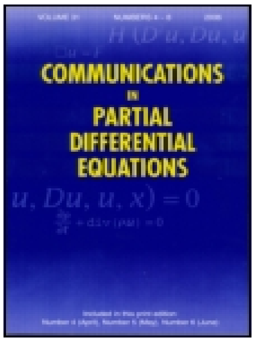

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Two-Dimensional Div-Curl Results: Application to the Lack of Nonlocal Effects in Homogenization

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In this paper, we study the asymptotic behaviour of sequences of conduction problems and sequences of the associated diffusion energies. We prove that, contrary to the three-dimensional case, the boundedness of the conductivity sequence in L^1 combined with its equi-coerciveness prevents from the appearance of nonlocal effects in dimension two. More precisely, in the two-dimensional case we extend the Murat–Tartar H-convergence which holds for uniformly bounded and equi-coercive conductivity sequences, as well as the compactness result which holds for bounded and equi-integrable conductivity sequences in L^1. Our homogenization results are based on extensions of the classical div-curl lemma, which are also specific to the dimension two.

Keywords Dirichlet forms; Div-curl results; Elliptic problems; Homogenization; Γ-convergence; Unbounded coefficients.

Mathematics Subject Classification Primary 35R05, 35B40; Secondary 35B27.

1. Introduction

This paper is devoted to the asymptotic analysis of sequences of conduction problems and sequences of the associated diffusion energies.

Let Ω be a bounded open set of $\mathbb{R}^2$, and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of equi-coercive matrix-valued functions in $L^\infty(\Omega)^{2\times 2}$. We consider the conduction problem

$$\begin{cases} -\mathrm{div}(A_n\nabla u_n) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \Omega. \end{cases} \tag{1.1}$$

Concerning the energy associated with the conduction problem (1.1) we restrict ourselves to a symmetric conductivity matrix, but in the same time, we extend the framework by replacing the Lebesgue measure by a sequence of Radon measures.

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Let $(\mu_n)_{n\in\mathbb{N}}$ be a bounded sequence of Radon measures defined on Ω and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of equi-coercive matrix-valued functions in $L^\infty_{\mu_n}(\Omega)^{2\times 2}$. We consider the quadratic functional defined in $L^2(\Omega)$ by

$$F_n(u) := \begin{cases} \int_\Omega A_n \nabla u \cdot \nabla u \, d\mu_n & \text{if } u \in C^1_0(\Omega) \\ +\infty & \text{elsewhere.} \end{cases} \tag{1.2}$$

The first issue is to determine the asymptotic behaviour of the conduction problem (1.1) and of the diffusion energy (1.2). The second issue is to predict the appearance of nonlocal effects from the knowledge of the conductivity sequence A_n in (1.1) or $A_n\, d\mu_n$ in (1.2).

The topic has been widely studied for the last thirty years. The pioneer work is due to Spagnolo (1968) in the end of the sixties. He treated the homogenization of problem (1.1) by introducing the G-convergence theory. So, under the assumption that A_n is symmetric, equi-coercive, and equi-bounded, he proved that the limit problem is of same nature. In the seventies, Tartar (1977) and Murat (1998) extended with the H-convergence theory the Spagnolo result by getting rid of the symmetry assumption and by giving a corrector result.

A few years later, Buttazzo and Dal Maso (1980) as well as Carbone and Sbordone (1979) obtained a compactness result, in the context of the Γ-convergence, for any sequence of quadratic functionals (1.2) with $d\mu_n = dx$ and A_n symmetric but without equi-coerciveness assumption. Under the assumption that $|A_n|$ is bounded and equi-integrable in $L^1(\Omega)$, they proved that any sequence F_n, Γ-converges for the strong topology of $L^2(\Omega)$, up to a subsequence, to a quadratic form of the same type (1.2) at least on the set of regular functions. More recently, starting from the Beurling–Deny theory of the Dirichlet forms (Beurling and Deny, 1958), Mosco (1994) proved that any sequence of quadratic forms of type (1.2), which is asymptotically regular (i.e., for any $u \in C^1_0(\Omega)$, there exists a sequence u_n with $\liminf F_n(u_n) < +\infty$ which strongly converges to u in $L^2(\Omega)$), Γ-converges for the strong topology of $L^2(\Omega)$, up to a subsequence, to a Dirichlet form F on $L^2(\Omega)$, which satisfies the Beurling–Deny representation formula (Beurling and Deny, 1958), i.e., for every $u \in L^2(\Omega)$ with $F(u) < +\infty$,

$$F(u) = \underbrace{F_d(u)}_{\text{strongly local term}} + \underbrace{\int_\Omega u^2\, dk}_{\text{simply local term}} + \underbrace{\int_{\Omega\times\Omega\setminus\text{diag}} (u(x) - u(y))^2 dj}_{\text{nonlocal term}}, \tag{1.3}$$

where k is a Radon measure on Ω and j is a Radon measure on $\Omega \times \Omega \setminus$ diag. Moreover, there exists a measure μ on Ω and a matrix-valued function A in $L^1_\mu(\Omega)^{2\times 2}$ such that

$$F_d(u) = \int_\Omega A \nabla u \cdot \nabla u \, d\mu, \quad \forall u \in C^1_0(\Omega).$$

However, there is no simple way to compute the limit measures μ, k, j from the sequence μ_n. In particular, there is no general result concerning the appearance of simply local and nonlocal terms. Moreover, assuming that $\mu_n = \mu$ is independent of n and $|A_n|$ is equi-integrable in $L^1_\mu(\Omega)$, Mosco (1994) extended the results (Buttazzo and Dal Maso, 1980; Carbone and Sbordone, 1979) by proving that the Γ-limit F (1.3) is strongly local, i.e., $k = 0$ and $j = 0$.

If one removes the equi-integrability assumption, then the Γ-limit of F_n (1.2) is not necessarily of the same type. So, simply local and nonlocal terms may appear in accordance with the Beurling–Deny representation formula (1.3). Fenchenko and Khruslov (1981) were the first to obtain nonlocal effects in three-dimensional conduction from non-uniformly bounded conductivity sequences A_n in (1.1). Their model example, which consists in a periodic lattice of high-conductivity and thin-diameter fibers embedded in a medium of conductivity 1, has been revisited and extended by several authors in Bellieud and Bouchitté (1998), Briane and Tchou (2001), Briane (2002; 2003). More recently, Camar-Eddine and Seppecher (2002) proved that, in dimension three, the Γ-closure of the set of diffusion energies (1.2) is exactly the set of the Dirichlet forms satisfying the Beurling–Deny formula (1.3). Their proof is partially based on suitable fiber-reinforced microstructures which allow them to derive particular, but rich enough, sets of nonlocal terms. Note that the situation is completely different in elasticity for which Camar-Eddine and Seppecher (2003) proved a spectacular closure result. Very recently, Alibert and Seppecher (2005) completely solved the one-dimensional conduction case.

In the particular case of periodic microstructures ($A_n(x) := B_n\left(\frac{x}{\varepsilon_n}\right)$, with B_n periodic and $\varepsilon_n \to 0$, being a highly-oscillating sequence of matrix-valued functions), the first author gave in Briane (2002) an asymptotic barrier below which nonlocal effects do not appear. More essentially, he obtained in Briane (2006) a periodic H-convergence result in dimension two under the assumption that the conductivity sequence A_n in (1.1) is equi-coercive but only bounded in $L^1(\Omega)^{2\times 2}$. Of course, the model example of Fenchenko and Khruslov (1981) and Khruslov (1991) shows that the L^1-boundedness does not prevent from the appearance of nonlocal effects in dimension three. This example points out the specificity of the two-dimensional result (Briane, 2006). Therefore, this result shows the gap between the dimension three and the dimension two concerning the appearance of nonlocal effects.

Our present contribution deals with the asymptotic behaviour of (1.1) and (1.2) in the very few studied two-dimensional case. We extend the result of Briane (2006) to a non-periodic framework by introducing measures. Generally speaking, we prove that simply local and nonlocal effects cannot appear in dimension two under the assumption that the conductivity sequence is equi-coercive but only bounded in $L^1(\Omega)^{2\times 2}$. On the one hand, we extend (see Theorem 2.14) the Murat–Tartar H-convergence relating to problem (1.1), assuming that the conductivity sequence A_n is equi-coercive but that $|A_n|$ only weakly $*$ converges in the sense of the Radon measures to a function in $L^\infty(\Omega)$. On the other hand, we obtain (see Theorem 2.9) a two-dimensional compactness result similar to that of Buttazzo and Dal Maso (1980), Carbone and Sbordone (1979), and Mosco (1994), assuming that the sequence A_n is equi-coercive and equi-bounded but that the sequence μ_n only weakly $*$ converges in the sense of the Radon measures. Shortly, in dimension two the Γ-limit of F_n (1.2) is still a strongly local Dirichlet form, without any equi-integrability condition but under the equi-coerciveness assumption.

The key-ingredients of the previous results are original two-dimensional div-curl lemmas. Recall that the classical Murat–Tartar div-curl lemma (Murat, 1978) claims that, for any bounded sequence ξ_n in $L^2(\Omega)^2$ with compact divergence in $H^{-1}(\Omega)$, and for any bounded sequence v_n in $H^1(\Omega)$, the sequence $\xi_n \cdot \nabla v_n$ converges in the distributions sense to the product of the limits $\xi \cdot \nabla v$. In the context of problem (1.1), replacing the $L^2(\Omega)^2$ bound of ξ_n with the assumption that the sequence $A_n^{-1}\xi_n \cdot \xi_n$ is bounded in $L^1(\Omega)$ while leaving the other assumptions unchanged, the previous convergence still holds true (see Theorem 2.1). The new assumption is weaker than the boundedness in $L^2(\Omega)^2$ since the conductivity sequence A_n is not uniformly bounded and its inverse is thus not equi-coercive. In the context of (1.2), an estimate satisfied by ξ_n replaces (more or less equivalently) the boundedness of $A_n^{-1}\xi_n \cdot \xi_n$ in $L^1(\Omega)$ (see Theorem 2.6). These div-curl results are also specific to the dimension two and are false in dimension three (see Example 2.4).

The paper is organized as follows. The second section is devoted to the statement of the results. We start by stating the two div-curl results (Theorems 2.1 and 2.6). Then, we give the two applications about the lack of nonlocal effects in two-dimensional homogenization (Theorems 2.9 and 2.14). The third section is devoted to the proof of the div-curl results and the homogenization results.

First of all, let us give a few notations:

Notations

- $\cdot$ denotes the scalar product in $\mathbb{R}^2$ and $|\cdot|$ the euclidian norm;
- for any matrix $A \in \mathbb{R}^{2\times 2}$, A^t denotes the transposed of A, $A^s := \frac{1}{2}(A + A^t)$ its symmetric part and $A^a := A - A^s$ its antisymmetric part;
- for any $A \in \mathbb{R}^{2\times 2}$, $|A| := \max_{|\lambda|=1} |A\lambda|$ denotes the matrix-norm associated with $|\cdot|$, which coincides with the spectral radius when A is symmetric;
- I denotes the unit matrix of $\mathbb{R}^{2\times 2}$ and $J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$;
- Ω denotes a bounded open set of $\mathbb{R}^2$;
- for any open subset ω of Ω, $\bar{\omega}$ denotes the closure of ω in $\mathbb{R}^2$ and the inclusion $\bar{\omega} \subset \Omega$ is denoted by $\omega \Subset \Omega$;
- a.e. means almost everywhere with respect to the Lebesgue measure, and μ-a.e. means almost everywhere with respect to the measure μ;
- dx denotes the integration with respect to the Lebesgue measure and $d\mu$ the integration with respect to the measure μ;
- for any measure space (X, μ) with $\mu(X) > 0$, and any $f \in L^1_\mu(X)$, we denote

$$⨍_X f\, d\mu := \frac{1}{\mu(X)} \int_X f\, d\mu;$$

- $\rightharpoonup$ denotes a weak convergence and $\longrightarrow$ a strong one;
- $C(\overline{\Omega})$ denotes the space of the continuous functions on $\overline{\Omega}$, and $C_0(\Omega)$ the space of the functions in $C(\overline{\Omega})$ which vanish on $\partial\Omega$, equipped with the usual norm;
- $C^1_0(\Omega)$ denotes the subspace of $C_0(\Omega)$ composed of the differentiable functions on $\overline{\Omega}$ the derivative of which belongs to $C_0(\Omega)^2$, and $C^k_c(\Omega)$ denotes the space of the k-differentiable functions on Ω with compact support in Ω;
- $\mathcal{M}(\Omega)$ (respectively $\mathcal{M}(\overline{\Omega})$) denotes the set of the Radon measures on Ω (respectively $\overline{\Omega}$), i.e., the dual of $C_0(\Omega)$ (respectively $C(\overline{\Omega})$);
- $|\mu|$ denotes the total variation of the measure μ in $\mathcal{M}(\Omega)$;
- the weak-$*$ convergence in the Radon measures sense of a sequence μ_n to μ in $\mathcal{M}(\Omega)$ (respectively $\mathcal{M}(\overline{\Omega})$) is denoted by $\mu_n \rightharpoonup \mu$ weakly in $\mathcal{M}(\Omega)*$ (respectively $\mathcal{M}(\overline{\Omega})*$), i.e.,

$$\int_\Omega \varphi\, d\mu_n \underset{n\to+\infty}{\longrightarrow} \int_\Omega \varphi\, d\mu, \quad \text{for any } \varphi \in C_0(\Omega) (\text{respectively } C(\overline{\Omega})).$$

- c denotes a suitable positive constant which may vary from line to line.

Now, let us recall a few results concerning the De Giorgi Γ-convergence theory, which will be used in the sequel. We refer to Dal Maso (1993) for a general presentation.

1.1. *Recalls of* Γ*-Convergence and Dirichlet Forms*

Definition 1.1. A sequence of functionals $F_n : L^2(\Omega) \longrightarrow [0, +\infty]$ is said to Γ-converge to $F : L^2(\Omega) \longrightarrow [0, +\infty]$ for the strong topology of $L^2(\Omega)$ if, for any u in $L^2(\Omega)$,

(i) the Γ-liminf inequality holds

$$\forall u_n \longrightarrow u \text{ strongly in } L^2(\Omega), \quad F(u) \leq \liminf_{n\to+\infty} F_n(u_n), \tag{1.4}$$

(ii) the Γ-limsup inequality holds

$$\exists \bar{u}_n \longrightarrow u \quad \text{strongly in } L^2(\Omega), \quad F(u) = \lim_{n\to+\infty} F_n(\bar{u}_n). \tag{1.5}$$

Any sequence satisfying (1.5) is called a recovery sequence.

