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Uniform convergence of sequences of solutions of two-dimensional linear elliptic equations with unbounded coefficients

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ABSTRACT

This paper deals with the behavior of two-dimensional linear elliptic equations with unbounded (and possibly infinite) coefficients. We prove the uniform convergence of the solutions by truncating the coefficients and using a pointwise estimate of the solutions combined with a two-dimensional capacitary estimate. We give two applications of this result: the continuity of the solutions of two-dimensional linear elliptic equations by a constructive approach, and the density of the continuous functions in the domain of the Γ-limit of equicoercive diffusion energies in dimension two. We also build two counter-examples which show that the previous results cannot be extended to dimension three.

1. Introduction

In this paper we study the behavior of degenerate linear elliptic equations posed in a bounded open subset Ω of $\mathbb{R}^N$, especially in the case $N=2$, of type

$$-\operatorname{div}(A\nabla u)=f \quad \text{in } \Omega, \tag{1.1}$$

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where the right-hand side f belongs to $H^{-1}(\Omega)$, and A is a coercive but not necessarily bounded matrix-valued function. Indeed, the quadratic form relating to A can even take infinite values (see Definition 2.1). Therefore, the solutions of (1.1) will be understood in the sense

$$\begin{cases} \displaystyle\int_\Omega A\nabla u \cdot \nabla u\, dx < \infty, \\ \displaystyle\int_\Omega A\nabla u \cdot \nabla v\, dx = \langle f, v\rangle, \quad \forall v \in H_0^1(\Omega) \text{ with } \int_\Omega A\nabla v \cdot \nabla v\, dx < \infty. \end{cases} \tag{1.2}$$

We will prove (see Proposition 2.3) that the solutions of this problem can be obtained as the limits of solutions of coercive problems with bounded coefficients, using some truncation of the matrix-valued A. In particular, this allows us to extend some classical properties of the solutions of linear elliptic equations with coercive and bounded operators, such as the maximum principle (see Theorem 2.4). The main results of the paper refer to the compactness of the solutions of (1.1) for the uniform convergence in dimension two.

Recall that for any bounded open subset Ω of $\mathbb{R}^N$, $N \geqslant 1$, any coercive matrix-valued function $A \in L^\infty(\Omega)^{N\times N}$, and any f in $W^{-1,q}(\Omega)$ with $q > 2$, the solutions of (1.1) are Hölder-continuous in Ω (see e.g. [14,18,22]). As a consequence, if u_n is a bounded sequence in $H^1(\Omega)$ of solutions of equations

$$-\operatorname{div}(A_n\nabla u_n) = f_n \quad \text{in } \Omega, \tag{1.3}$$

where f_n is bounded in $W^{-1,q}(\Omega)$ for some $q > 2$, and A_n is uniformly coercive and bounded in $L^\infty(\Omega)^{N\times N}$, then u_n is compact in $C^0(\Omega)$, i.e. uniformly convergent in any compact set of Ω. In the two-dimensional case the solutions of (1.3) are still continuous even if the diffusion matrix A_n is not bounded from above (see e.g. [15,17]). In general they are no longer Hölder-continuous (see Example 4.6), but we show in the present paper that the former uniform convergence result does subsist. More precisely, we prove (see Theorems 2.5 and 2.7) that the compactness of u_n in $C^0(\Omega)$ still holds true without assuming any bound from above on the equicoercive sequence A_n. Moreover, if Ω is Lipschitz and u_n is compact in $C^0(\partial\Omega)$, then the sequence u_n is compact in $C^0(\bar{\Omega})$.

The previous uniform convergence results are applied in two directions. On the one hand, we give an alternative proof (see Theorem 4.1) of the continuity of the solutions of (1.3) using the approximation by truncation of the matrix-valued A_n combined with the uniform convergence of the solutions of the equations with truncated coefficients (which are known to be continuous).

On the other hand, the asymptotic behavior of sequences of solutions of (1.3) is strongly connected to the homogenization theory. The homogenization of sequences of elliptic equations has formed the subject of several works since the end of sixties. Assuming the uniformly boundedness of the coefficients Spagnolo [24] and Murat and Tartar [21,25] proved that the limit problem of (1.3) has the same structure. These results were extended in [10,12] and [20] under the weaker assumption of the L^1-equiintegrability, as well as in the periodic case [5] under the control of a weighted Poincaré–Wirtinger inequality. Moreover, Fenchenko and Khruslov [16] proved that only L^1-bounded coefficients may induce nonlocal effects in dimension three through jumping measures which modify the nature of the limit problem (also see the recent book [19] on the topic and another approaches in [1,4,9,11,20]). Mosco [20] showed the nonlocal terms arise naturally in the limit process using the Beurling–Deny [2] representation of the Dirichlet forms. Completing the previous work Camar-Eddine and Seppecher [11] proved that in dimension three any jumping measure can be obtained by the homogenization of a suitable equicoercive sequence of conductivity matrices A_n. Recently, we showed first in the periodic case [6], then in the general case [7,8], that, contrary to the dimension three, the dimension two prevents from the appearance of nonlocal effects. In [7] we proved the compactness of Eqs. (1.3) assuming the L^1-boundedness of A_n and using two-dimensional div–curl lemmas. In [8] we extended the previous result without assuming any bound from above on A_n.

The uniform convergence results of the present paper allow us to complete and improve the results of [8]. More precisely, the approach of [8] is based on a (exclusively) two-dimensional capacitary estimate (see Lemma 3.2). This tool is now combined with two extra ingredients: the truncation principle mentioned above and a pointwise estimate satisfied by the solutions of linear elliptic equations (see Theorem 2.4) which extends a classical one (see e.g. Theorem 8.16 of [18]). The combination of these three ingredients yields the uniform convergence of the sequence u_n of solutions of (1.3) without assuming a priori the continuity of its limit as in [8]. As a consequence of this approach, we prove (see Theorem 4.3) that the continuous functions are dense in the domain (endowed with the intrinsic norm) of the Γ-limit of any equicoercive sequence of linear diffusion energies in dimension two, which is a new contribution in the topic up to our knowledge.

We conclude the paper with two counter-examples illustrating the gap between the dimension two and the higher one, concerning the former uniform convergence results. First, we give an example (see Proposition 4.7) of a sequence of solutions of three-dimensional Dirichlet problems, which does not converges uniformly. Then, we construct (see Proposition 4.9) a discontinuous solution of a three-dimensional linear elliptic equation with unbounded coefficients. The two counter-examples are based on fibers reinforced structures which were first used in [16] to derive nonlocal effects in homogenization.

2. Statement of the results

2.1. A pointwise estimate of solutions of linear elliptic equations

We start the paper by giving a consistent definition of unbounded non-symmetric matrix-valued functions, with possibly infinite values:

Definition 2.1. Let $\alpha > 0$ and let Ω be a bounded open subset of $\mathbb{R}^N$. Let $\mathcal{A}(\alpha, \Omega)$ be the set of the matrix-valued functions A such that there exist $P \in L^\infty(\Omega)^{N\times N}$ with values on the set of orthogonal matrices, N measurable functions $d_1, \dots, d_N : \Omega \to [\alpha, \infty]$, a constant $\beta > 0$, and a measurable skew-symmetric matrix-valued function $B : \Omega \to \mathbb{R}^{N\times N}$, with

$$|B|_\infty := \max_{1\leqslant i,j\leqslant N} |B_{ij}| \leqslant \beta \min\{d_1, \dots, d_N\}, \quad \text{a.e. in } \Omega,$$

which satisfy for a.e. $x \in \Omega$,

$$A(x)\xi = P(x)^t D(x)P(x)\xi + B(x)\xi, \quad \forall \xi \in V_x := \left\{\xi \in \mathbb{R}^N\colon \ \left|DP(x)\xi\right| < \infty\right\}, \tag{2.1}$$

where D is the diagonal matrix-valued function $\operatorname{diag}(d_1, \dots, d_N)$.

