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Average radial integrability spaces of analytic functions [☆]



Tanausú Aguilar-Hernández ^a, Manuel D. Contreras ^{a,*},
Luis Rodríguez-Piazza ^b

^a *Departamento de Matemática Aplicada II and IMUS, Escuela Técnica Superior de Ingeniería, Universidad de Sevilla, Camino de los Descubrimientos, s/n 41092, Sevilla, Spain*

^b *Departamento de Análisis Matemático and IMUS, Facultad de Matemáticas, Universidad de Sevilla, Calle Tarfia, s/n 41012, Sevilla, Spain*

ARTICLE INFO

Article history:

Received 27 February 2020

Accepted 28 September 2021

Available online 12 October 2021

Communicated by K. Seip

MSC:

primary 30H20, 47B33, 47D06

secondary 46E15, 47G10

Keywords:

Mixed norm spaces

Radial integrability

Bergman projection

ABSTRACT

In this paper we introduce the family of spaces $RM(p, q)$, $1 \leq p, q \leq +\infty$. They are spaces of holomorphic functions in the unit disc with average radial integrability. This family contains the classical Hardy spaces (when $p = \infty$) and Bergman spaces (when $p = q$). We characterize the inclusion between $RM(p_1, q_1)$ and $RM(p_2, q_2)$ depending on the parameters. For $1 < p, q < \infty$, our main result provides a characterization of the dual spaces of $RM(p, q)$ by means of the boundedness of the Bergman projection. We show that $RM(p, q)$ is separable if and only if $q < +\infty$. In fact, we provide a method to build isomorphic copies of ℓ^∞ in $RM(p, \infty)$.

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[☆] This research was supported in part by Ministerio de Economía y Competitividad, Spain, and the European Union (FEDER), project PGC2018-094215-B-I00, and Junta de Andalucía, FQM133 and FQM-104.

* Corresponding author.

E-mail addresses: taguilar@us.es (T. Aguilar-Hernández), contreras@us.es (M.D. Contreras), piazza@us.es (L. Rodríguez-Piazza).

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1. Introduction

In 1923, the classical Hardy spaces H^p were introduced by F. Riesz [16]. He named those spaces after the article of G.H. Hardy [13]. Subsequently, Bergman developed his theory of reproducing kernels in A^2 spaces of general domains [4] and the theory of A^p spaces continued developing throughout the 20th century by the works of many authors; see monographs [8] and [14]. Since then, great progress has been made in the study of these and other spaces of analytic functions in the unit disc. In most of the cases, the belonging to the space is given in terms of boundedness (or integrability) of a certain average of the function on circles centered at the origin or in terms of the integrability with respect to the Lebesgue area, maybe with a certain weight. There are many other good books about these spaces, but we single out [7,10,15].

In other less studied cases, the belonging is determined by the average radial integrability. Maybe the most well-known space in this situation is the space of analytic functions of bounded radial variation BRV, a topic that goes back to Zygmund and where many different authors have worked (see, e.g., the papers of Bourgain [5], Rudin [17], and Zygmund [20]). The space BRV of analytic functions with bounded radial variation consists of those holomorphic functions $g \in \mathcal{H}(\mathbb{D})$ such that

$$\sup_{\theta} \int_0^1 |g'(te^{i\theta})| dt < \infty.$$

Other different situation where the radial integrability plays an important role is in the Riesz-Féjér Theorem which says that there is a constant $C_p > 0$ such that if f belongs to the Hardy space H^p then

$$\sup_{\theta} \left(\int_0^1 |f(re^{i\theta})|^p dr \right)^{1/p} \leq C_p \|f\|_{H^p}. \quad (1.1)$$

The left-hand side of (1.1), considered as a function in the variables θ and r , is the norm of f in the space $L^\infty(\mathbb{T}, L^p[0, 1])$. This paper is devoted to introducing and studying the family of spaces $RM(p, q)$ of analytic functions on the disk \mathbb{D} such that $f \in L^q(\mathbb{T}, L^p[0, 1])$ (Definition 2.1). This family of spaces contains the Bergman spaces (when $p = q$) and Hardy spaces (when $p = \infty$).

As far as we know, there is no systematic study of spaces of average radial integrability. A second part of this research will appear in [1] where Littlewood-Paley type inequalities and integration operators are analyzed in the setting of these spaces.

In Section 2, we introduce the family of spaces $RM(p, q)$ and show a range of examples. Among them, we point out Proposition 2.5 where we characterize lacunary series belonging to $RM(p, q)$. We analyze other properties such as boundedness of point-evaluation functionals and separability. We show that $RM(p, q)$ is separable if and only if $q < +\infty$ (see Proposition 2.13 and Theorem 2.18). In fact, $RM(p, \infty)$ always contains a subspace isomorphic to ℓ^∞ (Theorem 2.18).

The main results of the paper appear in Section 3 and 4. In Section 3 we provide a complete characterization of when one of such spaces is included in another one (Theorem 3.3) and, in such a case, we characterize when the inclusion mapping is compact (Theorem 3.8). As a byproduct of such characterization, we see that the converse of (1.1) does not hold, that is, there are holomorphic functions f in \mathbb{D} such that $\sup_\theta \left(\int_0^1 |f(re^{i\theta})|^p dr \right)^{1/p} < +\infty$ but $f \notin H^p$.

In the last section of this article we show the boundedness of the Bergman projection from $L^q(\mathbb{T}, L^p[0, 1])$ onto our spaces $RM(p, q)$ when $1 < p, q < \infty$. This allows us to identify the dual space of $RM(p, q)$ for $1 < p, q < +\infty$ (Corollary 4.8). The proof of the boundedness of the Bergman projection depends on techniques and tools coming from Harmonic Analysis. In particular, we use a classical result of C. Fefferman and E. Stein. The case $p = q$ gives the well-known boundedness of the Bergman projection from $L^p(\mathbb{D})$ onto the Bergman space A^p , which is usually proved by different techniques that do not work in our situation.

Throughout the paper the letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \leq Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we will write $a \asymp b$.

Acknowledgment. The authors are grateful to the anonymous referee for a very careful reading of the manuscript and a number of helpful comments.

2. Definition and first properties

We start this section introducing the spaces which are the goals of our study and providing some different kind of functions that belongs to them. In addition, we deal with some properties of such spaces, as for instance the separability.

Definition 2.1. Let $0 < p, q \leq +\infty$. We define the spaces of analytic functions

$$RM(p, q) = \{f \in \mathcal{H}(\mathbb{D}) : \rho_{p,q}(f) < +\infty\}$$

where

$$\rho_{p,q}(f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^1 |f(re^{it})|^p dr \right)^{q/p} dt \right)^{1/q}, \quad \text{if } p, q < +\infty,$$

$$\rho_{p,\infty}(f) = \operatorname{ess\,sup}_{t \in [0, 2\pi)} \left(\int_0^1 |f(re^{it})|^p dr \right)^{1/p}, \quad \text{if } p < +\infty,$$

$$\rho_{\infty,q}(f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\sup_{r \in [0,1)} |f(re^{it})| \right)^q dt \right)^{1/q}, \quad \text{if } q < +\infty,$$

$$\rho_{\infty,\infty}(f) = \|f\|_{H^\infty}.$$

Remark 2.2. In the definition of $\rho_{p,\infty}$ the essential supremum can be replaced by the supremum. Fix $\theta \in [0, 2\pi]$. Since the set of $t \in [0, 2\pi]$ such that $\left(\int_0^1 |f(re^{it})|^p dr \right)^{1/p} \leq \rho_{p,\infty}(f)$ is dense in $[0, 2\pi]$, we can extract a sequence $\{t_n\}$ in this set such that $t_n \rightarrow \theta$. Using Fatou's lemma it follows that $\left(\int_0^1 |f(re^{i\theta})|^p dr \right)^{1/p} \leq \rho_{p,\infty}(f)$.

One can easily check that if $1 \leq p, q \leq +\infty$, then $RM(p, q)$ is a Banach space when we endow it with the norm $\rho_{p,q}$. In fact in this paper, we will be interested only in these cases. So, we will stand most of our results for $1 \leq p, q \leq +\infty$. Nevertheless, sometimes in the proofs of such results considering other values of p and q will help us. In these cases, $\rho_{p,q}$ is a quasi-norm and $RM(p, q)$ is a quasi-Banach space.

For certain parameters p, q these spaces $RM(p, q)$ are well known spaces. Namely, it is clear that $RM(p, p)$ is nothing but the Bergman space A^p , for $0 < p < \infty$. Considering the case $p = +\infty$, one can check that $RM(\infty, q)$ is contained in the Hardy space H^q . On the other hand, by [18, Theorem 17.11(a), p. 340], there is a constant $C = C(q)$ such that if $f \in H^q$, then

$$\int_0^{2\pi} \sup_{r \in [0,1]} |f(re^{i\theta})|^q d\theta \leq \int_0^{2\pi} \sup\{|f(z)|^q : |e^{i\theta} - z| < 3(1 - |z|)\} d\theta \leq C \|f\|_{H^q}^q, \quad (2.1)$$

so that we get that $RM(\infty, q) = H^q$ for all $q \in (0, +\infty]$. Another interesting space that fits in this family is the space of analytic functions of bounded radial variation BRV (see, e.g., [5]), that is the space of analytic functions such that $f' \in RM(1, \infty)$.

2.1. First examples

Example 2.3. For $\alpha > 0$, consider the function $f_\alpha(z) = (1 - z)^{-\alpha}$, where we are using the main branch of the logarithm to define $w^{-\alpha}$. Then f_α belongs to $RM(p, q)$ if and only if $\alpha < \frac{1}{p} + \frac{1}{q}$.

Proof. Assume that $0 < p, q < \infty$. Write $I(t) = \int_0^1 |f_\alpha(re^{it})|^p dr$. Since I is even and decreasing in $[0, \pi]$, we have

$$\int_0^{\pi/4} I(t)^{q/p} dt \leq \rho_{p,q}^q(f_\alpha) = 2 \int_0^{\pi} I(t)^{q/p} dt \leq 8 \int_0^{\pi/4} I(t)^{q/p} dt.$$

In addition, for $t \in [0, \pi/4]$, we have that $1 - \cos(t) \asymp t^2/2$. Therefore,

$$\rho_{p,q}^q(f_\alpha) \asymp \int_0^{\pi/4} \left[\int_0^1 \frac{1}{((1-r)^2 + rt^2)^{\alpha p/2}} dr \right]^{q/p} dt.$$

With a similar argument, we can reduce the integral in r to the interval $[1/2, 1]$ and using that when r runs through this interval, the function rt^2 is equivalent to t^2 we have

$$\rho_{p,q}^q(f_\alpha) \asymp \int_0^{\pi/4} \left[\int_{1/2}^1 \frac{1}{((1-r)^2 + t^2)^{\alpha p/2}} dr \right]^{q/p} dt. \tag{2.2}$$

If $\alpha \geq \frac{1}{p} + \frac{1}{q}$, and $t \in [0, 1/2]$, then

$$\int_{1/2}^1 \frac{1}{((1-r)^2 + t^2)^{\alpha p/2}} dr \geq \int_{1-t}^1 \frac{1}{((1-r)^2 + t^2)^{\alpha p/2}} dr \geq \int_{1-t}^1 \frac{1}{(2t^2)^{\alpha p/2}} dr = \frac{1}{2^{\alpha p/2}} \frac{1}{t^{\alpha p-1}}.$$

Thus

$$\int_0^{\pi/4} \left[\int_{1/2}^1 \frac{1}{((1-r)^2 + t^2)^{\alpha p/2}} dr \right]^{q/p} dt \geq \frac{1}{2^{\alpha q/2}} \int_0^{1/2} \left[\frac{1}{t^{\alpha p-1}} \right]^{q/p} dt = +\infty,$$

and so, by (2.2), f_α does not belong to $RM(p, q)$.