In the sequel, we will always consider the Γ-convergence with respect to the strong topology of $L^2(\Omega)$. Consequently, this topology will be not necessarily mentioned. We then denote the Γ-convergence of F_n to F by

$$F_n \overset{\Gamma - L^2(\Omega)}{\longrightarrow} F.$$

Recall a few properties of the Γ-convergence:

Properties 1.2.

a) Since $L^2(\Omega)$ is separable, any sequence of functionals $F_n : L^2(\Omega) \longrightarrow [0, +\infty]$ has a subsequence which Γ-converges with respect to the strong topology of $L^2(\Omega)$.
b) Let $F_n : L^2(\Omega) \longrightarrow [0, +\infty]$ be a sequence of functionals which Γ-converges to F. Then, the Γ-limit F coincides with the Γ-limit of the lower semicontinuous envelope of F_n with respect to the strong topology of $L^2(\Omega)$.
c) Let $F_n : L^2(\Omega) \longrightarrow [0, +\infty]$ be a sequence of functionals which Γ-converges to F. Assume that every functional F_n is Markovian, i.e.,

$$\forall T \in C^1(\mathbb{R}) \text{ with } T(0) = 0 \quad \text{and} \quad \|T'\|_{L^\infty(\mathbb{R})} \leq 1, \;\; F_n \circ T \leq F_n,$$

and that $u \in L^\infty(\Omega)$. Let $T_u \in C^1(\mathbb{R})$ be a smooth truncation function with $\|T_u'\|_{L^\infty(\mathbb{R})} \leq 1$, satisfying

$$T_u(t) := \begin{cases} t & \text{if } |t| \leq M \\ M & \text{if } t \geq M + 2 \\ -M & \text{if } t \leq -M - 2, \end{cases} \quad \text{where } M := \|u\|_{L^\infty(\Omega)} \, .$$

Then, if $\bar{u}_n$ is a recovery sequence satisfying (1.5), so is $T_u(\bar{u}_n)$ since $T_u(\bar{u}_n)$ strongly converges to $T_u(u) = u$ in $L^2(\Omega)$ and thus

$$F(u) \leq \liminf_{n \to +\infty} F_n(T_u(\bar{u}_n)) \leq \limsup_{n \to +\infty} F_n(T_u(\bar{u}_n)) \leq \limsup_{n \to +\infty} F_n(\bar{u}_n) = F(u).$$

Therefore, we may always suppose that $\bar{u}_n$ is bounded in $L^\infty(\Omega)$ if $u \in L^\infty(\Omega)$.

d) Let $F_n : L^2(\Omega) \longrightarrow [0, +\infty]$ be a sequence of quadratic forms which Γ-converges to F. Then, F is a quadratic form on $L^2(\Omega)$ which is lower semicontinuous with respect to the strong topology of $L^2(\Omega)$.

e) Let $F_n : L^2(\Omega) \longrightarrow [0, +\infty]$ be a sequence of quadratic forms which Γ-converges to F. Then, for any u in the domain of F, i.e., $F(u) < +\infty$, the polarized Ψ_n of the quadratic form F_n and any recovery sequence $\bar{u}_n$ (1.5), satisfy

$$\forall v_n \longrightarrow v \quad \text{strongly in } L^2(\Omega), \quad F_n(v_n) \leq c \Longrightarrow \Psi_n(\bar{u}_n, v_n) \underset{n \to +\infty}{\longrightarrow} \Psi(u, v), \tag{1.6}$$

where Ψ is the polarized of the Γ-limit F. Property (1.6) can be considered as a kind of asymptotic Euler equation.

Let us conclude this section by a few notions about the Dirichlet forms defined in $L^2(\Omega)$, which will be used in the statement of Theorem 2.9 and 2.12. We refer to Mosco (1994) for more details in connection with the Γ-convergence.

Definition 1.3. Let $F : L^2(\Omega) \longrightarrow [0, +\infty]$ be a quadratic form of domain

$$D(F) := \{u \in L^2(\Omega) : F(u) < +\infty\}.$$

(i) The form F is said to be *closed* if it is lower semicontinuous with respect to the $L^2(\Omega)$-norm. The form F is said to be *closable* if there exists a closed extension $\widetilde{F}$ of F in $L^2(\Omega)$ such that $D(F) \subset D(\widetilde{F})$. The *closure* of a closable form is its smallest closed extension in $L^2(\Omega)$.

(ii) A *Dirichlet form* on $L^2(\Omega)$ is a closed Markovian quadratic form defined in $L^2(\Omega)$.

(iii) The form F is said to be *regular* if there exists a subset of $D(F) \cap C_0(\Omega)$, which is dense both in $C_0(\Omega)$ and in $D(F)$ with the norm $(F + \|\cdot\|_{L^2(\Omega)})^{1/2}$.

(iv) The form F is said to be *local* if its polarized Ψ satisfies

$$\Psi(u, v) = 0, \quad \forall u, v \in D(F), \ \text{with } \operatorname{supp}(u) \cap \operatorname{supp}(v) = \varnothing.$$

The form F is said to be *strongly local* if

$$\Psi(u, v) = 0, \quad \forall u, v \in D(F), \, u \text{ constant in a neighbourhood of } \operatorname{supp}(v). \tag{1.7}$$

Thanks to the Beurling–Deny theory (Beurling and Deny, 1958) any regular Dirichlet form F on $L^2(\Omega)$ can be split in the form (1.3).

2. Statement of the Results

In the first subsection, we state two div-curl results. The first one (Theorem 2.1) actually reads as an extension of the Murat–Tartar (Murat, 1978) div-curl lemma for unbounded sequences in L^2. The second one (Theorem 2.6) deals with the more general case where the conductivity sequence only converges in the weak-$*$ sense of the Radon measures.

In the second section, we give two applications in homogenization of the previous div-curl results. The first result (Theorem 2.9), as an application of Theorem 2.6, claims that any sequence of strongly local quadratic forms Γ-converges for the strong topology of L^2, to a strongly local Dirichlet form if the conductivity sequence converges in the weak-$*$ sense of the Radon measures. The second result (Theorem 2.14), as an application of Theorem 2.1, is an extension of the Murat–Tartar H-convergence (Murat, 1998) when the conductivity sequence is not uniformly bounded but only converges to a bounded function in the weak-$*$ sense of the Radon measures.

2.1. *Two-Dimensional Div-Curl Results*

2.1.1. An extension of the Murat–Tartar Div-Curl Lemma. We have the following result:

Theorem 2.1. *Let Ω be a bounded open subset of $\mathbb{R}^2$. Let $\alpha > 0$, let $\bar{a} \in L^\infty(\Omega)$ and let A_n be a sequence of symmetric matrix-valued functions in $L^\infty(\Omega)^{2\times 2}$ satisfying*

$$A_n \geq \alpha I \text{ a.e. in } \Omega \quad \text{and} \quad |A_n| \rightharpoonup \bar{a} \text{ weakly in } \mathcal{M}(\Omega)*. \tag{2.1}$$

Let ξ_n be a sequence in $L^2(\Omega)^2$ and v_n be a sequence in $H^1(\Omega)$ satisfying the following assumptions:

(i) *ξ_n and v_n satisfy the estimate*

$$\int_\Omega A_n^{-1}\xi_n \cdot \xi_n dx + \|v_n\|_{H^1(\Omega)} \leq c; \tag{2.2}$$

(ii) *ξ_n satisfies the classical condition*

$$\operatorname{div} \xi_n \quad \text{is compact in } H^{-1}(\Omega). \tag{2.3}$$

Then, there exists $\xi \in L^2(\Omega)^2$ and $v \in H^1(\Omega)$ such that the following convergences hold true up to a subsequence

$$\xi_n \rightharpoonup \xi \text{ weakly in } \mathcal{M}(\Omega)^2 * \quad \text{and} \quad \nabla v_n \rightharpoonup \nabla v \text{ weakly in } L^2(\Omega)^2. \tag{2.4}$$

Moreover, we have the following convergence in the distributions sense

$$\xi_n \cdot \nabla v_n \rightharpoonup \xi \cdot \nabla v \quad \text{in } \mathcal{D}'(\Omega). \tag{2.5}$$

Remark 2.2. The only but fundamental difference between Theorem 2.1 and the Murat–Tartar div-curl lemma is that the sequence $A_n^{1/2}\xi_n$ is bounded in $L^2(\Omega)^2$ in Theorem 2.1, while ξ_n is bounded in $L^2(\Omega)^2$ in the classical result.

Remark 2.3. Theorem 2.1 is false in dimension three. At this end, let us consider the model example of nonlocal effects in conduction due to Fenchenko and Khruslov (1981) and extended in Bellieud and Bouchitté (1998) and Briane and Tchou (2001).

Example 2.4. Let Ω' be a bounded open set of $\mathbb{R}^2$ and let Ω be the vertical (parallel to the x_3-axis) cylinder defined by $\Omega := \Omega' \times (0, 1)$. Let ω_n be a $\frac{1}{n}$-periodic lattice of thin vertical cylinders of radius $\frac{1}{n}e^{-n^2}$. Let a_n be the conductivity function defined by

$$a_n := \begin{cases} 2e^{2n^2} & \text{in } \omega_n \\ 1 & \text{in } \Omega \setminus \omega_n. \end{cases}$$

For a fixed f in $L^2(\Omega)$, let u_n be the solution in $H_0^1(\Omega)$ of the equation

$$-\text{div}(a_n \nabla u_n) = f \quad \text{in } \mathscr{D}'(\Omega),$$

and let $\widehat{V}_n$ be the Y-periodic function on $\mathbb{R}^3$ defined on $Y := (-1, 1)^3$ and for $R \in (0, 1)$, by

$$\widehat{V}_n(y) := \begin{cases} \dfrac{\ln r + n^2}{\ln R + n^2} & \text{if } r := \sqrt{y_1^2 + y_2^2} \in (e^{-n^2}, R) \\ 0 & \text{if } r \leq e^{-n^2} \text{ (region of high conductivity)} \\ 1 & \text{if } r \geq R. \end{cases}.$$

It is easy to check that the sequences $\xi_n := a_n \nabla u_n$ and $\hat{v}_n(x) := \widehat{V}_n(nx)$ satisfy the assumptions (2.1)–(2.3) of Theorem 2.1 and that $\hat{v}_n$ weakly converges to 1 in $H^1(\Omega)$. However, we can prove that (see Briane and Tchou, 2001 for details)

$$\xi_n \cdot \nabla \hat{v}_n \rightharpoonup 2\pi(u - v),$$

where the weak limit u of u_n in $H_0^1(\Omega)$ and $v \in H_0^1(0, 1; L^2(\Omega'))$ satisfy the coupled system

$$\begin{cases} -\Delta u + 2\pi(u - v) = f & \text{in } \Omega \\ -\dfrac{\partial^2 v}{\partial x_3^2} + v - u = 0 & \text{in } \Omega, \end{cases}$$

for which $u \neq v$ as soon as $f \neq 0$. Therefore, convergence (2.5) does not hold true. In this example, the function v reads as an integral in u, which yields a nonlocal term in the limit equation satisfied by u. We will see that such nonlocal effects are not possible in dimension two under the only assumption (2.1).

2.1.2. An Extension in the Case of Measures. Let Ω be a bounded open subset of $\mathbb{R}^2$. We consider a bounded sequence of nonnegative measures μ_n in $\mathscr{M}(\Omega)$ and a sequence of symmetric matrix-valued functions A_n in $L^\infty_{\mu_n}(\Omega)^{2\times 2}$, and we assume that there exist two constants $\alpha, \beta > 0$ such that, for any $n \in \mathbb{N}$,

$$\int_E \inf_{|\lambda|=1} A_n(x)\lambda \cdot \lambda \, d\mu_n(x) \geq \alpha|E|, \quad \text{for any Borel set } E \subset \Omega, \tag{2.6}$$

where $|E|$ denotes the Lebesgue measure of E, and

$$|A_n(x)| \leq \beta, \quad \mu_n\text{-a.e. } x \in \Omega. \tag{2.7}$$

Note that, due to the dimension two, we have the explicit formula

$$\inf_{|\lambda|=1} A_n\lambda\cdot\lambda = \frac{1}{2}\Big(\mathrm{tr}A_n - \sqrt{(\mathrm{tr}A_n)^2 - 4\det A_n}\Big) \in L^\infty_{\mu_n}(\Omega) \quad \text{since } A_n \in L^\infty_{\mu_n}(\Omega)^{2\times 2}.$$

Remark 2.5. Denoting by $A_{n,r}$ the density function of the regular part of $A_n\, d\mu_n$ with respect to the Lebesgue measure, and taking into account the measure derivation theorem which establishes

$$A_{n,r}(x)\lambda\cdot\lambda = \lim_{r\to 0}\left(⨍_{B(x,r)} A_n\lambda\cdot\lambda\, d\mu_n\right), \quad \forall\lambda\in\mathbb{R}^2, \ \text{a.e. } x\in\Omega,$$

it is easy to check that assumption (2.6) is equivalent to suppose that A_n is nonnegative and that the smaller eigenvalue of $A_{n,r}$ is greater or equal than α. On the other hand, note that, thanks to the symmetry of A_n, the nonnegativity of A_n μ_n-a.e. in Ω and (2.7) are equivalent to

$$|A_n(x)\lambda|^2 \le \beta A_n(x)\lambda\cdot\lambda, \quad \forall\lambda\in\mathbb{R}^2, \ \mu_n\text{-a.e. } x\in\Omega. \tag{2.8}$$

The fact that (2.8) implies the first assertion is immediate. In order to show the direct implication, it is enough to use the Cauchy–Schwarz combined with the nonnegativity of A_n, and (2.7) which imply that

$$\begin{aligned}|A_n(x)\lambda|^2 = A_n(x)A_n(x)\lambda\cdot\lambda &\le (A_n(x)A_n(x)\lambda\cdot A_n(x)\lambda)^{\frac{1}{2}}(A_n(x)\lambda\cdot\lambda)^{\frac{1}{2}}\\ &\le (\beta|A_n(x)\lambda|^2)^{\frac{1}{2}}(A_n(x)\lambda\cdot\lambda)^{\frac{1}{2}},\end{aligned}$$

for any $\lambda\in\mathbb{R}^2$ and μ-a.e. $x\in\Omega$. This gives (2.8).