Remark 2.2. Definition 2.1 allows us to give a sense to $A(x)\xi \cdot \xi$, for a.e. $x \in \Omega$ and any $\xi \in \mathbb{R}^N$, by

$$A(x)\xi \cdot \xi := \begin{cases} D(x)P(x)\xi \cdot P(x)\xi & \text{if } \xi \in V_x, \\ \infty & \text{if } \xi \in \mathbb{R}^2 \setminus V_x. \end{cases} \tag{2.2}$$

In the particular case of symmetric matrices this leads us to the following extensions of classical definitions:

- For any $A, B \in \mathcal{A}(\alpha, \Omega)$, $A \leqslant B$ means that

$$A(x)\xi \cdot \xi \leqslant B(x)\xi \cdot \xi \leqslant \infty, \quad \text{for a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^2. \tag{2.3}$$

- A sequence $A_n \in \mathcal{A}(\alpha, \Omega)$ is said to converge a.e. to $A \in \mathcal{A}(\alpha, \Omega)$ if

$$A(x)\xi \cdot \xi = \lim_{n\to\infty} A_n(x)\xi \cdot \xi \in [0, \infty], \quad \text{for a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^2. \tag{2.4}$$

Let $A \in \mathcal{A}(\alpha, \Omega)$ and denote by A^s its symmetric part. The conditions $d_1, \ldots, d_N : \Omega \to [\alpha, \infty]$ and $|B|_\infty \leqslant \beta \min\{d_1, \ldots, d_N\}$ a.e. in Ω, read as, according to (2.3),

$$\alpha I \leqslant A^s \quad \text{and} \quad |B|_\infty I \leqslant \beta A^s, \quad \text{a.e. in } \Omega, \tag{2.5}$$

where I is the unit matrix of $\mathbb{R}^{2\times 2}$.

Denote by T_κ, for $\kappa \geqslant 0$, the truncation at size κ, i.e. $T_\kappa(t) := \min(\kappa, \max(-\kappa, t))$ for any $t \in \mathbb{R}$. Then, we have the following truncation result:

Proposition 2.3. *Let A be an element of $\mathcal{A}(\alpha, \Omega)$, and consider P, D, a, β as in Definition* 2.1. *For $n \geqslant \alpha$, let D_n be the diagonal matrix-valued function with entries $T_n(d_1), \ldots, T_n(d_N)$, let B_n be the skew-symmetric matrix-valued function with entries $(B_n)_{ij} := T_{n\beta}(B_{ij})$, and let A_n be the element of $\mathcal{A}(\alpha, \Omega)$ defined by* (2.1) *with P, D_n and B_n. Then, for any $f \in H^{-1}(\Omega)$ and any $u \in H^1(\Omega)$ which satisfies* (1.2)*, the solution u_n of*

$$\begin{cases} -\operatorname{div}(A_n \nabla u_n) = f & \text{in } \Omega, \\ u_n - u \in H_0^1(\Omega), \end{cases} \tag{2.6}$$

converges strongly to u in $H^1(\Omega)$. Moreover, we have

$$\lim_{n\to\infty} \int_\Omega A_n \nabla(u_n - u) \cdot \nabla(u_n - u)\, dx = 0. \tag{2.7}$$

A consequence of this truncation result is the following pointwise estimate satisfied by the solutions of (1.2):

Theorem 2.4. *For any $\alpha > 0$ and any $q > 2$, there exists a constant $C > 0$ with the following property*:

For any bounded open subset Ω of $\mathbb{R}^N$, $A \in \mathcal{A}(\alpha, \Omega)$, $f \in W^{-1,q}(\Omega)$, $q > 2$, $m, M \in \mathbb{R}$ with $m < M$, and any $u \in H^1(\Omega)$ satisfying (1.2) *with $(m - u)^+, (u - M)^+ \in H_0^1(\Omega)$, we have*

$$m - C|\Omega|^{\frac{Nq-N-q}{Nq}} \|f\|_{W^{-1,q}(\Omega)} \leqslant u \leqslant M + C|\Omega|^{\frac{Nq-N-q}{Nq}} \|f\|_{W^{-1,q}(\Omega)}, \quad \text{a.e. in } \Omega. \tag{2.8}$$

2.2. Uniform convergence results

The main results of the paper are the following compactness result and its refinement in Theorem 2.7:

Theorem 2.5. *Let $\alpha > 0$, let Ω be a bounded open subset of $\mathbb{R}^2$, with a Lipschitz boundary, and let Γ be a relatively open subset of $\partial\Omega$. Consider a sequence A_n in $\mathcal{A}(\alpha, \Omega)$, a sequence u_n in $H^1(\Omega) \cap C^0(\Omega \cup \Gamma)$ which converges weakly in $W^{1,p}(\Omega)$, with $p \in (1, 2]$, and uniformly in the closed subsets of Γ to a function $u \in W^{1,p}(\Omega) \cap C^0(\Omega \cup \Gamma)$, and consider a sequence f_n in $W^{-1,q}(\Omega)$, with $q > 2$. Assume that A_n, u_n and f_n satisfy the following conditions*:

$$\forall n \in \mathbb{N}, \quad \begin{cases} \displaystyle\int_\Omega A_n \nabla u_n \cdot \nabla u_n\, dx < \infty, \\ \displaystyle\int_\Omega A_n \nabla u_n \cdot \nabla v\, dx = \langle f_n, v\rangle, \quad \forall v \in H_0^1(\Omega) \text{ with } \int_\Omega A_n \nabla v \cdot \nabla v\, dx < \infty. \end{cases} \tag{2.9}$$

Then, the sequence u_n converges to u in $C^0(\Omega \cup \Gamma)$, i.e. uniformly in any compact set of $\Omega \cup \Gamma$.

We introduce the following notations:

- We denote by $B(x,r)$ the open ball of center $x\in\mathbb{R}^2$ and of radius $r>0$, by $\bar B(x,r)$ its closure, and the corresponding half-balls by

$$B(x,r)^+ := B(x,r)\cap\{y\in\mathbb{R}^2\colon\ y_2>x_2\}\quad\text{and}\quad \bar B(x,r)^+ := \bar B(x,r)\cap\{y\in\mathbb{R}^2\colon\ y_2\geqslant x_2\}.$$

- For any bounded continuous function u in a subset E of $\mathbb{R}^2$, we set

$$\operatorname*{osc}_E u := \sup_E u - \inf_E u.$$

As a consequence of Theorem 2.7, we have the following local estimates for the solutions of (1.2):

Corollary 2.6. *Let $\alpha,\varepsilon>0$, $p>1$ and $q>2$. Then, there exists a constant $C>0$ such that the two following assertions hold*:

(i) *For any $x_0\in\mathbb{R}^2$, $\delta>0$, $A\in\mathcal{A}(\alpha,B(x_0,2\delta))$, $f\in W^{-1,q}(B(x_0,2\delta))$, and any function u in $H^1(B(x_0,2\delta))\cap C^0(B(x_0,2\delta))$ satisfying* (1.2) *with $\Omega=B(x_0,2\delta)$, we have*

$$\operatorname*{osc}_{\bar B(x_0,\delta)} u\leqslant \varepsilon\big(\delta^{\frac{q-2}{q}}\|f\|_{W^{-1,q}(B(x_0,2\delta))}+\delta^{\frac{p-2}{p}}\|\nabla u\|_{L^p(B(x_0,2\delta))^2}\big)$$
$$+\frac{C}{\delta^2}\bigg\|u-\frac{1}{4\pi\delta^2}\int\limits_{B(x_0,2\delta)}u\,dx\bigg\|_{L^1(B(x_0,2\delta))}. \tag{2.10}$$

(ii) *For any $x_0\in\mathbb{R}^2$, $\delta>0$, $A\in\mathcal{A}(\alpha,B(x_0,2\delta)^+)$, $f\in W^{-1,q}(B(x_0,2\delta)^+)$, and any function u in $H^1(B(x_0,2\delta)^+)\cap C^0(\bar B(x_0,2\delta)^+)$ satisfying* (1.2) *with $\Omega=B(x_0,2\delta)^+$, we have*

$$\operatorname*{osc}_{\bar B(x_0,\delta)^+} u\leqslant \varepsilon\big(\delta^{\frac{q-2}{q}}\|f\|_{W^{-1,q}(B(x_0,2\delta)^+)}+\delta^{\frac{p-2}{p}}\|\nabla u\|_{L^p(B(x_0,2\delta)^+)^2}\big)$$
$$+C\Bigg(\frac{1}{\delta^2}\bigg\|u-\frac{1}{2\delta}\int_{-\delta}^{\delta}u(t,0)\,dt\bigg\|_{L^1(B(x_0,2\delta)^+)}+\operatorname*{osc}_{[-2\delta,2\delta]\times\{0\}}u\Bigg). \tag{2.11}$$

Using Corollary 2.6 the following result is a refinement of Theorem 2.5 in the case $p=2$, without assuming the continuity of the limit:

Theorem 2.7. *Let $\alpha>0$, let Ω be a bounded open subset of $\mathbb{R}^2$, with a Lipschitz boundary, and let Γ be a relatively open subset of $\partial\Omega$. Consider a sequence A_n in $\mathcal{A}(\alpha,\Omega)$, a sequence u_n in $H^1(\Omega)\cap C^0(\Omega\cup\Gamma)$ converging weakly in $H^1(\Omega)$ and uniformly in the closed subsets of Γ to a function u, and consider a bounded sequence f_n in $W^{-1,q}(\Omega)$, with $q>2$. Assume that A_n, u_n and f_n satisfy* (2.9)*. Then, the sequence u_n converges to u in $C^0(\Omega\cup\Gamma)$.*

Note that in Theorem 2.5 the continuity of the limit cannot be removed when $p<2$, as shown in Example 4.5 below.

