Random Dynamics and Limiting Behaviors for 3D Globally Modified Navier-Stokes Equations Driven by Colored Noise

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Abstract. This paper is mainly concerned with the long-term random dynamics for the non-autonomous 3D globally modified Navier-Stokes equations with nonlinear colored noise. We first prove the existence of random attractors of the non-autonomous random dynamical system generated by the solution operators of such equations. Then we establish the existence of invariant measures supported on the random attractors of the underlying system. Random Liouville type theorem is also derived for such invariant measures. Moreover, we further investigate the limiting relationship of invariant measures between the above equations and the corresponding limiting equations when the noise intensity approaches to zero. In addition, we show the invariant measures of such equations with additive white noise can be approximated by those of the corresponding equations with additive colored noise as the correlation time of the colored noise goes to zero.

Keywords. Globally modified Navier-Stokes equations, random attractor, invariant measure, random Liouville type theorem, limit measure.

AMS 2020 Mathematics Subject Classification. 37L40, 37L55, 35B41, 76F20.

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1 Introduction

In this paper, we investigate the pullback random attractors and invariant measures of the following non-autonomous 3D globally modified Navier-Stokes equations (GMNSE) with nonlinear colored noise:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + F_N(\|u\|)\left[(u(t)\cdot\nabla)u\right] + \nabla p = f(t) + \varepsilon G(u)\mathcal{G}_\delta(\theta_t\omega) \text{ in } \mathcal{O}\times(\tau,\infty), \\ \text{div } u = 0 \text{ in } \mathcal{O}\times(\tau,\infty), \\ u = 0 \text{ on } \Gamma\times(\tau,\infty), \\ u(x,\tau) = u_\tau(x),\ x\in\mathcal{O}, \end{cases} \tag{1.1}$$

where $\varepsilon \in (0,1]$, $\mathcal{O}\subset\mathbb{R}^3$ is an open bounded set with regular boundary Γ, $\nu > 0$ is the viscosity coefficient, u denotes the velocity field, $F_N(\cdot) : (0,\infty) \to (0,1]$ is defined as $F_N(r) = \min\left\{1, \dfrac{N}{r}\right\}$ where $r, N > 0$, p denotes the pressure, $f \in L^2_{loc}(\mathbb{R}; H)$ is an external force field, G is a nonlinear diffusion term, $\mathcal{G}_\delta$ is a colored noise with correlation time $\delta \in (0,1]$, which will be specified later.

Navier-Stokes equations can be used to describe the time evolution of an incompressible fluid, which are important in fluid and mechanics and turbulence. The well-posedness, regularity and asymptotic behaviors of solutions of 2D Navier-Stokes equations have been studied in the literature, see, e.g., [30, 32] and references therein. However, as we all know, the uniqueness of weak solutions and the existence of strong solutions of 3D Navier-Stokes equations are still open problems due to the nonlinear convection term. It is worth mentioning that in order to overcome the difficulties caused by the nonlinear convection term, T. Caraballo and his collaborators processed a class of 3D globally modified Navier-Stokes equations in [3], and proved the existence and uniqueness of solutions as well as the existence of V-attractors. After that, the existence and asymptotic behaviors of solutions of 3D GMNSE have been investigated in [16, 22, 28, 45]. And the invariant measures as well as the Liouville type theorem of 3D GMNSE have been examined in [45, 49] and the references therein.

Since noise is ubiquitous in many physical systems due to the lack of knowledge of certain physical parameters as well as measurement errors arising in experiments or modeling, it is of great significance to consider noise influences on Navier-Stokes equations. For stochastic 2D Navier-Stokes equations, the existence of solutions, random attractors and invariant measures have been reported in [1, 13, 14, 31]. But there are a few results on the 3D GMNSE with stochastic perturbations. Recently, reference [15] investigated the well-posedness and the convergence of solutions for 3D GMNSE with white noise. In addition, it is also significant to study the pathwise dynamics of such system, for which we need to define a pathwise random dynamical system via the solution operators. However, the existence of such a random dynamical system is unknown for nonlinear white noise until now, which results in the pathwise random attractors theory cannot be applied to this case. Therefore, systems driven by colored noise have drawn considerable attention in the literature, see, e.g. [10, 19, 38, 39, 42] and the references therein. In fact, many physical systems can be simulated by colored noise rather than white noise [29]. The colored noise is a kind of auto-correlated noise, and can be used to approximate the Wiener process [33]. From the perspective of theoretical and practical value, we aim to study the non-autonomous 3D GMNSE with nonlinear colored noise (1.1), which can be considered as an approximation of the corresponding system with white noise in some sense. We also would like to mention some research on the approximation of stochastic differential equations driven by white noise [18, 20, 43].

In this paper, we will first investigate the existence and uniqueness of pullback random attractors of (1.1). Recently, the pullback random attractors for 2D Navier-Stokes equations with colored noise have been reported in [19]. However, the argument therein cannot be directly applied to 3D Navier-Stokes equations with colored noise due to differences regarding Sobolev embedding inequalities for nonlinear convection term. Thanks to the modifying factor $F_N(\cdot)$ in (1.1), we are able to establish the existence of pullback random attractors for 3D GMNS with colored noise. More precisely, by the Galerkin method and the property of the modifying factor $F_N(\cdot)$, the well-posedness of (1.1) can be shown, which leads to the existence of a random dynamical system. Then, based on the uniform estimates of solutions in spaces H and V as well as the compactness of injection of V into H, we examine the pullback asymptotic compactness of solutions of (1.1), which together with [36, Theorem 2.23] induces the existence of random attractors. For the long-term behaviors of stochastic partial differential equation, the reader is referred to [2, 39, 42, 44] and the reference therein.

Another major goal of this paper is to study the invariant measures of non-autonomous 3D

GMNSE with nonlinear colored noise (1.1), which can provide important information about long-term dynamics and be used to identify statistical equilibrium. Based on the existence of random attractors of (1.1), we will first prove the existence of invariant measures supported on such attractors. In the deterministic case, the existence of invariant measures has been studied by many experts, see, e.g., [4, 23] for autonomous systems and [17, 27] for non-autonomous systems. It is worth mentioning that [13, 48] investigated the existence of invariant measures for autonomous random dynamical systems. However, these results in above papers cannot be applied to non-autonomous system with colored noise (1.1). Recently, by virtue of the theory of non-autonomous random dynamical system in [36] together with the generalized Banach limit in [17], reference [9] constructed a family of invariant measures for general non-autonomous random dynamical systems. In order to apply such abstract theory to system (1.1), we need to verify the system (1.1) is jointly continuous in initial time and initial value (see Lemma 3.1), by which we can obtain the existence of invariant measures supported on the random attractors (see Theorem 3.2). In addition, if the non-autonomous term $f(t)$ is periodic in t, we further show such invariant measure is also periodic. We also refer the readers to [1, 7, 11, 14, 21, 24, 26, 37, 41] for the existence of invariant measures and periodic measures of Markov semigroups generated by infinite-dimensional stochastic differential equations.

Then we also derives that the invariant measures constructed in above satisfy a random Liouville type equation. For deterministic differential systems, it has been shown that the time average invariant measures satisfy a Liouville type equation in [12, 17, 27, 46, 47]. Hereafter, reference [48] established that invariant measures of autonomous 2D Navier-Stokes equations with additive white noise also satisfy a random Liouville type equation. Despite some advances in this direction, there is no relevant result for non-autonomous GMNSE with nonlinear colored noise. Indeed, it is difficult to verify the continuity of the mapping $u \mapsto \langle F(t, \theta_t\omega, u(t)), \Psi'(u) \rangle$ in H for given test function Ψ in order to use the invariant property of invariant measures, where F is given in (4.2). To overcome this issue, what we first do is to approximate $\Psi(u)$ by using the Galerkin projection technique. Then together with the form of the construction of invariant measures and the cocycle property of the non-autonomous random dynamical system, and taking the limit, we will show that the invariant measures supported on the pullback random attractors satisfy a random Liouville type equation (see Theorem 4.1).

In order to show the limiting relationship of invariant measures between GMNSE with colored noise and the corresponding deterministic equations, we further study the limiting behaviors of invariant measures of (1.1) as the noise intensity parameter ε goes to zero, which also means the zero-noise limit is observable as noise is non-negligible in real world. Note that the limiting behaviors of invariant measures have been investigated in [34, 35] for deterministic equations, in [5, 6, 8, 25] for autonomous stochastic systems, and in [40] for non-autonomous stochastic systems.

Different from [40] where discussed the convergence of invariant measures of time inhomogeneous transition operators, we will study the convergence of invariant measures of the non-autonomous cocycle associated with (1.1). For that purpose, we will first prove the set of all random attractors of (1.1) for $\varepsilon \in (0,1]$ is precompact (see Lemma 5.1). And then, together with [9, Theorem 4.1] and the convergence of solutions with respect to noise intensity (see Lemma 5.2), we can obtain that any limit of invariant measures of (1.1) must be an invariant measure of the corresponding deterministic equations as $\varepsilon \to 0$.

On the other hand, we also interested in the approximation of invariant measures of GMNSE with additive white noise by those of GMNSE with additive colored noise. Therefore, we consider the limiting behaviors of invariant measures of GMNSE with additive colored noise as the correlation time δ of the colored noise tends to zero. More precisely, we will consider the following GMNSE with additive white noise:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + F_N(\|u\|)\left[(u(t)\cdot\nabla)u\right] + \nabla p = f(t) + \beta \dfrac{dW}{dt} \text{ in } \mathcal{O}\times(\tau,\infty), \\ \text{div } u = 0 \text{ in } \mathcal{O}\times(\tau,\infty), \\ u = 0 \text{ on } \Gamma\times(\tau,\infty), \\ u(x,\tau) = u_\tau(x),\ x\in\mathcal{O}, \end{cases} \tag{1.2}$$

where $\beta \in D(A)$ which will be specified later in Section 2. We prove the equation (1.2) has a family of invariant measures supported on the pullback random attractors (see Theorem 5.2), which can be considered as the limit of the invariant measures of the following random equations with colored noise as $\delta \to 0$:

$$\begin{cases} \dfrac{\partial u_\delta}{\partial t} - \nu \Delta u_\delta + F_N(\|u_\delta\|)\left[(u_\delta(t)\cdot\nabla)u_\delta\right] + \nabla p = f(t) + \beta \mathcal{G}_\delta(\theta_t\omega) \text{ in } \mathcal{O}\times(\tau,\infty), \\ \text{div } u_\delta = 0 \text{ in } \mathcal{O}\times(\tau,\infty), \\ u_\delta = 0 \text{ on } \Gamma\times(\tau,\infty), \\ u_\delta(x,\tau) = u_\tau(x),\ x\in\mathcal{O}, \end{cases} \tag{1.3}$$

which is a particular case of (1.1). Indeed, we show that any limit of a sequence of invariant measures of the random equations (1.3) must be an invariant measure of the stochastic equations (1.2) as $\delta \to 0$.

This paper is organized as follows. In the next section, we show the existence of pullback random attractors for GMNSE with nonlinear colored noise (1.1). In Section 3, we prove the existence of invariant measures of (1.1). In Section 4, we derive that such invariant measures satisfy a random Liouville type equation. Section 5 is devoted to the limiting behaviors of invariant measures with respect to noise intensity ε and correlation time δ.

2 Existence of random attractors

In this section, we will investigate the existence of pullback random attractors of GMNSE with nonlinear colored noise (1.1). For this purpose, let us first recall some results from the theory of random dynamical system, see [36] for more details.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is its Borel σ-algebra, and $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$. There exists a classical group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(\Omega, \mathcal{F}, \mathbb{P})$ is given by $\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t)$ for any $t \in \mathbb{R}$ and $\omega \in \Omega$.

Let (X, d) be a complete separable metric space with Borel σ-algebra $\mathcal{B}(X)$. Let $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of nonempty subsets of X satisfying $\lim_{t\to-\infty} e^{ct} \sup_{u\in D(\tau+t,\theta_t\omega)} d(x, 0) = 0$ for every $c > 0$. Such D is said to be tempered in X. And denote by $\mathcal{D}$ the class of all tempered families of nonempty subsets of X.

Definition 2.1 ([36]). *A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$,*

(i) *$\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;*

(ii) *$\Phi(0, \tau, \omega, \cdot)$ is the identity on X;*

(iii) *$\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$;*

(iv) *$\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous.*

If, in addition, there exists $T > 0$ such that for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi(t, \tau + T, \omega, \cdot) = \Phi(t, \tau, \omega, \cdot),$$

then Φ is called a periodic cocycle on X with period T.

Definition 2.2 ([36]). *A family $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback absorbing set for Φ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and for every $D \in \mathcal{D}$, there exists $T_0 = T_0(D, \tau, \omega) > 0$ such that*

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega), \quad \forall\ t \geqslant T_0.$$

If, in addition, for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $K(\tau, \omega)$ is a closed nonempty subset of X and K is measurable in ω with respect to $\mathcal{F}$, then we say K is a closed measurable $\mathcal{D}$-pullback absorbing set for Φ.

Definition 2.3 ([36]). *A family $\mathcal{A} = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a $\mathcal{D}$-pullback random attractor for Φ if for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

(i) *$\mathcal{A}$ is measurable in ω with respect to $\mathcal{F}$, and $A(\tau, \omega)$ is compact in X;*

(ii) *$\mathcal{A}$ is invariant, that is, for all $t \geqslant 0$, $\Phi(t, \tau, \omega, A(\tau, \omega)) = A(\tau + t, \theta_t\omega)$;*

(iii) *$\mathcal{A}$ attracts all sets in $\mathcal{D}$, that is, for all $D \in \mathcal{D}$,*

$$\lim_{t\to+\infty} d_X(\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)), A(\tau, \omega)) = 0,$$

where d_X is the Hausdorff semi-metric given by $d_X(A,B) = \sup_{a\in A}\inf_{b\in B} d(a,b)$ for any $A \subseteq X$ and $B \subseteq X$.

If, in addition, there exists $T > 0$ such that for any $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $A(\tau + T, \omega) = A(\tau, \omega)$, then we say $\mathcal{A}$ is periodic with period T.

Next, the colored noise will be introduced. Given $\delta > 0$, consider the one-dimensional stochastic equation

$$d\mathcal{G}_\delta + \frac{1}{\delta}\mathcal{G}_\delta dt = \frac{1}{\delta} dW, \tag{2.1}$$

where W is a two-sided real-valued Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}_\delta : \Omega \to \mathbb{R}$ be a random variable given by

$$\mathcal{G}_\delta(\omega) = \frac{1}{\delta}\int_{-\infty}^{0} e^{\frac{s}{\delta}} dW, \ \forall\ \omega \in \Omega.$$

Then, one can check that $\mathcal{G}_\delta(\theta_t\omega)$ is the unique stationary solution of (2.1). The process $\mathcal{G}_\delta(\theta_t\omega)$ is called a real-valued colored noise which is a stationary Gaussian process with mean zero and variance $\frac{1}{2\delta}$. And there exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset (still denoted by) Ω of full measure, such that for any $\omega \in \Omega$,

$$\lim_{t\to\pm\infty} \frac{|\mathcal{G}_\delta(\theta_t\omega)|}{t} = 0, \text{ for any } 0 < \delta \leqslant 1, \tag{2.2}$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \mathcal{G}_\delta(\theta_s\omega) ds = \mathbb{E}[\mathcal{G}_\delta] = 0, \text{ uniformly for } 0 < \delta \leqslant 1. \tag{2.3}$$

Now, let us recall some abstract spaces which will be frequently used in the sequel.
$\mathcal{V} = \big\{\phi \in (\mathcal{C}_0^\infty(\mathcal{O}))^3 : \nabla \cdot \phi = 0\big\}$,
H = closure of $\mathcal{V}$ in $(L^2(\mathcal{O}))^3$ with inner product $(\cdot,\cdot)$ and associate norm $|\cdot|$,
V = closure of $\mathcal{V}$ in $(H_0^1(\mathcal{O}))^3$ with inner product $((\cdot,\cdot))$ and associate norm $\|\cdot\|$,
H' = dual space of H, V' = dual space of V with norm $\|\cdot\|_{V'}$,
$\langle\cdot,\cdot\rangle$ denotes the dual pairing between V and V'.
It is clear that $V \subset H \equiv H' \subset V'$, where the injections are dense and compact.

Next, consider the operator $A : V \to V'$ defined by $\langle Au, v\rangle = ((u, v))$. And denote $D(A) = (H^2(\mathcal{O}))^3 \cap V$. Then for any $u \in D(A)$, $Au = -P\triangle u$ is the Stokes operator, where P is the ortho-projector from $(L^2(\mathcal{O}))^3$ onto H. We denote by $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ the eigenvalues of A. In this context, the Poincaré inequality reads

$$\lambda_1|u|^2 \leqslant \|u\|^2, \ \forall\ u \in V. \tag{2.4}$$

Then define a trilinear form $b(\cdot,\cdot,\cdot)$ as follows,

$$b(u, v, w) = \sum_{i,j=1}^{3}\int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad \forall\ u, v, w \in V,$$

and set

$$\langle B_N(u,v), w\rangle = b_N(u,v,w) = F_N(\|v\|)b(u,v,w).$$

Recall that there exists a constant $C > 0$ depending only on $\mathcal{O}$ such that

$$|b(u,v,w)| \leqslant C\|u\|\|v\||w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}, \qquad \forall\, u,v,w \in V, \tag{2.5}$$

$$|b(u,v,w)| \leqslant C\|u\|\|v\|^{\frac{1}{2}}|Av|^{\frac{1}{2}}|w|, \qquad \forall\, u \in V, v \in D(A), w \in H. \tag{2.6}$$

For any $p, r, N > 0$, one can obtain that

$$|F_N(p) - F_N(r)| \leqslant \frac{1}{N}F_N(p)F_N(r)|p-r|, \tag{2.7}$$

$$|F_N(p) - F_N(r)| \leqslant \frac{|p-r|}{r}, \tag{2.8}$$

$$|F_M(p) - F_N(p)| \leqslant \frac{|M-N|}{p}. \tag{2.9}$$

Moreover, we assume the nonlinear diffusion term $G(u) = \alpha u + h(u) + \beta$ with $\alpha \geqslant 0$, $\beta \in H$ and $h : H \to H$ satisfying that

(G1) there exists $L_h > 0$ such that for all $u, v \in H$,

$$|h(u) - h(v)| \leqslant L_h|u-v|,$$

(G2) there exist $\beta_1, \beta_2 > 0$ and $\kappa \in [0,1)$ such that for all $u \in H$,

$$|(h(u), u)| \leqslant \beta_1|u|^{1+\kappa} + \beta_2.$$

In order to obtain some energy estimates, we need an additional assumption.