Associated with A_n we define the sequence of quadratic functionals $F_n : L^2(\Omega) \longrightarrow [0,+\infty]$ by

$$F_n(u) = \begin{cases} \displaystyle\int_\Omega A_n\nabla u\cdot\nabla u\, d\mu_n & \text{if } u\in C^1_0(\Omega)\\ +\infty & \text{elsewhere.}\end{cases} \tag{2.9}$$

Since μ_n is bounded and $L^2(\Omega)$ is separable, we may assume (up to an extraction of a subsequence) that there exist $\mu\in\mathcal{M}(\Omega)$ and $F : L^2(\Omega)\longrightarrow[0,+\infty]$, such that

$$\mu_n \rightharpoonup \mu \quad \text{weakly in } \mathcal{M}(\Omega)*, \tag{2.10}$$

$$F_n \overset{\Gamma - L^2(\Omega)}{\longrightarrow} F. \tag{2.11}$$

Note that the relaxed functional of F_n with respect to the $L^2(\Omega)$-norm (the domain of which contains $C^1_0(\Omega)$) Γ-converges to the same limit F. By Properties 1.2, F is a nonnegative quadratic functional which is lower semicontinuous in $L^2(\Omega)$. Moreover, thanks to (2.6) the functional F_n satisfies the coerciveness

$$F_n(u) \ge \alpha\int_\Omega |\nabla u|^2\, dx, \quad \forall\, u\in C^1_0(\Omega), \tag{2.12}$$

so does its Γ-limit F on its domain by the Γ-limsup inequality combined with the lower semicontinuity of the $H_0^1(\Omega)$-norm. Therefore, the domain $D(F)$ of F is a Hilbert subspace of $H_0^1(\Omega)$ equipped with the norm $\sqrt{F}$. Moreover, by the Γ-liminf inequality combined with (2.7) and (2.10) we have, for any $u \in C_0^1(\Omega)$,

$$F(u) \le \liminf_{n\to+\infty} F_n(u) \le \liminf_{n\to+\infty} \left(\beta\|u\|^2_{C_0^1(\Omega)}\|\mu_n\|_{\mathcal{M}(\Omega)}\right) < +\infty, \tag{2.13}$$

hence the domain $D(F)$ of F contains the space $C_0^1(\Omega)$. Finally, since $D(F)$ (which contains $C_0^1(\Omega)$) is dense in $L^2(\Omega)$ and the embedding of $D(F)$ in $L^2(\Omega)$ is continuous (since F satisfies (2.12)), the usual identification of $L^2(\Omega)$ with its dual implies that

$$L^2(\Omega) \quad \text{is dense in } D(F)'. \tag{2.14}$$

In this context, we have the following extension of the div-curl lemma:

Theorem 2.6. *Assume that the conditions* (2.6), (2.7), *and* (2.10) *hold true.*
Let $\xi_n : \Omega \longrightarrow \mathbb{R}^2$ be a sequence of μ_n-measurable functions with the following properties:

(i) *there exists a constant $C > 0$ such that*

$$\int_\Omega \varphi|\xi_n|d\mu_n \le C\|\varphi\|^{\frac{1}{2}}_{C_0(\Omega)}\left(\int_\Omega \varphi\, d\mu_n\right)^{\frac{1}{2}}, \quad \forall\varphi \in C_0(\Omega), \ \ \varphi \ge 0; \tag{2.15}$$

(ii) *there exists $\xi \in L^1_\mu(\Omega)^2$ such that*

$$\xi_n d\mu_n \rightharpoonup \xi d\mu \quad \textit{weakly in } \mathcal{M}(\Omega)^2*; \tag{2.16}$$

(iii) *there exists $L \in D(F)'$ such that, for any sequence v_n in $C_0^1(\Omega)$ and any function $v \in D(F) \cap C_0(\Omega)$ satisfying*

$$v_n \to v \textit{ strongly in } L^2(\Omega), \quad \sup_{n\in\mathbb{N}}\left(\int_\Omega A_n\nabla v_n\cdot\nabla v_n d\mu_n + \|v_n\|_{C(\bar\Omega)}\right) < +\infty, \tag{2.17}$$

we have

$$\lim_{n\to+\infty}\int_\Omega \xi_n\cdot\nabla v_n\, d\mu_n = L(v). \tag{2.18}$$

Then, for any sequence $v_n \in C^1(\bar\Omega)$ and any $v \in C(\bar\Omega)$ satisfying (2.17) *and the boundedness condition*

$$\xi_n\cdot\nabla v_n d\mu_n \quad \textit{is bounded in } \mathcal{M}(\Omega), \tag{2.19}$$

and for any $\varphi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \xi_n\cdot\nabla v_n\,\varphi\, d\mu_n \underset{n\to+\infty}{\longrightarrow} L(\varphi v) - \int_\Omega \xi\cdot\nabla\varphi v\, d\mu. \tag{2.20}$$

Remark 2.7. If in Theorem 2.6. the function v also belongs to $C^1(\Omega)$, then from (2.16) and (2.18) with $v_n := \varphi v$ for any $n \in \mathbb{N}$, we deduce that

$$\int_\Omega \xi \cdot \nabla(\varphi v) d\mu = \lim_{n\to+\infty} \int_\Omega \xi_n \cdot \nabla(\varphi v) d\mu_n = L(\varphi v). \tag{2.21}$$

Therefore, in this case (2.20) reads as

$$\int_\Omega \xi_n \cdot \nabla v_n \varphi \, d\mu_n \underset{n\to+\infty}{\longrightarrow} \int_\Omega \xi \cdot (\nabla(\varphi v) - \nabla\varphi v) d\mu = \int_\Omega \xi \cdot \nabla v \varphi \, d\mu, \quad \text{for any } \varphi \in C_c^\infty(\Omega),$$

or equivalently,

$$\xi_n \cdot \nabla v_n \, d\mu_n \rightharpoonup \xi \cdot \nabla v \, d\mu \quad \text{in } \mathscr{D}'(\Omega), \tag{2.22}$$

similarly to the classical div-curl lemma.

Remark 2.8. We can localize inequality (2.15). More precisely, for any $\varphi \in C_0(\Omega)$ and any compact set $K \subset \Omega$, we have

$$\int_K |\xi_n||\varphi| d\mu_n \le C \|\varphi\|_{C_0(\Omega)}^{\frac{1}{2}} \left(\int_K |\varphi| d\mu_n \right)^{\frac{1}{2}}. \tag{2.23}$$

Indeed, let us consider $\psi \in C_0(\Omega)$ such that $0 \le \psi \le 1$ and $\psi = 1$ in K. Then, from (2.15) we deduce that

$$\int_K |\xi_n||\varphi| d\mu_n \le \int_\Omega |\xi_n||\varphi|\psi \, d\mu_n \le C \|\varphi\|_{C_0(\Omega)}^{\frac{1}{2}} \left(\int_\Omega |\varphi|\psi \, d\mu_n \right)^{\frac{1}{2}}.$$

Now, minimizing the previous inequality with respect to ψ yields estimate (2.23).

2.2. *Applications in Homogenization*

2.2.1. A Homogenization Result in the Case of Measures. We have the following homogenization result:

Theorem 2.9. *Let Ω be a bounded open subset of $\mathbb{R}^2$. Let μ_n be a sequence of Radon measures in $\mathscr{M}(\overline{\Omega})$ satisfying convergence* (2.10). *Let $\alpha, \beta > 0$ and let A_n be a sequence of symmetric matrix-valued functions in $L^\infty_{\mu_n}(\overline{\Omega})^{2\times 2}$ satisfying the bounds* (2.6) *and* (2.7). *Then, there exist a matrix-valued function A in $L^\infty_\mu(\overline{\Omega})^{2\times 2}$, a subsequence of n, still denoted by n, and a sequence of matrix-valued functions $P_n \in C(\bar{\Omega})^{2\times 2}$ which satisfy*

$$\int_E \inf_{|\lambda|=1} A(x)\lambda \cdot \lambda \, d\mu(x) \ge \alpha|E|, \quad \textit{for any Borel set } E \subset \Omega, \tag{2.24}$$

$$|A(x)| \le \beta, \quad \mu\textit{-a.e. } x \in \Omega, \tag{2.25}$$

each column of P_n is a gradient,

$$P_n \rightharpoonup I \textit{ weakly in } L^2(\Omega)^{2\times 2} \quad \textit{and} \quad A_n P_n d\mu_n \rightharpoonup A \, d\mu \textit{ weakly in } \mathscr{M}(\Omega) * .$$

For every open set $\omega \subset \Omega$, the Γ-limit G of the sequence of functionals (which exists up to a subsequence)

$$G_n(u) := \begin{cases} \int_\omega A_n \nabla u \cdot \nabla u \, d\mu_n & \text{if } u \in C_0^1(\omega) \\ +\infty & \text{if } u \in L^2(\omega) \setminus C_0^1(\omega), \end{cases} \tag{2.26}$$

is a strongly local Dirichlet form on ω, which satisfies

$$G(u) = \int_\omega A \nabla u \cdot \nabla u \, d\mu, \quad \forall u \in C_0^1(\omega). \tag{2.27}$$

Moreover, for any $u \in C_0^1(\omega)$ and any sequence $u_n \in C_0^1(\omega)$ satisfying

$$u_n \longrightarrow u \text{ strongly in } L^2(\Omega) \quad \text{and} \quad G(u) = \lim_{n\to+\infty} G_n(u_n), \tag{2.28}$$

we have

$$A_n \nabla u_n d\mu_n \rightharpoonup A \nabla u \, d\mu \quad \text{weakly in } \mathcal{M}(\Omega)^2 *, \tag{2.29}$$

and the corrector result

$$\lim_{n\to+\infty} \int_\omega A_n(\nabla u_n - P_n \nabla u) \cdot (\nabla u_n - P_n \nabla u) d\mu_n = 0. \tag{2.30}$$

Remark 2.10. In Theorem 2.9 the matrix-valued function A does not depend on ω, which means that the Γ-limit is a local process in our case. Namely, the value of A at a point x only depends on the behaviour of A_n in a neighbourhood of x, that we can choose as small as we wish. Clearly, for $\omega := \Omega$, Theorem 2.9 provides an integral representation of the Γ-limit F of the sequence F_n (2.9) restricted to $C_0^1(\Omega)$.

Remark 2.11. A particular case of Theorem 2.9 is given by the following framework:

$$d\mu_n := |B_n| dx \quad \text{and} \quad A_n := \frac{B_n}{|B_n|},$$

where B_n is a symmetric matrix-valued function in $L^1(\Omega)^{2\times 2}$ satisfying, for given $\alpha > 0$ and $\bar{b} \in L^1(\Omega)$,

$$\forall n \in \mathbb{N}, \; B_n \geq \alpha I \quad \text{a.e. in } \Omega \quad \text{and} \quad |B_n| \rightharpoonup \bar{b} \quad \text{weakly in } \mathcal{M}(\Omega) * .$$

Then, the assumptions (2.10) (up to a subsequence), (2.6) and (2.7) hold true. In this case, the Γ-limit F is such that

$$F(u) = \int_\Omega B \nabla u \cdot \nabla u \, dx, \quad \text{for any } u \in C_0^1(\Omega),$$

where B is a symmetric matrix-valued function in $L^1(\Omega)^{2\times 2}$ satisfying

$$B \geq \alpha I \text{ a.e. in } \Omega \quad \text{and} \quad B \in L^1(\Omega)^{2\times 2}.$$

Remark 2.12. From Theorem 2.9 the Γ-limit G of the sequence G_n defined by (2.26) is a strongly local Dirichlet form (see Definition 1.3). Its restriction to $D(G) \cap C_0(\omega)$, denoted by G_C, is then strongly local, Markovian and closable. The closure $\overline{G}_C$ of G_C is thus a strongly local Dirichlet form (see e.g., Theorem 4.1.2 of Mosco, 1994). Moreover, from the Beurling–Deny theory (Beurling and Deny, 1958) $\overline{G}_C$ is a diffusion, i.e., the killing measure and the jumping measure which appear in (1.3) are zero in this case. Therefore, we can directly conclude to the existence of a matrix-valued A (which could depend on ω) such that the representation (2.27) holds (see e.g., Section 3 of Mosco, 1994). However, in Section 3.2. we give a more constructive proof of (2.27), which is based on the second div-curl lemma (Theorem 2.6). In general, we do not know if $D(G) \cap C_0(\omega)$ is dense in $D(G)$, and thus, we do not know if $\overline{G}_C$ agrees with G.

Remark 2.13. Let u_n be a sequence in $C_0^1(\Omega)$ such that $F_n(u_n) \leq C$. Then, by the Cauchy–Schwarz inequality combined with (2.7) we have, for any $\varphi \in C_0(\Omega)$, $\varphi \geq 0$,

$$\begin{aligned} \int_\Omega |A_n \nabla u_n| \varphi \, d\mu_n &\leq \|\varphi\|_{C_0(\Omega)}^{\frac{1}{2}} \left(\int_\Omega A_n \nabla u_n \cdot \nabla u_n d\mu_n \right)^{\frac{1}{2}} \left(\int_\Omega |A_n| \varphi \, d\mu_n \right)^{\frac{1}{2}} \\ &\leq (C\beta)^{\frac{1}{2}} \|\varphi\|_{C_0(\Omega)}^{\frac{1}{2}} \left(\int_\Omega \varphi \, d\mu_n \right)^{\frac{1}{2}}. \end{aligned} \tag{2.31}$$

Therefore, the sequence $A_n \nabla u_n$ satisfies estimate (2.15). Moreover, the sequence of measures $A_n \nabla u_n d\mu_n$ is bounded in $\mathcal{M}(\Omega)^2$ and we have (up to a subsequence)

$$A_n \nabla u_n d\mu_n \rightharpoonup \nu \quad \text{weakly in } \mathcal{M}(\Omega)^2 * .$$

The limit measure ν is absolutely continuous with respect to the measure μ since (2.31) implies that, for any $\varphi \in C_0(\Omega)$, $\varphi \geq 0$,

$$\left| \int_\Omega \varphi \, d\nu \right| \leq \limsup_{n \to +\infty} \int_\Omega |A_n \nabla u_n| \varphi \, d\mu_n \leq (C\beta)^{\frac{1}{2}} \|\varphi\|_{C_0(\Omega)}^{\frac{1}{2}} \left(\int_\Omega \varphi \, d\mu \right)^{\frac{1}{2}}.$$

2.2.2. *An Extension of the H-Convergence.* We have the following result:

Theorem 2.14. *Let Ω be a bounded open subset of $\mathbb{R}^2$ such that its boundary $\partial\Omega$ has a zero Lebesgue measure. Let $\alpha, \gamma > 0$, let $\bar{a} \in L^\infty(\Omega)$ and let A_n be a sequence of (non necessarily symmetric) matrix-valued functions in $L^\infty(\Omega)^{2\times 2}$ such that the symmetric part A_n^s of A_n and the antisymmetric one A_n^a satisfy*

$$|a_n| I \leq \gamma A_n^s, \quad \text{where } A_n^a := a_n J, \tag{2.32}$$

$$A_n^s \geq \alpha I \text{ a.e. in } \Omega \quad \text{and} \quad |A_n^s| \rightharpoonup \bar{a} \text{ weakly in } \mathcal{M}(\overline{\Omega}) * . \tag{2.33}$$

Then, there exist a subsequence, still denoted by n, and a matrix-valued A in $L^\infty(\Omega)^{2\times 2}$ satisfying

$$A \geq \alpha I \text{ and } A^{-1} \geq [(1+\gamma)^2 \|\bar{a}\|_{L^\infty(\Omega)}]^{-1} I \quad \text{a.e. in } \Omega, \tag{2.34}$$

such that, for any f in $H^{-1}(\Omega)$, the solution u_n in $H_0^1(\Omega)$ of the equation

$$-\mathrm{div}(A_n \nabla u_n) = f \quad \text{in } \mathscr{D}'(\Omega), \tag{2.35}$$

satisfies the weak convergences

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } H_0^1(\Omega) \\ A_n \nabla u_n \rightharpoonup A\nabla u & \text{weakly in } \mathscr{M}(\Omega)^2 *, \end{cases} \tag{2.36}$$

where u is the solution in $H_0^1(\Omega)$ of the equation

$$-\mathrm{div}(A\nabla u) = f \quad \text{in } \mathscr{D}'(\Omega). \tag{2.37}$$

Moreover, there exists a matrix-valued function P_n in $L^2(\Omega)^{2\times 2}$ satisfying

$$\begin{cases} P_n \rightharpoonup I & \text{weakly in } L^2(\Omega)^{2\times 2} \\ P_n\lambda & \text{is a gradient for any } \lambda \in \mathbb{R}^2 \\ \mathrm{div}(A_n P_n) & \text{is compact in } H^{-1}(\Omega), \end{cases} \tag{2.38}$$

$$A_n P_n \rightharpoonup A \quad \text{weakly in } \mathscr{M}(\Omega)^{2\times 2} *, \tag{2.39}$$

and such that, for any f in $H^{-1}(\Omega)$, the solutions u_n of (2.35) *and u of* (2.37) *satisfy the strong convergence*

$$\nabla u_n - P_n \nabla u \longrightarrow 0 \quad \text{strongly in } L^1(\Omega)^{2\times 2}. \tag{2.40}$$

Remark 2.15. Theorem 2.14 extends the H-convergence result of Murat (1998), which holds in any dimension but for a uniformly bounded conductivity sequence. Since the conductivity sequence A_n is not uniformly bounded, contrary to the classical H-convergence, we have to assume with (2.32), that the antisymmetric part of A_n is controlled by the symmetric one.