3. Proof of the results

Proof of Proposition 2.3. First of all, note that the symmetric part A_n^s of A_n is a nondecreasing sequence according to (2.3), which satisfies $A_n^s \geqslant \alpha I$ by (2.5), and A_n^s converges to the symmetric part A^s of A according to (2.4). Taking $u_n - u$ as test function in (2.6) we get that there exists $C > 0$ with

$$\int_\Omega A_n \nabla u_n \cdot \nabla u_n \, dx \leqslant C, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Due to the α-coerciveness of A_n, the sequence $u_n - u$ is bounded in $H_0^1(\Omega)$. Hence, up to extracting a subsequence, there exists $u^* \in H^1(\Omega)$, with $u^* - u \in H_0^1(\Omega)$, such that u_n converges weakly to u^* in $H^1(\Omega)$. By semicontinuity and the non-decrease of A_n^s we have for any $k \geqslant \alpha$,

$$\int_\Omega A_k \nabla u^* \cdot \nabla u^* \, dx \leqslant \liminf_{n\to\infty} \int_\Omega A_k \nabla u_n \cdot \nabla u_n \, dx \leqslant \liminf_{n\to\infty} \int_\Omega A_n \nabla u_n \cdot \nabla u_n \, dx \leqslant C.$$

Therefore, since A_k^s converges to A^s in a nondecreasing way, we deduce from the monotone convergence theorem that

$$\int_\Omega A \nabla u^* \cdot \nabla u^* \, dx = \lim_{k\to\infty} \int_\Omega A_k \nabla u^* \cdot \nabla u^* \, dx \leqslant C. \tag{3.2}$$

Now, consider $v \in H_0^1(\Omega)$ such that

$$\int_\Omega A \nabla v \cdot \nabla v \, dx < \infty. \tag{3.3}$$

Taking v as test function in (2.6) we obtain for any n and k,

$$\langle f, v \rangle = \int_\Omega A_n \nabla u_n \cdot \nabla v \, dx = \int_\Omega (A_n - A_k) \nabla u_n \cdot \nabla v \, dx + \int_\Omega A_k \nabla u_n \cdot \nabla v \, dx. \tag{3.4}$$

We will pass to the limit in the right-hand side of this equality first in $n \geqslant k$ then in k. Consider the second term of the right-hand side of (3.4). Since by (3.2) $A\nabla u^* \cdot \nabla u^* < \infty$ a.e. in Ω, by (2.2) we have $|DP\nabla u^*| < \infty$ a.e. in Ω, hence $A_k \nabla u^* \cdot \nabla v$ converges to $A \nabla u^* \cdot \nabla v$ a.e. in Ω. Moreover, by the Cauchy–Schwarz inequality combined with the inequalities $A_k^s \leqslant A^s$, $|B_k| \leqslant |B|$ and (2.5), we have

$$\begin{aligned}
2|A_k \nabla u^* \cdot \nabla v| &\leqslant 2|D_k P \nabla u^* \cdot P \nabla v| + 2|B_k \nabla u^* \cdot \nabla v| \\
&\leqslant A_k \nabla u^* \cdot \nabla u^* + A_k \nabla v \cdot \nabla v + N|B_k|_\infty |\nabla u^*|^2 + N|B_k|_\infty |\nabla v|^2 \\
&\leqslant A \nabla u^* \cdot \nabla u^* + A \nabla v \cdot \nabla v + N|B|_\infty |\nabla u^*|^2 + N|B|_\infty |\nabla v|^2 \\
&\leqslant (1 + N\beta)(A \nabla u^* \cdot \nabla u^* + A \nabla v \cdot \nabla v) \in L^1(\Omega).
\end{aligned}$$

Therefore, by the weak convergence of u_n to u^* in $H^1(\Omega)$, and the Lebesgue convergence theorem we get

$$\lim_{k\to\infty} \lim_{n\to\infty} \int_\Omega A_k \nabla u_n \cdot \nabla v \, dx = \lim_{k\to\infty} \int_\Omega A_k \nabla u^* \cdot \nabla v \, dx = \int_\Omega A \nabla u^* \cdot \nabla v \, dx.$$

For the first term of the right-hand side of (3.4), the decomposition (2.1) for A_n, the Cauchy–Schwarz inequality, the inequalities $A_k^s \leqslant A_n^s \leqslant A^s$ and $|B_k|_\infty I \leqslant |B_n|_\infty I \leqslant \beta A_n^s$ (as a consequence of (2.5)), and estimate (3.1) yield

$$\begin{aligned}\left|\int_\Omega (A_n - A_k)\nabla u_n \cdot \nabla v\,dx\right| &\leqslant \left(\int_\Omega (A_n - A_k)\nabla u_n \cdot \nabla u_n\,dx\right)^{\frac{1}{2}}\left(\int_\Omega (A_n - A_k)\nabla v \cdot \nabla v\,dx\right)^{\frac{1}{2}}\\ &\quad + \left(\int_\Omega N|B_n - B_k|_\infty |\nabla u_n|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega N|B_n - B_k|_\infty |\nabla v|^2\,dx\right)^{\frac{1}{2}}\\ &\leqslant \left(\int_\Omega A_n\nabla u_n \cdot \nabla u_n\,dx\right)^{\frac{1}{2}}\left(\int_\Omega (A - A_k)\nabla v \cdot \nabla v\,dx\right)^{\frac{1}{2}}\\ &\quad + \left(\int_\Omega 2N\beta A_n\nabla u_n \cdot \nabla u_n\,dx\right)^{\frac{1}{2}}\left(\int_\Omega N|B_n - B_k|_\infty |\nabla v|^2\,dx\right)^{\frac{1}{2}}\\ &\leqslant c\left(\int_\Omega (A - A_k)\nabla v \cdot \nabla v\,dx\right)^{\frac{1}{2}} + c\left(\int_\Omega |B_n - B_k|_\infty |\nabla v|^2\,dx\right)^{\frac{1}{2}}.\end{aligned}$$

Since $A^s - A_k^s \leqslant A^s$ and $|B_n - B_k|_\infty I \leqslant \beta A^s$, we deduce from the Lebesgue convergence theorem that the two last terms of the previous inequality converge to zero as $n \geqslant k$ then k tend to ∞. So, we get

$$\lim_{k\to\infty}\limsup_{n\to\infty}\left|\int_\Omega (A_n - A_k)\nabla u_n \cdot \nabla v\,dx\right| = 0,$$

and thus, by (3.4) we obtain

$$\langle f, v\rangle = \int_\Omega A\nabla u^* \cdot \nabla v\,dx, \tag{3.5}$$

for any $v \in H_0^1(\Omega)$ which satisfies (3.3). Taking $u - u^*$ as test function in the difference of (1.2) and (3.5) we deduce that $u^* = u$. Then, the whole sequence u_n converges weakly to u in $H_0^1(\Omega)$.