(F1) Let $\gamma \in (0, \nu\lambda_1)$ be a fixed number and $\int_{-\infty}^{t} e^{\gamma s}|f(s)|^2 ds < +\infty$ for all $t \in \mathbb{R}$.

Sometimes, the following tempered condition is also needed.

(F2) For every $c > 0$, $\lim_{t\to-\infty} e^{ct}\int_{-\infty}^{0} e^{\gamma s}|f(s+t)|^2 ds = 0.$

A solution of equation (1.1) will be considered in the following sense.

Definition 2.4. *Suppose $u_\tau \in H$. A function $u(\cdot)$ is called a weak solution of (1.1) if $u \in L^2(\tau, T; V) \cap C([\tau, T], H)$ for any $T > \tau$, such that for all $v \in V$ and $t \geqslant \tau$,*

$$\begin{aligned}(u(t), v) &+ \nu\int_\tau^t \langle Au(s), v\rangle ds + \int_\tau^t F_N(\|u(s)\|)b(u(s), u(s), v)ds \\ &= (u_\tau, v) + \int_\tau^t (f(s), v)\, ds + \varepsilon\int_\tau^t \mathcal{G}_\delta(\theta_s\omega)\,(G(u(s)), v)\, ds,\end{aligned}$$

in the sense of distribution on $[\tau, +\infty)$.

2.1 Cocycle for GMNSE with colored noise

In this subsection, we will investigate the well-posedness of (1.1) and define a non-autonomous cocycle based on the solution operators. To that end, we first show the existence and uniqueness of solutions to (1.1). In fact, given $\omega \in \Omega$, (1.1) can be viewed as a deterministic problem. Thus, by the Galerkin method as in [3], one can verify that for any $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $u_\tau \in H$, if (G1) and (G2) hold, problem (1.1) has a unique solution $u(\cdot, \tau, \omega, u_\tau)$ in the sense of of Definition 2.4. And this solution is continuous with respect to initial data in H and is $(\mathcal{F}, \mathcal{B}(H))$-measurable in $\omega \in \Omega$. Moreover, the solution u satisfies the energy equality

$$\begin{aligned} |u(t)|^2 + 2\nu \int_s^t \|u(r)\|^2 dr &= |u(s)|^2 + 2\int_s^t (f(r), u(r)) dr \\ &+ 2\varepsilon \int_s^t \mathcal{G}_\delta(\theta_r\omega)(G(u(r)), u(r)) dr, \ \forall\, \tau \leqslant s \leqslant t. \end{aligned} \tag{2.10}$$

Now, given $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_\tau \in H$, we define a mapping U by

$$U(t, \tau, \omega, u_\tau) = u(t+\tau, \tau, \theta_{-\tau}\omega, u_\tau), \tag{2.11}$$

where we recall that $u(\cdot, \tau, \theta_{-\tau}\omega, u_\tau)$ is the solution of (1.1) corresponding to τ, $\theta_{-\tau}\omega$ and u_τ. This mapping U is a continuous non-autonomous cocycle in H.

Denote by $\mathcal{D}$ the class of all families of nonempty subsets $D = \{D(t,\omega) : t \in \mathbb{R}, \omega \in \Omega\} \subset \mathcal{P}(H)$ such that for any $c > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{t \to -\infty} \left(e^{ct} \sup_{\xi \in D(\tau + t, \theta_t\omega)} |\xi|^2 \right) = 0,$$

where $\mathcal{P}(H)$ denotes the family of all nonempty subsets of H.

2.2 Construction of pullback random absorbing sets

In this subsection, we will find a $\mathcal{D}$-pullback random absorbing set for the cocycle U associated with (1.1). We start with the uniform estimates on the solutions of (1.1) in H.

Lemma 2.1. *Suppose (G1), (G2) and (F1) hold. Then for every $\varepsilon, \delta \in (0,1]$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(t,\omega) : t \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T_1 = T_1(\tau, \omega, D) > 0$ such that, for all $t \geqslant T_1$ and $k \in [-1, 0]$,*

$$\begin{aligned} &|u(\tau + k, \tau - t, \theta_{-\tau}\omega, u_{\tau - t})|^2 \\ &\leqslant 1 + M \int_{-\infty}^k e^{\int_k^s (\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_r\omega)) dr} \Big(|f(s+\tau)|^2 + |\mathcal{G}_\delta(\theta_s\omega)| + |\mathcal{G}_\delta(\theta_s\omega)|^{\frac{2}{1-\kappa}} + |\mathcal{G}_\delta(\theta_s\omega)|^2 \Big) ds \\ &:= R_H(\tau, \omega, \varepsilon, \delta, k), \end{aligned} \tag{2.12}$$

where $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ and $M > 1$ is independent of τ, ω, D and $\varepsilon, \delta \in (0,1]$.

Proof. It follows from the energy equality (2.10) that

$$\frac{d}{dt}|u|^2 + 2\nu\|u\|^2 = 2(f(t), u) + 2\varepsilon\mathcal{G}_\delta(\theta_t\omega)(G(u), u). \tag{2.13}$$

For the last term of (2.13), by (G2) and Young's inequality, we obtain

$$\begin{aligned}
&2\varepsilon\mathcal{G}_\delta(\theta_t\omega)(G(u), u)\\
&\leqslant 2\varepsilon\alpha\mathcal{G}_\delta(\theta_t\omega)|u|^2 + \frac{\nu\lambda_1}{4}|u|^2 + 2\beta_2|\mathcal{G}_\delta(\theta_t\omega)| + C_\nu|\mathcal{G}_\delta(\theta_t\omega)|^{\frac{2}{1-\kappa}} + \frac{8}{\nu\lambda_1}|\mathcal{G}_\delta(\theta_t\omega)|^2|\beta|^2,
\end{aligned}$$

which, together with (2.4) and (2.13), shows that

$$\begin{aligned}
&\frac{d}{dt}|u|^2 + \left(\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_t\omega)\right)|u|^2 + \frac{\nu}{2}\|u\|^2\\
&\leqslant \frac{4}{\nu\lambda_1}|f(t)|^2 + 2\beta_2|\mathcal{G}_\delta(\theta_t\omega)| + C_\nu|\mathcal{G}_\delta(\theta_t\omega)|^{\frac{2}{1-\kappa}} + \frac{8}{\nu\lambda_1}|\mathcal{G}_\delta(\theta_t\omega)|^2|\beta|^2.
\end{aligned} \tag{2.14}$$

By Gronwall's inequality to (2.14), integrating on $(\tau - t, \tau + k)$ with $k \in [-1, 0]$ and $t \geqslant -k$, after replacing ω by $\theta_{-\tau}\omega$, we obtain

$$\begin{aligned}
&|u(\tau + k, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})|^2 + \frac{\nu}{2}\int_{\tau-t}^{\tau+k} e^{\int_{\tau+k}^{s}(\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_{r-\tau}\omega))dr}\|u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 ds\\
&\leqslant e^{\int_k^{-t}(\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_r\omega))dr}|u_{\tau-t}|^2 + \frac{4}{\nu\lambda_1}\int_{-\infty}^{k} e^{\int_k^s(\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_r\omega))dr}|f(s+\tau)|^2 ds\\
&\quad + C'_\nu\int_{-\infty}^{k} e^{\int_k^s(\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_r\omega))dr}\left(|\mathcal{G}_\delta(\theta_s\omega)| + |\mathcal{G}_\delta(\theta_s\omega)|^{\frac{2}{1-\kappa}} + |\mathcal{G}_\delta(\theta_s\omega)|^2\right)ds\\
&:= I_1 + I_2 + I_3,
\end{aligned} \tag{2.15}$$

where $C'_\nu = \max\left\{2\beta_2, C_\nu, \frac{8|\beta|^2}{\nu\lambda_1}\right\}$. We now estimate all terms on the right-hand side of (2.15). By (2.3), we obtain

$$\lim_{s\to-\infty}\frac{1}{s}\int_0^s\left(\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_r\omega)\right)dr = \nu\lambda_1 - 2\varepsilon\alpha\mathbb{E}[\mathcal{G}_\delta] = \nu\lambda_1 > \gamma,$$

which shows that there exists $s_0 = s_0(\omega) < 0$ such that for any $s \leqslant s_0$,

$$\int_0^s\left(\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_r\omega)\right)dr \leqslant \gamma s. \tag{2.16}$$

It follows from (2.16) that

$$\int_{-\infty}^{s_0} e^{\int_0^s(\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_r\omega))dr}|f(s+\tau)|^2 ds \leqslant \int_{-\infty}^{s_0} e^{\gamma s}|f(s+\tau)|^2 ds,$$

which, together with (F1), implies

$$I_2 < +\infty. \tag{2.17}$$

By (2.2) and (2.16), we can deduce that

$$I_3 < +\infty. \tag{2.18}$$

In addition, noting $u_{\tau-t} \in D(\tau-t, \theta_{-t}\omega)$ and $D \in \mathcal{D}$, by using (2.16) again, we obtain

$$I_1 = e^{\int_k^{-t}(\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_r\omega))dr}|u_{\tau-t}|^2 \to 0, \quad \text{as } t \to +\infty,$$

which yields that there exists $T_1 = T_1(\tau, \omega, \mathcal{D}) > 0$ such that for all $t \geqslant T_1$ and $k \in [-1, 0]$,

$$I_1 \leqslant 1,$$

which, together with (2.17) and (2.18), concludes the proof. □

As a consequence of Lemma 2.1, U possesses a $\mathcal{D}$-pullback random absorbing set.

Lemma 2.2. *Suppose (G1), (G2), (F1) and (F2) hold. Then the continuous cocycle U associated with problem (1.1) possesses a closed measurable $\mathcal{D}$-pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ given by*

$$K(\tau, \omega) = \left\{u \in H : |u|^2 \leqslant R_H(\tau, \omega, \varepsilon, \delta, 0)\right\}, \tag{2.19}$$

where $R_H(\tau, \omega, \varepsilon, \delta, 0) > 0$ is defined in Lemma 2.1 with k being replaced by 0.

Proof. It follows from Lemma 2.1 that, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D \in \mathcal{D}$, there exists $T_1 = T_1(\tau, \omega, D) > 0$ such that, for any $t \geqslant T_1$,

$$U(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega). \tag{2.20}$$

And one can find that for any $c > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\varepsilon, \delta \in (0, 1]$,

$$\begin{aligned}\lim_{t\to-\infty} e^{ct}R_H(\tau + t, \theta_t\omega, \varepsilon, \delta, 0) =& M \lim_{t\to-\infty} e^{ct} \int_{-\infty}^0 e^{\nu\lambda_1 s - \int_0^{s+t} 2\varepsilon\alpha\mathcal{G}_\delta(\theta_r\omega)dr + \int_0^t 2\varepsilon\alpha\mathcal{G}_\delta(\theta_r\omega)dr}\Big(|f(s+\tau+t)|^2 \\ &+ |\mathcal{G}_\delta(\theta_{s+t}\omega)| + |\mathcal{G}_\delta(\theta_{s+t}\omega)|^{\frac{2}{1-\kappa}} + |\mathcal{G}_\delta(\theta_{s+t}\omega)|^2\Big)ds,\end{aligned}$$

which, together with (2.3), shows that there exists $\epsilon > 0$ satisfying $\epsilon < \min\{\nu\lambda_1 - \gamma, \frac{c}{4}\}$ such that

$$\begin{aligned}\lim_{t\to-\infty} e^{ct}R_H(\tau + t, \theta_t\omega, \varepsilon, \delta, 0) \leqslant& M \lim_{t\to-\infty} e^{(c-2\epsilon)t} \int_{-\infty}^0 e^{(\nu\lambda_1-\epsilon)s}\Big(|f(s+\tau+t)|^2 \\ &+ |\mathcal{G}_\delta(\theta_{s+t}\omega)| + |\mathcal{G}_\delta(\theta_{s+t}\omega)|^{\frac{2}{1-\kappa}} + |\mathcal{G}_\delta(\theta_{s+t}\omega)|^2\Big)ds.\end{aligned} \tag{2.21}$$

It then follows from (F2) and (2.2) that

$$\lim_{t\to-\infty} e^{ct}R_H(\tau + t, \theta_t\omega, \varepsilon, \delta, 0) = 0. \tag{2.22}$$

Therefore, by (2.20) and (2.22), $K \in \mathcal{D}$ is a closed measurable $\mathcal{D}$-pullback absorbing set for U, as desired. □

2.3 Existence of random attractors

In this subsection, we will prove the existence of pullback random attractors of (1.1). The uniform estimates on the solutions of (1.1) in V will be first derived, which will be useful to prove the pullback asymptotic compactness of solutions to (1.1).

Lemma 2.3. *Suppose (G1), (G2) and (F1) hold. Then for every $\varepsilon, \delta \in (0,1]$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(t,\omega) : t \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T_1 = T_1(\tau, \omega, D) > 0$ such that for all $t \geqslant T_1$,*

$$\begin{aligned}
&\|u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 \\
&\leqslant (1 + \widetilde{C}) L_V(\tau, \omega, \varepsilon, \delta) + \frac{4(\alpha^2 + \beta_1^2) R_H(\tau, \omega, \varepsilon, \delta, 0)}{\nu} \int_{-1}^{0} |\mathcal{G}_\delta(\theta_s\omega)|^2 ds \\
&\quad + \frac{8}{\nu} \int_{\tau-1}^{\tau} |f(s)|^2 ds + \frac{4(|\beta|^2 + \beta_1^2)}{\nu} \int_{-1}^{0} |\mathcal{G}_\delta(\theta_s\omega)|^2 ds := R_V(\tau, \omega, \varepsilon, \delta),
\end{aligned} \tag{2.23}$$

where $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$, $L_V(\tau, \omega, \varepsilon, \delta) > 0$, and $\widetilde{C} = \widetilde{C}(\nu, N) > 0$ may vary from line to line.

Proof. Let us denote $u(s) = u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})$, for any $s \geqslant \tau - t$. It follows from (2.14) that

$$\begin{aligned}
&|u(\tau, \tau - 1, \theta_{-\tau}\omega, u(\tau - 1))|^2 + \frac{\nu}{2} \int_{\tau-1}^{\tau} \|u(s)\|^2 ds \\
&\leqslant |u(\tau - 1)|^2 + \int_{\tau}^{\tau-1} (\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_{s-\tau}\omega)) |u(s)|^2 ds + \frac{4}{\nu\lambda_1} \int_{\tau-1}^{\tau} |f(s)|^2 ds \\
&\quad + C'_\nu \int_{\tau-1}^{\tau} \left(|\mathcal{G}_\delta(\theta_{s-\tau}\omega)| + |\mathcal{G}_\delta(\theta_{s-\tau}\omega)|^{\frac{2}{1-\kappa}} + |\mathcal{G}_\delta(\theta_{s-\tau}\omega)|^2 \right) ds.
\end{aligned} \tag{2.24}$$

Note that

$$\begin{aligned}
\max_{-1 \leqslant k \leqslant 0} R_H(\tau, \omega, \varepsilon, \delta, k) \leqslant& 1 + M \int_{-\infty}^{0} \max_{-1 \leqslant k \leqslant 0} e^{\int_k^0 (\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_r\omega)) dr} e^{\int_0^s (\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_r\omega)) dr} \\
&\times \left(|f(s + \tau)|^2 + |\mathcal{G}_\delta(\theta_s\omega)| + |\mathcal{G}_\delta(\theta_s\omega)|^{\frac{2}{1-\kappa}} + |\mathcal{G}_\delta(\theta_s\omega)|^2 \right) ds \\
\leqslant& 1 + M e^{\int_{-1}^{0} |\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_r\omega)| dr} \int_{-\infty}^{0} e^{\int_0^s (\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_r\omega)) dr} \\
&\times \left(|f(s + \tau)|^2 + |\mathcal{G}_\delta(\theta_s\omega)| + |\mathcal{G}_\delta(\theta_s\omega)|^{\frac{2}{1-\kappa}} + |\mathcal{G}_\delta(\theta_s\omega)|^2 \right) ds,
\end{aligned}$$

which, together with (2.16) and (F1), implies that

$$\max_{-1 \leqslant k \leqslant 0} R_H(\tau, \omega, \varepsilon, \delta, k) < +\infty. \tag{2.25}$$

