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Backstepping-Based Exponential Stabilization of Timoshenko Beam with Prescribed Decay Rate

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Abstract: In this paper, we present a rapid boundary stabilization of a Timoshenko beam with anti-damping and anti-stiffness at the uncontrolled boundary, by using PDE backstepping. We introduce a transformation to map the Timoshenko beam states into a $(2+2)\times(2+2)$ hyperbolic PIDE-ODE system. Then backstepping is applied to obtain a control law guaranteeing closed-loop stability of the origin in the H^1 sense. Arbitrarily rapid stabilization can be achieved by adjusting control parameters. Finally, a numerical simulation shows that the proposed controller can rapidly stabilize the Timoshenko beam. This result extends a previous work which considered a slender Timoshenko beam with Kelvin-Voigt damping, allowing destabilizing boundary conditions at the uncontrolled boundary and attaining an arbitrarily rapid convergence rate.

1. INTRODUCTION

Flexible beams are widely used in many applications ranging from aerospace to civil structures. Correspondingly, beam stabilization has become an important research topic. Among all the beam models, the Timoshenko model, as the most realistic of the 1D distributed parameter models, takes into account both the rotatory inertia of the beam cross-sections and the deflection due to shear effect.

We next give an overview of past results on control of Timoshenko beams. More than three decades ago, Kim and Renardy [15] used a classical boundary damper feedback which required both the space and time derivatives at the tip of the beam. Later, considering a clamped-free Timoshenko beam, Morgul [24] proposed a more general dynamic boundary feedback. Balakrishnan [2; 3] considered boundary conditions leading to superstability (vanishing of the beam states in finite time), for clamped boundary conditions on the uncontrolled end. Taylor and Yau [30] studied a rotating Timoshenko beam that can be stabilized by both applying a force at the free end and a torque at the pivoted end. Xu et al. [33] investigated the use of pointwise feedback controls based on asymptotic analysis of eigenvalues and the eigenfunctions. Soufyane et al. [29] achieved uniform stabilization by using a locally distributed damping; in this case, stability can be guaranteed if and only if the two wave equations have the same speeds. Macchelli et al. [21] used a distributed port Hamiltonian (dpH) approach to describe Timoshenko beams and proposed a finite dimensional passive controller that shapes the beam's total energy. This approach has also been followed by other authors; for instance, Siuka et al. [25] also adopted a dpH model and proposed an invariant-based method to achieve stabilization and Wu et al. [32] used a passive LQG control design method. Xu [34] presented a boundary feedback design for the exponential stabilization of a Timoshenko beam with both ends free, and gave an explicit asymptotic formula of eigenvalues of the closed loop system . Considering a Timoshenko beam with local Kelvin Voigt damping, Zhao et al. [36] obtained exponential stability under some additional hypotheses. Krstic et al [16; 26]. extended the backstepping method, by using a singular perturbation approach, to controller and observer design for a slender Timoshenko beam, with actuation only at the beam base and sensing only at the beam tip. For a nonuniform Timoshenko beam with spatial-varying parameters, Ammar-Khodja et al. [1] studied the stabilization for both internal and boundary cases with one control force. He et al. [12] designed an output-feedback control law using a Lyapunov-based approach with a disturbance observer; the Lyapunov approach is a powerful tool in design of control laws for beams, not only for the Timoshenko model (see e.g. [7]). Extending the approach, He et al. [13] proposed an adaptive integral-Barrier Lyapunov function boundary control for inhomogeneous Timoshenko beams with constraints. Considering both system uncertainties and uncertain input backlash nonlinearity, He et al. [11] gave vibration boundary control law using a disturbance observer. Allowing for hysteresis of the boundary control input, Liu and Xu [19] proposed a dynamic feedback control law that exponentially stabilized the beam with distributed delay. Yildirim et al. [35] proposed a novel optimal piezoelectric control approach for suppressing vibrations. Finally, to cite several very recent contributions, Ma et al. [20] introduced a prescribed performance function restricting within an arbitrarily small residual set, Guo and Meng [9] consider a two-dimensional robust output tracking for a Timoshenko beam equation

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by using an observer-based error feedback control approach, and Mattioni et al. [22] address a beam clamped on a moving inertia actuated by an external torque and force with the dpH method using strong dissipation feedback, and also in the case of having a mass at the controlled end [23].

In recent years, the backstepping method has proven itself as a powerful design method for control of infinite dimensional systems. However, beyond the results in [16; 26], backstepping has not been fully exploited for Timoshenko beam control, even though it has produced results for the shear beam model [17] and the Euler-Bernouilli model [27; 31]. In the present paper, we aim to achieve rapid stabilization of a Timoshenko beam with anti-damping and anti-stiffness at the uncontrolled boundary. The decay rate can be prescribed arbitrarily by setting the controller parameters. Specifically, we propose an initial transformation of the Timoshenko beam states to a new set of variables governed by a system of hyperbolic PIDEs-ODEs. Then the backstepping method is directly applied to controller design of the new system, by extending previously-developed tools [8].

Thus, the main contribution of this paper with respect to previous results is allowing destabilizing boundary conditions at the uncontrolled boundary (numerous works consider simple clamped conditions) and attaining an arbitrarily rapid convergence rate. The paper is organized as follows: Section 2 presents the Timoshenko beam model. Section 3 gives the boundary controller and the main result. Then, Section 4 analyzes the resulting controller. Section 5 studies the closed-loop stability. Finally, Section 6 validates the effectiveness of the proposed controller by numerical simulation.