Now, taking $u_n - u$ as test function in (2.6) and passing to the limit in n we obtain

$$\lim_{n\to\infty}\int_\Omega A_n\nabla u_n \cdot \nabla(u_n - u)\,dx = \lim_{n\to\infty}\langle f, u_n - u\rangle = 0. \tag{3.6}$$

By (3.2) and reasoning similarly as in (3.4) we have

$$\lim_{n\to\infty}\int_\Omega A_n\nabla u \cdot \nabla(u_n - u)\,dx = 0,$$

which combined with (3.6) yields the desired limit (2.7). □

Proof of Theorem 2.4. Set $\delta := |\Omega|^{\frac{1}{N}}$, and define $\Omega^\delta := \delta^{-1}\Omega$. For a measurable function h in Ω, we denote by h^δ the measurable function defined by

$$h^\delta(y) := h(\delta y), \quad \text{a.e. } y \in \Omega^\delta.$$

Then, taking α, q, A, f and u as in the statement of Theorem 2.4, we define A^δ, u^δ by the previous rule, and $f^\delta \in W^{-1,q}(\Omega^\delta)$ by

$$\langle f^\delta, \varphi^\delta\rangle_{W^{-1,q}(\Omega^\delta), W_0^{1,q'}(\Omega^\delta)} := \langle f, \varphi\rangle_{W^{-1,q}(\Omega), W_0^{1,q'}(\Omega)}, \quad \forall \varphi \in W_0^{1,q'}(\Omega).$$

We have that the measure of Ω^δ is equal to 1. Moreover, making the former change of variables in (1.2), the function u^δ satisfies

$$\begin{cases} (m-u^\delta)^+, (u^\delta - M)^+ \in H_0^1(\Omega^\delta), & \displaystyle\int_{\Omega^\delta} A^\delta \nabla u^\delta \cdot \nabla u^\delta \, dy < \infty, \\ \displaystyle\int_{\Omega^\delta} A^\delta \nabla u^\delta \cdot \nabla v^\delta \, dy = \delta^{2-N}\langle f^\delta, v^\delta\rangle, & \forall v^\delta \in H_0^1(\Omega^\delta) \text{ with } \displaystyle\int_{\Omega^\delta} A^\delta \nabla v^\delta \cdot \nabla v^\delta \, dy < \infty. \end{cases}$$

On the other hand, let A_n^δ be the truncated of A^δ defined as in Proposition 2.3, and let u_n^δ be the solution of (2.6) with A_n, f, u replaced respectively by A_n^δ, f^δ, u^δ. Since $(m-u^\delta)^+$, $(u^\delta - M)^+$ and $u_n^\delta - u^\delta$ belong to $H_0^1(\Omega^\delta)$, so do $(m-u_n^\delta)^+$ and $(u_n^\delta - M)^+$. Then, taking into account $|\Omega^\delta| = 1$, Theorem 8.16 of [18] implies the existence of a constant $C > 0$ only depending on α and q, such that

$$m - C\|f^\delta\|_{W^{-1,q}(\Omega^\delta)} \leqslant u_n^\delta \leqslant M + C\|f^\delta\|_{W^{-1,q}(\Omega^\delta)}, \quad \text{a.e. in } \Omega^\delta, \ \forall n \in \mathbb{N}.$$

However, by Proposition 2.3 the sequence u_n^δ converges strongly to u^δ in $H^1(\Omega)$, thus in particular a.e. in Ω (up to a subsequence). Therefore, passing to the pointwise limit in the previous inequalities we get

$$m - C\|f^\delta\|_{W^{-1,q}(\Omega^\delta)} \leqslant u^\delta \leqslant M + C\|f^\delta\|_{W^{-1,q}(\Omega^\delta)}, \quad \text{a.e. in } \Omega^\delta. \tag{3.7}$$

Finally, noting that

$$\begin{aligned} \|f^\delta\|_{W^{-1,q}(\Omega^\delta)} &= \inf_{\varphi^\delta \neq 0} \frac{\langle f^\delta, \varphi^\delta\rangle_{W^{-1,q}(\Omega^\delta), W_0^{1,q'}(\Omega^\delta)}}{\|\varphi^\delta\|_{W_0^{1,q'}(\Omega^\delta)}} \\ &= \inf_{\varphi \neq 0} \frac{\langle f, \varphi\rangle_{W^{-1,q}(\Omega), W_0^{1,q'}(\Omega)}}{\delta^{\frac{N}{q}-N+1}\|\varphi\|_{W_0^{1,q'}(\Omega)}} = \delta^{N-1-\frac{N}{q}}\|f\|_{W^{-1,q}(\Omega)}, \end{aligned}$$

we easily deduce (2.8). □

Let us now prove the compactness results stated in Section 2.2. First of all recall the definition of the r-capacity:

Definition 3.1. Let $r \in (1,2)$. The r-capacity of a subset E of $\mathbb{R}^2$ is defined by

$$C_r(E) := \inf\left\{ \int_{\mathbb{R}^2} |\nabla u|^r \, dx\text{:} \ u \in D^{1,r}(\mathbb{R}^2), \ u \geqslant 1 \text{ a.e. in a neighborhood of } E \right\},$$

where $D^{1,r}(\mathbb{R}^2)$ denotes the space of the functions u in $L^{\frac{2r}{2-r}}(\mathbb{R}^2)$, with $\nabla u \in L^r(\mathbb{R}^2)^2$.

We need the following capacitary estimate which is proved in [8]:

Lemma 3.2. *For any $r \in (1,2)$, there exists a constant $R_r > 0$ such that for any $a, b \in \mathbb{R}^2$, and any curve L of extremities a, b, we have*

$$C_r(L) \geqslant R_r |a-b|^{2-r}. \tag{3.8}$$

Proof of Theorem 2.5. For $\delta > 0$, we define Ω^δ by

$$\Omega^\delta := \begin{cases} \{x \in \Omega \cup \Gamma\colon\ \operatorname{dist}(x, \partial\Omega \setminus \Gamma) \geqslant \delta\} & \text{if } \Gamma \neq \partial\Omega, \\ \bar{\Omega} & \text{if } \Gamma = \partial\Omega. \end{cases}$$

Then, for any $l \in \mathbb{N}$, we take $\delta_l > 0$ which converges to zero such that

$$\left|u(x) - u(y)\right| < \frac{1}{2l}, \quad \forall x, y \in \bar{\Omega}_{\delta_l} \text{ with } |x-y| \leqslant \delta_l. \tag{3.9}$$

Now, since u_n converges weakly in $W^{1,p}(\Omega)$ and Ω is regular, we can take $r \in (1,p)$ and a subsequence of u_n, still denoted by u_n, which converges C_r-quasi-uniformly to u. Also using that u_n converges to u in $C^0(\Gamma)$, we can choose this sequence in such a way that there exists an open subset G_l of Ω, which satisfies (R_r defined in Lemma 3.2)

$$C_r(G_l) < R_r \delta_l^{2-r}, \tag{3.10}$$

$$\left|u_n(x) - u(x)\right| < \frac{1}{2l}, \quad \forall x \in (\Omega \setminus G_l) \cup (\Gamma \cap \Omega_{\delta_l}),\ \forall n \geqslant l. \tag{3.11}$$

Let us prove that this subsequence u_n converges to u in $C^0(\Omega \cup \Gamma)$. This will prove that the whole sequence u_n converges to u in $C^0(\Omega \cup \Gamma)$. We fix $\delta > 0$ and we take l such that $2\delta_l < \delta$.

Consider a connected component O of G_l such that $O \cap \Omega^\delta \neq \emptyset$. Since O is connected by curves, for any $y_1, y_2 \in O$, there exists a curve $L \subset O$, which contains y_1, y_2. By Lemma 3.2 and (3.10) we have

$$R_r |y_1 - y_2|^{2-r} \leqslant C_r(L) \leqslant C_r(O) \leqslant C_r(G_l) \leqslant R_r \delta_l^{2-r},$$

hence $\operatorname{diam}(O) \leqslant \delta_l$. Therefore, since $2\delta_l < \delta$, we have $\bar{O} \subset \Omega_{\delta_l}$, which implies in particular that $\partial O \subset (\Omega \setminus G_l) \cup (\Gamma \cap \Omega_{\delta_l})$.