It follows from (2.12), (2.24) and (2.25) that there exists $T_1 = T_1(\tau, \omega, D) > 0$ such that, for any $t \geqslant T_1$,

$$\int_{\tau-1}^{\tau} \|u(s)\|^2 ds \leqslant \frac{2}{\nu} \left(1 + \int_{-1}^{0} |\nu\lambda_1 - 2\varepsilon\alpha\mathcal{G}_\delta(\theta_s\omega)| \, ds \right) \max_{-1 \leqslant k \leqslant 0} R_H(\tau, \omega, \varepsilon, \delta, k) + \frac{8}{\nu^2\lambda_1} \int_{\tau-1}^{\tau} |f(s)|^2 ds$$

$$
\begin{aligned}
&+\frac{2C'_\nu}{\nu}\int_{-1}^{0}\left(|\mathcal{G}_\delta(\theta_s\omega)|+|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{2}{1-\kappa}}+|\mathcal{G}_\delta(\theta_s\omega)|^2\right)ds\\
:=&L_V(\tau,\omega,\varepsilon,\delta).
\end{aligned}\tag{2.26}
$$

In addition, we can obtain that

$$
\frac{d}{dt}\|u\|^2+2\nu|Au|^2+2F_N(\|u\|)b(u,u,Au)=2(f(t),Au)+2\varepsilon\mathcal{G}_\delta(\theta_t\omega)(G(u),Au).\tag{2.27}
$$

For the third term on the left-hand side of (2.27), by (2.6), we have

$$
2\left|F_N(\|u\|)b(u,u,Au)\right|\leqslant\frac{\nu}{8}|Au|^2+\widetilde{C}\|u\|^2,\tag{2.28}
$$

where $\widetilde{C}=\widetilde{C}(\nu,N)>0$ is independent of ε and δ. For the first term on the right-hand side of (2.27), by Young's inequality, we obtain

$$
2(f(t),Au)\leqslant\frac{\nu}{8}|Au|^2+\frac{8}{\nu}|f(t)|^2.\tag{2.29}
$$

For the second term on the right-hand side of (2.27), by (G1) and Young's inequality, we deduce

$$
\begin{aligned}
&2\varepsilon\mathcal{G}_\delta(\theta_t\omega)\left(G(u),Au\right)=2\varepsilon\mathcal{G}_\delta(\theta_t\omega)\left(\alpha u+h(u)+\beta,Au\right)\\
&\leqslant\frac{3\nu}{4}|Au|^2+\frac{4\alpha^2}{\nu}|\mathcal{G}_\delta(\theta_t\omega)|^2|u|^2+\frac{8L_h^2}{\nu}|\mathcal{G}_\delta(\theta_t\omega)|^2|u|^2+\frac{8}{\nu}|h(0)|^2+\frac{4|\beta|^2}{\nu}|\mathcal{G}_\delta(\theta_t\omega)|^2,
\end{aligned}
$$

which, together with (2.27)-(2.29), shows that

$$
\begin{aligned}
&\frac{d}{dt}\|u\|^2+\nu|Au|^2\\
&\leqslant\widetilde{C}\|u\|^2+\frac{4(\alpha^2+2L_h^2)}{\nu}|\mathcal{G}_\delta(\theta_t\omega)|^2|u|^2+\frac{8}{\nu}(|f(t)|^2+|h(0)|^2)+\frac{4|\beta|^2}{\nu}|\mathcal{G}_\delta(\theta_t\omega)|^2.
\end{aligned}\tag{2.30}
$$

It then follows from (2.12), (2.26) and (2.30) that for any $r\in[\tau-1,\tau]$,

$$
\begin{aligned}
&\|u\left(\tau,r,\theta_{-\tau}\omega,u(r)\right)\|^2+\nu\int_r^\tau|Au(s)|^2ds\\
&\leqslant\|u(r)\|^2+\widetilde{C}\int_{\tau-1}^{\tau}\|u(s)\|^2ds+\frac{4(\alpha^2+2L_h^2)}{\nu}\int_{\tau-1}^{\tau}|\mathcal{G}_\delta(\theta_{s-\tau}\omega)|^2|u(s)|^2ds\\
&\quad+\frac{8}{\nu}\int_{\tau-1}^{\tau}\left(|f(s)|^2+|h(0)|^2\right)ds+\frac{4|\beta|^2}{\nu}\int_{\tau-1}^{\tau}|\mathcal{G}_\delta(\theta_{s-\tau}\omega)|^2ds\\
&\leqslant\|u(r)\|^2+\widetilde{C}L_V(\tau,\omega,\varepsilon,\delta)+\frac{4(\alpha^2+2L_h^2)\max\limits_{-1\leqslant k\leqslant0}R_H(\tau,\omega,\varepsilon,\delta,k)}{\nu}\int_{-1}^{0}|\mathcal{G}_\delta(\theta_s\omega)|^2ds\\
&\quad+\frac{8}{\nu}\int_{\tau-1}^{\tau}\left(|f(s)|^2+|h(0)|^2\right)ds+\frac{4|\beta|^2}{\nu}\int_{-1}^{0}|\mathcal{G}_\delta(\theta_s\omega)|^2ds.
\end{aligned}\tag{2.31}
$$

Integrating in r, and using again (2.26), we obtain that

$$
\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2
$$

$$\leqslant (1+\widetilde{C})L_V(\tau,\omega,\varepsilon,\delta) + \frac{4(\alpha^2+2L_h^2)\max\limits_{-1\leqslant k\leqslant 0} R_H(\tau,\omega,\varepsilon,\delta,k)}{\nu}\int_{-1}^{0}|\mathcal{G}_\delta(\theta_s\omega)|^2 ds$$
$$+\frac{8}{\nu}\int_{\tau-1}^{\tau}\left(|f(s)|^2+|h(0)|^2\right)ds+\frac{4|\beta|^2}{\nu}\int_{-1}^{0}|\mathcal{G}_\delta(\theta_s\omega)|^2 ds. \tag{2.32}$$

The proof is complete. □

Now, let us present the existence of $\mathcal{D}$-pullback random attractors for (1.1).

Theorem 2.1. *Suppose (G1), (G2), (F1) and (F2) hold. Then the continuous non-autonomous cocycle U has a unique $\mathcal{D}$-pullback random attractor in H.*

If, in addition, there exists $T>0$ such that f is T-periodic in t, then the attractor $\mathcal{A}$ is also T-periodic.

Proof. It follows from Lemma 2.2 that U has a closed measurable $\mathcal{D}$-pullback absorbing set in H. On the other hand, by Lemma 2.3 and the compactness of injection of V into H, we can obtain that U is $\mathcal{D}$-pullback asymptotically compact in H. Thanks to [36, Theorem 2.23], the existence and unique of $\mathcal{D}$-pullback random attractor of U can be obtained.

In addition, if $f:\mathbb{R}\to H$ is T-periodic, then we infer that the cocycle U is T-periodic. And by (2.12), we see that the $\mathcal{D}$-pullback absorbing set K of U given by (2.19) is also T-periodic. Therefore, the periodicity of $\mathcal{D}$-pullback attractor of U follows from [36, Theorem 2.24] immediately. □

3 Existence of invariant measures

In this section, we will investigate the existence of invariant probability measures of (1.1). Moreover, the existence of periodic invariant probability measures will also be shown when the nonlinear function f is periodic with respect to t. In order to do that, we first recall some useful definitions and theorems.

Definition 3.1 ([17], The generalized Banach limit)**.** *A generalized Banach limit is any linear functional, which can be denoted by $\mathrm{LIM}_{t\to+\infty}$, defined on the space of all bounded real-valued functions on $[0,+\infty)$ that satisfies*

(1) $\mathrm{LIM}_{t\to+\infty} f(t)\geqslant 0$ *for nonnegative functions f.*
(2) $\mathrm{LIM}_{t\to+\infty} f(t)=\lim_{t\to+\infty} f(t)$ *if the usual limit $\lim_{t\to+\infty} f(t)$ exists.*

Definition 3.2 ([9])**.** *A mapping $(t,\omega)\in\mathbb{R}\times\Omega\mapsto\mu_{t,\omega}\in\mathcal{P}_r(X)$, is called an invariant measure for random dynamical system Φ if for any real-valued continuous functional φ on X,*

$$\int_X\varphi(v)\mu_{t,\omega}(dv)=\int_X\varphi(\Phi(t-\tau,\tau,\theta_{\tau-t}\omega,v))\mu_{\tau,\theta_{\tau-t}\omega}(dv),\quad\forall\, t\geqslant\tau, \tag{3.1}$$

where $\mathcal{P}_r(X)$ is the space of all probability measures on Banach space X.

Given $T > 0$, $\mu_{t,\omega}$ is called T-periodic if

$$\mu_{t,\omega} = \mu_{t+T,\omega}, \quad \forall\, t \in \mathbb{R} \text{ and } \mathbb{P}\text{-a.s. } \omega \in \Omega. \tag{3.2}$$

$\mu_{t,\omega}$ is called a T-periodic invariant measure if it is invariant and T-periodic.

Theorem 3.1 ([9]). *Assume that a non-autonomous random dynamical system Φ satisfies that $\Phi(-t, \tau + t, \theta_t\omega, v)$ is continuous in $(t, v) \in (-\infty, 0] \times X$ and has a unique $\mathcal{D}$-pullback attractor $\mathcal{A} = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$. Fix a generalized Banach limit $\mathrm{LIM}_{t\to-\infty}$ and let $v(\cdot) : \mathbb{R} \to X$ be a continuous mapping such that $\{v(r)\}_{r\in\mathbb{R}} \in \mathcal{D}$. Then, there exists a family of Borel probability measures $\{\mu_{\tau,\omega}\}_{\tau\in\mathbb{R},\omega\in\Omega}$ such that $\mu_{\tau,\omega}$ is an invariant measure for random dynamical system Φ and is supported on $A(\tau, \omega)$, and for any real-valued continuous functional Υ on X, $\tau \in \mathbb{R}$ and $\mathbb{P}$-a.s. $\omega \in \Omega$,*

$$\begin{aligned}
&\mathop{\mathrm{LIM}}_{t\to-\infty} \frac{1}{-t}\int_t^0 \Upsilon\left(\Phi(-s, \tau + s, \theta_s\omega, v(\tau + s))\right) ds\\
&= \int_X \Upsilon(u)\mu_{\tau,\omega}(\mathrm{d}u) = \int_{A(\tau,\omega)} \Upsilon(u)\mu_{\tau,\omega}(du)\\
&= \mathop{\mathrm{LIM}}_{t\to-\infty} \frac{1}{-t}\int_t^0 \left[\int_{A(\tau+s,\theta_s\omega)} \Upsilon\left(\Phi(-s, \tau + s, \theta_s\omega, u)\right) \mu_{\tau+s,\theta_s\omega}(du)\right] ds.
\end{aligned} \tag{3.3}$$

If, in addition, Φ is a T-periodic random dynamical system, and $v(\cdot) : \mathbb{R} \to X$ is T-periodic and continuous, then the invariant measure $\mu_{\tau,\omega}$ is T-periodic.

Next, we will show the continuity of U associated with (1.1), which will be useful for constructing invariant measures.

Lemma 3.1. *Suppose (G1) and (G2) hold. Then, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the mapping $U(t, \tau - t, \theta_{-t}\omega, u_0)$ is continuous in $(t, u_0) \in [0, +\infty) \times H$.*

Proof. Let $\tau \in \mathbb{R}$ and $(t^*, u_0^*) \in [0, +\infty) \times H$ be fixed. In order to prove such continuity, we need to show for any $\epsilon > 0$, there exists $0 < \zeta < 1$ such that for any $(t, u_0) \in [0, +\infty) \times H$ with $|t - t^*| < \zeta$ and $|u_0 - u_0^*| < \zeta$,

$$|U(t, \tau - t, \theta_{-t}\omega, u_0) - U(t^*, \tau - t^*, \theta_{-t^*}\omega, u_0^*)|^2 < \epsilon.$$

By (2.11), we see that

$$\begin{aligned}
&|U(t, \tau - t, \theta_{-t}\omega, u_0) - U(t^*, \tau - t^*, \theta_{-t^*}\omega, u_0^*)|^2\\
&= |u(\tau, \tau - t, \theta_{-\tau}\omega, u_0) - u(\tau, \tau - t^*, \theta_{-\tau}\omega, u_0^*)|^2\\
&\leqslant 2|u(\tau, \tau - t, \theta_{-\tau}\omega, u_0) - u(\tau, \tau - t, \theta_{-\tau}\omega, u_0^*)|^2\\
&\quad + 2|u(\tau, \tau - t, \theta_{-\tau}\omega, u_0^*) - u(\tau, \tau - t^*, \theta_{-\tau}\omega, u_0^*)|^2.
\end{aligned} \tag{3.4}$$

We now estimate the first term on the right-hand side of (3.4). For short, $u(s,\tau-t,\theta_{-\tau}\omega,u_0)$ and $u(s,\tau-t,\theta_{-\tau}\omega,u_0^*)$ are denoted by $u(s)$ and $\hat{u}(s)$, respectively. By (1.1), we have

$$\begin{aligned}
&2|u(\tau)-\hat{u}(\tau)|^2+4\nu\int_{\tau-t}^{\tau}\|u(s)-\hat{u}(s)\|^2ds\\
&=2|u_0-u_0^*|^2+4\varepsilon\int_{\tau-t}^{\tau}\mathcal{G}_\delta(\theta_s\omega)\big(G(u(s))-G(\hat{u}(s)),u(s)-\hat{u}(s)\big)ds\\
&\quad+4\int_{\tau-t}^{\tau}\big(F_N(\|\hat{u}(s)\|)b(\hat{u}(s),\hat{u}(s),u(s)-\hat{u}(s))-F_N(\|u(s)\|)b(u(s),u(s),u(s)-\hat{u}(s))\big)ds.
\end{aligned}\tag{3.5}$$

For the second term on the right-hand side of (3.5), by (G1) and Hölder's inequality, we obtain

$$\begin{aligned}
&4\varepsilon\int_{\tau-t}^{\tau}\mathcal{G}_\delta(\theta_s\omega)\big(G(u(s))-G(\hat{u}(s)),u(s)-\hat{u}(s)\big)ds\\
&\leqslant 4\alpha\int_{\tau-t}^{\tau}\mathcal{G}_\delta(\theta_s\omega)|u(s)-\hat{u}(s)|^2ds+4L_h\int_{\tau-t}^{\tau}|\mathcal{G}_\delta(\theta_s\omega)||u(s)-\hat{u}(s)|^2ds.
\end{aligned}\tag{3.6}$$

For the third term on the right-hand side of (3.5), it follows from (2.5), (2.7) and Young's inequality that

$$\begin{aligned}
&4\int_{\tau-t}^{\tau}\big(F_N(\|\hat{u}(s)\|)b(\hat{u}(s),\hat{u}(s),u(s)-\hat{u}(s))-F_N(\|u(s)\|)b(u(s),u(s),u(s)-\hat{u}(s))\big)ds\\
&=4\int_{\tau-t}^{\tau}\big(F_N(\|\hat{u}(s)\|)b(\hat{u}(s)-u(s),\hat{u}(s),u(s)-\hat{u}(s))\\
&\quad+\left[F_N(\|\hat{u}(s)\|)-F_N(\|u(s)\|)\right]b(u(s),\hat{u}(s),u(s)-\hat{u}(s))\big)ds\\
&\leqslant 2\nu\int_{\tau-t}^{\tau}\|u(s)-\hat{u}(s)\|^2ds+2\widetilde{C}\int_{\tau-t}^{\tau}|u(s)-\hat{u}(s)|^2ds,
\end{aligned}$$

which, together with (3.5) and (3.6), shows that

$$\begin{aligned}
&2|u(\tau)-\hat{u}(\tau)|^2+2\nu\int_{\tau-t}^{\tau}\|u(s)-\hat{u}(s)\|^2ds\\
&\leqslant 2|u_0-u_0^*|^2+2\int_{\tau-t}^{\tau}\left[2\left(\alpha+L_h\right)|\mathcal{G}_\delta(\theta_s\omega)|+\widetilde{C}\right]|u(s)-\hat{u}(s)|^2ds.
\end{aligned}$$

Then by Gronwall's inequality, we have,

$$2|u(\tau)-\hat{u}(\tau)|^2\leqslant 2|u_0-u_0^*|^2e^{\int_{\tau-t}^{\tau}\left[2(\alpha+L_h)|\mathcal{G}_\delta(\theta_s\omega)|+\widetilde{C}\right]ds},$$

which implies that for any $t\geqslant 0$ with $|t-t^*|<1$,

$$2|u(\tau)-\hat{u}(\tau)|^2\leqslant 2|u_0-u_0^*|^2e^{\int_{\tau-t^*-1}^{\tau}\left[2(\alpha+L_h)|\mathcal{G}_\delta(\theta_s\omega)|+\widetilde{C}\right]ds}.$$

Therefore, there exists $0<\zeta_1<1$ such that for any $(t,u_0)\in[0,+\infty)\times H$ with $|t-t^*|<\zeta_1$ and $|u_0-u_0^*|<\zeta_1$,

$$2|u(\tau)-\hat{u}(\tau)|^2<\frac{\epsilon}{2}.\tag{3.7}$$

Next, the last term of (3.4) will be estimated. Without loss of generality, we assume $t^* < t$. Since u is the solution of (1.1), we have

$$\begin{aligned} &2|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0^*) - u(\tau,\tau-t^*,\theta_{-\tau}\omega,u_0^*)|^2 \\ &= 2|u(\tau,\tau-t^*,\theta_{-\tau}\omega,u(\tau-t^*,\tau-t,\theta_{-\tau}\omega,u_0^*)) - u(\tau,\tau-t^*,\theta_{-\tau}\omega,u_0^*)|^2. \end{aligned} \tag{3.8}$$