2. PROBLEM STATEMENT

Our goal is to exponentially stabilize (with arbitrary convergence rate) the equilibrium at the origin of the following Timoshenko beam model

$$\varepsilon u_{tt} = u_{xx} - \alpha_x, \tag{1}$$

$$\mu\alpha_{tt} = \alpha_{xx} + \frac{a}{\varepsilon}\left(u_x - \alpha\right), \tag{2}$$

where $u(x,t)$ denotes the displacement, $\alpha(x,t)$ denotes the angle of rotation, for $x \in (0,1)$, $t > 0$. We omit time and space dependency except where necessary for clarity. The coefficients $\varepsilon, \mu > 0$, $a \in \mathbb{R}$ are non-dimensional physical parameters (see [10]). The boundary conditions are

$$u_x(0,t) = \alpha(0,t) - \theta u_t(0,t) - \xi u(0,t), \tag{3}$$

$$u_x(1,t) = V_1(t),\ \alpha_x(0,t) = 0,\ \alpha_x(1,t) = V_2(t), \tag{4}$$

with $\theta, \xi \in \mathbb{R}$ (respectively, anti-damping and anti-stiffness), and $V_1(t)$, $V_2(t)$ the actuation variables. The initial conditions for (1)–(4) are denoted by $u_0(x) = u(x,0)$, $\alpha_0(x) = \alpha(x,0)$, $u_{0t} = u_t(x,0)$, $\alpha_{0t} = \alpha_t(x,0)$.

Assumption 1. The anti-damping coefficient θ appearing in (3) verifies $\theta \neq \sqrt{\varepsilon}$.

This assumption is critical in what follows. To understand its underlying reason, consider just a simple wave equation $\varepsilon u_{tt} = u_{xx}$; then, a boundary condition of the type $u_x(0,t) = -\sqrt{\epsilon}u_t(0,t)$ can be seen to be ill-posed.

3. CONTROLLER DESIGN AND MAIN RESULT

As a first step, the Timoshenko beam is transformed into a first-order hyperbolic integro-differential system coupled with ODEs. This represents an alternative, novel idea to design a controller for this plant, since it opens the door to apply 1-D hyperbolic system control designs. The system becomes a $(2+2) \times (2+2)$ heterodirectional system of hyperbolic PIDEs, coupled with two ODEs, by using the following transformations

$$p = u_x + \sqrt{\varepsilon}u_t,\ q = u_x - \sqrt{\varepsilon}u_t, \tag{5}$$

$$r = \alpha_x + \sqrt{\mu}\alpha_t,\ s = \alpha_x - \sqrt{\mu}\alpha_t, \tag{6}$$

$$x_1 = u(0,t),\ x_2 = \alpha(0,t). \tag{7}$$

Then (1)–(4) is equivalent to the PIDE-ODE system

$$p_t = \frac{1}{\sqrt{\varepsilon}}p_x - \frac{1}{2\sqrt{\varepsilon}}\left(r+s\right), \tag{8}$$

$$r_t = \frac{1}{\sqrt{\mu}}r_x + \frac{a}{2\varepsilon\sqrt{\mu}}\left(p+q\right) - \frac{a}{2\varepsilon\sqrt{\mu}}\left[\int_0^x \left(r\left(y,t\right)+s\left(y,t\right)\right)dy + 2x_2\right], \tag{9}$$

$$q_t = -\frac{1}{\sqrt{\varepsilon}}q_x - \frac{1}{2\sqrt{\varepsilon}}\left(r+s\right), \tag{10}$$

$$s_t = -\frac{1}{\sqrt{\mu}}s_x + \frac{a}{2\varepsilon\sqrt{\mu}}\left(p+q\right) - \frac{a}{2\varepsilon\sqrt{\mu}}\left[\int_0^x \left(r\left(y,t\right)+s\left(y,t\right)\right)dy + 2x_2\right], \tag{11}$$

$$\dot{x}_1 = \frac{1}{\sqrt{\varepsilon}-\theta}\left[\xi x_1 - x_2 + p(0,t)\right], \tag{12}$$

$$\dot{x}_2 = -\frac{1}{\sqrt{\mu}}s\left(0,t\right), \tag{13}$$

with boundary conditions

$$q(0,t) = -\frac{(\sqrt{\varepsilon}+\theta)}{\sqrt{\varepsilon}-\theta}p(0,t) - \frac{2\sqrt{\varepsilon}}{\sqrt{\varepsilon}-\theta}(\xi x_1 - x_2), \tag{14}$$

$$s(0,t) = -r(0,t),\ p\left(1,t\right) = V_p(t),\ r\left(1,t\right) = V_r(t), \tag{15}$$

where $V_p(t) = V_1(t) + \sqrt{\varepsilon}u_t(1,t)$ and $V_r(t) = V_2(t) + \sqrt{\mu}\alpha_t(1,t)$ are the redefined control variables for this plant.