Assume now that $\alpha < \frac{1}{p} + \frac{1}{q}$. If $\alpha p < 1$, then

$$\int_{1/2}^1 \frac{1}{((1-r)^2 + t^2)^{\alpha p/2}} dr \leq \int_{1/2}^1 \frac{1}{(1-r)^{\alpha p}} dr < +\infty,$$

so that, by (2.2), $f_\alpha \in RM(p, q)$. If $\alpha p = 1$, then we obtain

$$\int_{1/2}^1 \frac{1}{((1-r)^2 + t^2)^{\frac{\alpha p}{2}}} dr \leq \int_{1-t}^1 \frac{1}{t} dr + \int_{1/2}^{1-t} \frac{1}{1-r} dr \leq \ln\left(\frac{e}{2t}\right).$$

Integrating with respect to t it follows that $\rho_{p,q}(f_\alpha) < +\infty$.

It remains to see what happens if $1 < \alpha p < 1 + \frac{p}{q}$. In this case, if $t \in [0, \pi/4]$, we have

$$\int_{1/2}^1 \frac{1}{((1-r)^2 + t^2)^{\alpha p/2}} dr \leq \int_{1/2}^{1-t} \frac{1}{(1-r)^{\alpha p}} dr + \int_{1-t}^1 \frac{1}{t^{\alpha p}} dr \leq \frac{\alpha p}{\alpha p - 1} \frac{1}{t^{\alpha p - 1}}.$$

Therefore, by (2.2),

$$\rho_{p,q}^q(f_\alpha) \lesssim \left(\frac{\alpha p}{\alpha p - 1}\right)^{q/p} \int_0^{\pi/4} \frac{1}{t^{\alpha q - \frac{q}{p}}} dt < +\infty.$$

Summing up, the result holds if both p and q are finite. For $p = \infty$, since $RM(\infty, q) = H^q$, the result is well-known (see, e.g., [7, Page 13]).

For $q = \infty$, arguing as above we have

$$\rho_{p,\infty}^p(f_\alpha) \asymp \sup_{0 \leq t \leq \pi/2} \int_{1/2}^1 \frac{1}{((1-r)^2 + t^2)^{\alpha p/2}} dr = \int_{1/2}^1 \frac{dr}{(1-r)^{\alpha p}} < +\infty \tag{2.3}$$

if and only if $\alpha < \frac{1}{p}$.

Finally it is clear that the function f_α does not belong to H^∞ . \square

Example 2.4. Let $1 \leq p, q < \infty$, $n \geq 1$ and take α such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha}$. The $RM(p, q)$ -norm of the holomorphic function

$$f_{n,\alpha}(z) = \left(\sum_{k=0}^n z^k\right)^{1/\alpha} = \left(\frac{1 - z^{n+1}}{1 - z}\right)^{1/\alpha},$$

where we are using the main branch of the logarithm to define $w^{1/\alpha}$, can be estimated as

$$\rho_{p,q}(f_{n,\alpha}) \lesssim \left(\frac{p}{p - \alpha}\right)^{1/p} \ln^{1/q}(n + 1). \tag{2.4}$$

Proof. Clearly, $f_{n,\alpha}$ is well-defined. Since $\rho_{p,q}(f_{n,\alpha}) = \rho_{p/\alpha, q/\alpha}^{1/\alpha}(f_{n,1})$, the proof of (2.4) is reduced to the case $\alpha = 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Notice that, in this case, $p > 1$.

Since

$$\int_{\pi/4}^{\pi} \left(\int_0^1 |f_{n,1}(re^{i\theta})|^p dr \right)^{q/p} d\theta \leq \int_{\pi/4}^{\pi} \left(\int_0^1 |4|^p dr \right)^{q/p} d\theta = \frac{3\pi}{4} 4^q,$$

we have

$$2\pi \rho_{p,q}(f_{n,1})^q \leq 2 \int_0^{\pi/4} \left(\int_0^1 |f_{n,1}(re^{i\theta})|^p dr \right)^{q/p} d\theta + \frac{3\pi}{4} 4^q.$$

If $1 - \theta \leq r \leq 1$ and $\theta \in [0, \pi/4]$, arguing as in Example 2.3, we obtain

$$|f_{n,1}(re^{i\theta})| \lesssim \frac{2}{\sqrt{(1-r)^2 + r\theta^2}} \leq \frac{2}{\theta\sqrt{1-\theta}} \leq \frac{2}{\theta\sqrt{1-\frac{\pi}{4}}} < \frac{5}{\theta}.$$

Therefore, there is a constant $C > 2$ such that

$$\begin{aligned} & \int_{\frac{1}{n+1}}^{\pi/4} \left(\int_0^1 |f_{n,1}(re^{i\theta})|^p dr \right)^{q/p} d\theta = \\ & \leq \int_{\frac{1}{n+1}}^{\pi/4} \left(\int_0^{1-\theta} \frac{2^p}{(1-r)^p} dr + \int_{1-\theta}^1 \frac{C^p}{\theta^p} dr \right)^{q/p} d\theta \\ & \leq C^q \int_{\frac{1}{n+1}}^{\pi/4} \left(\frac{p}{p-1} \theta^{-p+1} \right)^{q/p} d\theta = C^q \left(\frac{p}{p-1} \right)^{q/p} (\ln(\pi/4) + \ln(n+1)) \end{aligned}$$

and

$$\begin{aligned} & \int_0^{\frac{1}{n+1}} \left(\int_0^1 |f_{n,1}(re^{i\theta})|^p dr \right)^{q/p} d\theta = \\ & \leq 2^q \int_0^{\frac{1}{n+1}} \left(\int_0^{1-\frac{1}{n+1}} \frac{1}{(1-r)^p} dr + \int_{1-\frac{1}{n+1}}^1 (n+1)^p dr \right)^{q/p} d\theta \leq 2^q \left(\frac{p}{p-1} \right)^{q/p}. \end{aligned}$$

Putting altogether, we get the estimation of $\rho_{p,q}(f_{n,\alpha})$. \square

Next example provides the lacunary series that belong to $RM(p, q)$. For $p = \infty$, that is for Hardy spaces, the characterization is different and it can be seen in [15,

Theorem 6.2.2] for $q < +\infty$ and in [21, Vol. I, p. 247] for $q = \infty$. We will say that a sequence of positive numbers $\{x_k\}$ is a lacunary sequence if there is a constant λ such that $\frac{x_{k+1}}{x_k} \geq \lambda > 1$.

Proposition 2.5. *Let $\{n_k\}_{k=0}^\infty$ be a lacunary sequence of positive integer numbers, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Then*

$$f(z) = \sum_{k=0}^\infty \alpha_k z^{n_k}$$

belongs to $RM(p, q)$ if and only if

$$\sum_{k=0}^\infty \frac{|\alpha_k|^p}{n_k} < +\infty.$$

Moreover, it is satisfied that

$$\rho_{p,q}(f) \asymp \left(\sum_{k=0}^\infty \frac{|\alpha_k|^p}{n_k} \right)^{1/p}. \tag{2.5}$$

Remark 2.6. Notice that the term on the right in (2.5) does not depend on q .

Proof. Notice that $\left\{n_k + \frac{1}{p}\right\}_{k \geq 0}$ is also a lacunary sequence. The proof of this result is based on a characterization of bases on $L^p[0, 1]$ due to Gurariĭ and Macaev [12]. Namely they proved that, fixed $p \in [1, +\infty)$, if a sequence $\{n_k\}_{k \geq 0}$ is lacunary then there exist two positive constants A and B such that

$$A \left(\sum_{k=0}^\infty |\beta_k|^p \right)^{1/p} \leq \left\| \sum_{k=0}^\infty \beta_k \sqrt[p]{n_k + 1/p} t^{n_k} \right\|_{L^p} \leq B \left(\sum_{k=0}^\infty |\beta_k|^p \right)^{1/p}, \tag{2.6}$$

for every $\{\beta_k\} \in \ell^p$.

Take now $f(z) = \sum_{k=0}^\infty \alpha_k z^{n_k}$ a holomorphic function in the unit disc. Fix $\theta \in [0, 2\pi]$ and write $\beta_k := \frac{\alpha_k}{\sqrt[p]{n_k + 1/p}} e^{i\theta n_k}$ if $k \geq 0$. By (2.6),

$$\begin{aligned} A \left(\sum_{k=0}^\infty \frac{|\alpha_k|^p}{n_k + 1/p} \right)^{1/p} &\leq \left\| \sum_{k=0}^\infty \alpha_k r^{n_k} e^{i\theta n_k} \right\|_{L^p} = \left(\int_0^1 |f(re^{i\theta})|^p dr \right)^{1/p} \\ &\leq B \left(\sum_{k=0}^\infty \frac{|\alpha_k|^p}{n_k + 1/p} \right)^{1/p}. \end{aligned}$$

Now, looking at the very definition of $\rho_{p,q}$ we get the result. \square

2.2. Point-evaluation functionals

This subsection is devoted to the functionals $f \mapsto f(z)$ and $f \mapsto f'(z)$. We prove that both of them are bounded and estimate their norms. We will need the following inclusion.

Proposition 2.7. *Let $0 < s \leq +\infty$. Then $H^s \subset RM(p, q)$ if and only if $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{s}$.*

Proof. The result is clear for $s = +\infty$ since H^∞ is a subspace of $RM(p, q)$ for every p, q . Thus, from now on we consider the case $s < +\infty$. Assume that $H^s \subset RM(p, q)$ and suppose that $\frac{1}{s} > \frac{1}{p} + \frac{1}{q}$. Take $\frac{1}{s} > \alpha > \frac{1}{p} + \frac{1}{q}$. Then the function f_α defined in Example 2.3 belongs to $RM(\infty, s) = H^s$ and not to $RM(p, q)$. A contradiction. Thus $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{s}$.

To see that converse implication we claim that $H^s \subset RM(p_1, q_1)$ whenever $1 \leq s < +\infty$ and $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{s}$. Assume for the moment that the claim holds. Fix p and q such that $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{s}$. We consider $p_1 \geq p$ and $q_1 \geq q$ such as $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{s}$. By the claim $H^s \subset RM(p_1, q_1)$. Moreover, it is easy to prove that $RM(p_1, q_1) \subset RM(p, q)$ using Hölder’s inequality twice for $p_1 \geq p$ and $q_1 \geq q$.