The unboundedness of the sequence A_n implies that the second convergence of (2.36) and convergence (2.39) only hold true in the weak-$*$ sense of the Radon measures.

We also have the corrector result (2.40) which is quite similar to the one of the classical H-convergence.

Remark 2.16. Thanks to Theorem 2.1 combined with (2.36) the H-convergence according to Theorem 2.14 inherits the local property of the classical H-convergence, which is itself a consequence of the div-curl lemma. More precisely, if A_n, B_n are two sequences in $L^\infty(\Omega)$ satisfying (2.32), (2.33), and converging respectively to A, B in the sense of Theorem 2.14, then we have

$$\text{for any open set } \omega \Subset \Omega, \quad A_n = B_n \text{ a.e. in } \omega \Longrightarrow A = B \text{ a.e. in } \omega.$$

Similarly, if A_n converges to A in the sense of Theorem 2.14, so does A_n^t to A^t.

Example 2.17. Let us consider the case where A_n is a $(0, \varepsilon_n)^2$-periodic symmetric matrix-valued function. More precisely, assume that

$$A_n(x) := B_n\left(\frac{x}{\varepsilon_n}\right) \quad \text{a.e. } x \in \Omega,$$

where B_n is a Y-periodic, $Y := (0, 1)^2$, symmetric matrix-valued function and ε_n a positive sequence converging to 0 as $n \to +\infty$. Then, assumption (2.33) is equivalent to

$$\sup_{n\in\mathbb{N}} \|B_n\|_{L^1(Y)^{2\times 2}} < +\infty. \tag{2.41}$$

Indeed, assumption (2.33) implies that the sequence $|A_n|$ is bounded in $L^1_{\text{loc}}(\Omega)$, hence the boundedness (2.41). Inversely, it is easy to check that (2.41) implies the convergence (up to a subsequence)

$$|A_n| \rightharpoonup \bar{a} \text{ weakly in } \mathscr{M}(\overline{\Omega})*, \quad \text{where } \bar{a} := \lim_{n\to+\infty} \|B_n\|_{L^1(Y)^{2\times 2}}.$$

Under this assumption the first author proved in Briane (2006) that the sequence A_n converges in the sense of Theorem 2.14 to a constant matrix.

3. Proof of the Results

3.1. *Proof of the Div-Curl Results*

We start by a self-contained proof of Theorem 2.1 to present in a simpler framework the main ideas of these two-dimensional div-curl lemmas. Then, we extend the first proof to the case of measures in order to prove Theorem 2.6.

3.1.1. Proof of Theorem 2.1. The key-ingredient of the proof consists in subtracting to the "good-divergence" sequence ξ_n a compact sequence of gradients ∇u_n in such a way that the difference $\xi_n - \nabla u_n$ is divergence-free. Therefore, the function ξ_n reads as $\xi_n = \nabla u_n + J\nabla w_n$, where w_n is a stream function. Then, considering a piecewise constant approximation of w_n we prove that the sequence w_n strongly converges in $L^2_{\text{loc}}(\Omega)$. The strong approximation of w_n is based on the embedding of $W^{1,1}(Q)/\mathbb{R}$ in $L^2(Q)$, the constant of which is independent of any square Q. Finally, replacing ξ_n by $\nabla u_n + J\nabla w_n$, we conclude owing to integrations by parts. So, the proof of Theorem 2.1 is divided in five steps:

- In the first step, we prove the weak convergences (2.4).
- In the second step, we introduce the stream function w_n.
- In the third step, we establish a strong approximation of w_n by a piecewise constant function.
- In the fourth step, we prove the strong convergence of w_n in $L^2_{\text{loc}}(\Omega)$.
- The fifth step is devoted to the proof of convergence (2.5).

First step: Proof of convergences (2.4).

The sequence ξ_n is bounded in $L^1(\Omega)^{2\times 2}$ since the Cauchy–Schwarz inequality combined with the weak-$*$ convergence of (2.1) (which implies the boundedness of $|A_n|$ in $L^1(\Omega)$ thanks to the Banach Steinhaus theorem) and (2.2) yields

$$\int_\Omega |\xi_n| dx \leq \left(\int_\Omega |A_n| dx\right)^{\frac{1}{2}} \left(\int_\Omega A_n^{-1}\xi_n \cdot \xi_n dx\right)^{\frac{1}{2}} \leq c.$$

Therefore, ξ_n converges up to a subsequence to some $\xi \in \mathscr{M}(\Omega)^2$ in the weak-$*$ sense of the measures. Let us prove that the vector-valued measure ξ is actually in $L^2(\Omega)^2$. Again by the Cauchy–Schwarz inequality combined with (2.1) and (2.2) we have, for any $\Phi \in C_0(\Omega)^2$,

$$\begin{aligned} \left|\int_\Omega \xi(dx)\cdot\Phi\right| &= \lim_{n\to+\infty}\left|\int_\Omega \xi_n\cdot\Phi\,dx\right| \\ &\leq \limsup_{n\to+\infty}\left(\int_\Omega A_n^{-1}\xi_n\cdot\xi_n dx\right)^{\frac{1}{2}}\left(\int_\Omega |A_n||\Phi|^2 dx\right)^{\frac{1}{2}} \leq c\left(\int_\Omega \bar{a}|\Phi|^2 dx\right)^{\frac{1}{2}}, \end{aligned}$$

which implies that ξ is absolutely continuous with respect to the Lebesgue measure. Since $\bar{a} \in L^\infty(\Omega)$, we also get

$$\left|\int_\Omega \xi\cdot\Phi\,dx\right| \leq c\|\Phi\|_{L^2(\Omega)^2}, \quad \text{for any } \Phi \in C_0(\Omega)^2,$$

hence $\xi \in L^2(\Omega)^2$. Therefore, the first convergence of (2.4) holds true with its limit in $L^2(\Omega)^2$. The second one is immediate.

Second step: Introduction of a stream function.

By (2.3) the sequence u_n in $H_0^1(\Omega)$ defined by $u_n := \Delta^{-1}(\operatorname{div}\xi_n)$ strongly converges in $H_0^1(\Omega)$. Let ω be a regular simply connected open set such that $\omega \Subset \Omega$. Since by definition $\xi_n - \nabla u_n$ is a divergence-free function in $L^2(\omega)$, there exists (see e.g. Girault and Raviart, 1979) a unique stream function $w_n \in H^1(\omega)$ with zero ω-average such that

$$\xi_n = \nabla u_n + J\nabla w_n \quad \text{a.e. in } \omega.$$

The boundedness of ξ_n in $L^1(\Omega)^2$ (showed in the first step) and of ∇u_n in $L^2(\Omega)^2$ imply that the sequence w_n is bounded in $W^{1,1}(\omega)$. Hence, thanks to the embedding of $W^{1,1}(\omega)$ in $L^2(\omega)$ the sequence w_n is bounded in $L^2(\omega)$. Therefore, w_n weakly converges up to a subsequence to some w in $L^2(\omega)$. However, since the Sobolev exponent 2 is critical in dimension two, we cannot directly conclude to the compactness of w_n in $L^2_{\text{loc}}(\omega)$. This is the aim of the two following steps.

Third step: Approximation of w_n by a piecewise constant function.

Let Q be an open set such that $Q \Subset \omega$. For a given $h > 0$, let $(Q_k^h)_{1\leq k\leq N_h}$ be a partition of open squares satisfying

$$|Q_k^h| = h^2, \quad Q_j^h \cap Q_k^h = \emptyset \quad \text{for any } j \neq k, \text{ and } Q \subset \bigcup_{k=1}^{N_h} \overline{Q}_k^h \subset \omega. \tag{3.1}$$

Let $\bar{w}_n^h$ be the piecewise constant function associated with this partition by

$$\bar{w}_n^h := \sum_{k=1}^{N_h} \left(\fint_{Q_k^h} w_n \right) 1_{Q_k^h}.$$

Since w_n weakly converges to w in $L^2(\omega)$, the sequence $\bar{w}_n^h$, for a fixed h, strongly converges in $L^\infty(\omega)$ to the piecewise constant function defined by

$$\bar{w}^h := \sum_{k=1}^{N_h} \left(\fint_{Q_k^h} w \right) 1_{Q_k^h}. \tag{3.2}$$

Now, let us estimate the difference $w_n - \bar{w}_n^h$ in $L^2(Q)$-norm. Applying the Sobolev–Poincaré inequality in each square Q_k^h, relating to the embedding of $W^{1,1}(Q_k^h)$ in $L^2(Q_k^h)$, whose constant C_S is independent of k and h (since it is invariant by translations and similarities), it follows

$$\int_Q (w_n - \bar{w}_n^h)^2 dx \le \sum_{k=1}^{N_h} \int_{Q_k^h} \left| w_n - \fint_{Q_k^h} w_n \right|^2 dx \le C_S \sum_{k=1}^{N_h} \left(\int_{Q_k^h} |\nabla w_n| \, dx \right)^2.$$

Set $\widetilde{A}_n := J^{-1} A_n^{-1} J$, we have $|\widetilde{A}_n^{-1}| = |A_n|$. Then, the Cauchy–Schwarz inequality implies that

$$\begin{aligned}\left(\int_{Q_h^k} |\nabla w_n| dx \right)^2 &\le \left(\int_{Q_h^k} |\widetilde{A}_n^{-1/2}| |\widetilde{A}_n^{1/2} \nabla w_n| dx \right)^2 \\ &\le \left(\int_{Q_k^h} |A_n| \, dx \right) \left(\int_{Q_k^h} \widetilde{A}_n \nabla w_n \cdot \nabla w_n dx \right).\end{aligned}$$

Hence, the former inequalities yield

$$\int_Q (w_n - \bar{w}_n^h)^2 \, dx \le C_S \left(\sup_{1 \le k \le N_h} \int_{Q_k^h} |A_n| \, dx \right) \sum_{k=1}^{N_h} \int_{Q_k^h} \widetilde{A}_n \nabla \, dx.$$

Moreover, noting that

$$\widetilde{A}_n \nabla w_n \cdot \nabla w_n = A_n^{-1} (\xi_n - \nabla u_n) \cdot (\xi_n - \nabla u_n) \le 2 \big(A_n^{-1} \xi_n \cdot \xi_n + A_n^{-1} \nabla u_n \cdot \nabla u_n \big),$$

the estimate (2.2) and the equi-coerciveness of A_n in (2.1) combined with the boundedness of ∇u_n in $L^2(\Omega)^2$ imply that

$$\sum_{k=1}^{N_h} \int_{Q_k^h} \widetilde{A}_n^{-1} \nabla w_n \cdot \nabla w_n dx \le 2 \int_\Omega \big(A_n^{-1} \xi_n \cdot \xi_n + A_n^{-1} \nabla u_n \cdot \nabla u_n \big) dx \le c.$$

Therefore, there exists a constant $c > 0$ such that

$$\int_Q (w_n - \bar{w}_n^h)^2 dx \le c \sup_{1 \le k \le N_h} \int_{Q_k^h} |A_n| dx.$$

Finally, passing to the lim sup as $n \to +\infty$ and using the convergence of $|A_n|$ in (2.1), we get

$$\limsup_{n\to+\infty} \int_Q (w_n - \bar{w}_n^h)^2 dx \le c \sup_{1\le k\le N_h} \int_{Q_k^h} \tilde{a}\, dx = O(h^2), \tag{3.3}$$

since $\tilde{a} \in L^\infty(\Omega)$.

Fourth step: Strong convergence of w_n in $L^2(Q)$.
From the equality

$$w_n - w = w_n - \bar{w}_n^h + \bar{w}_n^h - \bar{w}^h + \bar{w}^h - w,$$

where $\bar{w}^h$ is the strong $L^\infty(Q)$-limit of $\bar{w}_n^h$ defined by (3.2), and from estimate (3.3) we deduce that

$$\limsup_{n\to+\infty} \|w_n - w\|_{L^2(Q)} \le O(h) + \|\bar{w}^h - w\|_{L^2(Q)}. \tag{3.4}$$

Moreover, considering a $L^2(\omega)$-strong approximation of w by $\psi \in C(\bar{\omega})$, and using the inequality (as a consequence of the Cauchy–Schwarz inequality)

$$\left\| \sum_{k=1}^{N_h} \left(\fint_{Q_k^h} (w - \psi) \right) 1_{Q_k^h} \right\|_{L^2(Q)} \le \|w - \psi\|_{L^2(\omega)},$$

and the uniform convergence

$$\lim_{h\to 0} \left\| \psi - \sum_{k=1}^{N_h} \left(\fint_{Q_k^h} \psi \right) 1_{Q_k^h} \right\|_{L^\infty(Q)} = 0,$$

we obtain that $\bar{w}^h$ strongly converges to w in $L^2(Q)$ as $h \to 0$. This combined with estimate (3.4) yields the desired convergence.