The system (8)–(15) is similar to the one stabilized with backstepping in [8]. Thus, the method therein can be easily adapted. Assuming $\frac{1}{\sqrt{\varepsilon}} > \frac{1}{\sqrt{\mu}}$ (the other cases can be analogously treated), define

$$Z = \begin{bmatrix} p \\ r \end{bmatrix}, Y = \begin{bmatrix} q \\ s \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, V = \begin{bmatrix} V_p \\ V_r \end{bmatrix}. \tag{16}$$

Then, (8)–(15) can be written in the following simplified matrix form

$$Z_t = \Sigma Z_x + \Lambda_1(Z+Y) + \Lambda_2 X + \int_0^x F[Z(y,t)+Y(y,t)]dy \tag{17}$$

$$Y_t = -\Sigma Y_x + \Lambda_1(Y+Z) + \Lambda_2 X + \int_0^x F[Z(y,t)+Y(y,t)]dy \tag{18}$$

$$\dot{X} = (A + B_2 D)X + (B_1 + B_2 C)Z(0,t) \tag{19}$$

with boundary conditions

$$Z(1,t) = V,\ Y(0,t) = CZ(0,t) + DX \tag{20}$$

where

$$\Sigma=\begin{bmatrix}\frac{1}{\sqrt{\varepsilon}} & 0\\ 0 & \frac{1}{\sqrt{\mu}}\end{bmatrix},\ \Lambda_1=\begin{bmatrix}0 & -\frac{1}{2\sqrt{\epsilon}}\\ \frac{a}{2\varepsilon\sqrt{\mu}} & 0\end{bmatrix}, \tag{21}$$

$$\Lambda_2=\begin{bmatrix}0 & 0\\ 0 & -\frac{a}{\varepsilon\sqrt{\mu}}\end{bmatrix},\ F=\begin{bmatrix}0 & 0\\ 0 & -\frac{a}{2\varepsilon\sqrt{\mu}}\end{bmatrix}, \tag{22}$$

$$A=\begin{bmatrix}\frac{\xi}{\sqrt{\varepsilon}-\theta} & -\frac{1}{\sqrt{\varepsilon}-\theta}\\ 0 & 0\end{bmatrix}\quad B_1=\begin{bmatrix}\frac{1}{\sqrt{\varepsilon}-\theta} & 0\\ 0 & 0\end{bmatrix}, \tag{23}$$

$$B_2=\begin{bmatrix}0 & 0\\ 0 & -\frac{1}{\sqrt{\mu}}\end{bmatrix},\ C=\begin{bmatrix}-\frac{\sqrt{\varepsilon}+\theta}{\sqrt{\varepsilon}-\theta} & 0\\ 0 & -1\end{bmatrix}, \tag{24}$$

$$D=\begin{bmatrix}-\frac{2\sqrt{\varepsilon}\xi}{\sqrt{\varepsilon}-\theta} & \frac{2\sqrt{\varepsilon}}{\sqrt{\varepsilon}-\theta}\\ 0 & 0\end{bmatrix}. \tag{25}$$

The system (17)–(20), differently from [8], contains integral coupling terms, and the states of ODEs appearing inside the domain of the PDEs.

For system (17)–(20), the following control law is obtained in Section 4.

$$V=\int_0^1 K\left(1,y\right)Z\left(y,t\right)dy+\int_0^1 L\left(1,y\right)Y\left(y,t\right)dy+\Phi(1)X, \tag{26}$$

whose gain kernels satisfy the following equations

$$\Sigma K_x+K_y\Sigma=(K+L)\Lambda_1-\Omega(x)K-F+\int_y^x\left[K(x,s)+L(x,s)\right]Fds, \tag{27}$$

$$\Sigma L_x-L_y\Sigma=(K+L)\Lambda_1-\Omega(x)L-F+\int_y^x\left[K(x,s)+L(x,s)\right]Fds, \tag{28}$$

$$\begin{aligned}\Phi_x=&\Sigma^{-1}\Phi A-\Sigma^{-1}\Lambda_2+\Sigma^{-1}\Phi B_2D\\ &-\Sigma^{-1}\Omega(x)\Phi+\int_0^x\Sigma^{-1}(K-L)\Lambda_2dy\\ &+\Sigma^{-1}L(x,0)\Sigma D,\end{aligned} \tag{29}$$

with boundary conditions for K and L

$$\Sigma L(x,x)+L(x,x)\Sigma=-\Lambda_1, \tag{30}$$

$$\Sigma K(x,x)-K(x,x)\Sigma=-\Lambda_1+\Omega(x), \tag{31}$$

$$K(x,0)-(x,0)\Sigma C\Sigma^{-1}=\Phi B_1\Sigma^{-1}+\Phi B_2C\Sigma^{-1}, \tag{32}$$

$$\Phi(0)=\begin{bmatrix}-\xi-\frac{\delta_1}{\kappa} & 1+\frac{1}{\sqrt{\mu}}\\ 0 & -\delta_2\sqrt{\mu}\end{bmatrix}, \tag{33}$$

with

$$\Omega(x)=\begin{bmatrix}0 & 0\\ \omega_{21} & 0\end{bmatrix}, \tag{34}$$

where $\omega_{21}(x,t)=(\frac{1}{\sqrt{\mu}}-\frac{1}{\sqrt{\varepsilon}})k_{21}(x,x)+\frac{a}{2\varepsilon}$ and $\kappa=1/(\sqrt{\varepsilon}-\theta)$. The parameters δ_1,δ_2 are arbitrary values which directly determine the decay rate of the closed-loop controlled Timoshenko beam (see Section 5). The well-posedness of the kernel equations for $K(x,y),L(x,y),\Phi(x)$ is stated in Theorem 2 in Section 4.