Similarly to (3.5), by (3.8), we find that

$$\begin{aligned} &2|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0^*) - u(\tau,\tau-t^*,\theta_{-\tau}\omega,u_0^*)|^2 \\ &\leqslant 2|u(\tau-t^*,\tau-t,\theta_{-\tau}\omega,u_0^*) - u_0^*|^2 e^{\int_{\tau-t^*}^{\tau}\left[2(\alpha+L_h)|\mathcal{G}_\delta(\theta_s\omega)|+\widetilde{C}\right]ds}. \end{aligned}$$

Note that

$$\begin{aligned} |u(\tau-t^*,\tau-t,\theta_{-\tau}\omega,u_0^*) - u_0^*|^2 &= |u(\tau-t^*,\tau-t,\theta_{-\tau}\omega,u_0^*)|^2 - |u_0^*|^2 \\ &\quad - 2\big(u(\tau-t^*,\tau-t,\theta_{-\tau}\omega,u_0^*) - u_0^*, u_0^*\big). \end{aligned} \tag{3.9}$$

For the first and second terms on the right-hand side of (3.9), by (2.14), we have

$$\begin{aligned} &|u(\tau-t^*,\tau-t,\theta_{-\tau}\omega,u_0^*)|^2 - |u_0^*|^2 \\ &\leqslant \int_{\tau-t}^{\tau-t^*} \left(2\varepsilon\alpha\mathcal{G}_\delta(\theta_{s-\tau}\omega) - \nu\lambda_1\right)|u(s,\tau-t,\theta_{-\tau}\omega,u_0^*)|^2 ds + \frac{4}{\nu\lambda_1}\int_{\tau-t}^{\tau-t^*}|f(s)|^2 ds \\ &\quad + \int_{-t}^{-t^*}(2\beta_2|\mathcal{G}_\delta(\theta_s\omega)| + C_\nu|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{2}{1-\kappa}} + \frac{8}{\nu\lambda_1}|\mathcal{G}_\delta(\theta_s\omega)|^2|\beta|^2)ds, \end{aligned} \tag{3.10}$$

which, together with Gronwall's inequality and the continuity of $\mathcal{G}_\delta(\theta_t\omega)$ in t, implies that there exists $M_1(\tau,t^*,\omega,u_0^*) > 0$ such that for any $t \geqslant 0$ with $|t-t^*| < 1$,

$$\sup_{\tau-t-1\leqslant s\leqslant\tau-t^*} |u(s,\tau-t,\theta_{-\tau}\omega)| + |u_0^*| \leqslant M_1(\tau,t^*,\omega,u_0^*). \tag{3.11}$$

It then follows from (3.10) and (3.11) that there exists $0 < \zeta_2 < \zeta_1$ such that for any $t \geqslant 0$ with $|t-t^*| < \zeta_2$,

$$|u(\tau-t^*,\tau-t,\theta_{-\tau}\omega,u_0^*)|^2 - |u_0^*|^2 < \frac{\epsilon}{8M_2(\tau,t^*,\omega)}, \tag{3.12}$$

where $M_2(\tau,t^*,\omega) = e^{\int_{\tau-t^*}^{\tau}\left[2(\alpha+L_h)|\mathcal{G}_\delta(\theta_s\omega)|+\widetilde{C}\right]ds}$.

For the last term of (3.9), by the density of V in H, we deduce that there exists an element $\widetilde{u_0^*} \in V$ such that $|\widetilde{u_0^*} - u_0^*| < \dfrac{\epsilon}{64M_1(\tau,t^*,\omega,u_0^*)M_2(\tau,t^*,\omega)}$. Therefore, we have for any $t \geqslant 0$ with $|t-t^*| < \zeta_2$,

$$\begin{aligned} &\left|\big(u(\tau-t^*,\tau-t,\theta_{-\tau}\omega,u_0^*) - u_0^*, u_0^*\big)\right| \\ &\leqslant \left|\big(u(\tau-t^*,\tau-t,\theta_{-\tau}\omega,u_0^*) - u_0^*, \widetilde{u_0^*}\big)\right| + \left|\big(u(\tau-t^*,\tau-t,\theta_{-\tau}\omega,u_0^*) - u_0^*, u_0^* - \widetilde{u_0^*}\big)\right| \end{aligned}$$

$$\leqslant \|\widetilde{u_0^*}\| \left(\int_{\tau-t}^{\tau-t^*} \|\frac{d}{ds}u(s,\tau-t,\theta_{-\tau}\omega,u_0^*)\|_{V^*}^2 ds \right)^{\frac{1}{2}} (t-t^*)^{\frac{1}{2}} + \frac{\epsilon}{32M_2(\tau,t^*,\omega)}. \tag{3.13}$$

By (2.5), (3.11), (G1) and the fact that $\mathcal{G}_\delta(\theta_t\omega)$ is continuous in t, we can deduce that

$$\int_{\tau-t}^{\tau-t^*} \|\frac{d}{ds}u(s,\tau-t,\theta_{-\tau}\omega,u_0^*)\|_{V^*}^2 ds < +\infty,$$

which, along with (3.13), implies that there exists $0 < \zeta_3 < \zeta_2$ such that, for any $t \geqslant 0$ with $|t-t^*| < \zeta_3$,

$$\left|\left(u(\tau-t^*,\tau-t,\theta_{-\tau}\omega,u_0^*) - u_0^*, u_0^*\right)\right| \leqslant \frac{\epsilon}{16M_2(\tau,t^*,\omega)}. \tag{3.14}$$

Therefore, from (3.4), (3.7), (3.9), (3.12) and (3.14), it follows that there exists $\zeta = \zeta_3 \in (0,1)$ such that, for any $(t,u_0) \in [0,+\infty) \times H$ with $|t-t^*| < \zeta$ and $|u_0 - u_0^*| < \zeta$,

$$|U(t,\tau-t,\theta_{-t}\omega,u_0) - U(t^*,\tau-t^*,\theta_{-t^*}\omega,u_0^*)|^2 < \epsilon,$$

as desired. □

Now, we are going to present the main result of this section.

Theorem 3.2. *Suppose (G1), (G2), (F1) and (F2) hold. Fix a generalized Banach limit* $\mathrm{LIM}_{t\to-\infty}$ *and let* $\xi(\cdot) : \mathbb{R} \to H$ *be a continuous mapping such that* $\{\xi(t)\}_{t\in\mathbb{R}} \in \mathcal{D}$*. Then, for any* $N > 0$ *and* $\varepsilon, \delta \in (0,1]$*, there exists a family of probability measures* $\{\mu_{t,\omega}\}_{t\in\mathbb{R},\omega\in\Omega}$ *of process* U *such that* $\mu_{t,\omega}$ *is an invariant measure for* U *and is supported on* $A(t,\omega)$*, and for any real-valued continuous functional* φ *on* H*,* $\tau \in \mathbb{R}$ *and* $\mathbb{P}$*-a.s.* $\omega \in \Omega$*,*

$$\begin{aligned}
&\mathop{\mathrm{LIM}}_{t\to-\infty} \frac{1}{-t}\int_t^0 \varphi\left(U(-s,\tau+s,\theta_s\omega,\xi(\tau+s))\right) ds \\
&= \int_H \varphi(u)\mu_{\tau,\omega}(du) = \int_{A(\tau,\omega)} \varphi(u)\mu_{\tau,\omega}(du) \\
&= \mathop{\mathrm{LIM}}_{t\to-\infty} \frac{1}{-t}\int_t^0 \left[\int_{A(\tau+s,\theta_s\omega)} \varphi\left(U(-s,\tau+s,\theta_s\omega,u)\right)\mu_{\tau+s,\theta_s\omega}(du)\right] ds.
\end{aligned}$$

If, in addition, $f(t)$ *is a* T*-periodic function in* t *and* $\xi(\cdot) : \mathbb{R} \to H$ *is a* T*-periodic continuous map in* t*, then the invariant measure* $\mu_{t,\omega}$ *is* T*-periodic.*

Proof. It follows from Theorem 2.1 that the non-autonomous random dynamical system U associated with (1.1) has a $\mathcal{D}$-pullback random attractor $\mathcal{A}$ in H. Then, by Lemma 3.1 and Theorem 3.1, the existence of invariant measures can be obtained. In addition, if $f(t,\cdot)$ is a T-periodic function in t, then we can deduce that the system U and the random attractor $\mathcal{A}$ are also periodic with T. By using Theorem 3.1 again, the periodicity of such invariant measures can be derived successfully. □

Proposition 3.1. *Let $\mu_{\tau,\omega}$ be an invariant probability measure for $\Phi(-t,\tau+t,\theta_t\omega,v)$ on X, and let $K(\tau,\omega)$ be a closed set that is absorbing for $\Phi(-t,\tau+t,\theta_t\omega,v)$. Then, $\mu_{\tau,\omega}\left(X\setminus K(\tau,\omega)\right)=0$.*

Proof. For every $r>0$, let $B_r(t,\omega)=\{u(t,\omega)\in X:\|u(t,\omega)\|_X\leqslant r\}$. Thanks to the fact that $K(\tau,\omega)$ is a closed absorbing set, there is a time $t_r\geqslant 0$ such that $\Phi(-t,\tau+t,\theta_t\omega,\cdot)B_r(\tau+t,\theta_t\omega)\subset K(\tau,\omega)$ for all $t\geqslant t_r$, which implies that $B_r(\tau+t,\theta_t\omega)\subset\Phi(-t,\tau+t,\theta_t\omega,\cdot)^{-1}K(\tau,\omega)$. Therefore, $\mu_{\tau+t,\theta_t\omega}\left(\Phi(-t,\tau+t,\theta_t\omega,\cdot)^{-1}K(\tau,\omega)\right)\geqslant\mu_{\tau+t,\theta_t\omega}\left(B_r(\tau+t,\theta_t\omega)\right)$. Noting that $\mu_{\tau+t,\theta_t\omega}(\Phi(-t,\tau+t,\theta_t\omega,\cdot)^{-1}K(\tau,\omega))=\mu_{\tau,\omega}(K(\tau,\omega))$, hence

$$1\geqslant\mu_{\tau,\omega}(K(\tau,\omega))=\mu_{\tau+t,\theta_t\omega}\left(\Phi(-t,\tau+t,\theta_t\omega,\cdot)^{-1}K(\tau,\omega)\right)\geqslant\mu_{\tau+t,\theta_t\omega}(B_r(\tau+t,\theta_t\omega)).$$

In addition, $X=\bigcup_{r>0}B_r(\tau,\omega)$ with $B_{r_1}(\tau,\omega)\subset B_{r_2}(\tau,\omega)$ if $r_1\leqslant r_2$, then $\mu_{\tau,\omega}(B_r(\tau,\omega))\to\mu_{\tau,\omega}(X)$ as $r\to+\infty$. Therefore, $\mu_{\tau,\omega}(K(\tau,\omega))=1$ which implies the desired result. □

Remark 3.1. *From Lemma 2.3, it follows that U has a closed pullback absorbing set $\widetilde{K}=\{\widetilde{K}(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ in V, where $\widetilde{K}(\tau,\omega)=\{u\in V:\|u\|^2\leqslant R_V(\tau,\omega,\varepsilon,\delta)\}$, and $R_V(\tau,\omega,\varepsilon,\delta)$ is a constant given in Lemma 2.3. Then together with Proposition 3.1, we have that $\mu_{t,\omega}(H\setminus\widetilde{K}(t,\omega))=0$ for any $t\in\mathbb{R}$ and $\omega\in\Omega$. Let us denote $\|v\|=+\infty$ if $v\in H\setminus V$. Then note that $\widetilde{K}(\tau,\omega)\subset V$, therefore, one can deduce that for any $t\in\mathbb{R}$ and $\omega\in\Omega$,*

$$\int_H\|u\|^2\mu_{t,\omega}(du)=\int_{\widetilde{K}(t,\omega)}\|u\|^2\mu_{t,\omega}(du)<+\infty.$$

4 Random Liouville type theorem

In this section, we will investigate that the family of measures $\{\mu_{t,\omega}\}_{t\in\mathbb{R},\omega\in\Omega}$ constructed in Section 3 satisfies a random Liouville equation. To that end, let $\mathcal{T}$ denote the class of real-valued functionals Ψ on H, which are bounded on bounded subsets of H and satisfy:

(1) for any $u\in V$, the first-order Fréchet derivative $\Psi'(u)$ exists,

(2) $\Psi'(u)\in V$ for all $u\in V$, and $u\longmapsto\Psi'(u)$ is continuous and bounded as a functional from V to V,

(3) for every global solution $u(t):=u(t,t_0,\omega,u_0)$ of (1.1), there holds for any $\omega\in\Omega$,

$$\frac{d}{dt}\Psi(u(t))=\left\langle F(t,\theta_t\omega,u(t)),\Psi'(u(t))\right\rangle,\tag{4.1}$$

where

$$F(t,\theta_t\omega,u)=-\nu Au-B_N(u,u)+f(t)+\varepsilon G(u)\mathcal{G}_\delta(\theta_t\omega).\tag{4.2}$$

Remark 4.1. *The above set $\mathcal{T}$ is not empty by referring [17] to construct some cylindrical test functionals. Indeed, for given $\psi\in C_0^1(\mathcal{O})$, $g_i\in V$, $i=1,2,\cdots,m$, let*

$$\Psi(u):=\psi((u,g_1),(u,g_2),\cdots,(u,g_m)),\ \forall\ u\in H.$$

Then by the chain rule of Fréchet derivative, we have

$$\Psi'(u) = \sum_{i=1}^{m} \partial_i \psi \left((u, g_1), (u, g_2), \cdots, (u, g_m) \right) g_i, \tag{4.3}$$

where $\partial_i \psi$ is the derivative of ψ with respect to the i-th variable. Obviously, it is continuous and bounded from V to V. In addition, if $u(t)$ is the solution of (1.1), it follows from (4.3) and the differential chain rule that (4.1) holds.

In order to use the invariance property of $\mu_{t,\omega}$, we need to verify that for any test function $\Psi \in \mathcal{T}$, the mapping $u \mapsto \left\langle F(t, \theta_t\omega, u), \Psi'(u) \right\rangle$ is continuous in H. But such mapping may not be continuous in H. Thus, what we first do is to approximate $\Psi(u)$ by $\Psi_m(u) = \Psi(P_m u)$ for any test function $\Psi \in \mathcal{T}$. And then letting $m \to +\infty$, the desired result can be obtained. We first state an auxiliary lemma.

Lemma 4.1. *Let $\Psi_m(u) = \Psi(P_m u)$, then for any $t \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\int_H \left\langle \Psi'(u), F(t, \theta_t\omega, u) \right\rangle \mu_{t,\omega}(du) = \lim_{m\to+\infty} \int_H \left\langle \Psi'_m(u), F(t, \theta_t\omega, u) \right\rangle \mu_{t,\omega}(du). \tag{4.4}$$

Also, we can obtain that the mapping

$$u \mapsto \left\langle \Psi'_m(u), F(t, \theta_t\omega, u) \right\rangle \tag{4.5}$$

is continuous on H.

Proof. **Step 1.** We will first claim that for any $u \in V$,

$$\lim_{m\to+\infty} \left\langle F(t, \theta_t\omega, u), \Psi'_m(u) \right\rangle = \left\langle F(t, \theta_t\omega, u), \Psi'(u) \right\rangle. \tag{4.6}$$

It is sufficient to prove that given $u \in V$ and $\epsilon > 0$, there is $N_0 > 0$ such that for any $m > N_0$,

$$\left| \left\langle F(t, \theta_t\omega, u), \Psi'(u) \right\rangle - \left\langle F(t, \theta_t\omega, u), \Psi'_m(u) \right\rangle \right| < \epsilon.$$

Indeed, for any $u \in V$, we have

$$\begin{aligned}
&\left\langle F(t, \theta_t\omega, u), \Psi'(u) \right\rangle - \left\langle F(t, \theta_t\omega, u), \Psi'_m(u) \right\rangle \\
&= \nu \left\langle \Psi'(u) - \Psi'(P_m u), -Au \right\rangle + \nu \left\langle \Psi'(P_m u), AP_m u - Au \right\rangle \\
&\quad + \left\langle -B_N(u, u), \Psi'(u) \right\rangle + \left\langle B_N(P_m u, P_m u), \Psi'(P_m u) \right\rangle \\
&\quad + \left(\Psi'(u) - \Psi'(P_m u), f(t) + \varepsilon G(u) \mathcal{G}_\delta(\theta_t\omega) \right) \\
&\quad + \left(\Psi'(P_m u), f(t) - P_m f(t) + \varepsilon \left(G(u) - G(P_m u) \right) \mathcal{G}_\delta(\theta_t\omega) \right).
\end{aligned} \tag{4.7}$$

For the first and second terms on the right-hand side of (4.7), we obtain that

$$\nu \left\langle \Psi'(u) - \Psi'(P_m u), -Au \right\rangle + \nu \left\langle \Psi'(P_m u), AP_m u - Au \right\rangle$$

$$\leqslant \nu \left\|\Psi'(u) - \Psi'(P_m u)\right\| \|u\| + \nu \sup_{u\in V} \left\|\Psi'(u)\right\| \|P_m u - u\| ,$$

which, together with the fact that $\lim_{m\to+\infty} \|u - P_m u\|^2 = 0$, and the continuity of Ψ' in V, implies that there exists $N_1 > 0$ such that, for any $m > N_1$,

$$\nu \left|\left\langle \Psi'(u) - \Psi'(P_m u), -Au\right\rangle\right| + \nu \left|\left\langle \Psi'(P_m u), AP_m u - Au\right\rangle\right| < \frac{\epsilon}{4}. \tag{4.8}$$