Fifth step: Proof of convergence (2.5).

Let $\varphi \in C_c^\infty(\Omega)$. Using a localization argument we can assume that the support of φ is contained in a regular simply connected open set ω such that $\omega \Subset \Omega$. Consider the stream function w_n of the second step, which satisfies the equality $\xi_n = \nabla u_n + J\nabla w_n$ and which converges to $w \in L^2(\omega)$ strongly in $L^2_{\text{loc}}(\omega)$ by the fourth step. Since the limit ξ of ξ_n in the weak-$*$ sense of the measures belongs to $L^2(\Omega)^2$ by the first step, and since $\xi = \nabla u + J\nabla w$ in $\mathscr{D}'(\omega)$, where u is the strong limit of u_n in $H_0^1(\Omega)$, we have $w \in H^1(\omega)$ and $\xi = \nabla u + J\nabla w$ a.e. in ω.

Integrating by parts and using that $J\nabla v_n$ is divergence-free we get

$$\begin{aligned}\int_\Omega \xi_n \cdot \nabla v_n \varphi\, dx &= \int_\Omega \nabla u_n \cdot \nabla v_n \varphi\, dx - \int_\omega J\nabla v_n \cdot \nabla w_n \varphi\, dx \\ &= \int_\omega \nabla u_n \cdot \nabla v_n \varphi\, dx + \int_\omega J\nabla v_n \cdot \nabla\varphi w_n\, dx.\end{aligned}$$

Therefore, the strong convergence of u_n to u in $H_0^1(\Omega)$, the weak convergence of v_n to v in $H^1(\Omega)$ and the strong convergence of w_n to w in $L^2_{\mathrm{loc}}(\omega)$ imply that

$$\lim_{n\to+\infty}\int_\Omega \xi_n\cdot\nabla v_n\varphi\,dx = \int_\omega \nabla u\cdot\nabla v\varphi\,dx + \int_\omega J\nabla v\cdot\nabla\varphi w\,dx$$
$$= \int_\Omega \nabla u\cdot\nabla v\varphi\,dx - \int_\omega J\nabla v\cdot\nabla w\varphi\,dx$$

by a new integration by parts. This combined with the equality $\xi = \nabla u + J\nabla w$ a.e. in ω, gives the thesis. Note that convergence (2.5) holds true for the whole sequence satisfying (2.4).

3.1.2. Proof of Theorem 2.6. In comparison to the former proof we now subtract to the sequence $\xi_n\mu_n$ a compact sequence of Radon measures ν_n^ε in such a way that the difference $\xi_n\mu_n - \nu_n^\varepsilon$ is divergence-free up to a small perturbation of order ε. Therefore, the measure $\xi_n\mu_n$ reads as

$$\xi_n\mu_n = \nu_n^\varepsilon + J\nabla w_n^\varepsilon + O(\varepsilon) \quad \text{in any open square } Q \Subset \Omega,$$

where the stream function w_n^ε belongs this time to $BV(Q)$. As previously, considering a piecewise constant approximation of w_n^ε we prove that w_n^ε satisfies the strong estimate (3.10) of type (3.3) in $L^2_{\mathrm{loc}}(Q)$. However, the bounded function $\bar{a}$ in (3.3) is now replaced by the measure μ in estimate (3.10). Therefore, we have to study the case where μ possibly loads some points of Ω (see the fifth step). To this end, we need the extra assumption (2.19).

Let $\varphi \in C_c^\infty(\Omega)$. In order to prove (2.20) we may assume, thanks to a partition of the unity, that the support of φ is contained in a open square Q such that $Q \Subset \Omega$. Then, the proof of Theorem 2.6 is divided in five steps:

- In the first step, we prove that, for a fixed $\varepsilon > 0$, there exist a function g^ε in $L^2(\Omega)$ satisfying

$$\|L - \operatorname{div} g^\varepsilon\|_{D(F)'} < \varepsilon, \tag{3.5}$$

a measure ν_n^ε in $\mathcal{M}(\Omega)^2$ satisfying

$$\limsup_{n\to+\infty} \|\nu_n^\varepsilon\|_{\mathcal{M}(\Omega)^2} = O(\varepsilon), \tag{3.6}$$

and a function w_n^ε in $BV(Q)$, with zero average in Q, such that

$$J\nabla w_n^\varepsilon = \xi_n\,\mu_n + g^\varepsilon - \nu_n^\varepsilon \quad \text{in } \mathcal{M}(Q)^2. \tag{3.7}$$

- In the second step, we prove that the weak limit w^ε of w_n^ε in $L^2(\Omega)$ and the weak-$*$ limit ν^ε of ν_n^ε in $\mathcal{M}(\Omega)^2$ satisfy

$$\|\nu^\varepsilon\|_{\mathcal{M}(\Omega)^2} = O(\varepsilon), \tag{3.8}$$

$$\int_Q \xi\cdot\nabla\varphi v\,d\mu = \int_Q w^\varepsilon\nabla v\cdot J\nabla\varphi\,dx - \int_Q v g^\varepsilon\cdot\nabla\varphi\,dx + \int_Q v\nabla\varphi\cdot d\nu^\varepsilon. \tag{3.9}$$

- In the third step, we prove that there exists a constant $C > 0$ such that, for any $\Phi \in C_0^1(\Omega)$,

$$\limsup_{n\to+\infty} \|(w_n^\varepsilon - w^\varepsilon)\Phi\|_{L^2(Q)}^2 \leq C\left[\varepsilon^2\|\Phi\|_{C_0(\Omega)}^2 + \|\Phi\|_{C_0(\Omega)}^{\frac{3}{2}} \sup_{x\in\overline{Q}}(|\Phi(x)|\mu(\{x\}))^{\frac{1}{2}}\right]. \tag{3.10}$$

- In the fourth step, we prove the estimate

$$\limsup_{n\to+\infty}\left|\int_\Omega \xi_n\cdot\nabla v_n\varphi\, d\mu_n - L(\varphi v) + \int_\Omega \xi\cdot\nabla\varphi v\, d\mu\right| \leq C\|\nabla\varphi\|_{C_0(\Omega)^2}^{\frac{3}{4}} \sup_{x\in\overline{Q}}\left(|\nabla\varphi|(x)\mu(\{x\})\right)^{\frac{1}{4}}. \tag{3.11}$$

- The fifth step is devoted to the proof of the convergence (2.20).

First step: Proof of (3.5)–(3.7).

By the density result in (2.14) there exists a function h^ε in $L^2(\Omega)$ such that

$$\|L - h^\varepsilon\|_{D(F)'} < \varepsilon.$$

Moreover, h^ε can be written as the divergence of a vector-valued function g^ε in $L^2(\Omega)^2$ (which can be chosen to be a gradient). This yields the first estimate (3.5).

Set

$$D := \left\{\nabla z : z \in C_0^1(\Omega)\right\}$$

endowed with the topology of $C_0(\Omega)^2$, and define $G_n^\varepsilon \in D'$ by

$$G_n^\varepsilon(\nabla z) := \int_\Omega \xi_n\cdot\nabla z\, d\mu_n + \int_\Omega g^\varepsilon\cdot\nabla z\, dx, \quad \text{for } z \in C_0^1(\Omega). \tag{3.12}$$

Thanks to the Hahn–Banach theorem, G_n^ε can be extended to a linear functional v_n^ε in $(C_0(\Omega)^2)' = \mathscr{M}(\Omega)^2$ such that

$$\|v_n^\varepsilon\|_{\mathscr{M}(\Omega)^2} = \|G_n^\varepsilon\|_{D'}.$$

Therefore, for any $n \in \mathbb{N}$, we can choose $z_n \in C_0^1(\Omega)$ with $\|\nabla z_n\|_{C_0(\Omega)^2} = 1$, such that

$$\|v_n^\varepsilon\|_{\mathscr{M}(\Omega)^2} \leq G_n^\varepsilon(\nabla z_n) + \frac{1}{n} = \int_\Omega \xi_n\cdot\nabla z_n d\mu_n + \int_\Omega g^\varepsilon\cdot\nabla z_n dx + \frac{1}{n}. \tag{3.13}$$

Since the sequence z_n belongs to $C_0^1(\Omega)$ and is bounded in $W^{1,\infty}(\Omega)$, there exists a function $\hat{z}$ in $W^{1,\infty}(\Omega)\cap C_0(\Omega)$ such that z_n weakly converges (up to a subsequence) to $\hat{z}$ both in $W^{1,\infty}(\Omega)*$ and in $H_0^1(\Omega)$. Moreover, using successively the Γ-liminf inequality of the Γ-convergence of F_n (2.9) to F, $\|\nabla z_n\|_{C_0(\Omega)^2} = 1$, and the boundedness (2.7) of A_n and (2.10) of μ_n, yields

$$F(\hat{z}) \leq \limsup_{n\to+\infty}\left(\int_\Omega A_n\nabla z_n\cdot\nabla z_n d\mu_n\right) \leq \limsup_{n\to+\infty}\left(\int_\Omega \beta\, d\mu_n\right) < +\infty,$$

hence $\hat{z} \in D(F) \cap C_0(\Omega)$. Therefore, the condition (2.17) holds true with $v_n := z_n$ and $v := \hat{z}$. Then, the convergence (2.18) satisfied by ξ_n combined with the weak convergence of ∇z_n and $\hat{z} \in H_0^1(\Omega)$ implies that

$$\lim_{n\to+\infty} \int_\Omega \xi_n \cdot \nabla z_n d\mu_n + \int_\Omega g^\varepsilon \cdot \nabla z_n dx = L(\hat{z}) + \int_\Omega g^\varepsilon \cdot \nabla \hat{z}\, dx = L(\hat{z}) - \int_\Omega h^\varepsilon \hat{z}\, dx. \tag{3.14}$$

Therefore, by (3.13), (3.14) and (3.5) we obtain

$$\limsup_{n\to+\infty} \|v_n^\varepsilon\|_{\mathscr{M}(\Omega)^2} \le \|L - h^\varepsilon\|_{D(F)'} \|\hat{z}\|_{D(F)} \le \varepsilon\sqrt{F(\hat{z})}.$$

which yields (3.6).

On the other hand, since v_n^ε is an extension of G_n^ε defined by (3.12), we have

$$\int_Q \xi_n \cdot \nabla z\, d\mu_n + \int_Q g^\varepsilon \cdot \nabla z\, dx - \int_Q \nabla z \cdot dv_n^\varepsilon = 0, \quad \text{for any } z \in C_0^1(Q),$$

or equivalently, the Radon vector-valued measure $\xi_n \mu_n + g^\varepsilon - v_n^\varepsilon$ is divergence-free in Q. This implies the existence of a stream function w_n^ε in $BV(Q)$, with zero average in Q, which satisfies the equality (3.7).

Second step: Proof of (3.8) and (3.9).

By (3.6) the sequence v_n^ε is bounded in $\mathscr{M}(\Omega)^2$, and thus weakly converges up to a subsequence to some v^ε in $\mathscr{M}(\Omega)^2*$. The estimate (3.8) is then an immediate consequence of (3.6).

By the equality (3.7) combined with (2.16) and (3.6) the sequence ∇w_n^ε is bounded in $\mathscr{M}(Q)^2$. Then, by the Sobolev–Poincaré inequality the sequence w_n^ε is also bounded in $L^2(Q)$. Therefore, there exists a function w^ε in $BV(Q)$ such that (up to a subsequence)

$$w_n^\varepsilon \rightharpoonup w^\varepsilon \text{ weakly in } L^2(Q) \quad \text{and} \quad \nabla w_n^\varepsilon \rightharpoonup \nabla w^\varepsilon \text{ weakly in } \mathscr{M}(Q)^2 * . \tag{3.15}$$

Passing to the limit as $n \to +\infty$ in (3.7) owing to (2.16) and (3.15) we get

$$J\nabla w^\varepsilon = \xi\mu + g^\varepsilon - v^\varepsilon \quad \text{in } \mathscr{M}(Q)^2 \quad \text{and} \quad \int_Q w^\varepsilon dx = 0. \tag{3.16}$$

Multiplying (3.16) by $v\nabla\varphi$ and integrating by parts we obtain

$$\begin{aligned}
\int_Q \xi \cdot \nabla\varphi v\, d\mu &= \int_Q J\nabla w^\varepsilon \cdot \nabla\varphi v - \int_Q g^\varepsilon \cdot \nabla\varphi v\, dx + \int_Q v\nabla\varphi \cdot dv^\varepsilon \\
&= \langle -\nabla(w^\varepsilon v) + w^\varepsilon \nabla v, J\nabla\varphi\rangle_{\mathscr{D}'(Q), C_c^\infty(Q)} - \int_Q g^\varepsilon \cdot \nabla\varphi v\, dx + \int_Q v\nabla\varphi \cdot dv^\varepsilon \\
&= \int_Q w^\varepsilon \nabla v \cdot J\nabla\varphi\, dx - \int_Q g^\varepsilon \cdot \nabla\varphi v\, dx + \int_Q v\nabla\varphi \cdot dv^\varepsilon
\end{aligned}$$

which is (3.9).

Third step: Proof of estimate (3.10).

Let $\Phi \in C_0^1(\Omega)$. Proceeding as in the third step of the proof of Theorem 2.1 owing to a piecewise approximation of w_n^ε, and using the bound (3.6) satisfied by v_n^ε, the estimate (2.23) satisfied by $|\xi_n|$, the convergence (2.10) of μ_n and the boundedness of $|\xi_n|\,d\mu_n$ in $\mathscr{M}(\Omega)$ (as a consequence of (2.16)) we get

$$\limsup_{n\to+\infty} \|(w_n^\varepsilon - w^\varepsilon)\Phi\|^2_{L^2(Q)} \\ \le c\left[\varepsilon^2\|\Phi\|^2_{C_0(\Omega)} + \|\Phi\|^{\frac{3}{2}}_{C_0(\Omega)} \sup_{1\le k\le N_h}\left(\int_{\overline{Q}_k^h}|\Phi|d\mu\right)^{\frac{1}{2}}\right]. \tag{3.17}$$

On the other hand, we denote by x_h the center of the square $Q_{j_h}^h$ such that

$$\sup_{1\le k\le N_h}\int_{\overline{Q}_k^h}|\Phi|d\mu = \int_{\overline{Q}_{j_h}^h}|\Phi|d\mu.$$

We may assume that x_h converges (up to an extraction of a subsequence) to $x_0 \in \overline{Q}$ as h tends to 0. Then, for any open set O which contains x_0, we have

$$\limsup_{h\to 0}\left(\sup_{1\le k\le N_h}\int_{\overline{Q}_k^h}|\Phi|d\mu\right) \le \int_O |\Phi|d\mu,$$

hence, minimizing with respect to O it follows

$$\limsup_{h\to 0}\left(\sup_{1\le k\le N_h}\int_{\overline{Q}_k^h}|\Phi|d\mu\right) \le |\Phi|(x_0)\mu(\{x_0\}).$$

Therefore, we get

$$\limsup_{h\to 0}\left(\sup_{1\le k\le N_h}\int_{\overline{Q}_k^h}|\Phi|d\mu\right) \le \sup_{x\in\overline{Q}}|\Phi|(x)\mu(\{x\}).$$

Then, passing to the lim sup as $h \to 0$ in inequality (3.17) and using the former inequality we obtain the desired estimate (3.10).