Expressing (26) using the Timoshenko beam variables:

$$\begin{aligned}V_1=&-\int_0^1\left(k_{11,y}(1,y)+l_{11,y}(1,y)\right)u(y,t)dy\\ &+\int_0^1\sqrt{\varepsilon}\left(k_{11}(1,y)+l_{11}(1,y)\,u_t(y,t)dy\right.\\ &-\int_0^1\left(k_{12,y}(1,y)+l_{12,y}(1,y)\right)\alpha(y,t)dy\\ &+\int_0^1\sqrt{\mu}\left(k_{12}(1,y)+l_{12}(1,y)\,\alpha_t(y,t)dy\right.\\ &+\left(k_{11}(1,1)+l_{11}(1,1)\right)u(1,t)\\ &-\left(k_{11}(1,0)+l_{11}(1,0)-\phi_{11}(1)\right)u(0,t)\\ &-(k_{12}(1,0)+l_{12}(1,0)-\phi_{12}(1))\alpha(0,t)\\ &+(k_{12}(1,1)+l_{12}(1,1))\alpha(1,t)-\sqrt{\varepsilon}u_t(1,t),\end{aligned} \tag{35}$$

$$\begin{aligned}V_2=&-\int_0^1(k_{21,y}(1,y)+l_{21,y}(1,y))u(y,t)dy\\ &+\int_0^1\sqrt{\varepsilon}(k_{21}(1,y)+l_{21}(1,y))u_t(y,t)dy\\ &-\int_0^1(k_{22,y}(1,y)+l_{22,y}(1,y))\alpha(y,t)dy\\ &+\int_0^1\sqrt{\mu}(k_{22}(1,y)+l_{22}(1,y))\alpha_t(y,t)dy\\ &+\left(k_{21}(1,1)+l_{21}(1,1)\right)u(1,t)\\ &-\left(k_{21}(1,0)+l_{21}(1,0)-\phi_{21}(1)\right)u(0,t)\\ &-\left(k_{22}(1,0)+l_{22}(1,0)-\phi_{22}(1)\right)\alpha(0,t)\\ &+\left(k_{22}(1,1)+l_{22}(1,1)\right)\alpha(1,t)-\sqrt{\mu}\alpha_t(1,t),\end{aligned} \tag{36}$$

The main result is stated next, where the spaces $L^2(0,1)$ and $H^1(0,1)$ are defined as usual and denoted simply as L^2 and H^1.

Theorem 1. Consider system (1)–(4), with initial conditions $u_0,\alpha_0\in H^1,u_{0t},\alpha_{0t}\in L^2$, under the control law (35)–(36). If the value of δ_1,δ_2 (the controller parameters appearing in (33)) are set large enough so that the constant

$$C_2=\min\{\delta_1,\delta_2\}-2-\max\left\{\frac{4}{\left(\sqrt{\varepsilon}-\theta\right)^2},\frac{1}{\mu}\right\}, \tag{37}$$

is positive, there exists a solution $u(\cdot,t),\alpha(\cdot,t)\in H^1$, $u_t(\cdot,t),\alpha_t(\cdot,t)\in L^2$ for $t>0$, and the following inequality is verified, guaranteeing the exponential stability of the equilibrium $u\equiv\alpha\equiv u_t\equiv\alpha_t\equiv 0$:

$$\begin{aligned}&\|u(\cdot,t)\|^2_{H^1}+\|\alpha(\cdot,t)\|^2_{H^1}+\|u_t(\cdot,t)\|^2_{L^2}+\|\alpha_t(\cdot,t)\|^2_{L^2}\\ &\leq C_1\mathrm{e}^{-C_2t}\Big(\|u_0\|^2_{H^1}+\|\alpha_0\|^2_{H^1}+\|u_{0t}\|^2_{L^2}+\|\alpha_{0t}\|^2_{L^2}\Big).\end{aligned} \tag{38}$$

The proof of Theorem 1 is given in Section 5.

4. CONTROLLER ANALYSIS

This section presents the steps leading to (26). We start by designing a target system as follows

$$\sigma_t=\Sigma\sigma_x+\Omega(x)\sigma, \tag{39}$$

$$\psi_t=-\Sigma\psi_x+\Lambda_1\left(\psi+\sigma\right)+\int_0^x\Xi_2(x,y)\sigma(y,t)dy$$

$$+\int_0^x \Xi_3(x,y)\psi(y,t)dy + \Xi_1(x)X, \tag{40}$$

$$\dot{X} = E_1 X + E_2\sigma(0,t), \tag{41}$$

with boundary conditions

$$\sigma(1,t) = 0,\ \psi(0,t) = E_3 X + C\sigma(0,t), \tag{42}$$

where

$$\sigma = \begin{bmatrix} \eta \\ \beta \end{bmatrix},\ E_1 = (B_1 + B_2 C)\Phi(0) + A + B_2 D, \tag{43}$$

$$E_2 = C\Phi(0) + D,\ E_3 = B_1 + B_2 C, \tag{44}$$

and where the values of $\Xi_1(x)$, $\Xi_2(x,y)$, and $\Xi_3(x,y)$ are obtained in terms of the inverse backstepping transformation (which is given subsequently). The stability of this target system is shown in Section 5.