By (2.9), the fact that $u_n \in C^0(\Omega \cap \Gamma)$, by Theorem 2.4 and $|O| \leqslant \pi \delta_l^2/4$, we have (for another constant C)

$$\min_{\partial O} u_n - C\delta_l^{\frac{q-2}{q}} \|f_n\|_{W^{-1,q}(\Omega)} \leqslant u_n \leqslant \max_{\partial O} u_n + C\delta_l^{\frac{q-2}{q}} \|f_n\|_{W^{-1,q}(\Omega)}, \quad \text{in } O,\ \forall n \geqslant l. \tag{3.12}$$

Moreover, thanks to (3.11) and (3.9) we have

$$\min_{\partial O} u_n \geqslant \min_{\partial O} u - \frac{1}{2l} \geqslant u(x) - \frac{1}{l}, \qquad \max_{\partial O} u_n \leqslant \max_{\partial O} u + \frac{1}{2l} \leqslant u(x) + \frac{1}{l}, \quad \forall x \in O,\ \forall n \geqslant l.$$

Hence, inequality (3.12) implies that

$$|u_n - u| \leqslant \frac{1}{l} + C\delta_l^{\frac{q-2}{q}} \|f_n\|_{W^{-1,q}(\Omega)}, \quad \text{in } O,\ \forall n \geqslant l.$$

Since G_l is equal to the union of its connected components, (3.11) and the previous estimates thus yield

$$|u_n - u| \leqslant \frac{1}{l} + C\delta_l^{\frac{q-2}{q}} \|f_n\|_{W^{-1,q}(\Omega)}, \quad \text{in } \Omega^\delta, \ \forall n \geqslant l.$$

This combined with the boundedness of $\|f_n\|_{W^{-1,q}(\Omega)}$ proves the convergence of u_n in $C^0(\Omega \cup \Gamma)$. □

Proof of Corollary 2.6. We will only prove (ii). The proof of (i) is quite similar.

First, let us prove that there exists a constant $C > 0$ such that for any $A \in \mathcal{A}(\alpha, B(0,2)^+)$, $f \in W^{-1,q}(B(0,2)^+)$, and any $u \in H^1(B(0,2)^+) \cap C^0(\bar{B}(0,2)^+)$ which satisfies (1.2) with $\Omega = B(0,2)^+$, we have

$$\left\| u - \frac{1}{2}\int_{-1}^{1} u(t,0)\,dt \right\|_{C^0(\bar{B}(0,1)^+)} \leqslant \varepsilon\big(\|f\|_{W^{-1,q}(B(0,2)^+)} + \|\nabla u\|_{L^p(B(0,2)^+)^2}\big) + C\left(\left\| u - \frac{1}{2}\int_{-1}^{1} u(t,0)\,dt \right\|_{L^1(B(0,2)^+)} + \operatorname*{osc}_{[-2,2]\times\{0\}} u \right). \tag{3.13}$$

We reason by contradiction. If (3.13) does not hold true, then for any $n \in \mathbb{N}$, there exist A_n in $\mathcal{A}(\alpha, B(0,2)^+)$, $f_n \in W^{-1,q}(B(0,2)^+)$, and $u_n \in H^1(B(0,2)^+) \cap C^0(\bar{B}(0,2\delta)^+)$ satisfying (1.2) with $\Omega = B(0,2)^+$, such that

$$\left\| u_n - \frac{1}{2}\int_{-1}^{1} u_n(t,0)\,dt \right\|_{C^0(\bar{B}(0,1)^+)} > \varepsilon\big(\|f_n\|_{W^{-1,q}(B(0,2)^+)} + \|\nabla u_n\|_{L^p(B(0,2)^+)^2}\big) + n\left(\left\| u_n - \frac{1}{2}\int_{-1}^{1} u_n(t,0)\,dt \right\|_{L^1(B(0,2)^+)} + \operatorname*{osc}_{[-2,2]\times\{0\}} u_n \right). \tag{3.14}$$

Therefore, the sequence

$$z_n := \frac{u_n - \frac{1}{2}\int_{-1}^{1} u_n(t,0)\,dt}{\|u_n - \frac{1}{2}\int_{-1}^{1} u_n(t,0)\,dt\|_{C^0(\bar{B}(0,1)^+)}} \in H^1\big(B(0,2)^+\big) \cap C^0\big(\bar{B}(0,2)^+\big)$$

satisfies (1.2) with Ω, u_n and f_n replaced respectively by $B(0,2)^+$, z_n and

$$g_n := \frac{f_n}{\|u_n - \frac{1}{2}\int_{-1}^{1} u_n(t,0)\,dt\|_{C^0(\bar{B}(0,1)^+)}}.$$

Moreover, z_n has zero average on $(-1,1)\times\{0\}$, $\|z_n\|_{C^0(\bar{B}(0,1)^+)} = 1$, and

$$1 > \varepsilon\big(\|g_n\|_{W^{-1,q}(B(0,2)^+)} + \|\nabla z_n\|_{L^p(B(0,2)^+)^2}\big) + n\Big(\|z_n\|_{L^1(B(0,2)^+)} + \operatorname*{osc}_{[-2,2]\times\{0\}} z_n\Big).$$

Thus, g_n is bounded in $W^{-1,q}(B(0,2)^+)$ and z_n converges weakly to zero in $W^{1,p}(B(0,2)^+)$ and strongly in $C^0([-2,2]\times\{0\})$. Hence, by Theorem 2.5 the sequence z_n converges to zero in $C^0(B(0,2)^+ \cup [(-2,2)\times\{0\}])$, which contradicts $\|z_n\|_{C^0(\bar{B}(0,1)^+)} = 1$.

Now, from the inequality

$$\left|u(x)-u(y)\right| \leqslant \left|u(x)-\frac{1}{2}\int_{-1}^{1}u(t,0)\,dt\right| + \left|u(y)-\frac{1}{2}\int_{-1}^{1}u(t,0)\,dt\right|$$
$$\leqslant 2\left\|u-\frac{1}{2}\int_{-1}^{1}u(t,0)\,dt\right\|_{C^0(\bar B(0,1)^+)}, \quad \forall x,y\in \bar B(0,1)^+,\ \forall u\in C^0\big(\bar B(0,1)^+\big),$$

combined with (3.13) we deduce the inequality (2.11) for $x_0=0$ and $\delta=1$. The general case follows easily using the change of variables $y=\delta^{-1}(x-x_0)$, which transforms $B(x_0,2\delta)$ in $B(0,2)$. □

Proof of Theorem 2.7. By virtue of Theorem 2.5 we just need to prove that u is continuous in $\Omega\cup\Gamma$. Due to the regularity of $\partial\Omega$, for any $x_0\in\Gamma$, there exist an open neighborhood O of x_0, and a one-to-one map φ, with φ and φ^{-1} Lipschitz, which transforms x_0 in 0, $O\cap\Omega$ in the half-ball $B(0,\varepsilon)^+$, and $O\cap\Gamma$ in the segment $B(0,\varepsilon)\cap(\mathbb{R}\times\{0\})$. We are thus led to the case $x_0=0$ with $O\cap\Gamma\subset\mathbb{R}\times\{0\}$. Then, from the estimate (2.11) with $p=2$, we deduce the existence of $\delta_0>0$ such that for any $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ which satisfies

$$\operatorname*{osc}_{\bar B(0,\delta)^+} u_n \leqslant \varepsilon\big(\delta^{\frac{q-2}{q}}\|f_n\|_{W^{-1,q}(B(0,2\delta)^+)}+\|\nabla u_n\|_{L^2(B(0,2\delta)^+)^2}\big)$$
$$+C_\varepsilon\left(\frac{1}{\delta^2}\left\|u_n-\frac{1}{2\delta}\int_{-\delta}^{\delta}u_n(t,0)\,dt\right\|_{L^1(B(0,2\delta)^+)}+\operatorname*{osc}_{[-2\delta,2\delta]\times\{0\}}u_n\right),$$

for any $\delta\leqslant\delta_0$ and any $n\in\mathbb{N}$. This estimate combined with the convergence of $u_n(0)$ (since $0\in\Gamma$) implies that the sequence u_n is bounded in $L^\infty(B(0,\delta)^+)$. Hence, u_n converges to u for the weak-∗ topology of $L^\infty(B(0,\delta)^+)$. Using the boundedness of $\|f_n\|_{W^{-1,q}(B(0,2\delta)^+)}$ and $\|\nabla u_n\|_{L^2(B(0,2\delta)^+)^2}$, the strong convergence of u_n in $L^1(\Omega)$ and in $C^0([-2\delta,2\delta]\times\{0\})$, and the lower semicontinuity of the $L^\infty(B(0,\delta)^+)$-norm, we deduce from the previous estimate that there exists a constant $M>0$ such that for any $\varepsilon>0$, we have

$$\big\|u-u(0)\big\|_{L^\infty(B(0,\delta)^+)}\leqslant M\varepsilon+C_\varepsilon\left(\frac{1}{\delta^2}\left\|u-\frac{1}{2\delta}\int_{-\delta}^{\delta}u(t,0)\,dt\right\|_{L^1(B(0,2\delta)^+)}+\operatorname*{osc}_{[-2\delta,2\delta]\times\{0\}}u\right)$$
$$\leqslant M\varepsilon+C_\varepsilon\Big(K\|\nabla u\|_{L^2(B(0,2\delta)^+)^2}+\operatorname*{osc}_{[-2\delta,2\delta]\times\{0\}}u\Big),$$

where the constant K does not depend on ε or δ. Therefore, u is continuous at 0. This also proves that u is continuous at any point of Γ.

