., G. R. Sell, *Lifting Properties in Skew–Product Flows with Application to Differential Equations*, Mem. Amer. Math. Soc. 11 (1977), no. 190.

[28] G. R. Sell, *Lectures on Topological Dynamics and Differential Equations,* Van Nostrand–Reinbold, London, 1971.

[29] G. R. Sell, A remark on an example of R. A. Johnson, *Proc. Amer. Math. Soc.* 82(1981), no. 2, 206–208.

[30] G. R. Sell, W. Shen, Y. Yi, Topological dynamics and differential equations., *Topological dynamics and applications* (Minneapolis, MN, 1995), 279–297, Contemp. Math., 215, Amer. Math. Soc., Providence, RI, 1998.

[31] B.A. Shcherbakov, *Topologic Dynamics and Poisson Stability of Solutions of Differential Equations*, Ştiinţa, Chişinău, 1972. (In Russian)

[32] B.A. Shcherbakov, The comparability of the motions of dynamical systems with regard to the nature of their recurrence, *Differential Equations* 11 (1975), no. 7, 1246–1255.

[33] B. A. Shcherbakov, The nature of the recurrence of the solutions of linear differential systems (in Russian), *An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.)* 21 (1975), 57–59.

[34] B.A. Shcherbakov, *Poisson Stability of Motions of Dynamical Systems and Solutions of Differential Equations*, Ştiinţa, Chişinău, 1985. (In Russian)

[35] W. Shen, Y. Yi, Almost automorphic and almost periodic dynamics in skew-product semiflows, *Mem. Amer. Math. Soc.* 136 (1998), no. 647.

[36] M. A. Subin, Almost periodic functions and partial differential operators, *Russ. Math. Surv.* 33 (1978), no. 2, 1-52.

[37] Shubin M., Local Favard theory, *Mosc. Univ. Math. Bul.,* 34 (1978), no.2, 32–37 .

[38] K.S. Sibirsky, *Introduction to Topological Dynamics,* Noordhoff, Leyden, 1975.

[39] V.V. Zhikov, B.M. Levitan, Favard theory, *Russian Math. Survey* 32 (1977), 129-180.

E-mail address, T. Caraballo: `caraball@us.es`

E-mail address, D. Cheban: `cheban@usm.md`

(T. Caraballo) Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. Correos 1160, 41080-Sevilla (Spain)

(D. Cheban) State University of Moldova, Department of Mathematics and Informatics, A. Mateevich Street 60, MD–2009 Chişinău, Moldova