Thus it remains to prove the claim. By F ej er-Riesz theorem [7, Theorem 3.13, p. 46], we have that for each $f \in H^s$ and θ ,

$$\int_0^1 |f(re^{i\theta})|^s dr \leq \frac{1}{2} \|f\|_{H^s}^s$$

(notice that, in particular, this implies that $H^s \subset RM(s, \infty)$). Now, since $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{s}$ we can take $\lambda \in [0, 1]$ such that $\frac{1}{p_1} = \frac{\lambda}{s}$ and $\frac{1}{q_1} = \frac{1-\lambda}{s}$. Then

$$\begin{aligned} \rho_{p_1, q_1}(f) &= \left(\int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^{p_1(1-\lambda)} |f(re^{i\theta})|^{p_1\lambda} dr \right)^{q_1/p_1} \frac{d\theta}{2\pi} \right)^{1/q_1} \\ &\leq \left(\int_0^{2\pi} \sup_r |f(re^{i\theta})|^{q_1(1-\lambda)} \left(\int_0^1 |f(re^{i\theta})|^s dr \right)^{q_1\lambda/s} \frac{d\theta}{2\pi} \right)^{1/q_1} \\ &\leq \left(\int_0^{2\pi} \sup_r |f(re^{i\theta})|^s \left(\int_0^1 |f(re^{i\theta})|^s dr \right)^{q_1\lambda/s} \frac{d\theta}{2\pi} \right)^{1/q_1} \\ &\lesssim \|f\|_{H^s}^\lambda \left(\int_0^{2\pi} \sup_r |f(re^{i\theta})|^s \frac{d\theta}{2\pi} \right)^{1/q_1} = \|f\|_{H^s}^\lambda \rho_{\infty, s}(f)^{1-\lambda} \lesssim \|f\|_{H^s}, \end{aligned}$$

where in the last inequality we have used (2.1). Hence, we have proved the claim and we are done. \square

Proposition 2.8. *Let $0 < p, q \leq \infty$ and $z \in \mathbb{D}$. The functional $\delta_z : RM(p, q) \rightarrow \mathbb{C}$ given by $\delta_z(f) := f(z)$, for all $f \in RM(p, q)$, is continuous and*

$$\|\delta_z\|_{(RM(p,q))^*} \asymp \frac{1}{(1 - |z|)^{\frac{1}{p} + \frac{1}{q}}},$$

where the underlying constants depend on p and q .

Proof. Assume $p, q < +\infty$. Given $p_0 > 0$, the subharmonicity of the function $|f|^{p_0}$ shows that for all $z \in \mathbb{D}$,

$$|f(z)|^{p_0} \leq \frac{1}{\pi r^2} \int_{B(z,r)} |f(w)|^{p_0} dA(w)$$

where $r = 1 - |z|$, $B(z, r)$ is the disc centered at z with radius r and $dA(w)$ means integration with respect to the Lebesgue measure on the unit disc \mathbb{D} .

Due to the rotational invariance of the space $RM(p, q)$ we can assume that z belongs to the interval $[0, 1)$. Take $f \in RM(p, q)$. Fix $p_0 > 0$. To prove the result we may assume that $\frac{1}{2} \leq z < 1$. Set $r = 1 - |z|$. Bearing in mind that

$$\arcsin\left(\frac{1 - z}{z}\right) \leq \pi(1 - z)$$

for $\frac{1}{2} \leq z < 1$, we have $|\text{Arg}(w)| \leq \pi r$ for $w \in B(z, r)$. It follows

$$\frac{1}{\pi r^2} \int_{B(z,r)} |f(w)|^{p_0} dA(w) \leq \frac{1}{\pi r^2} \int_{-\pi r}^{\pi r} \left(\int_{1-2r}^1 |f(\rho e^{i\theta})|^{p_0} d\rho \right) d\theta.$$

If $p, q \geq p_0$, applying Hölder’s inequality twice, we get

$$\begin{aligned} \frac{1}{\pi r^2} \int_{-\pi r}^{\pi r} \left(\int_{1-2r}^1 |f(\rho e^{i\theta})|^{p_0} d\rho \right) d\theta &\leq \frac{2}{r^2} \int_{-\pi r}^{\pi r} \left(\int_{1-2r}^1 |f(\rho e^{i\theta})|^p d\rho \right)^{\frac{p_0}{p}} (2r)^{1 - \frac{p_0}{p}} \frac{d\theta}{2\pi} \\ &\leq \frac{2^{2 - \frac{p_0}{p}}}{r^{1 + \frac{p_0}{p}}} \left(\int_{-\pi r}^{\pi r} \left(\int_{1-2r}^1 |f(\rho e^{i\theta})|^p d\rho \right)^{\frac{q}{p}} \frac{d\theta}{2\pi} \right)^{\frac{p_0}{q}} r^{1 - \frac{p_0}{q}} \leq \frac{2^{2 - \frac{p_0}{p}}}{(1 - |z|)^{p_0(\frac{1}{p} + \frac{1}{q})}} \rho_{p,q}^{p_0}(f). \end{aligned}$$

So that

$$|f(z)|^{p_0} \leq \frac{2^{2-\frac{p_0}{p}}}{(1-|z|)^{\frac{p_0}{p}+\frac{p_0}{q}}} \rho_{p,q}(f)^{p_0}. \tag{2.7}$$

Hence δ_z is continuous and $\|\delta_z\| \lesssim 1/(1-|z|)^{\frac{1}{p}+\frac{1}{q}}$.

This argument can be adapted if either p or q is infinite.

To see the converse inequality, take s such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$. Assume $s < +\infty$. By Proposition 2.7, $RM(\infty, s) = H^s \subset RM(p, q)$ and thus

$$\|\delta_z\|_{(RM(p,q))^*} \gtrsim \|\delta_z\|_{(RM(\infty,s))^*} \asymp \|\delta_z\|_{(H^s)^*} = \frac{1}{(1-|z|^2)^{1/s}},$$

where we have used [15, Exercise 2, p. 86] or [10, Exercise 5, p. 85]. If $s = +\infty$, the result follows using constant functions. \square

Proposition 2.9. *Let $1 \leq p, q \leq \infty$ and $z \in \mathbb{D}$. The functional $\delta'_z : RM(p, q) \rightarrow \mathbb{C}$ given by $\delta'_z(f) := f'(z)$, for all $f \in RM(p, q)$, is continuous and*

$$\|\delta'_z\|_{(RM(p,q))^*} \asymp \frac{1}{(1-|z|)^{\frac{1}{p}+\frac{1}{q}+1}},$$

where the underlying constants depend on p and q .

Proof. Again we assume that $z \in [0, 1)$. Fix $z \in [0, 1)$ and denote by C the boundary of the disc centered at z and with radius $(1-|z|)/2$. The Cauchy’s integral formula and the estimate of the point-evaluation functional given in Proposition 2.8 show

$$\begin{aligned} |f'(z)| &\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(z + \frac{1-z}{2}e^{i\theta})|}{((1-z)/2)^2} (1-z)/2 \, d\theta = \frac{1}{\pi} \int_0^{2\pi} \frac{|f(z + \frac{1-z}{2}e^{i\theta})|}{(1-z)} \, d\theta \\ &\lesssim \frac{1}{\pi(1-z)} \int_0^{2\pi} \frac{\rho_{p,q}(f)}{(1-|z + \frac{1-z}{2}e^{i\theta}|)^{\frac{1}{p}+\frac{1}{q}}} \, d\theta \leq \frac{2^{\frac{1}{p}+\frac{1}{q}+1}}{(1-z)^{\frac{1}{p}+\frac{1}{q}+1}} \rho_{p,q}(f). \end{aligned}$$

To prove the converse inequality, we will use a similar argument to the one given in Proposition 2.8. Using Proposition 2.7 we have that $RM(\infty, s) = H^s \subset RM(p, q)$ for $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$. So, it follows

$$\|\delta'_z\|_{(RM(p,q))^*} \gtrsim \|\delta'_z\|_{(RM(\infty,s))^*} \asymp \|\delta'_z\|_{(H^s)^*}.$$

On the one hand, if $s < +\infty$, since for the Hardy space H^s it is known that $\|\delta'_z\|_{(H^s)^*} \asymp \frac{1}{(1-|z|)^{\frac{1}{s}+1}}$ [10, Exercise 5, p. 85], we obtain

$$\|\delta'_z\|_{(RM(p,q))^*} \gtrsim \frac{1}{(1-|z|)^{\frac{1}{s}+1}}.$$

On the other hand, if $s = +\infty$, take the function $\varphi(w) := \frac{w-z}{1-\bar{z}w}$, $w \in \mathbb{D}$. Since φ is an automorphism of the unit disc, we have that $\|\varphi\|_{H^\infty} = 1$ and

$$\|\delta'_z\|_{(H^\infty)^*} \geq |\varphi'(z)| = \frac{1}{1-|z|^2} \geq \frac{1}{2(1-|z|)}.$$

And we end with a similar argument. \square

Combining Propositions 2.8 and 2.9, the next corollary follows.

Corollary 2.10. *Let $1 \leq p, q \leq +\infty$. If $z \in \mathbb{D}$, then*

$$\|\delta'_z\|_{(RM(p,q))^*} \asymp \frac{\|\delta_z\|_{(RM(p,q))^*}}{1-|z|}.$$

2.3. Density of polynomials and separability

Next lemma is obvious if f is continuous on $[0, 1]$ (and then uniformly continuous) and by density of such functions we extend to the whole space:

Lemma 2.11. *Let $1 \leq p < +\infty$ and $f \in L^p([0, 1])$. Then*

$$\lim_{\rho \rightarrow 1} \int_0^1 |f(x) - f(\rho x)|^p dx = 0. \tag{2.8}$$

Given a holomorphic function f in the unit disc and $0 < r < 1$, we define $f_r(z) := f(rz)$, for all $z \in \mathbb{D}$.