In order to prove the continuity of u in Ω, we proceed similarly by using inequality (2.10). Hence, there exists a constant $M>0$ such that for any $\varepsilon>0$, there exists $C_\varepsilon>0$ which satisfies

$$\operatorname*{sup\,ess}_{x,y\in B(x_0,\delta)}\big|u(x)-u(y)\big|\leqslant M\varepsilon+C_\varepsilon\|\nabla u\|_{L^2(B(0,2\delta))^2},$$

for any $x_0\in\Omega$ and any $\delta>0$ with $B(x_0,\delta)\subset\Omega$. This shows that u admits a continuous representative in Ω. □

4. Applications and counter-examples

4.1. Applications

In this section we give two applications of the compactness Theorem 2.7. First, we have the following continuity result:

Theorem 4.1. *Let Ω be a bounded open subset of $\mathbb{R}^2$, with a Lipschitz boundary, and let Γ be a relatively open subset of $\partial\Omega$. Let u be a function in $H^1(\Omega)$, which has a continuous trace on Γ. Assume that there exists $A \in \mathcal{A}(\alpha, \Omega)$ with $\alpha > 0$, and $f \in W^{-1,q}(\Omega)$ with $q > 2$, such that u is a solution of* (1.2). *Then, u is continuous in $\Omega \cup \Gamma$.*

Remark 4.2. The continuity of solutions of two-dimensional linear elliptic equations with unbounded coefficients (but which are controlled from below), is already known (see e.g. the first proposition of [17]). Our approach seems new and it is more constructive since such a solution is regarded as the uniform limit of solutions of equations with truncated coefficients.

The second application deals with the density of the continuous functions in the domain of the Γ-limit of a sequence of diffusion energies. We refer to [13] (see also [3]) for the definition of the Γ-convergence and its elementary properties. The following result improves the compactness results obtained in [8]:

Theorem 4.3. *Let $\alpha > 0$, and let Ω be a bounded open subset of $\mathbb{R}^2$, with a Lipschitz boundary. Let A_n be a sequence of symmetric matrix-valued functions in $\mathcal{A}(\alpha, \Omega)$, and define the sequence of quadratic functionals F_n in $L^2(\Omega)$ by*

$$F_n(u) := \begin{cases} \int_\Omega A_n \nabla u \cdot \nabla u \, dx & \text{if } u \in H_0^1(\Omega), \\ \infty & \text{if } u \in L^2(\Omega) \setminus H_0^1(\Omega). \end{cases} \tag{4.1}$$

Consider the Γ-limit F of a Γ-convergent subsequence of F_n for the strong topology of $L^2(\Omega)$ (such a limit does exist), the domain of which $D(F) := \{F < \infty\}$ is endowed with the norm $\sqrt{F}$. Then, $D(F) \cap C_0^0(\Omega)$ is dense in $D(F)$.

Remark 4.4. In [8] we studied the Γ-limit of any sequence of functionals F_n defined by (4.1), where A_n is an equicoercive sequence of symmetric matrix-valued functions in $\mathcal{A}(\alpha, \Omega) \cap L^\infty(\Omega)^{2\times 2}$, without any bound from above. Thanks to Proposition 2.3 all the results in [8] still hold true only assuming A_n in $\mathcal{A}(\alpha, \Omega)$ and symmetric.

Proof of Theorem 4.1. Let u_n be the solution of (2.6) with the truncation A_n of A. By the De Giorgi–Stampacchia theorem u_n is continuous in $\Omega \cap \Gamma$. Moreover, by Proposition 2.3 the sequence u_n converges strongly to u in $H^1(\Omega)$. Thus, Theorem 2.7 implies that u is continuous in $\Omega \cup \Gamma$. □

Proof of Theorem 4.3. It is well known that F is a quadratic form in $D(F)$ and $D(F)$ is a Hilbert space endowed with the norm $\sqrt{F}$. Let Φ be the bilinear form associated with F and defined in $D(F) \times D(F)$. Since $D(F)$ is continuously embedded in $H_0^1(\Omega)$ (as a consequence of the α-coerciveness of A_n and thus of F_n), for any $h \in L^2(\Omega)$, there exists a unique $u^h \in D(F)$ such that

$$\Phi(u^h, v) = \int_\Omega h v \, dx, \quad \forall v \in D(F).$$

We clearly have

$$\bigcap_{h\in L^2(\Omega)} \operatorname{Ker}\big(\Phi\big(u^h,\cdot\big)\big)=\{0\},$$

hence the set $\{u^h \in D(F)\colon\ h\in L^2(\Omega)\}$ is dense in $D(F)$. Therefore, in order to conclude it is enough to check that the functions u^h belong to $C^0_0(\Omega)$. To this end, note that, since $L^2(\Omega)$ is contained in $W^{-1,\infty}(\Omega)$ by the two-dimensional Sobolev embedding, Theorem 4.1 implies that the solution u_n of

$$\begin{cases} u_n\in H^1_0(\Omega), \quad \displaystyle\int_\Omega A_n\nabla u_n\cdot\nabla u_n\,dx<\infty,\\ \displaystyle\int_\Omega A_n\nabla u_n\cdot\nabla v\,dx=\int_\Omega hv\,dx, \quad \forall v\in H^1_0(\Omega) \text{ with } \int_\Omega A_n\nabla v\cdot\nabla v\,dx<\infty, \end{cases}$$

belongs to $C^0_0(\Omega)$. Moreover, by the Γ-convergence of F_n to F in $L^2(\Omega)$ combined with the α-coerciveness of F_n, the sequence u_n converges weakly to u^h in $H^1_0(\Omega)$. Therefore, thanks to Theorem 2.7 the function u^h belongs to $C^0_0(\Omega)$. □

4.2. Counter-examples

First of all, the following counter-example shows that Theorem 2.7 cannot be extended to the case $p\in(1,2)$. It is based on the famous Serrin [23] example:

Example 4.5. Let Ω be the unit disk of $\mathbb{R}^2$. Consider, for any integer $n\geqslant 1$, the matrix-valued function A_n and the function u_n defined by

$$A_n(x):=I+(n^2-1)\frac{x\otimes x}{|x|^2} \quad\text{and}\quad u_n(x):=\frac{x_1}{|x|^{1-1/n}}, \quad \text{a.e. } x\in\Omega. \tag{4.2}$$

The functions A_n and u_n satisfy for any $n\geqslant 1$ (see [23] for details),

$$I\leqslant A_n\leqslant n^2 I \quad \text{a.e. in } \Omega, \qquad \begin{cases} u_n\in H^1(\Omega)\cap C^0(\bar\Omega),\\ u_n(x)=x_1 \quad \text{for any } x\in\partial\Omega, \end{cases} \quad \text{and} \quad \operatorname{div}(A_n\nabla u_n)=0 \quad \text{in } \mathcal{D}'(\Omega).$$

Hence, the conditions of Theorem 2.5 are fulfilled with $\Gamma=\partial\Omega$. Moreover, the sequence u_n converges weakly to $u(x):=x_1/|x|$ in $W^{1,p}(\Omega)$, for any $p\in(1,2)$. Therefore, the limit u is not continuous in $\bar\Omega$, and the sequence $u_n\in C^0(\bar\Omega)$ does not converge uniformly to u in $\bar\Omega$.