For the third and fourth terms on the right-hand side of (4.7), by (2.5) and (2.7), we obtain that

$$\begin{aligned}
&\left\langle -B_N(u,u), \Psi'(u)\right\rangle + \left\langle B_N(P_m u, P_m u), \Psi'(P_m u)\right\rangle \\
&= -F_N(\|u\|) b\left(u - P_m u, u, \Psi'(u)\right) + \left(F_N(\|P_m u\|) - F_N(\|u\|)\right) b\left(P_m u, u, \Psi'(u)\right) \\
&\quad + F_N(\|P_m u\|)\Big(b\left(P_m u, P_m u, \Psi'(u)\right) - b\left(P_m u, u, \Psi'(u)\right)\Big) \\
&\quad + F_N(\|P_m u\|)\Big(b\left(P_m u, P_m u, \Psi'(P_m u)\right) - b\left(P_m u, P_m u, \Psi'(u)\right)\Big) \\
&\leqslant 3CN \|u - P_m u\| \sup_{u\in V} \left\|\Psi'(u)\right\| + CN \|u\| \left\|\Psi'(P_m u) - \Psi'(u)\right\| ,
\end{aligned} \tag{4.9}$$

which, along with the fact that $\lim_{m\to+\infty} \|u - P_m u\|^2 = 0$, and the continuity of Ψ' in V, shows that there exists $N_2 > N_1$ such that, for any $m > N_2$,

$$\left|\left\langle -B_N(u,u), \Psi'(u)\right\rangle\right| + \left|\left\langle B_N(P_m u, P_m u), \Psi'(P_m u)\right\rangle\right| < \frac{\epsilon}{4}. \tag{4.10}$$

For the fifth term on the right-hand side of (4.7), we obtain that

$$\begin{aligned}
&\left(\Psi'(u) - \Psi'(P_m u), f(t) + \varepsilon G(u)\mathcal{G}_\delta(\theta_t\omega)\right) \\
&\leqslant \frac{1}{\sqrt{\lambda_1}} \left\|\Psi'(u) - \Psi'(P_m u)\right\| \left[|f(t)| + \left(\frac{\alpha + L_h}{\sqrt{\lambda_1}}\|u\| + |h(0)| + |\beta|\right) |\mathcal{G}_\delta(\theta_t\omega)|\right],
\end{aligned}$$

which, together with the fact that $\lim_{m\to+\infty} \|u - P_m u\|^2 = 0$, and the continuity of Ψ' in V, implies that there exists $N_3 > N_2$ such that, for any $m > N_3$,

$$\left|\left(\Psi'(u) - \Psi'(P_m u), f(t) + \varepsilon G(u)\mathcal{G}_\delta(\theta_t\omega)\right)\right| < \frac{\epsilon}{4}. \tag{4.11}$$

For the last term of (4.7), by the fact that $\lim_{m\to+\infty} \|u - P_m u\|^2 = 0$, and the boundedness of Ψ' in V, we deduce that there exists $N_4 > N_3$ such that, for all $m > N_4$,

$$\left|\left(\Psi'(P_m u), f(t) - P_m f(t) + \varepsilon\left(G(u) - G(P_m u)\right)\mathcal{G}_\delta(\theta_t\omega)\right)\right| < \frac{\epsilon}{4}. \tag{4.12}$$

Therefore, by (4.7)-(4.12), the desired result can be obtained.

Step 2. We will first claim that for any $u \in V$,

$$\lim_{m\to+\infty} \int_H \left\langle F(t, \theta_t\omega, u), \Psi'_m(u)\right\rangle \mu_{t,\omega}(dv) = \int_H \left\langle F(t, \theta_t\omega, u), \Psi'(u)\right\rangle \mu_{t,\omega}(dv). \tag{4.13}$$

For any $m \in \mathbb{N}$ and $u \in V$, we have

$$
\begin{aligned}
&\left|\left\langle F(t,\theta_t\omega,u), \Psi_m'(u)\right\rangle\right| \\
&\leqslant (\nu + CN)\sup_{u\in V}\|\Psi'(u)\|\,\|u\| + \left[\left(\frac{\alpha+L_h}{\sqrt{\lambda_1}}\|u\| + |\beta|\right)|\mathcal{G}_\delta(\theta_t\omega)| + |f(t)|\right]\sup_{u\in V}\|\Psi'(u)\| \\
&:= g(t,\theta_t\omega,u).
\end{aligned}
$$

It then follows from Remark 3.1 that

$$
\int_H g(t,\theta_t\omega,u)\mu_{t,\omega}(du) = \int_{\widetilde{K}(t,\omega)} g(t,\theta_t\omega,u)\mu_{t,\omega}(du) < +\infty,
$$

which, together with (4.6) and the Lebesgue dominated convergence theorem, shows (4.13) holds.

Step 3. The mapping $u \mapsto \left\langle F(t,\theta_t\omega,u), \Psi_m'(u)\right\rangle$ is continuous in H.

In order to do that, we need to prove that given $u^0 \in H$ and $\epsilon > 0$, then there exists $\zeta \in (0,1]$ such that for any $u \in H$ with $|u-u^0| < \zeta$,

$$
\left|\left\langle F(t,\theta_t\omega,u), \Psi_m'(u)\right\rangle - \left\langle F(t,\theta_t\omega,u^0), \Psi_m'(u^0)\right\rangle\right| < \epsilon.
$$

Noting that $\Psi_m'(u) = \Psi'(P_m u)\Psi_m$, we have

$$
\begin{aligned}
&\left\langle -\nu Au - B_N(u,u) + f(t) + \varepsilon G(u)\mathcal{G}_\delta(\theta_t\omega), \Psi_m'(u)\right\rangle \\
&- \left\langle -\nu Au^0 - B_N(u^0,u^0) + f(t) + \varepsilon G(u^0)\mathcal{G}_\delta(\theta_t\omega), \Psi_m'(u^0)\right\rangle \\
=&\left\langle -\nu AP_m u, \Psi'(P_m u)\right\rangle - \left\langle -\nu AP_m u^0, \Psi'(P_m u^0)\right\rangle + \left\langle P_m f(t), \Psi'(P_m u) - \Psi'(P_m u^0)\right\rangle \\
&+ \left\langle -B_N(P_m u, P_m u), \Psi'(P_m u)\right\rangle - \left\langle -B_N(P_m u^0, P_m u^0), \Psi'(P_m u^0)\right\rangle \\
&+ \left\langle \varepsilon G(P_m u)\mathcal{G}_\delta(\theta_t\omega), \Psi'(P_m u)\right\rangle - \left\langle \varepsilon G(P_m u^0)\mathcal{G}_\delta(\theta_t\omega), \Psi'(P_m u^0)\right\rangle. \qquad (4.14)
\end{aligned}
$$

For the first and second terms on the right-hand side of (4.14), we obtain that

$$
\begin{aligned}
&\left\langle -\nu AP_m u, \Psi'(P_m u)\right\rangle - \left\langle -\nu AP_m u^0, \Psi'(P_m u^0)\right\rangle \\
&= \nu\left(\Psi'(P_m u^0) - \Psi'(P_m u), AP_m u\right) + \nu\left(\Psi'(P_m u^0), AP_m u^0 - AP_m u\right) \\
&\leqslant \frac{\nu\lambda_m(|u^0|+1)}{\sqrt{\lambda_1}}\left\|\Psi'(P_m u) - \Psi'(P_m u^0)\right\| + \sup_{u\in V}\|\Psi'(u)\|\frac{\nu\lambda_m}{\sqrt{\lambda_1}}|u-u^0|,
\end{aligned}
$$

which, together with the fact that $\|P_m u\|^2 \leqslant \lambda_m|u|^2$, and the continuity of Ψ' in V, implies that there exists $\zeta_1 \in (0,1]$ such that for any $u \in H$ with $|u-u^0| < \zeta_1$,

$$
\left\langle -\nu AP_m u, \Psi'(P_m u)\right\rangle - \left\langle -\nu AP_m u^0, \Psi'(P_m u^0)\right\rangle < \frac{\epsilon}{4}. \qquad (4.15)
$$

For the third term on the right-hand side of (4.14), it follows from the fact that $\|P_m u\|^2 \leqslant \lambda_m|u|^2$, and the continuity of Ψ' in V, that there exists $\zeta_2 \in (0,\zeta_1]$ such that, for all $u \in H$ with $|u-u^0| < \zeta_2$,

$$
\left\langle P_m f(t), \Psi'(P_m u) - \Psi'(P_m u^0)\right\rangle < \frac{\epsilon}{4}. \qquad (4.16)
$$

For the fourth and fifth terms on the right-hand side of (4.14), similar to (4.9), by (2.5) and (2.7), we obtain that

$$\begin{aligned}&\big\langle -B_N(P_m u, P_m u), \Psi'(P_m u)\big\rangle - \big\langle -B_N(P_m u^0, P_m u^0), \Psi'(P_m u^0)\big\rangle\\&\leqslant 3CN\sqrt{\lambda_m}\left|u-u^0\right|\sup_{u\in V}\left\|\Psi'(u)\right\| + CN\sqrt{\lambda_m}\left|u^0\right|\left\|\Psi'(P_m u^0)-\Psi'(P_m u)\right\|,\end{aligned}$$

which, together with the fact that $\|P_m u\|^2 \leqslant \lambda_m |u_m|^2$, and the continuity of Ψ' in V, implies that there exists $\zeta_3 \in (0, \zeta_2]$ such that, for any $u \in H$ with $|u - u^0| < \zeta_3$,

$$\big\langle -B_N(P_m u, P_m u), \Psi'(P_m u)\big\rangle - \big\langle -B_N(P_m u^0, P_m u^0), \Psi'(P_m u^0)\big\rangle < \frac{\epsilon}{4}. \tag{4.17}$$

For the last two terms of (4.14), we have

$$\begin{aligned}&\big\langle \varepsilon G(P_m u)\mathcal{G}_\delta(\theta_t\omega), \Psi'(P_m u)\big\rangle - \big\langle \varepsilon G(P_m u^0)\mathcal{G}_\delta(\theta_t\omega), \Psi'(P_m u^0)\big\rangle\\&\leqslant \left|G(P_m u) - G(P_m u^0)\right| |\mathcal{G}_\delta(\theta_t\omega)| \left|\Psi'(P_m u)\right| + \left|G(P_m u^0)\right| |\mathcal{G}_\delta(\theta_t\omega)| \left|\Psi'(P_m u^0) - \Psi'(P_m u)\right|\\&\leqslant \frac{|\mathcal{G}_\delta(\theta_t\omega)|}{\sqrt{\lambda_1}}\sup_{u\in V}\left\|\Psi'(u)\right\| \left|G(P_m u) - G(P_m u^0)\right|\\&\quad + \frac{|\mathcal{G}_\delta(\theta_t\omega)|}{\sqrt{\lambda_1}}\left((\alpha + L_h)|u^0| + |\beta| + |h(0)|\right)\left\|\Psi'(P_m u^0) - \Psi'(P_m u)\right\|,\end{aligned}$$

which, together with the fact that $\|P_m u\|^2 \leqslant \lambda_m |u_m|^2$, the continuity of Ψ' in V, and the continuity of G in H, shows that there exists $\zeta_4 \in (0, \zeta_3]$ such that for any $u \in H$ with $|u - u^0| < \zeta_4$,

$$\big\langle \varepsilon G(P_m u)\mathcal{G}_\delta(\theta_t\omega), \Psi'(P_m u)\big\rangle - \big\langle \varepsilon G(P_m u^0)\mathcal{G}_\delta(\theta_t\omega), \Psi'(P_m u^0)\big\rangle < \frac{\epsilon}{4}. \tag{4.18}$$

Therefore, by (4.14)-(4.18), the continuity of the mapping $u \mapsto \big\langle F(t, \theta_t\omega, u), \Psi'_m(u)\big\rangle$ in H can be obtained. □

Next, we will show the random Liouville type theorem for the non-autonomous random dynamical system U associated with (1.1).

Theorem 4.1. *Suppose (G1), (G2), (F1) and (F2) hold. Then for any $\Psi \in \mathcal{T}$, the family of measures $\{\mu_{t,\omega}\}_{t\in\mathbb{R},\omega\in\Omega}$ constructed in Section 3 satisfies the random Liouville equation:*

$$\begin{aligned}&\int_{A(\tau,\omega)} \Psi(v)\mu_{\tau,\omega}(dv) - \int_{A\left(\sigma,\theta_{-(\tau-\sigma)}\omega\right)} \Psi(v)\mu_{\sigma,\theta_{-(\tau-\sigma)}\omega}(dv)\\&= \int_\sigma^\tau \int_{A\left(\eta,\theta_{-(\tau-\eta)}\omega\right)} \big\langle F(\eta, \theta_{\eta-\tau}\omega, v), \Psi'(v)\big\rangle \mu_{\eta,\theta_{-(\tau-\eta)}\omega}(dv)\ d\eta,\end{aligned}$$

for all $\tau \geqslant \sigma$ and $\omega \in \Omega$.

Proof. For any $\tau \geqslant \sigma$, $s \leqslant 0$ and $\Psi \in \mathcal{T}$, by (4.1), we have

$$\Psi\left(u(\tau; \sigma + s, \theta_{-\tau}\omega, v)\right) - \Psi\left(u(\sigma; \sigma + s, \theta_{-\tau}\omega, v)\right)$$

$$= \int_{\sigma}^{\tau} \big\langle F(\eta, \theta_{\eta-\tau}\omega, u(\eta; \sigma+s, \theta_{-\tau}\omega, v)), \Psi'(u(\eta; \sigma+s, \theta_{-\tau}\omega, v)) \big\rangle \, d\eta. \tag{4.19}$$

Since $\tau \geqslant \sigma$, it follows from (2.11), (4.4), (4.19) and the definition of invariant measure that

$$\begin{aligned}
&\int_{A(\tau,\omega)} \Psi(v)\mu_{\tau,\omega}(dv) - \int_{A\left(\sigma,\theta_{-(\tau-\sigma)}\omega\right)} \Psi(v)\mu_{\sigma,\theta_{-(\tau-\sigma)}\omega}(dv) \\
&= \underset{t\to-\infty}{\mathrm{LIM}} \frac{1}{-t} \int_t^0 \int_H \Psi\left(U(\tau-\sigma-s, \sigma+s, \theta_{-(\tau-\sigma-s)}\omega, v)\right) \mu_{\sigma+s,\theta_{-(\tau-\sigma-s)}\omega}(dv)\ ds \\
&\quad - \underset{t\to-\infty}{\mathrm{LIM}} \frac{1}{-t} \int_t^0 \int_H \Psi\left(U(-s, \sigma+s, \theta_{-(\tau-\sigma-s)}\omega, v)\right) \mu_{\sigma+s,\theta_{-(\tau-\sigma-s)}\omega}(dv)\ ds \\
&= \underset{t\to-\infty}{\mathrm{LIM}} \frac{1}{-t} \int_t^0 \int_H \int_\sigma^\tau \big\langle F(\eta, \theta_{\eta-\tau}\omega, u(\eta; \sigma+s, \theta_{-\tau}\omega, v)), \\
&\qquad\qquad\qquad \Psi'(u(\eta;\sigma+s,\theta_{-\tau}\omega, v))\big\rangle d\eta\ \mu_{\sigma+s,\theta_{-(\tau-\sigma-s)}\omega}(dv)\ ds \\
&= \underset{t\to-\infty}{\mathrm{LIM}} \frac{1}{-t} \int_t^0 \int_\sigma^\tau \Big[\lim_{m\to+\infty} \int_H \big\langle F(\eta, \theta_{\eta-\tau}\omega, u(\eta; \sigma+s, \theta_{-\tau}\omega, v)), \\
&\qquad\qquad\qquad \Psi'_m(u(\eta;\sigma+s,\theta_{-\tau}\omega, v))\big\rangle \mu_{\sigma+s,\theta_{-(\tau-\sigma-s)}\omega}(dv)\Big]\ d\eta\ ds.
\end{aligned} \tag{4.20}$$

From (2.11) and the cocycle property of U, it follows that

$$\begin{aligned}
u(\eta; \sigma+s, \theta_{-\tau}\omega, v) &= U(\eta-\sigma-s, \sigma+s, \theta_{-(\tau-\sigma-s)}\omega, v) \\
&= U(\eta-\sigma, \sigma, \theta_{-(\tau-\sigma)}\omega, \cdot) \circ U(-s, \sigma+s, \theta_{-(\tau-\sigma-s)}\omega, v).
\end{aligned} \tag{4.21}$$

In addition, by (4.5) and Lemma 3.1, we can deduce that the mapping

$$v \mapsto \big\langle F\left(\eta, \theta_{\eta-\tau}\omega, U(\eta-\sigma, \sigma, \theta_{-(\tau-\sigma)}\omega, v)\right), \Psi'_m\left(U(\eta-\sigma,\sigma,\theta_{-(\tau-\sigma)}\omega, v)\right)\big\rangle \tag{4.22}$$

is continuous. Then, by (4.20)-(4.22), and Definition 3.2 about invariant measures, we obtain

$$\begin{aligned}
&\int_{A(\tau,\omega)} \Psi(v)\mu_{\tau,\omega}(dv) - \int_{A\left(\sigma,\theta_{-(\tau-\sigma)}\omega\right)} \Psi(v)\mu_{\sigma,\theta_{-(\tau-\sigma)}\omega}(dv) \\
&= \int_\sigma^\tau \Big[\lim_{m\to+\infty} \int_H \big\langle F(\eta, \theta_{\eta-\tau}\omega, U(\eta-\sigma, \sigma, \theta_{-(\tau-\sigma)}\omega, v)), \\
&\qquad\qquad \Psi'_m(U(\eta-\sigma,\sigma,\theta_{-(\tau-\sigma)}\omega, v))\big\rangle \mu_{\sigma,\theta_{-(\tau-\sigma)}\omega}(dv)\Big]\ d\eta \\
&= \int_\sigma^\tau \Big[\lim_{m\to+\infty} \int_H \big\langle F(\eta, \theta_{\eta-\tau}\omega, v), \Psi'_m(v)\big\rangle \mu_{\eta,\theta_{-(\tau-\eta)}\omega}(dv)\Big]\ d\eta.
\end{aligned} \tag{4.23}$$

Then, by using (4.5) again, it follows from (4.23) that

$$\begin{aligned}
&\int_{A(\tau,\omega)} \Psi(v)\mu_{\tau,\omega}(dv) - \int_{A\left(\sigma,\theta_{-(\tau-\sigma)}\omega\right)} \Psi(v)\mu_{\sigma,\theta_{-(\tau-\sigma)}\omega}(dv) \\
&= \int_\sigma^\tau \int_{A\left(\eta,\theta_{-(\tau-\eta)}\omega\right)} \big\langle F(\eta, \theta_{\eta-\tau}\omega, v), \Psi'(v)\big\rangle \mu_{\eta,\theta_{-(\tau-\eta)}\omega}(dv)\ d\eta.
\end{aligned}$$

The proof is complete. □

5 Limiting behaviors of invariant measures

In this section, we are devoted to the limiting behaviors of invariant measures with respect to noise intensity ε and correlation time δ, respectively.