Proposition 2.12. *Let $1 \leq p \leq +\infty$, $1 \leq q < +\infty$. If $f \in RM(p, q)$, then $\rho_{p,q}(f - f_r) \rightarrow 0$ when $r \rightarrow 1^-$.*

Proof. For H^p spaces, this result is known [7, Theorem 2.6, p. 21]. Therefore, we can assume that p is finite. We define

$$R_p(\theta, f) = \left(\int_0^1 |f(ue^{\theta i})|^p du \right)^{1/p},$$

what it is well-defined for almost every θ . Easily we can see that $R_p(\theta, f - f_r) \leq R_p(\theta, f_r) + R_p(\theta, f)$. Now we consider $r > 1/2$, then we have

$$R_p(\theta, f_r)^p = \int_0^1 |f(rue^{\theta i})|^p du = \int_0^r |f(ue^{\theta i})|^p \frac{du}{r} \leq \frac{1}{r} R_p(\theta, f)^p < 2R_p(\theta, f)^p.$$

Hence, we have that $R_p(\theta, f - f_r) \leq 3R_p(\theta, f)$. By Lemma 2.11, $R_p(\theta, f - f_r) \rightarrow 0$, when $r \rightarrow 1$. Since the function $[0, 2\pi] \ni \theta \mapsto R_p(\theta, f)$ is integrable, using the dominated convergence theorem we conclude the proof for $p < +\infty$. \square

Proposition 2.13. *Let $1 \leq p \leq +\infty$, $1 \leq q < +\infty$. Polynomials are dense in $RM(p, q)$. In particular, $RM(p, q)$ is a separable space.*

Proof. We will study the cases $1 \leq p, q < \infty$ since it is well-known that polynomials are dense in Hardy spaces $H^q = RM(\infty, q)$ for $0 < q < \infty$. Let $f \in RM(p, q)$. Let us fix $r < 1$. The function f_r is holomorphic on $\frac{1}{r}\mathbb{D}$. Since $\overline{\mathbb{D}} \subset \frac{1}{r}\mathbb{D}$ with $r \in (0, 1)$, the sequence of partial sums $\{P_n\}_n$ of the Taylor expansion of f_r converges uniformly to f_r in $\overline{\mathbb{D}}$. Therefore, polynomials $\{P_n\}_n$ converge in the topology of $RM(p, q)$ to f_r and together with Proposition 2.12 we obtain that polynomials are dense in $RM(p, q)$. This is enough to show the separability. \square

It is well-known that $H^\infty = RM(\infty, \infty)$ is a non-separable Banach space. In order to study the non-separability of $RM(p, \infty)$, for $p < +\infty$, we introduce:

Definition 2.14. Let $1 \leq p < +\infty$. We define the subspace $RM(p, 0)$ of $RM(p, \infty)$

$$RM(p, 0) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \limsup_{\rho \rightarrow 1} \sup_{\theta} \left(\int_{\rho}^1 |f(re^{i\theta})|^p dr \right)^{1/p} = 0 \right\}.$$

It can be proved that $RM(p, 0)$ is a closed subspace of $RM(p, \infty)$, so that it is a Banach space. We will show later that $RM(p, \infty) \neq RM(p, 0)$.

Now, we can provide an analogous statement to Proposition 2.12 for $q = \infty$:

Proposition 2.15. *Let $1 \leq p < +\infty$ and $f \in RM(p, \infty)$. Then $f_r \in RM(p, 0)$. Moreover, $f \in RM(p, 0)$ if and only if*

$$\rho_{p,\infty}(f - f_r) \rightarrow 0 \tag{2.9}$$

when $r \rightarrow 1$.

Proof. Assume that $f \in RM(p, \infty)$. Since $f_r \in H^\infty$, we have $f_r \in RM(p, 0)$. Bearing in mind that $RM(p, 0)$ is closed in $RM(p, \infty)$, we get $f \in RM(p, 0)$ if (2.9) holds.

Assume now that $f \in RM(p, 0)$, we have to see that $\rho_{p,\infty}(f - f_r) \rightarrow 0$. Fix $\varepsilon > 0$. Then there is $\rho_0 < 1$ such that

$$\sup_{\theta} \left(\int_{\rho}^1 |f(se^{i\theta})|^p ds \right)^{1/p} \leq \varepsilon \tag{2.10}$$

for all $\rho_0 \leq \rho < 1$. Take $\rho = (\rho_0 + 1)/2$ and $r < 1$ such that $r\rho > \rho_0$. Since f_r converges to f uniformly on the $\rho\overline{\mathbb{D}}$, we have

$$\lim_{r \rightarrow 1} \sup_{\theta} \int_0^{\rho} |f(se^{\theta i}) - f_r(se^{\theta i})|^p ds = 0.$$

Bearing in mind (2.10), for each θ , we have

$$\begin{aligned} \left(\int_{\rho}^1 |f(se^{\theta i}) - f_r(se^{\theta i})|^p ds \right)^{1/p} &\leq \left(\int_{\rho}^1 |f(se^{\theta i})|^p ds \right)^{1/p} + \left(\int_{\rho}^1 |f_r(se^{\theta i})|^p ds \right)^{1/p} \\ &\leq \varepsilon + \frac{1}{r^{1/p}} \left(\int_{r\rho}^r |f(se^{\theta i})|^p ds \right)^{1/p} \leq \varepsilon + \frac{1}{r^{1/p}} \varepsilon. \end{aligned}$$

This implies that $\limsup_{r \rightarrow 1} \rho_{p,\infty}(f - f_r) \leq 2\varepsilon$. Thus $\lim_{r \rightarrow 1} \rho_{p,\infty}(f - f_r) = 0$. \square

A density argument similar to the one used in Proposition 2.13 shows that:

Corollary 2.16. *Let $1 \leq p < +\infty$. Polynomials are dense in $RM(p, 0)$. In particular, $RM(p, 0)$ is a separable space.*

Corollary 2.17. *Let $1 \leq p < +\infty$. If $z \in \mathbb{D}$, then*

$$\|\delta_z\|_{(RM(p,0))^*} = \|\delta_z\|_{(RM(p,\infty))^*} \quad \text{and} \quad \|\delta'_z\|_{(RM(p,0))^*} = \|\delta'_z\|_{(RM(p,\infty))^*}.$$

In particular,

$$\|\delta'_z\|_{(RM(p,0))^*} \asymp \frac{\|\delta_z\|_{(RM(p,0))^*}}{1 - |z|^2}.$$

Proof. Since $RM(p, 0) \subset RM(p, \infty)$, we have that $\|\delta_z\|_{(RM(p,0))^*} \leq \|\delta_z\|_{(RM(p,\infty))^*}$ and $\|\delta'_z\|_{(RM(p,0))^*} \leq \|\delta'_z\|_{(RM(p,\infty))^*}$.

Let us see that $\|\delta_z\|_{(RM(p,0))^*} \geq \|\delta_z\|_{(RM(p,\infty))^*}$ and $\|\delta'_z\|_{(RM(p,0))^*} \geq \|\delta'_z\|_{(RM(p,\infty))^*}$. If $f \in RM(p, \infty)$ with $\rho_{p,\infty}(f) = 1$, by Proposition 2.15, $f_r \in RM(p, 0)$ and

$$\rho_{p,\infty}(f_r) = \sup_{\theta} \left(\int_0^1 |f(rue^{i\theta})|^p du \right)^{1/p} = \sup_{\theta} \left(\int_0^r |f(ue^{i\theta})|^p \frac{du}{r} \right)^{1/p} \leq \frac{1}{r^{1/p}} \rho_{p,\infty}(f)$$

Moreover, it is easy to see that $\delta_z(f_r) \rightarrow \delta_z(f)$ and $\delta'_z(f_r) \rightarrow \delta'_z(f)$, when $r \rightarrow 1^-$ for a fixed $z \in \mathbb{D}$.

Fixing $\varepsilon > 0$, there exists $f \in RM(p, \infty)$ with $\rho_{p,q}(f) = 1$ such that

$$|\delta_z(f)| \geq \|\delta_z\|_{(RM(p,\infty))^*} - \varepsilon.$$

In addition, we know that

$$\begin{aligned} |\delta_z(f)| &= |f(z)| = \lim_{r \rightarrow 1} |f(rz)| \\ &= \lim_{r \rightarrow 1} \frac{|\delta_z(f_r)|}{\rho_{p,\infty}(f_r)} \leq \lim_{r \rightarrow 1} \frac{\|\delta_z\|_{(RM(p,0))^*}}{r^{1/p}} = \|\delta_z\|_{(RM(p,0))^*}. \end{aligned}$$

Therefore, it satisfies, for all $\varepsilon > 0$,

$$\|\delta_z\|_{(RM(p,\infty))^*} \leq \|\delta_z\|_{(RM(p,0))^*} + \varepsilon,$$

that is, $\|\delta_z\|_{(RM(p,\infty))^*} \leq \|\delta_z\|_{(RM(p,0))^*}$. The proof for δ'_z can be done in a similar way. \square

The non-separability of $RM(p, \infty)$ is an easy consequence of the following much deeper result.

Theorem 2.18. *Let $1 \leq p < \infty$. Then $RM(p, \infty)$ has a subspace isomorphic to ℓ^∞ . Namely, there is a sequence $\{f_k\}$ of functions in $RM(p, 0)$ such that for every $\{\alpha_k\} \in \ell^\infty$ the series $\sum_{k=0}^\infty \alpha_k f_k$ converges uniformly on compact subsets of \mathbb{D} and the operator*

$$T : \ell^\infty \rightarrow RM(p, \infty) \quad \text{defined by} \quad T(\{\alpha_k\}) := \sum_{k=0}^\infty \alpha_k f_k$$

establishes an isomorphism between ℓ^∞ and $T(\ell^\infty)$. Moreover, $T(\{\alpha_k\}) \in RM(p, 0)$ if and only if $\{\alpha_k\} \in c_0$. In particular, $\sum_{k=0}^\infty f_k \in RM(p, \infty) \setminus RM(p, 0)$.

Proof. For each $k = 0, 1, 2, \dots$, take $r_k = 2^{-(k+1)}$, $a_k = 1 + 14^{-(k+1)}$, and

$$\varepsilon_k = \frac{\sqrt[p]{2p-1}}{2^{(k+1)(2-1/p)}} \frac{1}{\sqrt[p]{7^{(k+1)(2p-1)} - 1}}.$$

It is clear that $\sum_{k=0}^\infty r_k = 1$,

$$\begin{aligned} \sum_{k=0}^\infty \frac{\varepsilon_k}{r_k^2} &\leq \sqrt[p]{2p-1} \sum_{k=0}^\infty \frac{2^{(k+1)/p}}{\sqrt[p]{7^{(k+1)(2p-1)} - 1}} \leq \frac{7 \sqrt[p]{2p-1}}{6} \sum_{k=0}^\infty \frac{2^{k+1}}{\sqrt[p]{7^{(k+1)(2p-1)}}} \\ &\leq \frac{7 \sqrt[p]{2p-1}}{6} \sum_{k=0}^\infty \left(\frac{2}{7}\right)^{k+1} = \frac{7}{15} \sqrt[p]{2p-1} < 1, \end{aligned}$$

and

$$\int_{a_k-r_k}^1 \frac{\varepsilon_k^p}{|a_k-r|^{2p}} dr = \frac{\varepsilon_k^p}{2p-1} \left(\frac{1}{(a_k-1)^{2p-1}} - \frac{1}{r_k^{2p-1}} \right) = 1.$$

In addition we can find a sequence $\{\theta_k\}$ such that the disks $D(a_k e^{i\theta_k}, r_k)$ are pairwise disjoint. For that, we consider

$$\theta_k = \arcsin(r_k) + 2 \sum_{n=0}^{k-1} \arcsin(r_n).$$

It is easy to see that $D(a_k e^{i\theta_k}, r_k) \cap D(a_{k+1} e^{i\theta_{k+1}}, r_{k+1}) = \emptyset$, because

$$\theta_{k+1} - \theta_k = \arcsin(r_{k+1}) + \arcsin(r_k).$$

Moreover, it is also obtained that

$$|\theta_k| \leq \frac{\pi}{2} r_k + \pi \sum_{n=0}^{k-1} r_n < \pi \sum_{n=0}^k r_n < \pi.$$

Finally, take $f_k(z) := \frac{\varepsilon_k}{(z e^{-i\theta_k} - a_k)^2}$, $z \in \mathbb{C} \setminus \{a_k e^{i\theta_k}\}$. Since f_k is bounded in \mathbb{D} , it belongs to $RM(p, 0)$. In addition, we have that $|f_k(z)| \leq \frac{\varepsilon_k}{r_k^2}$ if $z \notin D(a_k e^{i\theta_k}, r_k)$.