Fourth step: Proof of estimate (3.11).

We start from the equality

$$\int_\Omega \xi_n\cdot\nabla v_n\varphi\,d\mu_n = \int_\Omega \xi_n\cdot\nabla(\varphi v_n)d\mu_n - \int_\Omega \xi_n\cdot\nabla\varphi v_n d\mu_n, \tag{3.18}$$

where thanks to assumption (2.18) we have

$$\int_\Omega \xi_n\cdot\nabla(\varphi v_n)d\mu_n \underset{n\to+\infty}{\longrightarrow} L(\varphi v). \tag{3.19}$$

For the second term of the right-hand side of (3.18) the definition (3.7) of w_n^ε implies that, similarly to (3.9),

$$\int_Q \xi_n\cdot\nabla\varphi v_n d\mu_n = \int_Q w_n^\varepsilon\nabla v_n\cdot J\nabla\varphi\,dx - \int_Q v_n g^\varepsilon\cdot\nabla\varphi\,dx + \int_Q v_n\nabla\varphi\cdot dv_n^\varepsilon. \tag{3.20}$$

Since v_n weakly converges in $L^2(\Omega)$, we have

$$\lim_{n\to+\infty} \int_Q v_n g^\varepsilon \cdot \nabla\varphi \, dx = \int_Q v g^\varepsilon \cdot \nabla\varphi \, dx, \tag{3.21}$$

Moreover, the bound (3.6) satisfied by v_n^ε combined with the boundedness (2.17) of v_n in $C_0(\Omega)$ implies that

$$\limsup_{n\to+\infty} \left| \int_Q v_n \nabla\varphi \cdot dv_n^\varepsilon \right| = c\varepsilon \|\nabla\varphi\|_{C_0(\Omega)^2}. \tag{3.22}$$

For the first term of the right-hand side of (3.20), the Cauchy–Schwarz inequality combined with the weak convergence of ∇v_n in $L^2(\Omega)^2$ (as a consequence of the equi-coerciveness (2.12) and the bound (2.17)) yields

$$\limsup_{n\to+\infty} \left| \int_Q w_n^\varepsilon \nabla v_n \cdot J\nabla\varphi \, dx - \int_Q w^\varepsilon \nabla v \cdot J\nabla\varphi \, dx \right| \leq c \limsup_{n\to+\infty} \|(w_n^\varepsilon - w^\varepsilon)\nabla\varphi\|_{L^2(Q)^2}. \tag{3.23}$$

Therefore, collecting (3.18)–(3.23) and the estimate (3.10) of the third step (Φ being each of the derivatives of φ) we obtain

$$\begin{aligned} &\limsup_{n\to+\infty} \left| \int_\Omega \xi_n \cdot \nabla v_n \varphi \, d\mu_n - L(\varphi v) - \int_Q v g^\varepsilon \cdot \nabla\varphi \, dx + \int_Q w^\varepsilon \nabla v \cdot J\nabla\varphi \, dx \right| \\ &\quad \leq c\left[\varepsilon \|\nabla\varphi\|_{C_0(\Omega)^2} + \|\nabla\varphi\|_{C_0(\Omega)^2}^{\frac{3}{4}} \sup_{x\in\bar{Q}} \left(|\nabla\varphi|(x)\mu(\{x\}) \right)^{\frac{1}{4}} \right], \end{aligned}$$

which combined with the estimate (3.8) and the equality (3.9) of the second step yields

$$\begin{aligned} &\limsup_{n\to+\infty} \left| \int_\Omega \xi_n \cdot \nabla v_n \varphi \, d\mu_n - L(\varphi v) + \int_\Omega \xi \cdot \nabla\varphi v \, d\mu \right| \\ &\quad \leq c\left[\varepsilon \|\nabla\varphi\|_{C_0(\Omega)^2} + \|\nabla\varphi\|_{C_0(\Omega)^2}^{\frac{3}{4}} \sup_{x\in\bar{Q}} \left(|\nabla\varphi|(x)\mu(\{x\}) \right)^{\frac{1}{4}} \right] + O(\varepsilon). \end{aligned}$$

Finally, the arbitrariness of ε in the last estimate implies (3.11).

Fifth step: Proof of (2.20).

The estimate (3.11) clearly implies convergence (2.20) if all the points in Ω have zero μ-measure. Otherwise, we denote by $\{x_i\}_{i\in\mathbb{N}}$ the set of points of Ω such that $\mu(\{x_i\}) > 0$, which is known to be at most a countable set. For $\varepsilon > 0$ and $i \in \mathbb{N}$, we consider a function ψ_i^ε in $C_c^\infty(\Omega)$ such that

$$\psi_i^\varepsilon = 1 \text{ in } B\left(x_i, \frac{\varepsilon}{2}\right), \quad \text{supp } \psi_i^\varepsilon \subset B(x_i, \varepsilon), \;\; 0 \leq \psi_i^\varepsilon \leq 1, \;\text{ and } |\nabla\psi_i^\varepsilon| < \frac{4}{\varepsilon}.$$

Then, for a fixed $j \in \mathbb{N}$, we choose $\varepsilon_j > 0$ small enough such that the supports of $\psi_i^{\varepsilon_j}$, $1 \leq i \leq j$, are disjoint and $\varepsilon_j < 1/j$. By considering the decomposition

$$\varphi = \varphi_j + \eta_j \quad \text{with } \varphi_j := \varphi \prod_{i=1}^{j} (1 - \psi_i^{\varepsilon_j}) + \sum_{i=1}^{j} \varphi(x_i) \psi_i^{\varepsilon_j} \text{ and } \eta_j := \sum_{i=1}^{j} \big(\varphi - \varphi(x_i)\big) \psi_i^{\varepsilon_j},$$

we have

$$\int_\Omega \xi_n \cdot \nabla v_n \varphi \, d\mu_n = \int_\Omega \xi_n \cdot \nabla v_n \varphi_j d\mu_n + \int_\Omega \xi_n \cdot \nabla v_n \eta_j d\mu_n.$$

Since the supports of the functions ψ_i^ε, for $1 \leq i \leq j$, are disjoint, we also have

$$|\nabla \varphi_j| = \left| \nabla \varphi \prod_{i=1}^{j} (1 - \psi_i^{\varepsilon_j}) - \sum_{i=1}^{j} \big[(\varphi - \varphi(x_i)) \nabla \psi_i^{\varepsilon_j} \big] \right| \leq 5 \|\nabla \varphi\|_{C_0(\Omega)^2}. \tag{3.24}$$

On the one hand, since $\nabla \varphi_j(x_i) = 0$ for any $i \leq j$, we can apply the estimate (3.11) with φ_j combined with (3.24) to deduce that

$$\begin{aligned} &\limsup_{n \to +\infty} \left| \int_\Omega \xi_n \cdot \nabla v_n \varphi_j d\mu_n - L(\varphi_j v) + \int_\Omega \xi \cdot \nabla \varphi_j v \, d\mu \right| \\ &\quad \leq c \|\nabla \varphi\|_{C_0(\Omega)^2} \sup_{i > j} \big(\mu(\{x_i\}) \big)^{\frac{1}{2}}. \end{aligned} \tag{3.25}$$

On the other hand, noting that

$$\|\eta_j\|_{C_0(\Omega)} \leq \varepsilon_j \|\nabla \varphi\|_{C_0(\Omega)^2} \leq \frac{1}{j} \|\nabla \varphi\|_{C_0(\Omega)^2},$$

the boundedness assumption (2.19) yields

$$\limsup_{n \to +\infty} \left| \int_\Omega \xi_n \cdot \nabla v_n \eta_j d\mu_n \right| \leq \frac{c}{j} \|\nabla \varphi\|_{C_0(\Omega)^2}. \tag{3.26}$$

Therefore, the estimates (3.25) and (3.26) imply the new one

$$\begin{aligned} &\limsup_{n \to +\infty} \left| \int_\Omega \xi_n \cdot \nabla v_n \varphi \, d\mu_n - L(\varphi_j v) + \int_\Omega \xi \cdot \nabla \varphi_j v \, d\mu \right| \\ &\quad \leq c \|\nabla \varphi\|_{C_0(\Omega)^2} \left[\sup_{i > j} \big(\mu(\{x_i\}) \big)^{\frac{1}{2}} + \frac{1}{j} \right]. \end{aligned} \tag{3.27}$$

It remains to pass to the limit in (3.27) as $j \to +\infty$. Since the series $\sum_{i=1}^{\infty} \mu(\{x_i\})$ is convergent, the right-hand side of (3.27) tends to zero as $j \to +\infty$.

For the left-hand side of (3.27), first note that the Γ-liminf inequality

$$F(\varphi_j v) \leq \liminf_{n \to +\infty} F_n(\varphi_j v_n) = \liminf_{n \to +\infty} \int_\Omega A_n \nabla(\varphi_j v_n) \cdot \nabla(\varphi_j v_n) d\mu_n$$

combined with the bounds (3.24) and (2.17), implies that $F(\varphi_j v)$ is bounded. Moreover, the convergence of $\varphi - \varphi_j = \eta_j$ to 0 in $C_0(\Omega)$ and (3.24) imply that φ_j

weakly converges to φ in $W^{1,\infty}(\Omega)*$. In particular, the sequence $\varphi_j v$ strongly converges to φv in $L^2(\Omega)$. Therefore, the sequence $\varphi_j v$ weakly converges to φv in $D(F)$, and the continuity of L ensures that

$$\lim_{j\to+\infty} L(\varphi_j v) = L(\varphi v). \tag{3.28}$$

On the other hand, if $v \in C^1(\Omega)$ then the formula (2.21) of L (see Remark 2.7) combined with the convergence of φ_j to φ in $C_0(\Omega)$ and (3.28) yields

$$\begin{aligned} \int_\Omega \xi \cdot \nabla\varphi_j v d\mu &= L(\varphi_j v) - \int_\Omega \xi \cdot \nabla v \varphi_j d\mu \\ &\underset{j\to+\infty}{\longrightarrow} L(\varphi v) - \int_\Omega \xi \cdot \nabla v \varphi \, d\mu \\ &= \int_\Omega \xi \cdot \nabla\varphi v \, d\mu. \end{aligned} \tag{3.29}$$

Otherwise, considering a $C^1(\Omega)$-regular approximation of v for the $C(\overline{\Omega})$-norm and using the uniform estimate (3.24) of $\nabla\varphi_j$, we still obtain convergence (3.28). Therefore, convergences (3.28) and (3.37) imply that

$$\lim_{j\to+\infty} \left(L(\varphi_j v) - \int_\Omega \xi \cdot \nabla\varphi_j v \, d\mu \right) = L(\varphi v) - \int_\Omega \xi \cdot \nabla\varphi v \, d\mu,$$

which combined with (3.27) yields the desired convergence (2.20). Theorem 2.6 is proved.

3.2. *Proof of the Homogenization Results*

Let us now prove Theorems 2.9 and 2.14. The proofs of these results are based on the previous div-curl lemmas (Theorems 2.1 and 2.6). Moreover, the proof of the compactness in Theorem 2.14 follows the scheme of the classical H-convergence.

3.2.1. Proof of Theorem 2.9. The proof is divided in four steps. In the first step, we prove that G is strongly local. In the second step, we construct the homogenized matrix-valued function A of (2.27) and the corrector P_n of (2.30). In the third step, we prove that A satisfies the properties (2.24) and (2.25). The fourth step is devoted to the proof of the homogenization results (2.27), (2.29) and the corrector result (2.30).

First step: Let u, v be two functions in $D(G)$ such that there exist $r \in \mathbb{R}$ and an open set $V \subset \omega$ with $u = r$ in V and $\operatorname{supp}(v) \Subset V$. Using a truncation argument we may assume that u, v belong to $L^\infty(\omega)$. Denoting by Ψ the polarized of the quadratic form G, we have to prove that $\Psi(u, v) = 0$. To this end, we consider two recovery sequences u_n, v_n in $C_0^1(\omega)$ relating to u, v respectively and bounded in $L^\infty(\omega)$, and three functions $\varphi_1, \varphi_2, \varphi_3$ in $C_c^\infty(V)$ satisfying

$$\varphi_1 = 1 \text{ in supp } (v), \quad \varphi_2 = 1 \text{ in supp } (\varphi_1), \quad \text{and} \quad \varphi_3 = 1 \text{ in supp}(\varphi_2).$$

By the Properties 1.2e) of a recovery sequence we have

$$\begin{aligned}\Psi(u, v) &= \lim_{n\to+\infty} \int_\omega A_n \nabla u_n \cdot \nabla v_n d\mu_n \\ &= \lim_{n\to+\infty} \left(\int_\omega A_n \nabla u_n \cdot \nabla v_n \, \varphi_2 d\mu_n + \int_\omega A_n \nabla u_n \cdot \nabla v_n \cdot (1-\varphi_2) d\mu_n \right). \end{aligned} \tag{3.30}$$

Using that $\varphi_3 = 1$ in $\operatorname{supp}(\varphi_2)$, the first term of the right-hand side of (3.30) reads as

$$\int_\omega A_n \nabla u_n \cdot \nabla v_n \varphi_2 d\mu_n = \int_\omega A_n \nabla v_n \cdot \nabla(u_n \varphi_3) \varphi_2 d\mu_n.$$

Up to a subsequence $A_n \nabla v_n \, d\mu_n$ weakly converges to some ξ_v in $\mathscr{M}(\Omega)*$. Then, since the sequence $u_n \varphi_3$ strongly converges to $r\varphi_3 \in C_0^1(\omega)$ in $L^2(\omega)$ and satisfies the estimate

$$\limsup_{n\to+\infty} \left(\int_\omega A_n \nabla(u_n\varphi_3) \cdot \nabla(u_n\varphi_3) d\mu_n + \|u_n\varphi_3\|_{L^\infty(\omega)} \right) < +\infty,$$

the convergence (2.22) of Remark 2.7 yields

$$\lim_{n\to+\infty} \int_\omega A_n \nabla v_n \cdot \nabla(u_n\varphi_3) \varphi_2 d\mu_n = \int_\omega \xi_v \cdot \nabla(r\varphi_3) \varphi_2 d\mu = 0.$$

For the second term of the right-hand side of (3.30), using that $\varphi_2 = 1$ in $\operatorname{supp}(\varphi_1)$, we have

$$\begin{aligned}\int_\omega A_n \nabla u_n \cdot \nabla v_n (1-\varphi_2) d\mu_n &= \int_\omega A_n \nabla u_n \cdot \nabla\big((1-\varphi_1)v_n\big)(1-\varphi_2) d\mu_n \\ &= \int_\omega A_n \nabla u_n \cdot \nabla\big((1-\varphi_1)v_n\big) d\mu_n \\ &\quad - \int_\omega A_n \nabla u_n \cdot \nabla\big((1-\varphi_1)v_n\big)\varphi_2 d\mu_n.\end{aligned}$$

Since $(1-\varphi_1)v_n$ strongly converges to $0 \in C_0^1(\Omega)$ in $L^2(\omega)$ and

$$\limsup_{n\to+\infty} \int_\omega A_n \nabla\big((1-\varphi_1)v_n\big) \cdot \nabla\big((1-\varphi_1)v_n\big) d\mu_n < +\infty,$$

the fact that u_n is a recovery sequence implies, in virtue of Properties 1.2e), that

$$\lim_{n\to+\infty} \int_\omega A_n \nabla u_n \cdot \nabla\big((1-\varphi_1)v_n\big) d\mu_n = \Psi(u, 0) = 0.$$

Moreover, we can apply convergence (2.22) as before to get

$$\lim_{n\to+\infty} \int_\omega A_n \nabla u_n \cdot \nabla\big((1-\varphi_1)v_n\big)\varphi_2 d\mu_n = 0.$$

Therefore, the right-hand side of (3.30) also tends to zero, hence $\Psi(u, v) = 0$.