5.1 Limiting behaviors of invariant measures as $\varepsilon \to 0$

In this subsection, we will discuss the limiting relationship of invariant measures between the GMNSE with colored noise and the corresponding deterministic GMNSE as parameter $\varepsilon \to 0$. To indicate the dependence of solutions on ε, we will write the solution of (1.1) as u^ε, and the corresponding cocycle as U^ε. As proved in the previous section, U^ε has a $\mathcal{D}$-pullback attractor $\mathcal{A}^\varepsilon \in \mathcal{D}$ in H and a closed measurable $\mathcal{D}$-pullback absorbing set K^ε given by, for any $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$K^\varepsilon(\tau,\omega) = \left\{ u \in H : |u|^2 \leqslant R_H(\tau,\omega,\varepsilon,\delta,0) \right\}, \tag{5.1}$$

where $R_H(\tau,\omega,\varepsilon,\delta,0)$ is as in Lemma 2.2. In addition, by (2.16), we can deduce that for any $\varepsilon \in (0,1]$,

$$\begin{aligned}
&R_H(\tau,\omega,\varepsilon,\delta,0)\\
&\leqslant 1 + M\int_{-\infty}^{s_0} e^{\gamma s}\Big(|f(s+\tau)|^2 + |\mathcal{G}_\delta(\theta_s\omega)| + |\mathcal{G}_\delta(\theta_s\omega)|^{\frac{2}{1-\kappa}} + |\mathcal{G}_\delta(\theta_s\omega)|^2\Big)ds\\
&\quad + M\int_{s_0}^{0} e^{\int_0^s (\nu\lambda_1 + 2\alpha|\mathcal{G}_\delta(\theta_r\omega)|)dr}\Big(|f(s+\tau)|^2 + |\mathcal{G}_\delta(\theta_s\omega)| + |\mathcal{G}_\delta(\theta_s\omega)|^{\frac{2}{1-\kappa}} + |\mathcal{G}_\delta(\theta_s\omega)|^2\Big)ds\\
&:= R_H(\tau,\omega,\delta,0),
\end{aligned} \tag{5.2}$$

where s_0 is as in (2.16), and $0 < R_H(\tau,\omega,\delta,0) < +\infty$ is independent of $\varepsilon \in (0,1]$. It then follows from (5.1) and (5.2) that

$$\bigcup_{0<\varepsilon\leqslant 1} \mathcal{A}^\varepsilon(\tau,\omega) \subseteq \bigcup_{0<\varepsilon\leqslant 1} K^\varepsilon(\tau,\omega) \subseteq B(\tau,\omega), \tag{5.3}$$

where $B(\tau,\omega)$, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, is given by

$$B(\tau,\omega) = \left\{ u \in H : |u|^2 \leqslant R_H(\tau,\omega,\delta,0) \right\}.$$

If $\varepsilon = 0$, then system (1.1) reduces to the following deterministic system:

$$\begin{cases}
\dfrac{\partial u}{\partial t} - \nu\Delta u + F_N(\|u\|)\left[(u(t)\cdot\nabla)u\right] + \nabla p = f(t) \text{ in } \mathcal{O}\times(\tau,\infty),\\
\text{div } u = 0 \text{ in } \mathcal{O}\times(\tau,\infty),\\
u = 0 \text{ on } \Gamma\times(\tau,\infty),\\
u(x,\tau) = u_\tau(x),\ x\in\mathcal{O}.
\end{cases} \tag{5.4}$$

Let u^0 and U^0 be the solution of (5.4) and the corresponding cocycle, respectively, and define a collection of families of deterministic nonempty subsets of H by

$$\mathcal{D}_0 = \left\{ D = \{D(\tau) : \tau \in \mathbb{R}\} : \lim_{t \to -\infty} e^{ct} |D(\tau + t)| = 0, \ \forall \ c > 0 \text{ and } \tau \in \mathbb{R} \right\},$$

where $|D(\tau + t)| = \sup_{u \in D(\tau+t)} |u|$. Observe that all estimates and results in the preceding sections are valid for $\varepsilon = 0$. Therefore, U^0 has a unique $\mathcal{D}_0$-pullback attractor $\mathcal{A}^0 = \{A^0(\tau) : \tau \in \mathbb{R}\}$ in H, and there exists a family of probability measures $\{\mu_t^0\}_{t \in \mathbb{R}}$ of process U^0 such that μ_t^0 is an invariant measure for U^0 and is supported on $A^0(t)$.

In what follows, we will show any limit of a sequence of invariant measures of (1.1) must be an invariant measure of the limiting system (5.4). To that end, we first prove the compactness of the set of all attractors of (1.1) for all $\varepsilon \in (0, 1]$.

Lemma 5.1. *Suppose (G1), (G2), (F1) and (F2) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the union $\bigcup_{0<\varepsilon \leqslant 1} A^\varepsilon(\tau, \omega)$ is precompact in H.*

Proof. Let $\left\{u^k\right\}_{k=1}^{\infty} \subseteq \bigcup_{0<\varepsilon \leqslant 1} A^\varepsilon(\tau, \omega)$. Then, for each k, there is $\varepsilon_k \in (0, 1]$ such that $u^k \in A^{\varepsilon_k}(\tau, \omega)$. Let $\{t_k\}_{k=1}^{+\infty}$ be a sequence of numbers with $t_k \to +\infty$. By the invariance of $\mathcal{A}^{\varepsilon_k}$, there exists $\hat{u}^k \in A^{\varepsilon_k}(\tau - t_k, \theta_{-t_k}\omega)$ such that

$$u^k = U^{\varepsilon_k}(t_k, \tau - t_k, \theta_{-t_k}\omega, \hat{u}^k).$$

It follows from (5.3) that $\hat{u}^k \in B(\tau - t_k, \theta_{\tau - t_k}\omega)$. In addition, by (2.16), we can deduce that there exists $R_V(\tau, \omega, \delta) > 0$ independent of $\varepsilon \in (0, 1]$ such that for any $\varepsilon \in (0, 1]$, $R_V(\tau, \omega, \varepsilon, \delta) \leqslant R_V(\tau, \omega, \delta)$. Then, by the same technique as in Lemma 2.3, we can deduce that u^k is uniformly bounded in V, which together with the compactness of embedding $V \hookrightarrow H$ shows that the sequence u^k is precompact in H. The proof is complete. $\square$

The following lemma is concerned with convergence of solutions of (1.1) with respect to ε.

Lemma 5.2. *Suppose (G1), (G2), (F1) and (F2) hold. Then, for any $\omega \in \Omega$, $\varepsilon_0 \in [0, 1]$, $T_0 > 0$ and bounded subset $K \subseteq H$, we have for any $0 \leqslant t \leqslant T_0$,*

$$\lim_{\varepsilon \to \varepsilon_0} \sup_{v \in K} |U^\varepsilon(t, \tau - t, \theta_{-t}\omega, v) - U^{\varepsilon_0}(t, \tau - t, \theta_{-t}\omega, v)| = 0.$$

Proof. It follows from (2.11) that

$$|U^\varepsilon(t, \tau - t, \theta_{-t}\omega, v) - U^{\varepsilon_0}(t, \tau - t, \theta_{-t}\omega, v)| = |u^\varepsilon(\tau, \tau - t, \theta_{-\tau}\omega, v) - u^{\varepsilon_0}(\tau, \tau - t, \theta_{-\tau}\omega, v)| . \quad (5.5)$$

For short, let $u^\varepsilon(s) = u^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v)$, $u^{\varepsilon_0}(s) = u^{\varepsilon_0}(s, \tau - t, \theta_{-\tau}\omega, v)$ and $z(\tau) = u^\varepsilon(\tau) - u^{\varepsilon_0}(\tau)$. It follows from (1.1), (2.5), (2.7) and (G1) that

$$\begin{aligned} &|u^\varepsilon(\tau) - u^{\varepsilon_0}(\tau)|^2 + 2\nu \int_{\tau-t}^{\tau} \|u^\varepsilon(s) - u^{\varepsilon_0}(s)\|^2 ds \\ &\leqslant \nu \int_{\tau-t}^{\tau} \|z(s)\|^2 ds + (1 + C_\nu + 2\alpha + 2L_h) \int_{\tau-t}^{\tau} |z(s)|^2 ds \\ &\quad + 16(\varepsilon - \varepsilon_0)^2 \int_{\tau-T_0}^{\tau} \left[\left(\alpha^2 + L_h^2\right) |u^{\varepsilon_0}(s)|^2 + |h(0)|^2 + |\beta|^2 \right] ds. \end{aligned} \tag{5.6}$$

By (2.15), we can deduce that there exists $M_1 = M_1(\omega, \tau, T_0, \varepsilon_0, K) > 0$ such that for all $t \in [0, T_0]$,

$$\int_{\tau-T_0}^{\tau} \left[\left(\alpha^2 + L_h^2\right) |u^{\varepsilon_0}(s)|^2 + |h(0)|^2 + |\beta|^2 \right] ds \leqslant M_1,$$

which, together with (5.6) and Gronwall's inequality, implies that, for any $t \in [0, T_0]$,

$$|u^\varepsilon(t) - u^{\varepsilon_0}(t)|^2 \leqslant 16 M_1 (\varepsilon - \varepsilon_0)^2 e^{(1 + C_\nu + 2\alpha + 2L_h)T_0}. \tag{5.7}$$

Therefore, by (5.5) and (5.7), the desire result can be obtained. □

Given $\varepsilon \in (0, 1]$, let $\mathcal{IM}_\varepsilon$ (respectively, $\mathcal{PIM}_\varepsilon$) be the set of all invariant (respectively, T-periodic invariant) probability measures of (1.1) with parameter ε satisfying that the support of such measure is contained in the pullback attractor. Through the above analysis, we find that $\mathcal{IM}_\varepsilon$ (respectively, $\mathcal{PIM}_\varepsilon$) is nonempty. Then the main results of this section are given below.

Theorem 5.1. *Suppose (G1), (G2), (F1) and (F2) hold. Let $\varepsilon_n \in (0, 1]$ for all $n \in \mathbb{N}$ satisfying that $\varepsilon_n \to 0$ as $n \to +\infty$. If $\mu_{t,\omega}^{\varepsilon_n} \in \mathcal{IM}_{\varepsilon_n}$ for every $t \in \mathbb{R}$ and $\omega \in \Omega$, then there exist a subsequence $\{\varepsilon_{n_k}\}_{k=1}^{+\infty}$ of $\{\varepsilon_n\}_{n=1}^{+\infty}$ depending on t and ω, and an invariant measure μ_t^0 of U^0 such that $\mu_{t,\omega}^{\varepsilon_{n_k}}$ converges weakly to μ_t^0 as $k \to +\infty$.*

If, in addition, $f(t)$ is a T-periodic function in t and $\mu_{t,\omega}^{\varepsilon_n} \in \mathcal{PIM}_{\varepsilon_n}$ for every $t \in \mathbb{R}$ and $\omega \in \Omega$, then there exist a subsequence $\{\varepsilon_{n_k}\}_{k=1}^{+\infty}$ of $\{\varepsilon_n\}_{n=1}^{+\infty}$ depending on t and ω, and a T-periodic invariant measure μ_t^0 of U^0 such that $\mu_{t,\omega}^{\varepsilon_{n_k}}$ converges weakly to μ_t^0 as $k \to +\infty$.

Proof. Taking $\varepsilon_0 = 0$ in Lemma 5.2, one can obtain that for any bounded subset $K \subseteq H$ and $0 \leqslant t \leqslant T_0$ ($T_0 > 0$),

$$\lim_{\varepsilon \to 0} \sup_{v \in K} |U^\varepsilon(t, \tau - t, \theta_{-t}\omega, v) - U^0(t, \tau - t, v)| = 0,$$

which, together with Lemma 5.1 and [9, Theorems 4.1 and 4.2], shows the desired results. □

5.2 Limiting behaviors of invariant measures as $\delta \to 0$

In this subsection, we will investigate the limiting relationships of invariant measures between the GMNSE with colored noise and GMNSE with white noise. More precisely, in that case, we consider the following GMNSE with additive colored noise:

$$\begin{cases} \dfrac{\partial u_\delta}{\partial t} - \nu \Delta u_\delta + F_N(\|u_\delta\|)\left[(u_\delta(t)\cdot\nabla)u_\delta\right] + \nabla p = f(t) + \beta \mathcal{G}_\delta(\theta_t\omega) \text{ in } \mathcal{O}\times(\tau,\infty), \\ \text{div } u_\delta = 0 \text{ in } \mathcal{O}\times(\tau,\infty), \\ u_\delta = 0 \text{ on } \Gamma\times(\tau,\infty), \\ u_\delta(x,\tau) = u_\tau(x),\ x\in\mathcal{O}, \end{cases} \tag{5.8}$$

where $\beta \in D(A)$. Then we will show that as $\delta \to 0$, the limit (periodic) invariant measures of (5.8) converges to (periodic) invariant measures of the following GMNSE driven by additive white noise:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + F_N(\|u\|)\left[(u(t)\cdot\nabla)u\right] + \nabla p = f(t) + \beta \dfrac{dW}{dt} \text{ in } \mathcal{O}\times(\tau,\infty), \\ \text{div } u = 0 \text{ in } \mathcal{O}\times(\tau,\infty), \\ u = 0 \text{ on } \Gamma\times(\tau,\infty), \\ u(x,\tau) = u_\tau(x),\ x\in\mathcal{O}. \end{cases} \tag{5.9}$$

5.2.1 Existence of invariant measures for GMNSE with additive white noise

Next, we will show the existence of invariant probability measures of (5.9). To do that, we first transform the stochastic system (5.9) into a pathwise deterministic one by

$$v(t,\tau,\omega,v_\tau) = u(t,\tau,\omega,u_\tau) - \beta z(\theta_t\omega), \tag{5.10}$$

where $z(\theta_t\omega)$ is the stationary solution of the one-dimensional Ornstein-Uhlenbeck equation $dz(\theta_t\omega) = -z(\theta_t\omega)dt + dW$. And there exists a θ_t-invariant set $\widetilde{\Omega} \subseteq \Omega$ with $\mathbb{P}(\widetilde{\Omega}) = 1$ such that for every $\omega \in \widetilde{\Omega}$, $z(\theta_t\omega)$ is continuous in t and

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{t} = 0 \text{ and } \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_s\omega)ds = 0. \tag{5.11}$$

From now on, we only consider the space $\widetilde{\Omega}$ rather than Ω, and write $\widetilde{\Omega}$ as Ω for convenience. By (5.9) and (5.10), we have

$$\frac{dv}{dt} + \nu A v + F_N(\|v+\beta z\|)\left[\big((v+\beta z)\cdot\nabla\big)(v+\beta z)\right] = f + \beta z - \nu z A\beta, \tag{5.12}$$

with initial condition $v_\tau = u_\tau - \beta z(\theta_\tau\omega)$.

Given $\omega \in \Omega$, (5.12) is a deterministic system. Therefore, for any $\omega\in\Omega$, $\tau\in\mathbb{R}$ and $v_\tau \in H$, (5.12) has a unique solution $v(\cdot,\tau,\omega,v_\tau)$ in the sense of Definition 2.4. Moreover, the solution v is

$(\mathcal{F}, \mathcal{B}(H))$-measurable in $\omega \in \Omega$, and depends continuously on v_τ in H. Based on this, a continuous non-autonomous cocycle $U_0 : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \to H$ for (5.9) can be defined by

$$U_0(t, \tau, \omega, u_\tau) = v(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau) + \beta z(\theta_t \omega) = u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau), \tag{5.13}$$

where $v_\tau = u_\tau - \beta z(\omega)$.

Next, the existence of $\mathcal{D}$-pullback absorbing set of U_0 will be given.