The second example shows that the Hölder-continuity of the solutions of Eqs. (1.2) does not hold in general when the matrix-valued function is not bounded:

Example 4.6. Let $\Omega:=B(0,\frac12)$ be the ball of $\mathbb{R}^2$ centered at the origin and of radius $\frac12$. Let A be the (unbounded) matrix-valued function in $\mathcal{A}((\ln 2)^2,\Omega)$ (see Definition 2.1) defined by

$$A(x):=2I+\big((\ln|x|)^2-2\big)\frac{x\otimes x}{|x|^2}, \quad \text{for } x\in\Omega\setminus\{0\}. \tag{4.3}$$

Then, the function $u\in H^1(\Omega)$ defined by

$$u(x) := \frac{x_2}{|x|(\ln|x|)^2}, \quad \text{for } x \in \Omega \setminus \{0\}, \tag{4.4}$$

is a solution of (1.2), with A of (4.3) and right-hand side 0, which is not Hölder-continuous in the neighborhood of the origin.

The proof is a simple computation using polar coordinates.

Now, we will give three-dimensional counter-examples to Theorems 2.7 and 4.1 respectively. The first result provides an example of lack of uniform convergence of the solutions of linear elliptic equations in dimension two:

Proposition 4.7. *There exist a regular bounded domain Ω of $\mathbb{R}^3$, and a sequence of functions a_n in $L^\infty(\Omega; [1, \infty))$, such that, for any non-zero function $f \in L^2(\Omega)$, the solution $u_n \in H^1_0(\Omega)$ of the equation $-\operatorname{div}(a_n \nabla u_n) = f$ in $\mathcal{D}'(\Omega)$, does not converge uniformly in $\bar{\Omega}$.*

Remark 4.8. The proof of Proposition 4.7 is based on the example model due to Fenchenko and Khruslov [16] of nonlocal effects arising in the homogenization of three-dimensional high-conductivity problems (see also [1,9] and [11] for alternative approaches).

Making the additional hypothesis $f \in W^{-1,q}(\Omega)$, with $q > 3$, the De Giorgi–Stampacchia regularity result (see e.g. Theorem 8.29 of [18]) ensures the continuity of u_n for a fixed n. Then, the present assumptions correspond to the ones of Theorem 2.5 for dimension three, but the conclusion of uniform convergence is no longer satisfied.

The second result provides an example of discontinuity of a solution of a two-dimensional linear elliptic equation with unbounded coefficients:

Proposition 4.9. *There exist a regular bounded domain Ω of $\mathbb{R}^3$, a function $a : \Omega \to [1, \infty)$ with $a = a(x_1, x_2) \in L^1(\Omega)$, and $f \in C^\infty_c(\Omega)$, such that the solution $u \in H^1_0(\Omega)$ of problem (1.2) with $A = aI$, is not continuous in Ω.*

Remark 4.10. An example of a discontinuous a-harmonic function (i.e. $f = 0$) with $a \geqslant 1$ and a exponentially integrable, is given in [17] solving a De Giorgi conjecture [15]. Here, we obtain a simpler and different counter-example with a non-zero right-hand side f but with a only integrable. The interest of this example is that it is based on the unidirectional fibers reinforcement principle used in the counter-example of Proposition 4.7. As a consequence, the three-dimensional conductivity a of our example depends only on two variables contrary to the one of [17].

Proof of Proposition 4.7. Let Ω' be a bounded open set of $\mathbb{R}^2$ and let Ω be the vertical (parallel to the x_3-axis) cylinder defined by $\Omega := \Omega' \times (0, 1)$. Let ω_n be a $\frac{1}{n}$-periodic lattice of thin vertical cylinders of radius $r_n := \frac{1}{n} e^{-n^2}$, and let a_n be the function defined by

$$a_n := \begin{cases} 2e^{2n^2} & \text{in } \omega_n, \\ 1 & \text{in } \Omega \setminus \omega_n. \end{cases}$$

For a fixed $f \in L^2(\Omega)$, let u_n be the solution in $H^1_0(\Omega)$ of the equation $-\operatorname{div}(a_n \nabla u_n) = f$ in $\mathcal{D}'(\Omega)$. By [9] the weak limit u of u_n in $H^1_0(\Omega)$ and the limit $v \in H^1_0((0, 1); L^2(\Omega'))$ of the rescaled function $v_n := \frac{1_{\omega_n}}{\pi r_n^2} u_n$ in the weak-$*$ sense of the Radon measures on Ω, satisfy the coupled system

$$\begin{cases} -\Delta u + 2\pi(u - v) = f & \text{in } \Omega, \\ -\dfrac{\partial^2 v}{\partial x_3^2} + v - u = 0 & \text{in } \Omega. \end{cases} \tag{4.5}$$

Assume that u_n converges uniformly to u in $\bar{\Omega}$. Then, since by the De Giorgi–Stampacchia theorem u_n is continuous in $\bar{\Omega}$ for any $n \geqslant 1$, so is its limit u. Hence, the rescaled function $\frac{1_{\omega_n}}{\pi r_n^2} u$ converges to u in the weak-$*$ sense of the Radon measures on Ω. We thus have for any $\varphi \in C_0^0(\Omega)$,

$$\int_\Omega \varphi(v-u)\,dx = \lim_{n\to\infty}\left[\frac{1}{\pi r_n^2}\int_{\omega_n}\varphi(u_n-u)\,dx\right] \leqslant \lim_{n\to\infty}\left[\frac{|\omega_n|}{\pi r_n^2}\|\varphi\|_{L^\infty(\Omega)}\|u_n-u\|_{L^\infty(\Omega)}\right] = 0,$$

which implies that $v-u=0$ a.e. in Ω. Putting this equality in system (4.5) we get the equalities $v=0$ and $f=0$ a.e in Ω. Therefore, if f is a non-zero function then the uniform convergence of u_n does not hold. □

Proof of Proposition 4.9. Define the open subsets of $\mathbb{R}^3$, $\Omega := (-1,1)\times(0,1)^2$, $\Omega_+ := (0,1)^3$, $\Omega_- := (-1,0)\times(0,1)^2$ and $\Gamma := \partial\Omega_+ \cap \partial\Omega_-$. Define the points of $\mathbb{R}^2$, $\tau_k^n := (2^{-n}, 2^{-n}k)$, for $n \in \mathbb{N}^*$ and $k \in \{1,\ldots,2^n-1\}$. Denote by $D(\tau, r)$ the disk of center $\tau \in \mathbb{R}^2$ and of radius $r>0$, and consider the subsets of Ω_+ defined by

$$\omega_n^k := D(\tau_k^n, r_n)\times(0,1), \qquad \hat{\omega}_n^k := D(\tau_k^n, 2^{-n}R)\times(0,1), \quad \text{where } r_n := e^{-4^n},\ R\in(0,1/3). \tag{4.6}$$

Note that $\bigcup_{k=1}^{2^n-1}\omega_n^k$ is composed of 2^n-1 very thin vertical cylinders uniformly arranged along the plane $x_1 = 2^{-n}$, which accumulate on the right-hand side of the boundary Γ. Let w be the function (independent of x_3) defined for $x=(x',x_3)\in\Omega$, by

$$w(x) := \begin{cases} 1 & \text{if } x\in\omega_n^k, \\ \frac{\ln|x'-\tau_k^n| - \ln(2^{-n}R)}{\ln(r_n)-\ln(2^{-n}R)} & \text{if } x \in \hat{\omega}_k^n \setminus \omega_n^k,\ \text{for some } n\in\mathbb{N}^*,\ 1\leqslant k\leqslant 2^n-1, \\ 0 & \text{elsewhere.}\end{cases} \tag{4.7}$$

Let a be the function defined by

$$a(x) := \begin{cases} \frac{1}{3^n|\omega_n^k|} & \text{if } x\in\omega_n^k,\ \text{for some } n\in\mathbb{N}^*,\ 1\leqslant k\leqslant 2^n-1, \\ 1 & \text{elsewhere.}\end{cases} \tag{4.8}$$