Since $\sum_{k=0}^{\infty} \frac{\varepsilon_k}{r_k^2} < \infty$, it is easy to see that, given a bounded sequence $\{\alpha_k\}$, the sequence $\{\sum_{n=0}^k \alpha_n f_n(z)\}$ converges uniformly on compacta of \mathbb{D} to $T(\{\alpha_k\}) := \sum_{k=0}^{\infty} \alpha_k f_k$, so that $T(\{\alpha_k\})$ is holomorphic in \mathbb{D} .

By construction, every radius $L_\theta = \{t e^{i\theta} : t \in [0, 1)\}$ only intersects one of the open disks. Let us see that $f = T(\{\alpha_k\}) \in RM(p, \infty)$. On the one hand, if $\theta \in [0, 2\pi]$ is such that there is k_0 with $L_\theta \cap D(a_{k_0} e^{i\theta_{k_0}}, r_{k_0}) \neq \emptyset$, Then

$$\begin{aligned} \left(\int_0^1 |f(re^{i\theta})|^p dr \right)^{1/p} &\leq |\alpha_{k_0}| \left(\int_0^1 |f_{k_0}(re^{i\theta})|^p dr \right)^{1/p} + \sum_{j=0}^{k_0-1} |\alpha_j| \frac{\varepsilon_j}{r_j^2} \\ &\leq \|\{\alpha_k\}\|_{\ell^\infty} \left(\left(\int_0^1 |f_{k_0}(re^{i\theta_{k_0}})|^p dr \right)^{1/p} + 1 \right) \\ &\leq \|\{\alpha_k\}\|_{\ell^\infty} \left(\left(1 + (a_{k_0} - r_{k_0}) \frac{\varepsilon_{k_0}^p}{r_{k_0}^{2p}} \right)^{1/p} + 1 \right) \leq 3 \|\{\alpha_k\}\|_{\ell^\infty}. \end{aligned}$$

On the other hand, if $\theta \in [0, 2\pi]$ is such that $e^{i\theta} \notin D(a_k e^{i\theta_k}, r_k)$ for all k , then

$$\left(\int_0^1 |f(re^{i\theta})|^p dr \right)^{1/p} \leq \| \{ \alpha_k \} \|_{\ell^\infty} \sum_{k=0}^\infty \frac{\varepsilon_k}{r_k^2} \leq \| \{ \alpha_k \} \|_{\ell^\infty}.$$

That is $f = T(\{ \alpha_k \}) \in RM(p, \infty)$ and, in particular, $T : \ell^\infty \rightarrow RM(p, \infty)$ is bounded.

Let us see that T is open from ℓ^∞ to $T(\ell^\infty)$ so that it establishes an isomorphism between ℓ^∞ and $T(\ell^\infty)$. For each n , it follows

$$\begin{aligned} \rho_{p,\infty}(T(\{ \alpha_k \})) &\geq \left(\int_0^1 |T(\{ \alpha_k \})(re^{i\theta_n})|^p dr \right)^{1/p} \geq \left(\int_{a_n-r_n}^1 |T(\{ \alpha_k \})(re^{i\theta_n})|^p dr \right)^{1/p} \\ &\geq |\alpha_n| - \| \{ \alpha_k \} \|_{\ell^\infty} \left(\sum_{j=0}^\infty \frac{\varepsilon_j}{r_j^2} \right) (1 - a_n + r_n)^{1/p} \geq |\alpha_n| - \| \{ \alpha_k \} \|_{\ell^\infty} \left(\sum_{j=0}^\infty \frac{\varepsilon_j}{r_j^2} \right). \end{aligned}$$

Therefore, $|\alpha_n| \leq \| \{ \alpha_k \} \|_{\ell^\infty} \left(\sum_{j=0}^\infty \frac{\varepsilon_j}{r_j^2} \right) + \rho_{p,\infty}(T(\{ \alpha_k \}))$ and taking supremum in n we obtain

$$\rho_{p,\infty}(T(\{ \alpha_k \})) \geq \left(1 - \sum_{j=0}^\infty \frac{\varepsilon_j}{r_j^2} \right) \| \{ \alpha_k \} \|_{\ell^\infty}.$$

Since $\sum_{j=0}^\infty \frac{\varepsilon_j}{r_j^2} < 1$, we get that T establishes an isomorphism between ℓ^∞ and $T(\ell^\infty)$.

To end the proof, we show that $T(\{ \alpha_k \}) \in RM(p, 0)$ if and only if $\{ \alpha_k \} \in c_0$.

Let $T(\{ \alpha_k \}) \in RM(p, 0)$. Then

$$\begin{aligned} \sup_\theta \left(\int_{a_n-r_n}^1 |T(\{ \alpha_k \})(re^{i\theta})|^p dr \right)^{1/p} &\geq \left(\int_{a_n-r_n}^1 |T(\{ \alpha_k \})(re^{i\theta_n})|^p dr \right)^{1/p} \\ &\geq |\alpha_n| - \| \{ \alpha_k \} \|_{\ell^\infty} \left(\sum_{j=0}^\infty \frac{\varepsilon_j}{r_j^2} \right) (1 - a_n + r_n)^{1/p}. \end{aligned}$$

Since $1 - a_n + r_n \rightarrow 0$ and $T(\{ \alpha_k \}) \in RM(p, 0)$, it follows that $\{ \alpha_k \} \in c_0$.

Conversely, let $\alpha = \{ \alpha_k \}_k \in c_0$ and let us prove that $T(\alpha) \in RM(p, 0)$. Since $f_k \in RM(p, 0)$, then $\sum_{k=1}^n \alpha_k f_k \in RM(p, 0)$ for all $n \in \mathbb{N}$. Moreover, $\sum_{k=1}^n \alpha_k f_k \rightarrow T(\alpha)$ because T is continuous and $(\alpha_1, \dots, \alpha_n, 0, 0, \dots) \rightarrow \alpha$ in ℓ^∞ . Finally, $T(\alpha) \in RM(p, 0)$ since $RM(p, 0)$ is a closed subspace of $RM(p, \infty)$. \square

3. Containment relationships

3.1. Inclusions

In this section we will give a characterization for the containment relationships between our spaces. To do this, we recall the notion of the Marcinkiewicz spaces $L^{p,\infty}$, also called the weak L^p spaces.

Definition 3.1. Let $0 < p < \infty$ and (X, Σ, μ) a measure space. We define the weak L^p space of measurable functions

$$L^{p,\infty} = \left\{ f : X \rightarrow \mathbb{C} \text{ measurable} : \|f\|_{p,\infty} := \sup_{t>0} t \lambda_f^{1/p}(t) < \infty \right\}$$

where

$$\lambda_f(t) = \mu(\{x \in X : |f(x)| > t\}).$$

Lemma 3.2. [11, Proposition 1.1.14, p. 8] Let $f \in L^{p_0,\infty} \cap L^{p_1,\infty}$ with $p_0 \neq p_1$. Then $f \in L^p$ for $\frac{1}{p} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}$, $\lambda \in (0, 1)$. Moreover, there exists a constant $C(p_0, p_1, \lambda) > 0$ such that

$$\|f\|_p \leq C(p_0, p_1, \lambda) \|f\|_{p_0,\infty}^{1-\lambda} \|f\|_{p_1,\infty}^\lambda,$$

for $\lambda \in (0, 1)$.

Theorem 3.3. Let $1 \leq p_0, q_0 \leq \infty$ and set

$$A(p_0, q_0) = \left\{ (p, q) \in (0, +\infty] \times (0, +\infty] : \frac{1}{p} + \frac{1}{q} \geq \frac{1}{p_0} + \frac{1}{q_0}, \quad p_0 \geq p \right\}.$$

- (1) If $p_0, q_0 < +\infty$, then $RM(p_0, q_0) \subset RM(p, q)$ if and only if $(p, q) \in A(p_0, q_0) \setminus \{(\beta, \infty)\}$, where $\beta = \frac{p_0 q_0}{p_0 + q_0}$.
- (2) If either p_0 or q_0 are $+\infty$, then $RM(p_0, q_0) \subset RM(p, q)$ if and only if $(p, q) \in A(p_0, q_0)$.

Before proving the result, it is worth showing a picture of the set $A(p_0, q_0)$. If $p_0, q_0 < +\infty$ then the set $\{(\frac{1}{p}, \frac{1}{q}) : (p, q) \in A(p_0, q_0)\}$ is the grey region (including its boundary) in Fig. 1A while if either p_0 or q_0 are $+\infty$, the set $\{(\frac{1}{p}, \frac{1}{q}) : (p, q) \in A(p_0, q_0)\}$ is the grey region (including its boundary) in Fig. 1B or Fig. 1C, respectively.

Proof. Bearing in mind that $H^{q_0} = RM(\infty, q_0)$, Proposition 2.7 is nothing but the case $p_0 = +\infty$. Therefore, from now on, we will assume that $p_0 < +\infty$. To clarify the exposition, we split the proof in several steps.

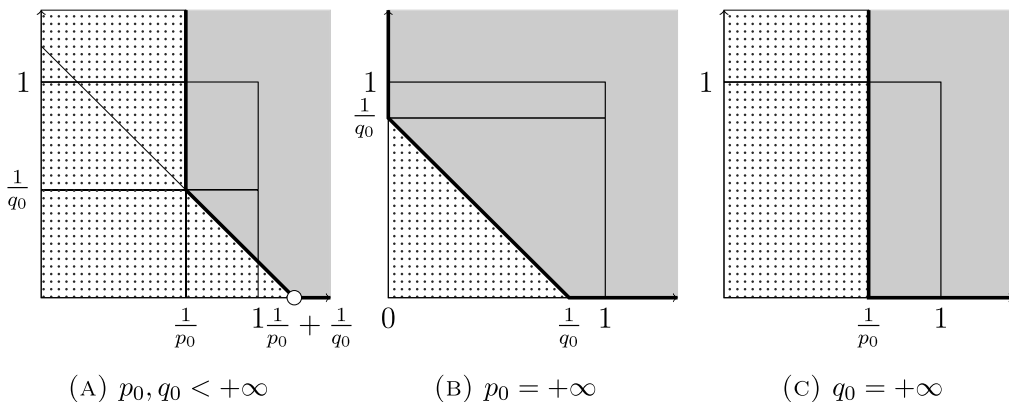


Fig. 1. Containment relationships.

Step 1. If $p_0, q_0 < +\infty$ (Fig. 1A) and (p, q) is such that $(1/p, 1/q)$ belongs to the open segment with end points $(1/p_0, 1/q_0)$ and $(\frac{1}{p_0} + \frac{1}{q_0}, 0)$, then $RM(p_0, q_0) \subset RM(p, q)$.