Lemma 5.3. *Suppose (F1) and (F2) hold. Then U_0 has a closed measurable $\mathcal{D}$-pullback absorbing set $K_0 = \{K_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ as given by*

$$K_0(\tau, \omega) = \left\{ u \in H : |u|^2 \leqslant R_{0,H}(\tau, \omega, 0) \right\},$$

where $R_{0,H}(\tau, \omega, 0) = 1 + 2|\beta z(\omega)|^2 + 2\widetilde{C} \int_{-\infty}^{0} e^{\nu\lambda_1 s} \left(|f(s + \tau)|^2 + |\beta z(\theta_s \omega)|^2 + \|\beta z(\theta_s \omega)\|^2 \right) ds.$

Proof. From (5.12), it follows that

$$\frac{d}{dt}|v|^2 + 2\nu\|v\|^2 + 2F_N(\|v + \beta z\|) b\left(v + \beta z, v + \beta z, v\right) = 2(f, v) + 2(\beta z - \nu z A\beta, v). \tag{5.14}$$

For the third term on the left-hand side of (5.14), by (2.5) and Young's inequality, we have

$$2F_N(\|v + \beta z\|) \left| b\left(v + \beta z, v + \beta z, v\right) \right| \leqslant 2CN\|\beta z\|\|v\| \leqslant \widetilde{C}\|\beta z\|^2 + \frac{1}{4}\|v\|^2,$$

which, together with (5.14), implies that

$$\frac{d}{dt}|v|^2 + \frac{\nu}{2}\|v\|^2 \leqslant -\nu\lambda_1|v|^2 + \widetilde{C}(|f|^2 + |\beta z|^2 + \|\beta z\|^2). \tag{5.15}$$

Then by applying Gronwall's inequality to (5.15), integrating on $(\tau - t, \tau + k)$ with $t > 1$ and $k \in [-1, 0]$, after replacing ω by $\theta_{-\tau}\omega$, we find that

$$\begin{aligned} &|v(\tau + k, \tau - t, \theta_{-\tau}\omega, v_{\tau - t})|^2 \\ &\leqslant e^{-\nu\lambda_1(t+k)}|v_{\tau-t}|^2 + \widetilde{C} \int_{-\infty}^{0} e^{-\nu\lambda_1(k-s)} \left(|f(s + \tau)|^2 + |\beta z(\theta_s \omega)|^2 + \|\beta z(\theta_s \omega)\|^2 \right) ds. \end{aligned} \tag{5.16}$$

By (F1) and (5.11), the last term of (5.16) is well-defined. Thanks to (5.10), we have

$$u(\tau + k, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}) = v(\tau + k, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}) + \beta z(\theta_k \omega), \tag{5.17}$$

where $v_{\tau-t} = u_{\tau-t} - \beta z(\theta_{-t}\omega)$. It follows from (5.16) and (5.17) that

$$\begin{aligned} &|u(\tau + k, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})|^2 \leqslant 2|v(\tau + k, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})|^2 + 2|\beta z(\theta_k \omega)|^2 \\ &\leqslant 4e^{-\nu\lambda_1(t+k)}|u_{\tau-t}|^2 + 4e^{-\nu\lambda_1(t+k)}|\beta z(\theta_{-t}\omega)|^2 + 2|\beta z(\theta_k \omega)|^2 \\ &\quad + 2\widetilde{C} \int_{-\infty}^{0} e^{-\nu\lambda_1(k-s)} \left(|f(s + \tau)|^2 + |\beta z(\theta_s \omega)|^2 + \|\beta z(\theta_s \omega)\|^2 \right) ds, \end{aligned}$$

which, together with (5.11), and the fact that $u_{\tau-t} \in D(\tau-t, \theta_{-t}\omega)$ and $D \in \mathcal{D}$, implies that there exists $T_1 = T_1(\tau, \omega, D) > 0$ such that for all $t \geqslant T_1$,

$$
\begin{aligned}
&|u(\tau+k, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|^2 \\
&\leqslant 1 + 2|\beta z(\theta_k \omega)|^2 + 2\widetilde{C} \int_{-\infty}^{0} e^{-\nu\lambda_1(k-s)} \left(|f(s+\tau)|^2 + |\beta z(\theta_s\omega)|^2 + \|\beta z(\theta_s\omega)\|^2\right) ds \\
&:= R_{0,H}(\tau, \omega, k),
\end{aligned}
\tag{5.18}
$$

which implies that for any $t \geqslant T_1$,

$$
u(\tau, \tau-t, \theta_{-\tau}\omega, D(\tau-t, \theta_{-\tau}\omega)) \subseteq K_0(\tau, \omega). \tag{5.19}
$$

In addition, by (F2) and (5.11), we can obtain that K_0 is tempered in H, which together with (5.19) completes the proof. □

The uniform estimate of (5.9) in V will be established, which is crucial to prove the pullback asymptotic compactness of U_0 in H.

Lemma 5.4. *Suppose (F1) and (F2) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(t,\omega) : t \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T_2 = T_2(\tau, \omega, D) > 0$ such that for all $t \geqslant T_2$,*

$$
\begin{aligned}
\|u(\tau, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 \leqslant& 2\left(1+\widetilde{C}\right) L_{0,V}(\tau,\omega) + 2\|\beta z(\omega)\|^2 + 2\widetilde{C}\int_{\tau-1}^{\tau} |f(s)|^2 ds \\
&+ 2\widetilde{C}\int_{-1}^{0} \left(|\beta z(\theta_s\omega)|^2 + \|\beta z(\theta_s\omega)\|^2 + \nu|z(\theta_s\omega)A\beta|^2\right) ds \\
:=& R_{0,V}(\tau,\omega),
\end{aligned}
$$

where $u_{\tau-t} \in D(\tau-t, \theta_{-t}\omega)$ and $0 < L_{0,V}(\tau,\omega) < +\infty$.

Proof. Let us denote $v(s) = v(s, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})$, for any $s \geqslant \tau - t$. It follows from (5.15) and (5.18) that there exists $T_1 = T_1(\tau, \omega, D) > 0$ such that, for any $t \geqslant T_1$,

$$
\begin{aligned}
&|v(\tau, \tau-1, \theta_{-\tau}\omega, v(\tau-1))|^2 + \frac{\nu}{2}\int_{\tau-1}^{\tau} \|v(s)\|^2 ds \\
&\leqslant R_{0,H}(\tau,\omega,-1) + \widetilde{C}\int_{\tau-1}^{\tau} |f(s)|^2 ds + \widetilde{C}\int_{-1}^{0} \left(|\beta z(\theta_s\omega)|^2 + \|\beta z(\theta_s\omega)\|^2\right) ds \\
&:= L_{0,V}(\tau,\omega),
\end{aligned}
\tag{5.20}
$$

where $R_{0,H}(\tau,\omega,-1) > 0$ given by (5.18) with k being replaced by -1. In addition, we can obtain

$$
\begin{aligned}
&\frac{d}{dt}\|v\|^2 + 2\nu|Av|^2 + 2F_N(\|v+\beta z\|)b(v+\beta z, v+\beta z, Av) \\
&= 2(f, Av) + 2(\beta z - \nu z A\beta, Av).
\end{aligned}
\tag{5.21}
$$

For the third term on the left-hand side of (5.21), by (2.6), we have

$$2\left|F_N(\|v+\beta z\|)b(v+\beta z, v+\beta z, Av)\right| \leqslant \frac{\nu}{4}|Av|^2 + \widetilde{C}\|v\|^2 + \widetilde{C}\|\beta z\|^2,$$

which, together with (5.21) and Young's inequality, shows that

$$\frac{d}{dt}\|v\|^2 + \nu|Av|^2 \leqslant \widetilde{C}\|v\|^2 + \widetilde{C}\left(|f(t)|^2 + |\beta z|^2 + \|\beta z\|^2 + \nu|zA\beta|^2\right)$$

It then follows from (5.20) that, for any $r \in [\tau-1, \tau]$ and $t \geqslant T_1$,

$$\begin{aligned}\|v\left(\tau, r, \theta_{-\tau}\omega, v(r)\right)\|^2 \leqslant& \|v(r)\|^2 + \widetilde{C}L_{0,V}(\tau,\omega) + \widetilde{C}\int_{\tau-1}^{\tau}|f(s)|^2 ds \\ &+ \widetilde{C}\int_{-1}^{0}\left(|\beta z(\theta_s\omega)|^2 + \|\beta z(\theta_s\omega)\|^2 + \nu|z(\theta_s\omega)A\beta|^2\right)ds.\end{aligned}$$

Integrating in r, and using again (5.20), we obtain for any $t \geqslant T_1$,

$$\begin{aligned}\|v(\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 \leqslant& \left(1+\widetilde{C}\right)L_{0,V}(\tau,\omega) + \widetilde{C}\int_{\tau-1}^{\tau}|f(s)|^2 ds \\ &+ \widetilde{C}\int_{-1}^{0}\left(|\beta z(\theta_s\omega)|^2 + \|\beta z(\theta_s\omega)\|^2 + \nu|z(\theta_s\omega)A\beta|^2\right)ds,\end{aligned}$$

which, together with (5.17), yields that

$$\begin{aligned}\|u(\tau, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 &\leqslant 2\left\|v(\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\right\|^2 + 2\|\beta z(\omega)\|^2 \\ &\leqslant 2\left(1+\widetilde{C}\right)L_{0,V}(\tau,\omega) + 2\|\beta z(\omega)\|^2 + 2\widetilde{C}\int_{\tau-1}^{\tau}|f(s)|^2 ds \\ &\quad + 2\widetilde{C}\int_{-1}^{0}\left(|\beta z(\theta_s\omega)|^2 + \|\beta z(\theta_s\omega)\|^2 + \nu|z(\theta_s\omega)A\beta|^2\right)ds.\end{aligned}$$

The proof is complete. □

Similar to the proof of Theorem 2.1, by Lemmas 5.3 and 5.4, we can show that U_0 has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}_0$ in H. If, in addition, there exists $T > 0$ such that f is T-periodic in t, then the attractor $\mathcal{A}_0$ is also T-periodic. Next, we will show the continuity of U_0, which is useful to prove the existence of invariant measures.

Lemma 5.5. *For every $\tau \in \mathbb{R}$, $\omega \in \Omega$, the mapping $U_0(t, \tau-t, \theta_{-t}\omega, u_0)$ is continuous in $(t, u_0) \in [0, +\infty) \times H$.*

Proof. Let $\tau \in \mathbb{R}$, $(t^*, u_0^*) \in [0, +\infty) \times H$ be fixed. In order to show such continuity, we need to prove for any $\epsilon > 0$, there exists $0 < \zeta < 1$ such that for all $|t - t^*| < \zeta$ and $|u_0 - u_0^*| < \zeta$,

$$|U_0(t, \tau-t, \theta_{-t}\omega, u_0) - U_0(t^*, \tau-t^*, \theta_{-t^*}\omega, u_0^*)|^2 < \epsilon.$$

By (5.13), we see that

$$
\begin{aligned}
&|U_0(t,\tau-t,\theta_{-t}\omega,u_0)-U_0(t^*,\tau-t^*,\theta_{-t^*}\omega,u_0^*)|^2\\
&=|u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)-u(\tau,\tau-t^*,\theta_{-\tau}\omega,u_0^*)|^2\\
&\leqslant 2|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)-v(\tau,\tau-t,\theta_{-\tau}\omega,v_0^*)|^2\\
&\quad+2|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0^*)-v(\tau,\tau-t^*,\theta_{-\tau}\omega,v_0^*)|^2,
\end{aligned}
\tag{5.22}
$$

where $v_0 = u_0 - \beta z(\theta_{-t}\omega)$ and $v_0^* = u_0^* - \beta z(\theta_{-t^*}\omega)$. We now estimate the first term on the right-hand side of (5.22). By (5.12), we have

$$
\begin{aligned}
&\frac{d}{dt}|v(\tau)-\hat{v}(\tau)|^2+2\nu\|v(t)-\hat{v}(t)\|^2\\
&=2\big(F_N(\|\hat{v}+\beta z\|)b(\hat{v}+\beta z,\hat{v}+\beta z,v-\hat{v})-F_N(\|v+\beta z\|)b(v+\beta z),v+\beta z,v-\hat{v})\big).
\end{aligned}
\tag{5.23}
$$

For the last term of (5.23), similar to (4.9), by (2.5), (2.7) and the Young inequality, we see that

$$
\begin{aligned}
&\big|F_N(\|\hat{v}+\beta z\|)b(\hat{v}+\beta z,\hat{v}+\beta z,v-\hat{v})-F_N(\|v+\beta z\|)b(v+\beta z,v+\beta z,v-\hat{v})\big|\\
&\leqslant \widetilde{C}|v-\hat{v}|^2+\nu\|v-\hat{v}\|^2,
\end{aligned}
$$

which, along with (5.23) and Gronwall's inequality, shows that for any $t>0$ with $|t-t^*|<1$,

$$
\begin{aligned}
&2\,|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)-v(\tau,\tau-t,\theta_{-\tau}\omega,v_0^*)|^2\\
&\leqslant 4\left(|u_0-u_0^*|^2+|\beta z(\theta_{-t}\omega)-\beta z(\theta_{-t^*}\omega)|^2\right)e^{\widetilde{C}(t^*+1)},
\end{aligned}
$$

which, together with the continuity of $z(\theta_t\omega)$ in t, shows that there exists $0<\zeta_1<1$ such that for any $|t-t^*|<\zeta_1$ and $|u_0-u_0^*|<\zeta_1$,

$$
2\,|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0)-v(\tau,\tau-t,\theta_{-\tau}\omega,v_0^*)|^2<\frac{\epsilon}{2}. \tag{5.24}
$$

Next, the last term of (5.22) will be estimated. Without loss of generality, we assume $t^*<t$. Since v is the solution of (5.12), we have

$$
\begin{aligned}
&2|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0^*)-v(\tau,\tau-t^*,\theta_{-\tau}\omega,v_0^*)|^2\\
&=2|v(\tau,\tau-t^*,\theta_{-\tau}\omega,v(\tau-t^*,\tau-t,\theta_{-\tau}\omega,v_0^*))-v(\tau,\tau-t^*,\theta_{-\tau}\omega,v_0^*)|^2,
\end{aligned}
$$

which, together with (5.23) and Gronwall's inequality, shows that

$$
\begin{aligned}
&2|v(\tau,\tau-t,\theta_{-\tau}\omega,v_0^*)-v(\tau,\tau-t^*,\theta_{-\tau}\omega,v_0^*)|^2\\
&\leqslant 2|v(\tau-t^*,\tau-t,\theta_{-\tau}\omega,v_0^*)-v_0^*|^2e^{\widetilde{C}t^*}\\
&=2\left(|v(\tau-t^*,\tau-t,\theta_{-\tau}\omega,v_0^*)|^2-|v_0^*|^2-2\big(v(\tau-t^*,\tau-t,\theta_{-\tau}\omega,v_0^*),v_0^*\big)\right)e^{\widetilde{C}t^*}.
\end{aligned}
$$

Then, by a similar technique to the one used in Lemma 3.1, we can derive that there exists $0 < \zeta_2 < \zeta_1$ such that, for any $|t - t^*| < \zeta_2$,

$$2|v(\tau, \tau - t, \theta_{-\tau}\omega, v_0^*) - v(\tau, \tau - t^*, \theta_{-\tau}\omega, v_0^*)|^2 < \frac{\epsilon}{2},$$

which, along with (5.22) and (5.24), shows the desired result. □

Now, we present the existence of invariant measures and periodic invariant measures of (5.9).

Theorem 5.2. *Suppose (F1) and (F2) hold. Fix a generalized Banach limit* $\mathrm{LIM}_{t\to-\infty}$ *and let* $\xi : \mathbb{R} \to H$ *be a continuous mapping such that* $\{\xi_t\}_{t\in\mathbb{R}} \in \mathcal{D}$. *Then for any* $N > 0$, *there exists a family of probability measures* $\{\mu^0_{t,\omega}\}_{t\in\mathbb{R},\omega\in\Omega}$ *of process* U_0 *such that* $\mu^0_{t,\omega}$ *is an invariant measure for* U_0 *and is supported on* $A_0(t, \omega)$.

If, in addition, $f(t)$ *is a* T*-periodic function in* t *and* $\xi : \mathbb{R} \to H$ *is a* T*-periodic continuous map in* t, *then the invariant measure* $\mu^0_{t,\omega}$ *is* T*-periodic.*

5.2.2 Convergence of invariant measures from colored noise to white noise

Next, the limiting behaviors of invariant measures of (5.8) will be investigated. Noting that system (5.8) is a particular case of (1.1), it follows from Theorem 3.2 that there exists an invariant measure $\mu^\delta_{t,\omega}$ supported on the attractor $A_\delta(t, \omega)$ for any $t \in \mathbb{R}$ and $\omega \in \Omega$. In order to investigate the limiting behaviors of invariant measures, we need to introduce a new transformation, which is useful to show the convergence of solutions of (5.8) as $\delta \to 0$. Let

$$v_\delta(t, \tau, \omega) = u_\delta(t, \tau, \omega) - \beta y_\delta(\theta_t\omega), \tag{5.25}$$

where y_δ is the solution of the following random equation driven by colored noise $dy_\delta = -y_\delta dt + \mathcal{G}_\delta(\theta_t\omega)dt$. By [19, Lemma 3.2], we have

$$\lim_{\delta\to0} y_\delta(\theta_t\omega) = z(\theta_t\omega), \text{ uniformly on } [\tau, \tau + T] \text{ with } \tau \in \mathbb{R} \text{ and } T > 0; \tag{5.26}$$

$$\lim_{t\to\pm\infty} \frac{|y_\delta(\theta_t\omega)|}{|t|} = 0, \text{ uniformly for } 0 < \delta \leqslant \frac{1}{2}; \tag{5.27}$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t y_\delta(\theta_s\omega)ds = 0, \text{ uniformly for } 0 < \delta \leqslant \frac{1}{2}; \tag{5.28}$$

$$\lim_{\delta\to0} \mathbb{E}[|y_\delta(\omega)|] = \mathbb{E}[|z(\omega)|]. \tag{5.29}$$

By (5.8) and (5.25), we have

$$\frac{dv_\delta}{dt} + \nu A v_\delta + F_N(\|v_\delta + \beta y_\delta\|)\left[\big((v_\delta + \beta y_\delta)\cdot\nabla\big)(v_\delta + \beta y_\delta)\right] = f + \beta y_\delta - \nu y_\delta A\beta, \tag{5.30}$$

with initial value $v_{\delta,\tau} = u_\tau - \beta y_\delta(\theta_\tau\omega)$.