Second step: Construction of A and P_n.
We define the functional $H_n : L^2(\Omega) \longrightarrow [0, +\infty]$ by

$$H_n(u) = \begin{cases} \int_\Omega A_n \nabla u \cdot \nabla u \, d\mu_n & \text{if } u \in C^1(\overline{\Omega}) \\ +\infty & \text{elsewhere .} \end{cases} \tag{3.31}$$

Up to an extraction of a subsequence we can assume the existence of the Γ-limit H of H_n. Since $C^1(\overline{\Omega})$ is contained in $D(H)$ (see (2.13)), we know that there exist $w_n^1, w_n^2 \in C^1(\overline{\Omega})$ which strongly converge in $L^2(\Omega)$ respectively to the functions w^1, w^2 defined by

$$w^1(x) = x_1, \quad w^2(x) = x_2, \ \forall x = (x_1, x_2) \in \overline{\Omega}, \tag{3.32}$$

such that, for $i = 1, 2$,

$$H(w^i) = \lim_{n \to +\infty} \int_\Omega A_n \nabla w_n^i \cdot \nabla w_n^i d\mu_n < +\infty. \tag{3.33}$$

Denote by Φ_H the polarized of the quadratic form H. Thanks to Properties 1.2e) combined with (3.33), for any $v \in D(H)$ and any sequence $v_n \in C^1(\overline{\Omega})$ strongly converging to v in $L^2(\Omega)$ and satisfying

$$\limsup_{n \to +\infty} \int_\Omega A_n \nabla v_n \cdot \nabla v_n \, d\mu_n < +\infty,$$

we have

$$\lim_{n \to +\infty} \int_\Omega A_n \nabla w_n^i \cdot \nabla v_n d\mu_n = \Phi_H(w^i, v). \tag{3.34}$$

Moreover, by the Cauchy–Schwarz inequality we have, for any $\varphi \in C(\overline{\Omega})$, $\varphi \geq 0$ on Ω, and for $i = 1, 2$,

$$\int_\Omega \varphi |A_n \nabla w_n^i| d\mu_n \leq \|\varphi\|_{C(\overline{\Omega})} \left(\int_\Omega |A_n| d\mu_n \right)^{\frac{1}{2}} \left(\int_\Omega A_n \nabla w_n^i \cdot \nabla w_n^i \varphi \, d\mu_n \right)^{\frac{1}{2}}. \tag{3.35}$$

In particular, the sequences $A_n \nabla w_n^i d\mu_n$, $i = 1, 2$, are bounded in $\mathcal{M}(\Omega)^2$. So, extracting a new subsequence (the one which appears in the statement of Theorem 2.9) we can assume that

$$A_n \nabla w_n^i d\mu_n \rightharpoonup \nu^i \quad \text{weakly in } \mathcal{M}(\Omega)^2 * .$$

Using (2.7) and (2.10) in (3.35) we get that the measures ν^i are absolutely continuous with respect to μ. Hence, there exist $A^i \in L^1_\mu(\Omega)^{2\times 2}$ such that $\nu^i = A^i \, d\mu$. We then define A and P_n by

$$A(x)\lambda := A^1(x)\lambda_1 + A^2(x)\lambda_2, \quad \forall \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2, \ \mu\text{-a.e. } x \in \Omega, \tag{3.36}$$

$$P_n(x)\lambda := \nabla w_n^1(x)\lambda_1 + \nabla w_n^2(x)\lambda_2, \quad \forall \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2, \ \forall x \in \overline{\Omega}. \tag{3.37}$$

Third step: Proof of (2.24) and (2.25).

Since H_n defined by (3.31) is less or equal than F_n defined by (2.9), its Γ-limit H is also less or equal than the Γ-limit F of F_n (which exists up to a subsequence). Therefore, the linear mappings L_i, for $i = 1, 2$,

$$\begin{aligned} L_i : D(F) &\longrightarrow \mathbb{R} \\ v &\longmapsto \Phi_H(w^i, v) \end{aligned}$$

belong to $D(F)'$. So, by (3.34) and (3.35) the sequences $\xi_n = A_n \nabla w_n^i$ and $v_n = w_n^i$, $i = 1, 2$, satisfy the assumptions of Theorem 2.6. From (2.22) and the definition (3.36) of A, we then deduce that, for any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and any $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$,

$$\begin{aligned} &\lim_{n\to+\infty} \int_\Omega A_n \nabla\Big(\sum_{i=1}^2 \lambda_i w_n^i\Big) \cdot \nabla\Big(\sum_{i=1}^2 \lambda_i w_n^i\Big) \varphi \, d\mu_n \\ &\quad = \lim_{n\to+\infty} \sum_{i,j=1}^2 \int_\Omega A_n \nabla w_n^i \cdot \nabla w_n^j \lambda_i \lambda_j \varphi \, d\mu_n = \int_\Omega A\lambda \cdot \lambda\varphi \, d\mu. \end{aligned} \tag{3.38}$$

On the other hand, using the equi-coerciveness (2.6) we have

$$\int_\Omega A_n \nabla\Big(\sum_{i=1}^2 \lambda_i w_n^i\Big) \cdot \nabla\Big(\sum_{i=1}^2 \lambda_i w_n^i\Big) \varphi \, d\mu_n \geq \alpha \int_\Omega \Big|\sum_{i=1}^2 \lambda_i \nabla w_n^i\Big|^2 \varphi \, dx.$$

In particular, the sequence w_n^i which strongly converges in $L^2(\Omega)$, is bounded in $H^1(\Omega)$ and thus weakly converges in $H^1(\Omega)$ to w^i. So, we get

$$\liminf_{n\to+\infty} \int_\Omega \Big|\sum_{i=1}^2 \lambda_i \nabla w_n^i\Big|^2 \varphi \, dx \geq |\lambda|^2 \int_\Omega \varphi \, dx.$$

This combined with (3.38) implies that

$$\int_\Omega A\lambda \cdot \lambda\varphi \, d\mu \geq \alpha|\lambda|^2 \int_\Omega \varphi \, dx, \quad \forall \lambda \in \mathbb{R}^2, \ \forall \varphi \in C_c^\infty(\Omega), \ \varphi \geq 0,$$

which is equivalent to (2.24).

In order to prove (2.25), we use the Cauchy–Schwarz inequality and (2.8) which give

$$\begin{aligned} \left|\int_\Omega A_n \nabla\Big(\sum_{i=1}^2 \lambda_i w_n^i\Big) \varphi \, d\mu_n\right| &\leq \left(\int_\Omega \left|A_n \nabla\Big(\sum_{i=1}^2 \lambda_i w_n^i\Big)\right|^2 \varphi \, d\mu_n\right)^{\frac{1}{2}} \left(\int_\Omega \varphi \, d\mu_n\right)^{\frac{1}{2}} \\ &\leq \left(\beta \int_\Omega A_n \nabla\Big(\sum_{i=1}^2 \lambda_i w_n^i\Big) \cdot \nabla\Big(\sum_{i=1}^2 \lambda_i w_n^i\Big) \varphi \, d\mu_n\right)^{\frac{1}{2}} \left(\int_\Omega \varphi \, d\mu_n\right)^{\frac{1}{2}}, \end{aligned}$$

for any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and any $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$. Thus, using (3.38) and the convergence of $A_n \nabla\big(\sum_{i=1}^2 \lambda_i w_n^i\big) d\mu_n$ and μ_n respectively to $A\lambda \, d\mu$ and μ in the weak-

$*$ sense of the measures in $\mathcal{M}(\Omega)^2$, and passing to the limit as $n \to +\infty$ in the above inequality, we obtain

$$\left| \int_\Omega A\lambda\varphi\, d\mu \right| \leq \left(\beta \int_\Omega A\lambda \cdot \lambda\varphi\, d\mu \right)^{\frac{1}{2}} \left(\int_\Omega \varphi\, d\mu \right)^{\frac{1}{2}},$$

for any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and any $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$. This implies that, for any closed ball $\overline{B}(x_0, r) \subset \Omega$ and any $\lambda \in \mathbb{R}^2$, we have

$$\left| \int_{\overline{B}(x_0,r)} A\lambda\, d\mu \right| \leq \left(\beta \int_{\overline{B}(x_0,r)} A\lambda \cdot \lambda\, d\mu \right)^{\frac{1}{2}} \mu\big(\overline{B}(x_0, r)\big)^{\frac{1}{2}},$$

or equivalently, for $\mu(\overline{B}(x_0, r)) > 0$,

$$\left| \fint_{\overline{B}(x_0,r)} A\lambda\, d\mu \right| \leq \left(\beta \fint_{\overline{B}(x_0,r)} A\lambda \cdot \lambda\, d\mu \right)^{\frac{1}{2}}.$$

Assuming that x_0 is a Lebesgue point of A with respect to μ (which holds for μ-a.e. $x_0 \in \Omega$) and letting r tend to 0 in the above inequality, we get

$$|A(x)\lambda| \leq \big(\beta A(x)\lambda \cdot \lambda\big)^{\frac{1}{2}}, \quad \forall \lambda \in \mathbb{R}^2, \ \mu\text{-a.e. } x \in \Omega.$$

By Remark 2.5 this proves (2.25).

Fourth step: Proof of (2.27), (2.29), and (2.30).

Let ω be an open subset of Ω and let G_n be defined by (2.26). Up to an extraction of a subsequence we can assume that G_n Γ-converges to a functional G. Consider $u \in C_0^1(\omega)$ and a recovery sequence $u_n \in C_0^1(\omega)$ such that

$$u_n \longrightarrow u \quad \text{strongly in } L^2(\Omega) \quad \text{and} \quad G(u) = \lim_{n\to+\infty} G_n(u_n) < \infty. \tag{3.39}$$

Denote by $\sigma\, d\mu$ the weak-$*$ limit of $A_n \nabla u_n\, d\mu_n$ in $\mathcal{M}(\omega)^2$, which holds up to a subsequence by Remark 2.13. Takings into account the results of the third step and Properties 1.2e) combined with (3.39), we may apply Theorem 2.6 (Ω being replaced by ω) with the choice $\xi_n = A_n \nabla u_n$ or $\xi_n = A_n \nabla w_n^i$, and $v_n = u_n$ or $v_n = w_n^i$, $i = 1, 2$. Then, from (2.22) we deduce that, for any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and any $\varphi \in C_c^\infty(\omega)$, $\varphi \geq 0$,

$$\lim_{n\to+\infty} \int_\omega A_n \nabla\left(u_n - \sum_{i=1}^2 \lambda_i w_n^i \right) \cdot \nabla\left(u_n - \sum_{i=1}^2 \lambda_i w_n^i \right) \varphi\, d\mu_n = \int_\omega (\sigma - A\lambda) \cdot (\nabla u - \lambda)\varphi\, d\mu, \tag{3.40}$$

which implies that

$$(\sigma(x) - A(x)\lambda) \cdot (\nabla u(x) - \lambda) \geq 0, \quad \forall \lambda \in \mathbb{R}^2, \ \forall x \in \omega \backslash N,$$

where N is a zero μ-measure subset of ω. For each $x \in \omega \backslash N$, setting $\lambda := \nabla u(x) + t\eta$, $t \in \mathbb{R}$, $\eta \in \mathbb{R}^2$, in the previous inequality yields

$$-t\big(\sigma(x) - A(x)(\nabla u(x) + t\eta)\big) \cdot \eta \geq 0.$$

To make use of the above inequality it is convenient to consider separately the cases $t > 0$ and $t < 0$. Then, dividing by t and letting t tend to 0 we get

$$\sigma(x) = A(x)\nabla u(x) \quad \mu\text{-a.e. } x \in \omega, \tag{3.41}$$

which implies convergence (2.29). Since $G_n(u_n - u)$ is bounded (by (3.39) and $u \in C_0^1(\omega)$) and $u_n - u$ strongly converges to zero in $L^2(\Omega)$, the Properties 1.2e) of the Γ-convergence implies that

$$\lim_{n\to+\infty} \int_\omega A_n \nabla u_n \cdot \nabla(u_n - u) d\mu_n = 0,$$

which combined with (3.41) and (3.39) yields

$$\begin{aligned} \int_\omega A\nabla u \cdot \nabla u \, d\mu &= \lim_{n\to+\infty} \int_\omega A_n \nabla u_n \cdot \nabla u \, d\mu_n \\ &= \lim_{n\to+\infty} \int_\omega A_n \nabla u_n \cdot \nabla u_n \, d\mu_n = G(u). \end{aligned}$$

This proves (2.27).