Write $(\frac{1}{p}, \frac{1}{q}) = \lambda(\frac{1}{p_0}, \frac{1}{q_0}) + (1 - \lambda)(\frac{1}{p_0} + \frac{1}{q_0}, 0)$ for some $\lambda \in (0, 1)$. Take $f \in RM(p_0, q_0)$ with $\rho_{p_0, q_0}(f) \leq 1$. For each θ , define $f_\theta(r) := f(re^{i\theta})$. Let us see that $f_\theta \in L^{p_0, \infty}([0, 1]) \cap L^{\alpha, \infty}([0, 1])$ for almost every $\theta \in [0, 2\pi]$, where $\frac{1}{p_0} + \frac{1}{q_0} = \frac{1}{\alpha}$. Since $f \in RM(p_0, q_0)$, by the very definition, we have that $f_\theta \in L^{p_0}([0, 1])$ for almost every θ and $\|f_\theta\|_{p_0, \infty} \leq \|f_\theta\|_{p_0}$. Moreover, by Proposition 2.8, there is a constant $C > 0$ such that $|f(z)| \leq C \frac{1}{(1 - |z|^2)^{\frac{1}{p_0} + \frac{1}{q_0}}}$, for all z , and thus

$$\begin{aligned} \|f_\theta\|_{\alpha, \infty} &= \sup_{t \geq 0} t m_1(\{r \in [0, 1] : |f_\theta(r)| > t\})^{1/\alpha} \\ &\leq \sup_{t \geq 0} t m_1\left(\left\{r \in [0, 1] : 1 - r \leq \frac{C^\alpha}{t^\alpha}\right\}\right)^{1/\alpha} = \sup_{t \geq 0} \min\{t, C\} \leq C, \end{aligned}$$

so that $f_\theta \in L^{\alpha, \infty}([0, 1])$ for all θ . Hence, applying Lemma 3.2 we have

$$\|f_\theta\|_p \leq C(p_0, \alpha, \lambda) \cdot \|f_\theta\|_{p_0, \infty}^\lambda \cdot \|f_\theta\|_{\alpha, \infty}^{1-\lambda}.$$

Thus

$$\begin{aligned} \rho_{p, q}(f) &\leq C(p_0, \alpha, \lambda) \left(\int_0^{2\pi} \|f_\theta\|_{p_0, \infty}^{\lambda q} \cdot \|f_\theta\|_{\alpha, \infty}^{(1-\lambda)q} d\theta \right)^{1/q} \\ &\leq C(p_0, \alpha, \lambda) C^{1-\lambda} \left(\int_0^{2\pi} \|f_\theta\|_{p_0}^{\lambda q} d\theta \right)^{1/q} = C(p_0, \alpha, \lambda) C^{1-\lambda} \left(\int_0^{2\pi} \|f_\theta\|_{p_0}^{q_0} d\theta \right)^{1/q} \\ &= C(p_0, \alpha, \lambda) C^{1-\lambda} \rho_{p_0, q_0}(f)^\lambda \leq C(p_0, \alpha, \lambda) C^{1-\lambda}. \end{aligned}$$

Step 2. If $\frac{1}{p} > \frac{1}{p_0} + \frac{1}{q_0}$, then $RM(p_0, q_0) \subset RM(p, \infty)$.

Take $f \in RM(p_0, q_0)$. By Proposition 2.8, there is $C > 0$ such that

$$\begin{aligned} \rho_{p,\infty}(f) &= \operatorname{ess\,sup}_{\theta \in [0, 2\pi)} \left(\int_0^1 |f(re^{i\theta})|^p \, dr \right)^{1/p} \\ &\leq C \operatorname{ess\,sup}_{\theta \in [0, 2\pi)} \left(\int_0^1 \frac{\rho_{p_0, q_0}(f)^p}{(1-r)^{\left(\frac{1}{p_0} + \frac{1}{q_0}\right)^p}} \, dr \right)^{1/p} < +\infty \end{aligned}$$

Step 3. If $p_0 \geq p$ and $q_0 \geq q$ then $RM(p_0, q_0) \subset RM(p, q)$.

This inclusion is a direct consequence of Höder’s inequality.

Denote by $B(p_0, q_0) = \{(p, q) \in \mathbb{R}^+ \times \mathbb{R}^+ : p_0 \geq p, q_0 \geq q\}$ and (p_λ, q_λ) the couple such that $\left(\frac{1}{p_\lambda}, \frac{1}{q_\lambda}\right) = \lambda \left(\frac{1}{p_0}, \frac{1}{q_0}\right) + (1-\lambda) \left(\frac{1}{p_0} + \frac{1}{q_0}, 0\right)$. Since

$$A(p_0, q_0) \setminus \left\{ \left(\frac{p_0 q_0}{p_0 + q_0}, \infty \right) \right\} = \cup_{\lambda \in (0, 1]} B(p_\lambda, q_\lambda) \cup \left\{ (p, \infty) : \frac{1}{p} > \frac{1}{p_0} + \frac{1}{q_0} \right\},$$

Steps 1, 2 and 3 give that if $p_0, q_0 < +\infty$ and $(p, q) \in A(p_0, q_0) \setminus \left\{ \left(\frac{p_0 q_0}{p_0 + q_0}, \infty \right) \right\}$ then $RM(p_0, q_0) \subset RM(p, q)$.

Step 4. If $RM(p_0, q_0) \subset RM(p, q)$ then $p_0 \geq p$.

By closed graph theorem there is a constant $C > 0$ such that $\rho_{p,q}(f) \leq C \rho_{p_0, q_0}(f)$ for all $f \in RM(p_0, q_0)$. Taking $f_n(z) = z^n$ we obtain

$$\rho_{p,q}(f_n) = (1 + np)^{-\frac{1}{p}} \leq C(1 + np_0)^{-\frac{1}{p_0}} = \rho_{p_0, q_0}(f_n)$$

and this inequality holds for all n if and only if $p_0 \geq p$.

Step 5. If $\frac{1}{p_0} + \frac{1}{q_0} > \frac{1}{p_1} + \frac{1}{q_1}$ then $RM(p_0, q_0) \not\subset RM(p_1, q_1)$

We consider a function f_α of Example 2.3 such that $\frac{1}{p_1} + \frac{1}{q_1} < \alpha < \frac{1}{p_0} + \frac{1}{q_0}$. Hence, we have a function f_α such that $f_\alpha \in RM(p_0, q_0) \setminus RM(p_1, q_1)$.

Step 6. If $p_0, q_0 < +\infty$, then $RM(p_0, q_0) \not\subset RM(\beta, \infty)$, where $\beta = \frac{p_0 q_0}{p_0 + q_0}$.

Assume that $RM(p_0, q_0) \subset RM(\beta, \infty)$. By closed graph theorem there is a positive constant $C > 0$ such that $\rho_{\beta, \infty}(f) \leq C \rho_{p_0, q_0}(f)$. For each n , consider the function $f_{n, \beta}$ introduced in Example 2.4. Then

$$\rho_{\beta, \infty}(f_{n, \beta}) \geq \left(\int_0^1 \sum_{k=0}^n r^k \, dr \right)^{\frac{1}{\beta}} = \left(\sum_{k=0}^n \frac{1}{k+1} \right)^{\frac{1}{\beta}} \geq \ln^{\frac{1}{\beta}}(n+1).$$

Thus, Example 2.4 would imply

$$\ln^{1/\beta}(n+1) \leq C \left(\frac{p_0}{p_0 - \beta} \right)^{1/p_0} \ln^{1/q_0}(n+1),$$

what is not possible if n is large enough. So $RM(p_0, q_0) \not\subset RM(\beta, \infty)$.

Clearly, Steps 4, 5 and 6 imply that if $RM(p_0, q_0) \subset RM(p, q)$ then $(p, q) \in A(p_0, q_0) \setminus \{(\beta, \infty)\}$. Therefore, statement (1) and (2) are proved. \square

A simple argument shows that if $q < +\infty$, the density of the polynomials in $RM(p, q)$ implies that if $RM(p, q) \subset RM(p_0, \infty)$ if and only if $RM(p, q) \subset RM(p_0, 0)$.

The situation is not so clear to study when $RM(p_0, 0)$ is contained in $RM(p, q)$. To characterize it, we need the following lemma.

Lemma 3.4. *Let $1 \leq p, q \leq +\infty$. If $\{f_n\}$ is a bounded sequence in $RM(p, q)$ that converges uniformly on compact subsets of the unit disc to f . Then $f \in RM(p, q)$.*

Proof. Clearly the function f is holomorphic. Assume that $p, q < +\infty$. By Fatou’s Lemma, for each θ we have

$$\int_0^1 |f(re^{i\theta})|^p dr \leq g(\theta) := \liminf_n g_n(\theta),$$

where, for each n , $g_n(\theta) := \int_0^1 |f_n(re^{i\theta})|^p dr$. Repeating again the argument, we have

$$\begin{aligned} \rho_{p,q}(f)^q &\leq \frac{1}{2\pi} \int_0^{2\pi} (g(\theta))^{q/p} d\theta \leq \liminf_n \frac{1}{2\pi} \int_0^{2\pi} (g_n(\theta))^{q/p} d\theta \\ &= \liminf_n \rho_{p,q}^q(f_n) \leq \sup_n \rho_{p,q}^q(f_n) < +\infty. \end{aligned}$$

A similar argument works in the remaining cases, so that we are done. \square

Proposition 3.5. *Let $1 \leq p_0 \leq +\infty$. Then $RM(p_0, \infty) \subset RM(p, q)$ if and only if $RM(p_0, 0) \subset RM(p, q)$.*

Proof. Assume that $RM(p_0, 0) \subset RM(p, q)$. Take $f \in RM(p_0, \infty)$. For each $r < 1$, the function f_r belongs to $RM(p_0, 0)$ and then to $RM(p, q)$. Since $\{f_r : r < 1\}$ is bounded in $RM(p_0, 0)$, it is also bounded in $RM(p, q)$. Since f_r converges uniformly on compact subset of \mathbb{D} to f , Lemma 3.4 guarantees that $f \in RM(p, q)$. \square

3.2. Compactness of the inclusions

Once the containment relationships of these spaces have been determined, we study when such inclusions are compact.

A standard argument shows the following characterization of compactness.

Lemma 3.6. *Let $1 \leq p_0, q_0 \leq +\infty$ and $1 \leq p, q \leq +\infty$. Then $i : RM(p_0, q_0) \rightarrow RM(p, q)$ is compact if and only if every bounded sequence $\{f_n\}$ in $RM(p_0, q_0)$ that converges to zero uniformly on compact subsets of the unit disc satisfies that $\lim_n \rho_{p,q}(f_n) = 0$.*

We will use this lemma several times in the proof of the next theorem without explicit reference. We also need the following result.

Proposition 3.7. *Let $1 \leq p < +\infty$, $f \in RM(p, \infty)$, and $\sigma \in \partial\mathbb{D}$. Then for the non-tangential limit we have $\angle \lim_{z \rightarrow \sigma} f(z)(1 - \bar{\sigma}z)^{1/p} = 0$.*

Proof. Without loss of generality we assume that $\sigma = 1$. Suppose that $\rho_{p,\infty}(f) \leq 1$ and consider the holomorphic function $h(z) = f(z)(1 - z)^{1/p}$. Fix $R > 1$ and the Stolz region $S(1, R) = \{z \in \mathbb{D} : |1 - z| < R(1 - |z|)\}$. Looking at (2.7) in the proof of Proposition 2.8, we see that there is a constant C such that

$$|h(z)| \leq R^{1/p} |f(z)| (1 - |z|)^{1/p} \leq CR^{1/p}, \quad z \in S(1, R).$$

That is, the function h is bounded on $S(1, R)$. Therefore, by Lindelöf’s Theorem [6, Theorem 1.5.7, p. 26], it is enough to prove that $\lim_{r \rightarrow 1^-} |f(r)|(1 - r)^{1/p} = 0$.