Next we introduce the backstepping transformation. Firstly, inspired by [18], we introduce

$$\sigma = Z - \int_0^x K\left(x,y\right)Z\left(y,t\right)dy - \int_0^x L\left(x,y\right)Y\left(y,t\right)dy - \Phi(x)X, \tag{45}$$

$$\psi = Y. \tag{46}$$

The kernel equations are deduced as usual, by a tedious but straightforward procedure of taking derivatives in the transformation, replacing the original and target equations, and integrating by parts. The details are skipped for brevity. The following result holds

Theorem 2. There exists a unique bounded solution $k_{ij}(x,y), l_{ij}(x,y), i = 1,2; j = 1,2,3,4$, coefficients of the matrices appearing in (27)–(32); in particular, there exists $M > 0$ such that

$$|k_{ij}(x,y)|, |l_{ij}(x,y)| \le M e^{Mx}. \tag{47}$$

The proof follows along the lines of [8] and is skipped; it is based on using the method of characteristics to write (27)–(32) in the form of integral equations and then posing a solution in terms of a successive approximation series, whose convergence is proven recursively. The derivations of [8] can be easily adapted to the presence of integral terms and the differences in the boundary conditions.

Since the kernels appearing in (45) are bounded, the transformation is invertible from the theory of Volterra integral equation. Thus one can define

$$Z = \sigma + \int_0^x \breve{K}\left(x,y\right)\sigma\left(y,t\right)dy + \int_0^x \breve{L}\left(x,y\right)\psi\left(y,t\right)dy + \breve{\Phi}\left(x\right)X \tag{48}$$

with bounded kernels. Both the transformation and its inverse, having bounded kernels, map L^2 functions into L^2 functions (see e.g. [14]). From (48), the kernels $\Xi_1(x)$, $\Xi_2(x,y)$, $\Xi_3(x,y)$ in (40) are

$$\Xi_1(x) = \Lambda_1\breve{\phi}(x) + \Lambda_2 + \int_0^x F\breve{\phi}(y)dy, \tag{49}$$

$$\Xi_2(x,y) = \Lambda_1\breve{K}(x,y) + F + \int_0^x F\breve{K}(s,y)ds, \tag{50}$$

$$\Xi_3(x,y) = \Lambda_1\breve{L}(x,y) + F + \int_0^x F\breve{L}(s,y)ds. \tag{51}$$

5. STABILITY OF CLOSED LOOP

This section proves Theorem 1; we start solving (39)–(42) with the method of characteristics. The solution of σ converges to zero in finite time $\sqrt{\mu}$. Thus, for $t > \sqrt{\mu}$,

$$\psi_t(x,t) = -\Sigma\psi_x(x,t) + \Lambda_1\psi(x,t) + \Xi_1(x)X + \int_0^x \Xi_3(x,y)\psi(y,t)dy, \tag{52}$$

$$\dot{X} = E_1 X, \tag{53}$$

with $\psi(0,t) = E_3 X$. Solving for X we get $X(t) = e^{E_1 t}X(0)$, where we used the matrix exponential. Then

$$\psi_t(x,t) = -\Sigma\psi_x(x,t) + \Lambda_1\psi(x,t) + \Xi_1(x)e^{E_1 t}X(0) + \int_0^x \Xi_3(x,y)\psi(y,t)dy, \tag{54}$$

with $\psi(0,t) = E_3 e^{E_1 t}X(0)$. By the method of characteristics, two Volterra-type integral equations can be found for the components of ψ. Details are skipped, but it is easy to see that one can always find a unique L^2 solution for ψ.

Obviously the only requirement for stability is that E_1 is Hurwitz as then the origin of the state is exponentially stable for (52). Nevertheless, for rapid arbitrary stabilization, the eigenvalues of E_1 need to be set. Indeed, since $D = -\sqrt{\varepsilon}A$ in (43), E_1 is rewritten as

$$E_1 = A\left(I - \sqrt{\varepsilon}B_2\right) + (B_1 + B_2 C)\Phi(0). \tag{55}$$

Which, remembering $\kappa = 1/(\sqrt{\varepsilon} - \theta)$, results in

$$E_1 = \begin{bmatrix} \kappa\xi + \kappa\phi_{11}(0) & -\kappa + \kappa\phi_{12}(0) \\ \dfrac{\phi_{21}(0)}{\sqrt{\mu}} & \dfrac{\phi_{22}(0)}{\sqrt{\mu}} \end{bmatrix}. \tag{56}$$

If we choose the boundary conditions $\Phi(0)$ as follows:

$$\phi_{11}(0) = -\xi - \frac{\delta_1}{\kappa},\ \phi_{12}(0) = 1, \tag{57}$$

$$\phi_{21}(0) = 0,\ \phi_{22}(0) = -\delta_2\sqrt{\mu}, \tag{58}$$

with $\delta_1, \delta_2 > 0$, then E_1 is a diagonal matrix with entries $-\delta_1$ and $-\delta_2$, which become its eigenvalues. The rate of convergence of X can be arbitrarily set by adjusting the value δ_1, δ_2 and will be equal to $c = \min\{\delta_1, \delta_2\}$.