It thus remains to prove (2.30). The definition (3.37) of P_n yields

$$\begin{aligned} &\int_\omega A_n(\nabla u_n - P_n \nabla u) \cdot (\nabla u_n - P_n \nabla u) d\mu_n \\ &= \int_\omega A_n\left(\nabla u_n - \sum_{i=1}^2 \frac{\partial u}{\partial x_i} \nabla w_n^i\right) \cdot \left(\nabla u_n - \sum_{i=1}^2 \frac{\partial u}{\partial x_i} \nabla w_n^i\right) d\mu_n. \end{aligned} \tag{3.42}$$

Since the sequences u_n and w_n^i, $i = 1, 2$, have bounded energy by (3.39) and (3.33), the convergences of $A_n \nabla u_n \cdot \nabla w_n^i d\mu_n$ and $A_n \nabla w_n^i \cdot \nabla w_n^j d\mu_n$, $i, j \in \{1, 2\}$, which hold in the distributions sense (by the same arguments used to prove (3.40)), also hold in $\mathcal{M}(\omega)*$, hence

$$\begin{cases} A_n \nabla u_n \cdot \nabla w_n^i d\mu_n \rightharpoonup A\nabla u \cdot \nabla w^i d\mu \\ A_n \nabla w_n^i \cdot \nabla w_n^j \, d\mu_n \rightharpoonup A\nabla w^i \cdot \nabla w^j \, d\mu \quad \text{weakly in } \mathcal{M}(\omega) * . \end{cases}$$

These convergences combined with the fact that the partial derivatives of u are suitable test-functions in $C_0(\omega)$, and (2.27), (2.28) imply that the sequence (3.42) tends to zero, hence (2.30).

3.2.2. Proof of Theorem 2.14. We will follow the scheme of the proof of the Murat-Tartar H-convergence. However, the unboundedness of A_n induces extra technical difficulties in comparison to the classical proof (in particular in the non symmetric case) which are detailed below.

The proof is divided in four steps. In the first step, we prove the convergence (up to a subsequence) of the sequence of operators $\operatorname{div}\big(A_n \nabla \cdot\big)$. In the second step, we construct the homogenized matrix-valued function A. In the third step, we prove, thanks to the div-curl lemma (Theorem 2.1), that the limit operator is of type $\operatorname{div}(A\nabla\cdot)$. In the fourth step, we prove that A satisfies (2.34). Finally, we refer to Murat (1998) for the proof of the corrector result (2.40) which is quite similar to the classical one.

First step: Convergence of the sequence $-\operatorname{div}(A_n\nabla\cdot)$.

Let $\mathscr{A}_n : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ be the linear operator defined by $\mathscr{A}_n := -\operatorname{div}(A_n\nabla\cdot)$ and let $\mathscr{B}_n := \mathscr{A}_n^{-1}$ be its inverse. Thanks to the α-coerciveness of A_n and to a diagonalization procedure using the separability of the space $H^{-1}(\Omega)$ (see Murat, 1998 for details), there exist a subsequence, still denoted by n, and a bounded linear operator $\mathscr{B} : H^{-1}(\Omega) \longrightarrow H_0^1(\Omega)$ such that

$$\forall f \in H^{-1}(\Omega), \quad \mathscr{B}_n f \rightharpoonup \mathscr{B} f \text{ weakly in } H_0^1(\Omega). \tag{3.43}$$

Let us prove that $\mathscr{B}$ is invertible. Let v_n be a sequence in $H^1(\Omega)$ and let $\Phi \in C_c^\infty(\Omega)^2$. By (2.32) and the Cauchy–Schwarz inequality we have

$$\begin{aligned}
\left|\int_\Omega A_n\nabla v_n \cdot \Phi\, dx\right| &\le \left|\int_\Omega A_n^s\nabla v_n \cdot \Phi\, dx\right| + \left|\int_\Omega a_n J\nabla v_n \cdot \Phi\, dx\right| \\
&\le \left(\int_\Omega |A_n^s||\Phi|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega A_n^s\nabla v_n \cdot \nabla v_n dx\right)^{\frac{1}{2}} \\
&\quad + \left(\int_\Omega |a_n||\Phi|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |a_n||\nabla v_n|^2 dx\right)^{\frac{1}{2}} \\
&\le (1+\gamma)\left(\int_\Omega |A_n^s||\Phi|^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega A_n\nabla v_n \cdot \nabla v_n\, dx\right)^{\frac{1}{2}}.
\end{aligned}$$

This combined with convergence (2.33) yields

$$\limsup_{n\to+\infty}\left|\int_\Omega A_n\nabla v_n \cdot \Phi\, dx\right| \le \sqrt{\beta}\|\Phi\|_{L^2(\Omega)^2} \limsup_{n\to+\infty}\left(\int_\Omega A_n\nabla v_n \cdot \nabla v_n dx\right)^{\frac{1}{2}},$$
$$\text{where } \beta := (1+\gamma)^2\, \|\bar{a}\|_{L^\infty(\Omega)}. \tag{3.44}$$

Let $f \in H^{-1}(\Omega)$. Applying inequality (3.44) to $v_n := \mathscr{B}_n f$ and $\Phi := \nabla\varphi$, for $\varphi \in C_c^\infty(\Omega)$, and using convergence (3.43) it follows

$$\begin{aligned}
|\langle f, \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}| &= \limsup_{n\to+\infty}\left|\int_\Omega A_n\nabla v_n \cdot \nabla\varphi\, dx\right| \\
&\le \sqrt{\beta}\|\nabla\varphi\|_{L^2(\Omega)^2} \limsup_{n\to+\infty}(\langle f, v_n\rangle_{H^{-1}(\Omega),H_0^1(\Omega)})^{\frac{1}{2}} \\
&= \sqrt{\beta}\|\varphi\|_{H_0^1(\Omega)}\big(\langle f, \mathscr{B} f\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\big)^{\frac{1}{2}},
\end{aligned}$$

which implies the inequality

$$\langle f, \mathscr{B} f\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \ge \beta^{-1}\|f\|^2_{H^{-1}(\Omega)}.$$

Therefore, the operator $\mathscr{B}$ is β^{-1}-coercive and by the Lax–Milgram theorem $\mathscr{B}$ is invertible. The inverse operator $\mathscr{A} := \mathscr{B}^{-1}$ satisfies $\|\mathscr{A}\| \leq \beta$.

Second step: Construction of the homogenized matrix A.

Let $\widetilde{\Omega}$ be an open subset of $\mathbb{R}^2$ such that $\Omega \Subset \widetilde{\Omega}$. Let $\widetilde{A}_n$ be the matrix-valued function defined by

$$\widetilde{A}_n := \begin{cases} A_n & \text{in } \Omega \\ \alpha I & \text{in } \widetilde{\Omega}\backslash\Omega. \end{cases} \tag{3.45}$$

Let $\widetilde{\mathscr{A}}_n : H_0^1(\widetilde{\Omega}) \longrightarrow H^{-1}(\widetilde{\Omega})$ be the operator defined by $\widetilde{\mathscr{A}}_n := -\text{div}\big(\widetilde{A}_n^t \nabla \cdot\big)$ and let $\widetilde{\mathscr{B}}_n$ be its inverse. Since $\partial\Omega$ has a zero Lebesgue measure, by (2.33) the sequence $|(\widetilde{A}_n^t)^s| = |\widetilde{A}_n^s|$ weakly converges in $\mathscr{M}(\widetilde{\Omega})*$ to the function $\tilde{a}$ defined by

$$\tilde{a} := \begin{cases} \bar{a} & \text{in } \Omega \\ \alpha & \text{in } \widetilde{\Omega}\backslash\Omega. \end{cases}$$

Therefore, the first step implies that the sequence $\widetilde{\mathscr{B}}_n$ converges in the sense of (3.43) to an operator $\widetilde{\mathscr{B}}$ satisfying $\|\widetilde{\mathscr{B}}\| \leq \alpha^{-1}$. Moreover, $\widetilde{\mathscr{A}} := \widetilde{\mathscr{B}}^{-1}$ is a bounded operator satisfying $\|\widetilde{\mathscr{A}}\| \leq \beta$, where β is defined in (3.44).

Let θ be a fixed cut-off function in $C_c^\infty(\widetilde{\Omega})$ such that $\theta = 1$ in Ω. Let $\tilde{w}_n^i$, $i = 1, 2$, be the function in $H_0^1(\widetilde{\Omega})$ defined by

$$\tilde{w}_n^i = \widetilde{\mathscr{B}}_n \circ \widetilde{\mathscr{A}}\big(\theta(x)x_i\big) \quad \text{or} \quad \text{div}\big(\widetilde{A}_n^t \nabla \tilde{w}_n^i\big) = \widetilde{\mathscr{A}}\big(\theta(x)x_i\big) \text{ in } \mathscr{D}'(\widetilde{\Omega}). \tag{3.46}$$

Since the sequence $\widetilde{\mathscr{B}}_n$ converges to $\widetilde{\mathscr{A}}^{-1}$ in the sense of (3.43), $\tilde{w}_n^i$ satisfies (up to a subsequence)

$$\tilde{w}_n^i \rightharpoonup \theta(x)x_i \quad \text{weakly in } H_0^1(\widetilde{\Omega}) \quad \text{and} \quad \int_{\widetilde{\Omega}} \widetilde{A}_n^t \nabla \tilde{w}_n^i \cdot \nabla \tilde{w}_n^i dx \leq c. \tag{3.47}$$

By the estimate (3.44) with $\widetilde{A}_n^t$, combined with (3.47) the sequence $\widetilde{A}_n^t \nabla \tilde{w}_n^i$ weakly converges (up to a subsequence) in $\mathscr{M}(\widetilde{\Omega})*$ to a Radon measure $\tilde{\xi}^i$ satisfying

$$\left| \int_{\widetilde{\Omega}} \tilde{\xi}^i(dx) \cdot \Phi \right| \leq c\|\Phi\|_{L^2(\Omega)^2}, \quad \text{for any } \Phi \in C_c^\infty(\Omega)^2,$$

hence $\tilde{\xi}^i$ is actually a function in $L^2(\widetilde{\Omega})^2$. Then, we can define the matrix-valued A in $L^2(\Omega)^{2\times 2}$ by the convergence

$$\widetilde{A}_n^t \nabla \tilde{w}_n^i \rightharpoonup \tilde{\xi}^i = A^t e_i \quad \text{weakly in } \mathscr{M}(\Omega)^2 * . \tag{3.48}$$

Third step: Determination of the limit operator $\mathscr{A}$.

Let $f \in H^{-1}(\Omega)$ and let $u_n := \mathscr{B}_n f$. As for $\tilde{w}_n^i$ in the second step, up to a subsequence, the sequence u_n weakly converges to some u in $H_0^1(\Omega)$ and the sequence $A_n \nabla u_n$ weakly converges to some $\xi \in L^2(\Omega)^2$ in $\mathscr{M}(\Omega)^2*$. The control (2.32) of the antisymmetric part of A_n by its symmetric part and the equality $\det A_n = \det A_n^s + a_n^2$, yield

$$(A_n^s)^{-1} = \frac{\det A_n}{\det A_n^s}(A_n^{-1})^s = \left(1 + \frac{a_n^2}{\det A_n^s}\right)(A_n^{-1})^s \le (1+\gamma^2)(A_n^{-1})^s,$$

and similarly for the transposed matrix,

$$(A_n^s)^{-1} = \big((A_n^t)^s\big)^{-1} \le (1+\gamma^2)\big((A_n^t)^{-1}\big)^s.$$

This implies that

$$\begin{cases} (A_n^s)^{-1}\xi\cdot\xi \le (1+\gamma^2)A_n^{-1}\xi\cdot\xi \\ (A_n^s)^{-1}\xi\cdot\xi \le (1+\gamma^2)(A_n^t)^{-1}\xi\cdot\xi \end{cases} \quad \text{for any } \xi \in L^1(\Omega)^2. \tag{3.49}$$

Then, by estimates (3.47) and (3.49) the sequences $\xi_n = A_n\nabla u_n$ (respectively $\xi_n = A_n^t\nabla\tilde{w}_n^i$) and $v_n = \tilde{w}_n^i$ (respectively $v_n = u_n$) satisfy the assumptions (2.2) and (2.3) of the div-curl lemma (Theorem 2.1) with the symmetric matrix-valued function A_n^s. Then, from the convergence (2.5) of Theorem 2.1 we deduce that

$$A_n\nabla u_n\cdot\nabla\tilde{w}_n^i \rightharpoonup \xi\cdot e_i \quad\text{and}\quad A_n^t\nabla\tilde{w}_n^i\cdot\nabla u_n \rightharpoonup A^te_i\cdot\nabla u \quad \text{in } \mathscr{D}'(\Omega),$$

hence $\xi = A\nabla u$. Therefore, the weak convergences (2.36) hold true for the whole subsequence defining A in (3.48). We have also established that $\mathscr{A} = -\mathrm{div}\big(A\nabla\cdot\big)$.

Fourth step: Proof of (2.34).

Let $u \in C_c^\infty(\Omega)$ and let u_n be the sequence in $H_0^1(\Omega)$ defined $u_n := \mathscr{B}_n \circ \mathscr{A}(u)$, which weakly converges to u in $H_0^1(\Omega)$. On the one hand, the α-coerciveness of A_n yields

$$\int_\Omega \alpha|\nabla u_n|^2dx \le \int_\Omega A_n\nabla u_n\cdot\nabla u_n dx = \int_\Omega A\nabla u\cdot\nabla u_n\,dx.$$

Therefore, using the lower semicontinuity of the $L^2(\Omega)$-norm we get

$$\int_\Omega A\nabla u\cdot\nabla u\,dx \ge \int_\Omega \alpha|\nabla u|^2dx, \quad \text{for any } u \in C_c^\infty(\Omega).$$

Taking (as in the proof of Lemma 22.5, p. 234 of Dal Maso, 1993) the test-functions $\varphi(x)\cos(t\lambda\cdot x)$ and $\varphi(x)\sin(t\lambda\cdot x)$, for arbitrary $\varphi \in C_c^\infty(\Omega)$, $\lambda \in \mathbb{R}^2$ and $t > 0$, in the previous inequality we obtain that $A\lambda\cdot\lambda \ge \alpha|\lambda|^2$ a.e. in Ω.

On the other hand, estimate (3.44) combined with the weak convergence (2.36) of $A_n\nabla u_n$ yields, for any $\Phi \in C_c^\infty(\Omega)^2$,

$$\left|\int_\Omega A\nabla u\cdot\Phi\,dx\right| \le \sqrt{\beta}\|\Phi\|_{L^2(\Omega)^2}\left(\int_\Omega A\nabla u\cdot\nabla u\,dx\right)^{\frac{1}{2}}.$$

This inequality also holds true for any $\Phi \in L^2(\Omega)^2$, thanks to the density of $C_c^\infty(\Omega)^2$ in $L^2(\Omega)^2$. Then, setting $\Phi := A\nabla u$ (recall that $A \in L^2(\Omega)^{2\times 2}$) in the previous inequality we get

$$\int_\Omega |A\nabla u|^2 dx \leq \int_\Omega \beta A\nabla u \cdot \nabla u \, dx, \quad \text{for any } u \in C_c^\infty(\Omega),$$

which implies that (again using the former test-functions)

$$|A\lambda|^2 \leq \beta A\lambda \cdot \lambda \text{ a.e. in } \Omega, \quad \text{for any } \lambda \in \mathbb{R}^2.$$

From this we deduce that $A^{-1} \geq \beta^{-1} I$ a.e. in Ω.

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