Assume by contradiction that there is a constant $c_1 > 0$ and a sequence $\{r_k\}$ where $r_k \rightarrow 1^-$ such that $c_1 \leq |f(r_k)|(1 - |r_k|)^{1/p}$ for all k . Write $\delta_k := 1 - r_k$. By Proposition 2.9, there is a constant C such that $|f'(x)| \leq \frac{C}{(1-x)^{1+\frac{1}{p}}}$ for all $x \in (0, 1)$. Choose $\varepsilon < \frac{c_1}{2C}$. Then, for $1 - (1 + \varepsilon)\delta_k < x < 1 - \delta_k$,

$$|f(x) - f(r_k)| \leq C|x - r_k| \frac{1}{(1 - r_k)^{1+1/p}} \leq C\varepsilon\delta_k \frac{1}{\delta_k^{1+1/p}} = \frac{C\varepsilon}{\delta_k^{1/p}} < \frac{c_1}{2\delta_k^{1/p}}.$$

Thus

$$|f(x)| \geq |f(r_k)| - |f(x) - f(r_k)| \geq \frac{c_1}{(1 - r_k)^{1/p}} - \frac{c_1}{2\delta_k^{1/p}} = \frac{c_1}{2\delta_k^{1/p}}$$

and

$$\left(\int_{1-(1+\varepsilon)\delta_k}^{1-\delta_k} |f(x)|^p dx \right)^{1/p} \geq (\varepsilon\delta_k)^{1/p} \frac{c_1}{2\delta_k^{1/p}} = \frac{c_1\varepsilon^{1/p}}{2}.$$

Notice that this lower bound does not depend on k . But, this is impossible because $\int_0^1 |f(x)|^p dx < +\infty$. \square

Theorem 3.8. *Let $1 \leq p_0, q_0 \leq +\infty$ and $1 \leq p, q \leq +\infty$. Then $i : RM(p_0, q_0) \rightarrow RM(p, q)$ is compact if and only if $\frac{1}{p} + \frac{1}{q} > \frac{1}{p_0} + \frac{1}{q_0}$ and $p < p_0$.*

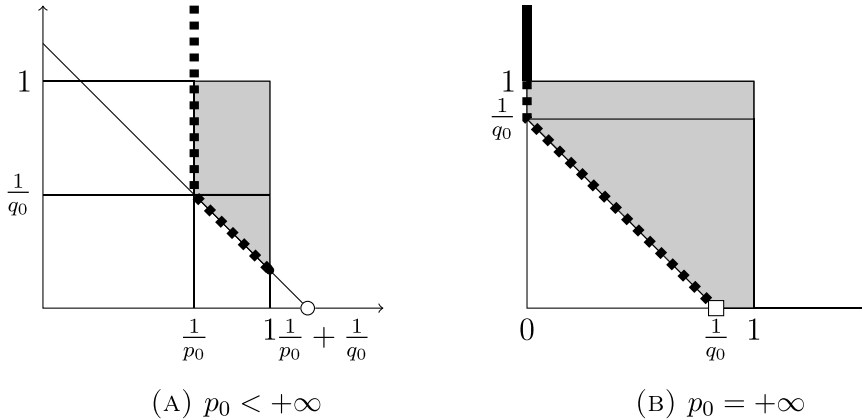


Fig. 2. Compactness of the inclusions.

As we can see in the Fig. 2, the grey region, removing this time the dotted lines, represents the spaces $RM(p, q)$ such as $i : RM(p_0, q_0) \rightarrow RM(p, q)$ is compact when $p_0 < +\infty$ in Fig. 2A and when $p_0 = +\infty$ in Fig. 2B.

Proof. Bearing in mind Theorem 3.3, we have to prove that the inclusion is compact if $\frac{1}{p} + \frac{1}{q} > \frac{1}{p_0} + \frac{1}{q_0}$ and $p < p_0$ and it is not compact if either $\frac{1}{p} + \frac{1}{q} = \frac{1}{p_0} + \frac{1}{q_0}$ or $p = p_0$.

Let us start by showing that it is not compact if $p = p_0$. For each n , consider the function $f_n(z) = (np_0 + 1)^{1/p_0} z^n, z \in \mathbb{D}$. A simple calculation shows that $\rho_{p_0, q_0}(f_n) = 1$ and that the sequence $\{f_n\}$ converges uniformly to zero on compact subsets of the unit disc. Assume that $i : RM(p_0, q_0) \rightarrow RM(p_0, q)$ is compact, then there exists a subsequence $\{f_{n_k}\}$ such that $\rho_{p_0, q}(f_{n_k})$ must go to 0 as k goes to ∞ . But this is not possible because $\rho_{p_0, q}(f_n) = 1$ for all n .

Now take p and q such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{p_0} + \frac{1}{q_0}$. Assume that $i : RM(p_0, q_0) \rightarrow RM(p, q)$ is compact. Then $i^* : (RM(p, q))^* \rightarrow (RM(p_0, q_0))^*$ is also a compact operator.

Assume firstly that $q < +\infty$. Let us see that $\frac{\delta_z}{\|\delta_z\|_{(RM(p, q))^*}} w^*$ -converges to 0 when $|z| \rightarrow 1$. Taking p a polynomial we obtain

$$\frac{|\delta_z(p)|}{\|\delta_z\|_{(RM(p, q))^*}} \asymp |p(z)|(1 - |z|^2)^{\frac{1}{p} + \frac{1}{q}} \leq \|p\|_\infty (1 - |z|^2)^{\frac{1}{p} + \frac{1}{q}},$$

which clearly goes to 0 when $|z| \rightarrow 1$. Since $q < +\infty$, by Proposition 2.13, polynomials are dense in $RM(p, q)$ and then $\frac{\delta_z}{\|\delta_z\|_{(RM(p, q))^*}} w^*$ -converges to 0 when $|z| \rightarrow 1$. Therefore, the compactness of i^* gives

$$\lim_{|z| \rightarrow 1} \left\| i^* \left(\frac{\delta_z}{\|\delta_z\|_{(RM(p, q))^*}} \right) \right\|_{(RM(p_0, q_0))^*} = 0.$$

However, this is impossible because, as we shall now see, such norm must be greater than a certain positive constant. Indeed, since $i^*(\delta_z) = \delta_z$, by Proposition 2.8,

$$\left\| i^* \left(\frac{\delta_z}{\|\delta_z\|_{(RM(p,q))^*}} \right) \right\|_{(RM(p_0,q_0))^*} \geq \frac{\|\delta_z\|_{(RM(p_0,q_0))^*}}{\|\delta_z\|_{(RM(p,q))^*}} \asymp 1.$$

The argument for the case $p_0 = q = \infty$ is the same. However, we consider a sequence $\{z_n\}$ in the Stolz region such that $|z_n| \rightarrow 1$. In this way, we obtain the w^* -convergence bearing in mind Proposition 3.7.

Assume now that $\frac{1}{p_1} > \frac{1}{p_0} + \frac{1}{q_0}$ and take a sequence $\{f_n\}$ in $RM(p_0, q_0)$ such that $\rho_{p_0,q_0}(f_n) \leq 1$ for all n and it converges to zero uniformly on compact subsets of \mathbb{D} . We claim that $\lim_n \rho_{p_1,\infty}(f_n) = 0$. Otherwise, there is $\epsilon > 0$ and a subsequence (that we denote in the same way) such that $\rho_{p_1,\infty}(f_n) > \epsilon$ for all n . Thus, we find $\{\theta_n\}$ such that

$$\int_0^1 |f_n(re^{i\theta_n})|^{p_1} dr \geq \epsilon^{p_1}, \tag{3.1}$$

for all $n \in \mathbb{N}$. For each n , we write $g_n(r) := f_n(re^{i\theta_n})$, $r \in [0, 1]$. Since $\rho_{p_0,q_0}(f_n) \leq 1$, by Proposition 2.8 there is a constant $C > 0$ such that

$$|g_n(r)| = |f_n(re^{i\theta_n})| \leq \frac{C}{(1-r^2)^{\frac{1}{p_0} + \frac{1}{q_0}}}.$$

Since the map $r \mapsto \frac{C}{(1-r^2)^{\frac{1}{p_0} + \frac{1}{q_0}}}$ belongs to $L^{p_1}([0, 1])$ and $\{g_n\}$ converges pointwise to zero, we get that it converges to zero in the norm of $L^{p_1}([0, 1])$ which contradicts (3.1). So that the claim holds.

Take now p, q such that there is $\lambda \in (0, 1)$ with $\left(\frac{1}{p}, \frac{1}{q}\right) = \lambda \left(\frac{1}{p_0}, \frac{1}{q_0}\right) + (1-\lambda) \left(\frac{1}{p_1}, 0\right)$. Then, for each $f \in RM(p, q)$, applying Hölder’s inequality we have

$$\begin{aligned} \int_0^1 |f(re^{i\theta})|^p dr &= \int_0^1 |f(re^{i\theta})|^{\lambda p} |f(re^{i\theta})|^{(1-\lambda)p} dr \\ &\leq \left(\int_0^1 |f(re^{i\theta})|^{p_0} dr \right)^{\lambda p/p_0} \left(\int_0^1 |f(re^{i\theta})|^{p_1} dr \right)^{(1-\lambda)p/p_1}. \end{aligned}$$

So that

$$\rho_{p,q}(f) \leq \rho_{p_0,q_0}(f)^\lambda \rho_{p_1,\infty}(f)^{1-\lambda}.$$

This inequality, the above claim and Lemma 3.6 show that $i : RM(p_0, q_0) \rightarrow RM(p, q)$ is compact whenever $\frac{1}{p} + \frac{1}{q} > \frac{1}{p_0} + \frac{1}{q_0}$ and $q_0 < q$.

Take now p, q such that $\frac{1}{p} + \frac{1}{q} > \frac{1}{p_0} + \frac{1}{q_0}$, $p < p_0$ and $q_0 \geq q$. Fix $\tilde{q} < q$ such that $\frac{1}{p_0} + \frac{1}{\tilde{q}} < \frac{1}{p} + \frac{1}{q}$. By the above argument, the inclusion map \tilde{i} from $RM(p_0, \tilde{q})$ into $RM(p, q)$ is compact. Since $i : RM(p_0, q_0) \rightarrow RM(p, q)$ factorizes through \tilde{i} , we get that i is compact. \square

4. Bergman projection

In the theory of Banach spaces of analytic functions, a useful integral operator is the Bergman projection

$$P(f)(z) = \int_{\mathbb{D}} K(z, w)f(w) dA(w), \quad z \in \mathbb{D},$$

with kernel

$$K(z, w) = (1 - z\bar{w})^{-2}, \quad z, w \in \mathbb{D}, \tag{4.1}$$

which is called the Bergman kernel. Such function is the reproducing kernel for the Bergman space A^2 .

This projection is well-defined on $L^1(\mathbb{D})$, mapping each function of $L^1(\mathbb{D})$ to an analytic function and mapping each function of the Bergman space A^1 into itself. Moreover, for $1 < p < \infty$ it is known that the Bergman projection is a bounded operator from $L^p(\mathbb{D})$ onto A^p . These properties allow to describe the dual of Bergman spaces A^p .