Next, we use a Lyapunov functional for the stability analysis of target system, to show exponential stability of the origin with a fixed convergence rate. Define, for $\zeta > 0$,

$$V = X^T X + \zeta\int_0^1 e^{\delta x}\sigma^T(x,t)\Sigma^{-1}\sigma(x,t)dx + \int_0^1 e^{-\delta x}\psi^T(x,t)\Sigma^{-1}\psi(x,t)dx. \tag{59}$$

Differentiating (59) with respect to t, and after a careful bounding process, we obtain

$$\dot{V} \le -c' X^T X - c'\zeta\int_0^1 e^{\delta x}\sigma^T(x,t)\Sigma^{-1}\sigma(x,t)dx - c'\int_0^1 e^{-\delta x}\psi^T(x,t)\Sigma^{-1}\psi(x,t)dx \le -c'V, \tag{60}$$

with $c' > \min\{\delta_1, \delta_2\} - 2 - \max\left\{\kappa^2, \frac{1}{\mu}\right\}$ (see details in [6]). Thus setting the controller parameters δ_1 and δ_2

sufficiently large, an arbitrary convergence rate $c' > 0$ is achieved for V.

From the Lyapunov inequality just obtained and using norm equivalences, and the boundedness of the kernels of both direct (45) and inverse (48) transformations, one obtains $\|p(\cdot,t)\|^2_{L^2} + \|q(\cdot,t)\|^2_{L^2} + \|r(\cdot,t)\|^2_{L^2} + \|s(\cdot,t)\|^2_{L^2} + x_1^2(t) + x_2^2(t) \leq K_1 e^{-c't}\Big(\|p_0\|^2_{L^2} + \|q_0\|^2_{L^2} + \|r_0\|^2_{L^2} + \|s_0\|^2_{L^2} + x_1^2(0) + x_2^2(0)\Big)$, for some $K_1 > 0$. When rewritten in terms of the physical Timoshenko beam states, the exponential stability bound of Theorem 1 follows, since

$$u(t,x) = x_1(t) + \frac{1}{2}\int_0^x (p(t,y) + q(t,y))dy, \tag{61}$$

$$\alpha(t,x) = x_2(t) + \frac{1}{2}\int_0^x (r(t,y) + s(t,y))dy, \tag{62}$$

$$u_t(t,x) = \frac{p(t,x) - q(t,x)}{2\sqrt{\epsilon}},\ \alpha_t(t,x) = \frac{r(t,x) - s(t,x)}{2\sqrt{\mu}}. \tag{63}$$

Finally, under the assumptions of Theorem 1, the initial conditions of p, q, r, s belong to L^2. Therefore the initial conditions of the transformed states are also L^2. It is easy to see that the target system is well-posed in L^2 (see Section IV.B.1); thus the original system will be as well, since the inverse transformation maps L^2 into L^2. This finishes the proof of Theorem 1, by applying (61)–(63).

6. NUMERICAL SIMULATION

To verify the effectiveness of the proposed boundary controller, (1)–(4) is simulated with $\epsilon = 1$, $\mu = 2$, $a = 1$, $\theta = -1$, $\xi = 1$. The initial values are set to $u_0 = 2.8 - 2.8x - 1.8x^2$, $u_{t0} = 0$, $\alpha_0 = x^2$, $\alpha_{t0} = 0$. We use the HPDE tool in MATLAB, in which the four equivalent first-order hyperbolic PDEs (8)–(11) and the ODEs (12)–(13) are solved, and the evolution of $u(x,t), \alpha(x,t)$ is obtained by using (61)–(63). The open-loop response is unstable, due to anti-damping (not shown for lack of space). Next, we apply the proposed controller (35)–(36) to the Timoshenko beam. The controller parameters are chosen as $\delta_1 = 5, \delta_2 = 2$. The feedback gains $K(1,y)$, $L(x,y)$ and $\Phi(x)$ are shown in Fig. 1 and were computed using a power series approach as in [5]. There is a discontinuity in the kernel function $k_{12}(1,y)$, which is typically present when applying the backstepping method to a $(2+2)\times(2+2)$ system and does not impact the result [14]. The variables $u(x,t)$, $u_t(x,t)$, $\alpha(x,t)$ and $\alpha_t(x,t)$ evolve as shown in Fig. 2, converging to zero exponentially, as expected from Theorem 1.

REFERENCES

[1] Ammar-Khodja, F., Kerbal, S., and Soufyane, A., "Stabilization of the nonuniform Timoshenko beam," *J. Math. Anal. Appl.,* 327(1), 525–538, 2007.

[2] Balakrishnan, A.V., "On Superstability of semigroups," In *Proceedings of the 18th IFIP TC-7*, pp. 12–19, 1997.

[3] Balakrishnan, A. V., "Superstability of systems," *Appl. Math. Comput.*, 164, pp. 321–326, 2005.

[4] Bastin, G. , and Coron, J.-M., *Stability and Boundary Stabilization of 1-D Hyperbolic Systems,* Basel: Birkhauser, 2016.

[5] Camacho-Solorio, L., Vazquez R., and Krstic, M.,"Boundary observers for coupled diffusion-reaction systems with prescribed convergence rate," *Systems & Control Letters* 135 : 104586, 2020.