Note that $a \geqslant 1$ a.e. in Ω, and

$$\int_\Omega a\,dx \leqslant |\Omega| + \sum_{n=1}^\infty\sum_{k=1}^{2^n-1}\frac{|\omega_n^k|}{3^n|\omega_n^k|} \leqslant |\Omega| + \sum_{n=1}^\infty \frac{2^n}{3^n} = |\Omega|+2,$$

$$\int_\Omega a|\nabla w|^2\,dx \leqslant \sum_{n=1}^\infty\sum_{k=1}^{2^n-1}\frac{2\pi}{\ln(2^{-n}R)-\ln(r_n)} \leqslant \sum_{n=1}^\infty\sum_{k=1}^{2^n-1}\frac{c}{4^n} \leqslant c\sum_{n=1}^\infty\frac{2^n}{4^n} \leqslant c.$$

For $\psi \in C_c^\infty(\Omega_-)$, $\psi \geqslant 0$ in Ω_-, ψ non-identically zero, let $u \in H_0^1(\Omega)$ be the solution of the problem

$$\int_\Omega a\nabla u\cdot\nabla v\,dx = \int_{\Omega_-}\psi v\,dx, \quad \forall v\in H_0^1(\Omega). \tag{4.9}$$

Taking $u^- := \max(-u, 0)$ as test function in (4.9), we deduce that $u^- = 0$ and so, $u \geqslant 0$ a.e. in Ω. Moreover, by Theorem 2.4 the function u belongs to $L^\infty(\Omega)$. Taking $u w^2 \chi_{\hat{\omega}_k^n}$, $n \in \mathbb{N}^*$, $1 \leqslant k \leqslant 2^n - 1$, as test function in (4.9), we also get

$$\int_{\hat{\omega}_k^n} a|\nabla u|^2 w^2\, dx + 2 \int_{\hat{\omega}_k^n} a\nabla u \cdot \nabla w\, w\, dx = 0,$$

which by Young's inequality implies

$$\int_{\hat{\omega}_k^n} a|\nabla u|^2 w^2\, dx \leqslant 4 \int_{\hat{\omega}_k^n} a|\nabla w|^2\, dx.$$

This combined with the definitions (4.8) of a and (4.7) of w, yields

$$\frac{1}{3^n} \fint_{\omega_n^k} |\nabla u|^2\, dx \leqslant \int_{\hat{\omega}_k^n} a|\nabla u|^2 w^2\, dx \leqslant 4 \int_{\hat{\omega}_k^n} a|\nabla w|^2\, dx \leqslant \frac{c}{4^n}.$$

Then, using that $u = 0$ on $\partial\Omega$, we have by the Cauchy–Schwarz inequality

$$\fint_{\omega_n^k} u\, dx = \fint_{\omega_n^k} \int_0^{x_3} \frac{\partial u}{\partial x_3}(x', t)\, dt\, dx \leqslant \left(\fint_{\omega_n^k} |\nabla u|^2\, dx \right)^{\frac{1}{2}} \leqslant c \left(\frac{\sqrt{3}}{2} \right)^n, \quad \forall n \in \mathbb{N}^*,\ 1 \leqslant k \leqslant 2^n - 1.$$

Now, assume that u is continuous in a neighborhood of the boundary Γ. Then, the previous estimate and $u \geqslant 0$ a.e. in Ω imply that $u = 0$ on Γ. Since by (4.9) and $\psi = 0$ in Ω_+, u is a harmonic function in $H_0^1(\Omega_+)$, we thus have $u = 0$ a.e. in Ω_+. On the other hand, since $a \in L^1(\Omega)$, for any $\varphi \in C_c^1(\Omega)$, the function $\varphi(1 - w)$ is a suitable test function for problem (4.9), which is equal to φ in Ω_-. Hence, taking into account that $u = 0$ a.e. in Ω_+, we obtain

$$\int_{\Omega_-} \nabla u \cdot \nabla\varphi\, dx = \int_{\Omega} a\nabla u \cdot \nabla\big(\varphi(1 - w)\big)\, dx = \int_{\Omega_-} \psi\varphi\, dx.$$

Therefore, u is solution of the Dirichlet problem

$$-\Delta u = \psi \neq 0 \quad \text{in } \mathcal{D}'(\Omega_-), \qquad u = 0 \quad \text{on } \partial\Omega_-, \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma,$$

which contradicts the Hopf maximum principle, i.e. the negativity of the normal derivative of u on Γ. As a consequence, the function u is not continuous in any neighborhood of Γ. □

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References

[1] M. Bellieud, G. Bouchitté, Homogenization of elliptic problems in a fiber reinforced structure. Non local effects, Ann. Sc. Norm. Super. Pisa Cl. Sci. 26 (3) (1998) 407–436.
[2] A. Beurling, J. Deny, Espaces de Dirichlet, Acta Math. 99 (1958) 203–224.
[3] A. Braides, Gamma-Convergence for Beginners, Oxford Univ. Press, 2002.
[4] M. Briane, Homogenization of high-conductivity periodic problems: Application to a general distribution of one-directional fibers, SIAM J. Math. Anal. 35 (1) (2003) 33–60.
[5] M. Briane, Homogenization of non-uniformly bounded operators: Critical barrier for nonlocal effects, Arch. Ration. Mech. Anal. 164 (2002) 73–101.
[6] M. Briane, Nonlocal effects in two-dimensional conductivity, Arch. Ration. Mech. Anal. 182 (2) (2006) 255–267.
[7] M. Briane, J. Casado-Díaz, Two-dimensional div–curl results. Application to the lack of nonlocal effects in homogenization, Comm. Partial Differential Equations 32 (2007) 935–969.
[8] M. Briane, J. Casado-Díaz, Asymptotic behaviour of equicoercive diffusion energies in dimension two, Calc. Var. Partial Differential Equations 29 (4) (2007) 455–479.
[9] M. Briane, N. Tchou, Fibered microstructures for some nonlocal Dirichlet forms, Ann. Sc. Norm. Super. Pisa Cl. Sci. 30 (4) (2001) 681–711.
[10] G. Buttazzo, G. Dal Maso, Γ-limits of integral functionals, J. Anal. Math. 37 (1980) 145–185.
[11] M. Camar-Eddine, P. Seppecher, Closure of the set of diffusion functionals with respect to the Mosco-convergence, Math. Models Methods Appl. Sci. 12 (8) (2002) 1153–1176.
[12] L. Carbone, C. Sbordone, Some properties of Γ-limits of integral functionals, Ann. Mat. Pura Appl. 122 (1979) 1–60.
[13] G. Dal Maso, An Introduction to Γ-Convergence, Birkhäuser, Boston, 1993.
[14] E. De Giorgi, Sulla differenziabilità et l'analiticità delle extremali degli integrali mulitipli regolari, Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. 3 (3) (1957) 25–43.
[15] E. De Giorgi, Congetture sulla continuità delle soluzioni di equazioni lineari ellittiche autoaggiunte, unpublished notes, 1995.
[16] V.N. Fenchenko, E.Ya. Khruslov, Asymptotic of solution of differential equations with strongly oscillating matrix of coefficients which does not satisfy the condition of uniform boundedness, Dokl. Acad. Nauk Ukr. SSR 4 (1981).
[17] B. Franchi, R. Serapioni, F. Serra Cassano, Irregular solutions of linear degenerate elliptic equations, Potential Anal. 9 (1998) 201–216.
[18] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, 1998.
[19] E.Ya. Khruslov, V.A. Marchenko, Homogenization of Partial Differential Equations, Prog. Math. Phys., vol. 46, Birkhäuser, Boston, 2006.
[20] U. Mosco, Composite media and asymptotic Dirichlet forms, J. Funct. Anal. 123 (2) (1994) 368–421.
[21] F. Murat, L. Tartar, H-convergence, in: L. Cherkaev, R.V. Kohn (Eds.), Topics in the Mathematical Modelling of Composite Materials, in: Progr. Nonlinear Differential Equations Appl., Birkhäuser, Boston, 1998, pp. 21–43.
[22] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. 80 (1958) 931–954.
[23] J. Serrin, Pathological solutions of elliptic differential equations, Ann. Sc. Norm. Super. Pisa Cl. Sci. 18 (3) (1964) 385–387.
[24] S. Spagnolo, Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche, Ann. Sc. Norm. Super. Pisa Cl. Sci. 22 (3) (1968) 571–597.
[25] L. Tartar, Cours Peccot, Collège de France, 1977, partly written in [21].