Given $\omega \in \Omega$, (5.30) is a deterministic equation. Therefore, under the same conditions as before, (5.30) has a unique solution v_δ in the sense of Definition 2.4. Moreover, the solution v_δ is $(\mathcal{F}, \mathcal{B}(H))$-measurable in $\omega \in \Omega$, and depends continuously on $v_{\delta,\tau}$ in H. Based on this, a continuous cocycle $U_\delta : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \to H$ for (5.8) can be defined by

$$U_\delta(t, \tau, \omega, u_\tau) = v_\delta(t + \tau, \tau, \theta_{-\tau}\omega, v_{\delta,\tau}) + \beta y_\delta(\theta_t\omega) = u_\delta(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau), \tag{5.31}$$

where $v_{\delta,\tau} = u_\tau - \beta y_\delta(\omega)$.

Next, the uniform estimates on the solutions of (5.8) will be established.

Lemma 5.6. *Suppose (F1) and (F2) hold. Then, for every $\delta \in (0, \frac{1}{2}]$, U_δ has a closed measurable $\mathcal{D}$-pullback absorbing set $K_\delta = \{K_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ as given by*

$$K_\delta(\tau, \omega) = \left\{u \in H : |u|^2 \leqslant R_{\delta,H}(\tau, \omega, 0)\right\},$$

where $R_{\delta,H}(\tau, \omega, 0) = 1 + 2|\beta y_\delta(\omega)|^2 + 2\widetilde{C} \int_{-\infty}^{0} e^{-\nu\lambda_1 s} \left(|f(s + \tau)|^2 + |\beta y_\delta(\theta_s\omega)|^2 + \|\beta y_\delta(\theta_s\omega)\|^2\right) ds$, and $\widetilde{C}$ is a positive constant independent of τ, ω and δ.

Proof. Similar to the proof of Lemma 5.3, we can derive that

$$\begin{aligned} &|v_\delta(\tau, \tau - t, \theta_{-\tau}\omega, v_{\delta,\tau-t})|^2 \\ &\leqslant e^{-\nu\lambda_1 t}|v_{\delta,\tau-t}|^2 + \widetilde{C} \int_{-\infty}^{0} e^{-\nu\lambda_1 s} \left(|f(s + \tau)|^2 + |\beta y_\delta(\theta_s\omega)|^2 + \|\beta y_\delta(\theta_s\omega)\|^2\right) ds. \end{aligned} \tag{5.32}$$

By (F1) and (5.27), the last term of (5.32) is well-defined. It follows from (5.25) and (5.32) that

$$\begin{aligned} &|u_\delta(\tau, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})|^2 \\ &\leqslant 4e^{-\nu\lambda_1 t}|u_{\delta,\tau-t}|^2 + 4e^{-\nu\lambda_1 t}|\beta y_\delta(\theta_{-t}\omega)|^2 + 2|\beta y_\delta(\omega)|^2 \\ &\quad + 2\widetilde{C} \int_{-\infty}^{0} e^{-\nu\lambda_1 s} \left(|f(s + \tau)|^2 + |\beta y_\delta(\theta_s\omega)|^2 + \|\beta y_\delta(\theta_s\omega)\|^2\right) ds, \end{aligned}$$

which, together with (5.11) and the fact that $u_{\delta,\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ and $D \in \mathcal{D}$, implies that there exists $T_2 = T_2(\tau, \omega, D) > 0$ such that, for all $t \geqslant T_2$,

$$\begin{aligned} &|u_\delta(\tau, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})|^2 \\ &\leqslant 1 + 2|\beta y_\delta(\omega)|^2 + 2\widetilde{C} \int_{-\infty}^{0} e^{-\nu\lambda_1 s} \left(|f(s + \tau)|^2 + |\beta y_\delta(\theta_s\omega)|^2 + \|\beta y_\delta(\theta_s\omega)\|^2\right) ds, \end{aligned}$$

which implies that, for any $t \geqslant T_2$,

$$u_\delta(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-\tau}\omega)) \subseteq K_\delta(\tau, \omega). \tag{5.33}$$

In addition, by (F2) and (5.27), we can obtain that K_δ is tempered in H, which, along with (5.33), completes the proof. □

Lemma 5.7. *Suppose (F1) and (F2) hold. Then for every* $\delta \in (0, \frac{1}{2}]$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D = \{D(t,\omega) : t \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *we have that, for all* $t \geqslant T_2$,

$$\begin{aligned}\|u_\delta(\tau, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})\|^2 \leqslant & 2\left(1 + \widetilde{C}\right) L_{\delta,V}(\tau,\omega) + 2\|\beta y_\delta(\omega)\|^2 + 2\widetilde{C}\int_{\tau-1}^{\tau} |f(s)|^2 ds \\ & + 2\widetilde{C}\int_{-1}^{0} \left(|\beta y_\delta(\theta_s\omega)|^2 + \|\beta y_\delta(\theta_s\omega)\|^2 + \nu |y_\delta(\theta_s\omega) A\beta|^2\right) ds \\ := & R_{\delta,V}(\tau,\omega), \end{aligned} \tag{5.34}$$

where $L_{\delta,V}(\tau,\omega) = R_{\delta,H}(\tau,\omega,-1) + \widetilde{C}\int_{\tau-1}^{\tau} |f(s)|^2 ds + \widetilde{C}\int_{-1}^{0}\left(|\beta y_\delta(\theta_s\omega)|^2 + \|\beta y_\delta(\theta_s\omega)\|^2\right) ds$, $u_{\delta,\tau-t} \in D(\tau - t, \theta_{-t}\omega)$, *and* $\widetilde{C} > 0$ *independent of* τ, ω, δ *and* D.

Proof. The proof is similar to Lemma 5.4 and the details are omitted. □

The following lemma is concerned with the convergence of solutions of (5.8) as $\delta \to 0$.

Lemma 5.8. *For every* $T_0 > 0$, $\omega \in \Omega$, *and bounded subset* $K \subseteq H$, *we have for any* $0 \leqslant t \leqslant T_0$,

$$\lim_{\delta \to 0} \sup_{u_0 \in K} |U_\delta(t, \tau - t, \theta_{-t}\omega, u_0) - U_0(t, \tau - t, \theta_{-t}\omega, u_0)| = 0.$$

Proof. It follows from (5.13) and (5.31) that

$$\begin{aligned} & |U_\delta(t, \tau - t, \theta_{-t}\omega, u_0) - U_0(t, \tau - t, \theta_{-t}\omega, u_0)| \\ = & |u_\delta(\tau, \tau - t, \theta_{-\tau}\omega, u_0) - u_0(\tau, \tau - t, \theta_{-\tau}\omega, u_0)| \\ \leqslant & |v_\delta(\tau, \tau - t, \theta_{-\tau}\omega, v_{\delta,0}) - v_0(\tau, \tau - t, \theta_{-\tau}\omega, v_0)| + |\beta z(\omega) - \beta y_\delta(\omega)|, \end{aligned} \tag{5.35}$$

where $v_{\delta,0} = u_0 - \beta y_\delta(\theta_{-t}\omega)$ and $v_0 = u_0 - \beta z(\theta_{-t}\omega)$. By (5.12) and (5.30), we have

$$\begin{aligned} & \frac{d}{dt}|v_\delta - v|^2 + 2\nu\|v_\delta - v\|^2 \\ = & -2F_N(\|v_\delta + \beta y_\delta\|) b(v_\delta + \beta y_\delta, v_\delta + \beta y_\delta, \Lambda) + 2F_N(\|v_0 + \beta z\|) b(v_0 + \beta z, v_0 + \beta z, \Lambda) \\ & + 2\big(\beta y_\delta - \beta z, \Lambda\big) - 2\nu\big((y_\delta - z)A\beta, \Lambda\big), \end{aligned} \tag{5.36}$$

where $\Lambda = v_\delta - v_0$. For the first and second terms on the right-hand side of (5.36), similar to (4.9), by (2.5), (2.7) and Young's inequality, we find

$$\begin{aligned} & \big| -2F_N(\|v_\delta + \beta y_\delta\|) b(v_\delta + \beta y_\delta, v_\delta + \beta y_\delta, \Lambda) + 2F_N(\|v_0 + \beta z\|) b(v_0 + \beta z, v_0 + \beta z, \Lambda)\big| \\ \leqslant & \frac{\nu}{2}\|\Lambda\|^2 + \widetilde{C}|\Lambda|^2 + \widetilde{C}\|\beta\|^2 |y_\delta - z|^2. \end{aligned} \tag{5.37}$$

For the last two terms of (5.36), we have

$$\big|2\big(\beta y_\delta - \beta z, \Lambda\big) - 2\nu\big((y_\delta - z)A\beta, \Lambda\big)\big| \leqslant 2|\beta|\, |y_\delta - z|\, |\Lambda| + 2\nu\|\beta\|\, \|\Lambda\|\, |y_\delta - z|$$

$$\leqslant \frac{\nu}{2}\|\Lambda\|^2 + \widetilde{C}|\Lambda|^2 + \widetilde{C}\left(|\beta|^2 + \|\beta\|^2\right)|y_\delta - z|^2 .$$

which, together with (5.36), (5.37) and Gronwall's inequality, implies that, for any $t \in [0, T_0]$,

$$\begin{aligned}
&|v_\delta(\tau, \tau - t, \theta_{-\tau}\omega, v_{\delta,0}) - v_0(\tau, \tau - t, \theta_{-\tau}\omega, v_0)|^2 \\
&\leqslant |v_{\delta,0} - v_0|^2 e^{\widetilde{C}T_0} + \widetilde{C}(|\beta|^2 + \|\beta\|^2) \int_{-T_0}^0 |y_\delta(\theta_s\omega) - z(\theta_s\omega)|^2 e^{-\widetilde{C}s} ds.
\end{aligned} \tag{5.38}$$

Therefore, by (5.35) and (5.38), we obtain that for any $t \in [0, T_0]$,

$$\begin{aligned}
&|U_\delta(t, \tau - t, \theta_{-t}\omega, u_0) - U_0(t, \tau - t, \theta_{-t}\omega, u_0)|^2 \\
&\leqslant 2\,|\beta z(\theta_{-t}\omega) - \beta y_\delta(\theta_{-t}\omega)|^2 e^{\widetilde{C}T_0} + 2|\beta z(\omega) - \beta y_\delta(\omega)|^2 \\
&\quad + 2\widetilde{C}(|\beta|^2 + \|\beta\|^2) e^{\widetilde{C}T_0} \int_{-T_0}^0 |y_\delta(\theta_s\omega) - z(\theta_s\omega)|^2 ds,
\end{aligned}$$

which, together with (5.26), shows the desired result. □

We also need the following compactness result.

Lemma 5.9. *Suppose (F1) and (F2) hold. Then, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exists $0 < \delta_0 \leqslant \frac{1}{2}$ such that the union $\bigcup_{0<\delta\leqslant\delta_0} A_\delta(\tau, \omega)$ is precompact in H.*

Proof. Note that

$$\begin{aligned}
&\int_{-\infty}^0 e^{-\nu\lambda_1 s}\left(|\beta y_\delta(\theta_s\omega)|^2 + \|\beta y_\delta(\theta_s\omega)\|^2\right) ds \\
&= (|\beta| + \|\beta\|^2) \int_{-\infty}^{s_1} e^{-\nu\lambda_1 s}|y_\delta(\theta_s\omega)|^2 ds + (|\beta| + \|\beta\|^2) \int_{s_1}^0 e^{-\nu\lambda_1 s}|y_\delta(\theta_s\omega)|^2 ds,
\end{aligned} \tag{5.39}$$

where $s_1 < 0$ will be determined later. By (5.27), we see that there exists $s_1 < 0$ such that, for any $s \leqslant s_1$ and $\delta \in (0, \frac{1}{2}]$, $|y_\delta(\theta_s\omega)| < |s|$. Therefore, one can obtain

$$(|\beta| + \|\beta\|^2) \int_{-\infty}^{s_1} e^{-\nu\lambda_1 s}|y_\delta(\theta_s\omega)|^2 ds \leqslant M_1, \tag{5.40}$$

where $M_1 > 0$ is independent of δ. It follows from (5.26) that there exists $\delta_0 \in (0, \frac{1}{2}]$ such that, for any $0 < \delta \leqslant \delta_0$,

$$(|\beta| + \|\beta\|^2) \int_{s_1}^0 e^{-\nu\lambda_1 s}|y_\delta(\theta_s\omega)|^2 ds \leqslant (|\beta| + \|\beta\|^2) \int_{s_1}^0 e^{-\nu\lambda_1 s}|z(\theta_s\omega)|^2 ds. \tag{5.41}$$

Therefore, by (5.26) and (5.39)-(5.41), we have for any $0 < \delta \leqslant \delta_0$,

$$R_{\delta,H}(\tau, \omega, 0) \leqslant 1 + 2\widetilde{C}M_1 + 2|z(\omega)|^2 + 2\widetilde{C} \int_{-\infty}^0 e^{-\nu\lambda_1 s}|f(s + \tau)|^2 ds$$

$$+ (|\beta| + \|\beta\|^2) \int_{s_1}^{0} e^{-\nu\lambda_1 s} |z(\theta_s\omega)|^2 ds := \widetilde{R}_H(\tau, \omega). \tag{5.42}$$

By using (5.26) again, we can deduce that there exists $M_2 = M_2(\tau, \omega) > 0$, independent of δ, such that

$$R_{\delta,V}(\tau, \omega) \leqslant M_2. \tag{5.43}$$

Let $\left\{u^k\right\}_{k=1}^{\infty} \subseteq \bigcup_{0<\delta\leqslant\delta_0} A_\delta(\tau, \omega)$. Then for each k, there is $\delta_k \in (0, \delta_0]$ such that $u^k \in A_{\delta_k}(\tau, \omega)$. Let $\{t_k\}_{k=1}^{\infty}$ be a sequence of numbers with $t_k \to +\infty$. By the invariance of $\mathcal{A}_{\delta_k}$, there exists $\hat{u}^k \in A_{\delta_k}(\tau - t_k, \theta_{-t_k}\omega)$ such that

$$u^k = U_{\delta_k}(t_k, \tau - t_k, \theta_{-t_k}\omega, \hat{u}^k).$$

It follows from (5.41) and Lemma 5.6 that $\hat{u}^k \in B_0(\tau - t_k, \theta_{\tau-t_k}\omega)$, where $B_0(\tau, \omega) = \{u \in H : |u|^2 \leqslant \widetilde{R}_H(\tau, \omega)\}$. Then by (5.43) and Lemma 5.7, we find that u^k is uniformly bounded in V, which, together with the compactness of embedding $V \hookrightarrow H$, shows that the sequence u^k is precompact in H. The proof is complete. □

Given $\delta \in (0, \frac{1}{2}]$, let $\mathcal{IM}_\delta$ (respectively, $\mathcal{PIM}_\delta$) be the set of all invariant (respectively, T-periodic invariant) probability measures of (5.8) with parameter δ satisfying that the support of such measure is contained in the pullback attractor. Through the above analysis, we find that $\mathcal{IM}_\delta$ (respectively, $\mathcal{PIM}_\delta$) is nonempty. The next result is concerned with the limiting behaviors of invariant measures of (5.8) with respect to δ.

Theorem 5.3. *Suppose (F1) and (F2) hold. Let $\delta_n \in (0, \frac{1}{2}]$ for all $n \in \mathbb{N}$ satisfying that $\delta_n \to 0$ as $n \to +\infty$. If $\mu_{t,\omega}^{\delta_n} \in \mathcal{IM}_{\delta_n}$ for every $t \in \mathbb{R}$ and $\omega \in \Omega$, then there exist a subsequence $\{\delta_{n_k}\}_{k=1}^{+\infty}$ of $\{\delta_n\}_{n=1}^{+\infty}$ depending on t and ω, and an invariant measure $\mu_{t,\omega}^0$ of U_0 such that $\mu_{t,\omega}^{\delta_{n_k}}$ converges weakly to $\mu_{t,\omega}^0$ as $k \to +\infty$.*

If, in addition, $f(t)$ is a T-periodic function in t and $\mu_{t,\omega}^{\delta_n} \in \mathcal{PIM}_{\delta_n}$ for every $t \in \mathbb{R}$ and $\omega \in \Omega$, then there exist a subsequence $\{\delta_{n_k}\}_{k=1}^{+\infty}$ of $\{\delta_n\}_{n=1}^{+\infty}$ depending on t and ω, and a T-periodic invariant measure $\mu_{t,\omega}^0$ of U_0 such that $\mu_{t,\omega}^{\delta_{n_k}}$ converges weakly to $\mu_{t,\omega}^0$ as $k \to +\infty$.

Proof. The proof follows from Lemmas 5.8 and 5.9. The details are similar to Theorem 5.1 and hence omitted. □

Acknowledgements

The work is partially supported by the NNSF of China (11471190, 11971260), the SDNSF (ZR2014AM002), the CSC (202006220197), the Spanish Ministerio de Ciencia e Innovación under project PDI2021-122991NB-C21, and Junta de Andalucía (Spain) under project P18-FR-4509.

Conflict of Interest

The authors declare that they have no competing interests.

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