[6] Chen, G., Vazquez, R., and Krstic, M.," Rapid Stabilization of Timoshenko Beam by PDE Backstepping," preprint submitted to ArXiv, `https://arxiv.org/abs/2207.04746`, 2022.

[7] Coron, J.M. and d'Andrea-Novel, B., "Stabilization of a rotating body beam without damping," *IEEE T. Automat. Contr.*, 43(5), pp.608–618, 1998.

[8] Di Meglio, F., Argomedo, F. B. , Hu, L., and Krstic, M., "Stabilization of coupled linear heterodirectional hyperbolic PDE-ODE systems," *Automatica*, vol. 87, pp. 281-289, 2018.

[9] Guo, B.Z. and Meng, T., "Robust output regulation for Timoshenko beam equation with two inputs and two outputs," *International Journal of Robust and Nonlinear Control*, 31(4), pp.1245–1269, 2021.

[10] Han, S. M., Benaroya, H., and Wei, T., "Dynamics of transversely vibrating beams using four engineering theories," *J. Sound Vib.*, vol. 225, pp. 935–988, 1999.

[11] He, W., and Liu, C., "Vibration control of a Timoshenko beam system with input backlash," *IET Control Theory & Applications,* 9(12), 1802–1809, 2015.

[12] He, W., Zhang, S., and Ge, S. S., "Boundary output-feedback stabilization of a Timoshenko beam using disturbance observer," *IEEE Transactions on Industrial Electronics*, 60(11), 5186–5194, 2012.

[13] He, W., Zhang, S., Ge, S. S., and Liu, C., "Adaptive boundary control for a class of inhomogeneous Timoshenko beam equations with constraints," *IET Control Theory & Applications*, 8(14), 1285–1292, 2014.

[14] Hu, L., Vazquez, R., Di Meglio, F., Krstic, M., "Boundary exponential stabilization of 1-D inhomogeneous quasilinear hyperbolic systems," *SIAM J. Contr. Optim.*, vol. 57, pp. 963–998, 2019.

[15] Kim, J. U., and Renardy, Y., "Boundary control of the Timoshenko beam," *SIAM J. Contr. Optim.*, 25(6), 1417–1429, 1987.

[16] Krstic, M., Siranosian, A. A., and Smyshlyaev, A., "Backstepping boundary controllers and observers for the slender Timoshenko beam: Part I-Design," *In 2006 ACC*, pp. 2412–2417, 2006.

[17] Krstic, M., Guo, B.Z., Balogh, A. and Smyshlyaev, A., "Control of a tip-force destabilized shear beam by observer-based boundary feedback," *SIAM J. Contr. Optim.*, 47(2), pp.553–574, 2008.

[18] Lingling, S., Wang, J.-M., and Krstic, M., "Boundary feedback stabilization of a class of coupled hyperbolic equations with nonlocal terms," *IEEE T. Automat. Contr.*, vol. 63, no. 8, pp. 2633-2640, 2017.

[19] Liu, X. F., and Xu, G. Q., "Exponential stabilization for Timoshenko beam with distributed delay in the boundary control," *Abstr. Appl. Anal.* , 726794, 2013.

[20] Ma, J., Wei, Z., Wen, H., and Jin, D., "Boundary control of a Timoshenko beam with prescribed performance," *Acta Mechanica,* 231, 3219–3234, 2020.

[21] Macchelli, A., and Melchiorri, C., "Modeling and control of the Timoshenko beam: The distributed port Hamiltonian approach," *SIAM J. Contr. Optim.,* 43(2), 743–767, 2004.

[22] Mattioni, A., Wu, Y., Le Gorrec, Y. and Zwart, H., "Stabilisation of a rotating beam clamped on a moving inertia with strong dissipation feedback," in *59th IEEE CDC*, pp. 5056–5061, 2020.

[23] Mattioni, A., Wu, Y. and Le Gorrec, Y., "Exponential stabilization of a clamped Timoshenko beam with actuation on a tip mass," in *60th IEEE CDC*, pp. 6200–6205, 2021.

[24] Morgul, O., "Dynamic boundary control of the Timoshenko beam," *Automatica,* 28(6), 1255–1260, 1992.

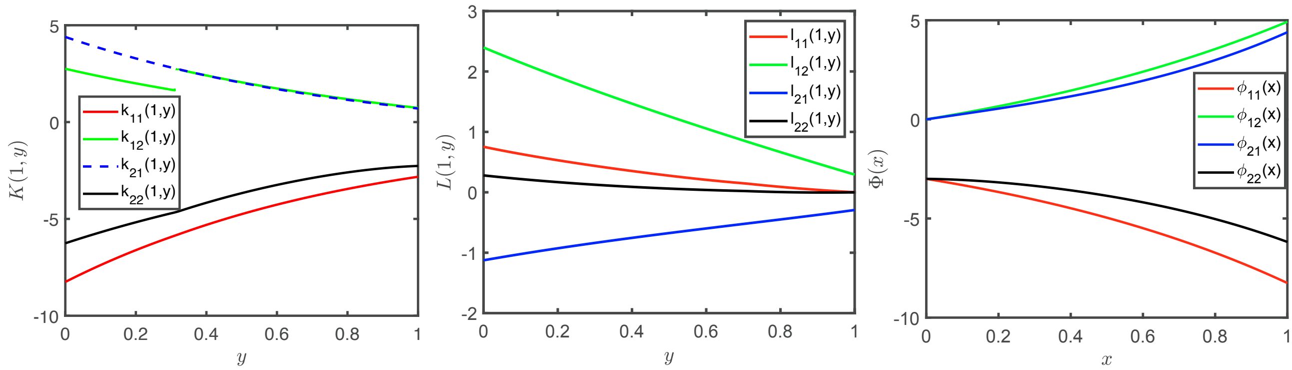

Fig. 1. Feedback gains $K(1, y)$, $L(1, y)$, $\Phi(x)$. Note the discontinuity in the kernel function $k_{12}(1, y)$.

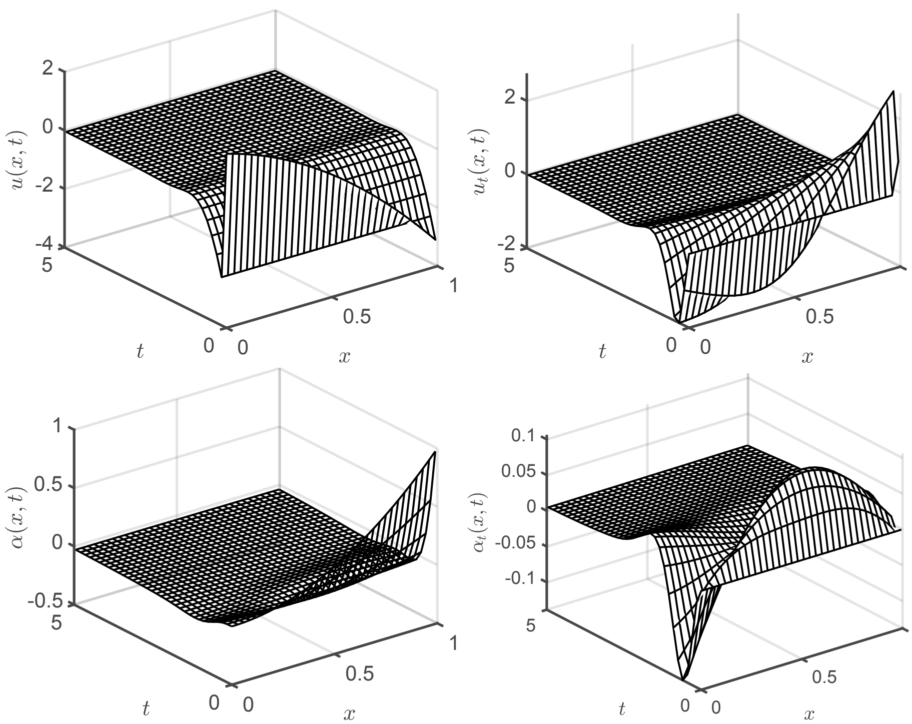

Fig. 2. Closed-loop evolution of Timoshenko beam states u, u_t, α, α_t over time.

[25] Siuka, A., Schoberl, M., and Schlacher, K., "Port-Hamiltonian modelling and energy-based control of the Timoshenko beam," *Acta Mech.,* 222(1), 69–89, 2011.

[26] Siranosian, A. A., Krstic, M., Smyshlyaev, A., and Bement, M., "Motion planning and tracking for tip displacement and deflection angle for flexible beams," *J. Dyn. Sys. Meas. Control,* 131(3), 2009

[27] Smyshlyaev, A., Guo, B.Z. and Krstic, M., "Arbitrary decay rate for Euler-Bernoulli beam by backstepping boundary feedback," *IEEE Transactions on Automatic Control*, 54(5), pp.1134–1140, 2009.

[28] Steeves, D., and Krstic, M., "Prescribed-time stabilization of ODEs with diffusive actuator dynamics," in *24th MTNS*, 2021.

[29] Soufyane, A., and Wehbe, A., "Uniform stabilization for the Timoshenko beam by a locally distributed damping," *Electron. J. Differ. Equ.*, 29, pp. 1–14, 2003.

[30] Taylor, S. W., and Yau, S. C., " Boundary control of a rotating Timoshenko beam," *ANZIAM Journal*, 44, E143–E184,2002.

[31] Wang, J.M. and Krstic, M., "Stability of an interconnected system of Euler-Bernouilli beam and heat equation with boundary coupling," *ESAIM: COCOV*, 21(4), pp.1029–1052, 2015.

[32] Wu, Y., Hamroun, B., Le Gorrec, Y. and Maschke, B., "Reduced order controller design for Timoshenko beam: A port Hamiltonian approach," *IFAC-PapersOnLine*, 50(1), pp.7121–7126, 2017.

[33] Xu, G. Q., and Yung, S. P., "Stabilization of Timoshenko beam by means of pointwise controls," *ESAIM: COCOV*, 9, 579–600, 2003.

[34] Xu G. Q., "Boundary feedback exponential stabilization of a Timoshenko beam with both ends free," *International Journal of Control,* 78(4), 286–297, 2005.

[35] Yildirim, K., and Kucuk, I., "Active piezoelectric vibration control for a Timoshenko beam," *J. Frankl. Inst.,* 353(1), 95–107, 2016.

[36] Zhao, H. L., Liu, K. S., and Zhang, C. G., "Stability for the Timoshenko beam system with local Kelvin Voigt damping," *Acta Math. Sin.,* 21(3), 655–666, 2005.