Theorem 4.1. *For $1 < p < +\infty$, the dual space of A^p can be identified with $A^{p'}$, where p' is the conjugated index, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. Such functional $\phi \in (A^p)^*$ has a unique representation*

$$\phi(f) = \phi_g(f) = \int_{\mathbb{D}} f(w)\overline{g(w)} dA(w), \quad f \in A^p,$$

for some $g \in A^{p'}$.

Mimicking this schedule for Bergman spaces (but with a deeper argument), in this section we prove the boundedness of the Bergman projection from $L^q(\mathbb{T}, L^p[0, 1])$ onto $RM(p, q)$, where $1 < p, q < \infty$ and, as a byproduct, we describe its dual.

To study the duality of $RM(p, q)$ spaces, the following theorem will be important because it provides a characterization of the dual space of $L^q(\mathbb{T}, L^p[0, 1])$, for $1 \leq p, q < +\infty$. Let us recall that a measurable function $f : \mathbb{T} \times [0, 1] \rightarrow \mathbb{C}$ belongs to $L^q(\mathbb{T}, L^p[0, 1])$ if $\rho_{p,q}(f) < +\infty$ (here, the definition of $\rho_{p,q}$ is the same as the one given in Definition 2.1 for holomorphic functions).

Theorem 4.2. *[3, Theorem 1, p. 304] Let $1 \leq p, q < +\infty$. $J(f)$ is a continuous functional on the normed space $L^q(\mathbb{T}, L^p[0, 1])$ if and only if it can be represented by*

$$J(f) = \int_{\mathbb{D}} h(w)f(w) dA(w)$$

where $h(w)$ is a uniquely determined function of $L^{q'}(\mathbb{T}, L^{p'}[0, 1])$ and $\|J\| = \rho_{p',q'}(h)$.

Theorem 4.3. *Let $1 < p, q < +\infty$. The Bergman projection P is bounded from the space $L^q(\mathbb{T}, L^p[0, 1])$ onto $RM(p, q)$.*

Since the restriction of P to $RM(p, q)$ is the identity and Pf is analytic for all $f \in L^q(\mathbb{T}, L^p[0, 1])$, in order to prove above theorem it is enough to show that P is bounded from $L^q(\mathbb{T}, L^p[0, 1])$ into itself.

Before going into the proof of this result, we introduce some necessary terminology. In general, given a measurable function $M : \mathbb{D} \times \mathbb{D} \mapsto \mathbb{C}$ we can define the integral operator

$$T_M(f)(re^{i\theta}) = \int_{\mathbb{D}} M(re^{i\theta}, w)f(w) dA(w) \\ = \int_0^{2\pi} \int_0^1 M(re^{i\theta}, \rho e^{i\varphi}) f(\rho e^{i\varphi}) \rho \frac{d\rho d\varphi}{\pi}, \quad r \in [0, 1), \theta \in [0, 2\pi],$$

whenever such integral exists.

From now on, with a little abuse of notation, $|\theta - \varphi|$ will denote the distance between θ and φ in the quotient group $\mathbb{R}/2\pi\mathbb{Z}$, that is, $\min_{k \in \mathbb{Z}} |\theta - \varphi + 2k\pi|$. Notice also that in order to prove the boundedness of $P = T_K$, it is sufficient to check the boundedness of $T_{\tilde{K}}$, where

$$\tilde{K}(re^{i\theta}, \rho e^{i\varphi}) := |K(re^{i\theta}, \rho e^{i\varphi})| \chi_{\{|\theta - \varphi| \leq 1\}},$$

because $|K| - \tilde{K}$ is a bounded function.

Moreover, by showing that $T_D : L^q(\mathbb{T}, L^p[0, 1]) \rightarrow L^q(\mathbb{T}, L^p[0, 1])$ is bounded, where

$$D(\theta, \varphi, r, \rho) = \begin{cases} 0, & \text{if } |\theta - \varphi| \geq 1, \\ \frac{1}{|\varphi - \theta|^2}, & \text{if } 1 \geq |\theta - \varphi| \geq 1 - r\rho, \\ \frac{1}{(1 - r\rho)^2}, & \text{if } |\theta - \varphi| \leq 1 - r\rho \end{cases}$$

we obtain the boundedness of $P : L^q(\mathbb{T}, L^p[0, 1]) \rightarrow L^q(\mathbb{T}, L^p[0, 1])$ since

$$|K(re^{i\theta}, \rho e^{i\varphi})| \chi_{\{|\theta - \varphi| \leq 1\}} \leq 4D(\theta, \varphi, r, \rho).$$

Bearing in mind the change of variable $x = 1 - r$ and $y = 1 - \rho$, it follows that $\frac{\tilde{H}(\theta, \varphi, x, y)}{4} \leq D(\theta, \varphi, 1 - x, 1 - y) \leq \tilde{H}(\theta, \varphi, x, y)$, $x, y \in [0, 1]$, with

$$\tilde{H}(\theta, \varphi, x, y) = \begin{cases} 0, & \text{if } |\theta - \varphi| \geq 1, \\ \frac{1}{|\theta - \varphi|^2}, & \text{if } 1 \geq |\theta - \varphi| \geq \max\{x, y\}, \\ \frac{1}{(\max\{x, y\})^2}, & \text{if } \max\{x, y\} \geq |\theta - \varphi|, \end{cases}$$

because $\max\{x, y\} \leq 1 - r\rho \leq 2 \max\{x, y\}$.

Finally, next lemma shows that the boundedness of the operator $T_{\tilde{H}} : L^q(\mathbb{T}, L^p[0, 1]) \rightarrow L^q(\mathbb{T}, L^p[0, 1])$ is equivalent to the boundedness of $T_H : L^q(\mathbb{T}, L^p[0, 1]) \rightarrow L^q(\mathbb{T}, L^p[0, 1])$, where

$$H(\theta, \varphi, x, y) = \begin{cases} 0, & \text{if } |\varphi - \theta| > 1 \text{ or } \max\{x, y\} > |\varphi - \theta|, \\ \frac{1}{|\varphi - \theta|^2}, & \text{if } 1 \geq |\varphi - \theta| \geq \max\{x, y\}. \end{cases}$$

Remark 4.4. Let $1 \leq p, q < +\infty$ and $a, b \in (0, 1]$. If we have the following relation $J(\theta, \varphi, x, y) = K(\theta, \varphi, ax, by)$ between the kernels J and K , then

$$\|T_J : L^q(\mathbb{T}, L^p[0, 1]) \rightarrow L^q(\mathbb{T}, L^p[0, 1])\| \leq \frac{b^{1/p}}{ba^{1/p}} \|T_K : L^q(\mathbb{T}, L^p[0, 1]) \rightarrow L^q(\mathbb{T}, L^p[0, 1])\|.$$

Lemma 4.5. *The operator $T_H : L^q(\mathbb{T}, L^p[0, 1]) \rightarrow L^q(\mathbb{T}, L^p[0, 1])$ is bounded if and only if the operator $T_{\tilde{H}} : L^q(\mathbb{T}, L^p[0, 1]) \rightarrow L^q(\mathbb{T}, L^p[0, 1])$ is bounded.*

Proof. Clearly, the boundedness of $T_{\tilde{H}}$ implies the boundedness of T_H because $0 \leq H \leq \tilde{H}$. Now, we show the converse implication. First of all, we define the dilated kernels $H_n(\theta, \varphi, x, y) := 2^{-2n}H(\theta, \varphi, 2^{-n}x, 2^{-n}y)$. Using Remark 4.4 and denoting by $\|\cdot\|$ the operator norm from $L^q(\mathbb{T}, L^p[0, 1])$ into itself, we have

$$\|T_{H_n}\| \leq 2^{-n}\|T_H\|. \tag{4.2}$$

Therefore, using the fact that for $\theta \neq \varphi$

$$\tilde{H}(\theta, \varphi, x, y) \leq 3 \sum_{n=0}^{\infty} H_n(\theta, \varphi, x, y)$$

and the previous inequality (4.2), we conclude

$$\|T_{\tilde{H}}\| \leq 3 \sum_{n=0}^{\infty} \|T_{H_n}\| \leq 3 \sum_{n=0}^{\infty} 2^{-n}\|T_H\| \leq 6\|T_H\|,$$

and we are done. \square

Lemma 4.6. *Let $f \in L^q(\mathbb{T}, L^p[0, 1])$, $g \in L^{q'}(\mathbb{T}, L^{p'}[0, 1])$ such that $f, g \geq 0$. Then*

$$\int_0^{2\pi} \int_0^1 (T_H f) g \, dx \, d\theta \leq \int_0^{2\pi} \int_0^1 Rf(\theta, |\varphi - \theta|) Rg(\varphi, |\varphi - \theta|) \, d\theta \, d\varphi,$$

where $Rf(\theta, x) = \begin{cases} \sup_{1 \geq t \geq x} \frac{1}{t} \int_0^t f(\theta, u) \, du, & \text{if } x < 1, \\ 0, & \text{if } x \geq 1. \end{cases}$

Proof. Using the definition of the kernel H and grouping terms, it follows

$$\begin{aligned} \int_0^{2\pi} \int_0^1 T_H f(\theta, x) g(\theta, x) dx d\theta &= \int_0^{2\pi} \int_0^{2\pi} \int_0^1 \int_0^1 H(\theta, \varphi, x, y) f(\varphi, y) g(\theta, x) dx y dy \frac{d\varphi}{\pi} d\theta \\ &\leq \iiint_{0 \leq x, y \leq |\theta - \varphi| \leq 1} \frac{1}{|\theta - \varphi|^2} f(\varphi, y) g(\theta, x) dx dy d\varphi d\theta \\ &= \iint_{|\varphi - \theta| \leq 1} \left(\frac{1}{|\theta - \varphi|} \int_0^{|\theta - \varphi|} f(\varphi, y) dy \right) \left(\frac{1}{|\theta - \varphi|} \int_0^{|\theta - \varphi|} g(\theta, x) dx \right) d\theta d\varphi \\ &\leq \int_0^{2\pi} \int_0^{2\pi} Rf(\varphi, |\varphi - \theta|) Rg(\theta, |\varphi - \theta|) d\theta d\varphi. \quad \square \end{aligned}$$

Remark 4.7. Let $1 < p < \infty$. Notice that if $0 \leq x \leq x_1 \leq 1$ then $Rf(\theta, x) \geq Rf(\theta, x_1)$. Moreover, for $e^{i\theta} \in \mathbb{T}$ fixed we define $f_\theta(x) := f(\theta, x)$. Therefore, since $Rf(\theta, x) \leq Mf_\theta(x)$, where M is the Hardy-Littlewood maximal function, there is a constant $C_p > 0$ such that $\|Rf(\theta, \cdot)\|_{L^p[0,1]} \leq C_p \|f_\theta\|_{L^p[0,1]}$.

Proof of Theorem 4.3. Bearing in mind the notation of the previous lemma, for $f \in L^q(\mathbb{T}, L^p[0, 1])$ and $g \in L^{q'}(\mathbb{T}, L^{p'}[0, 1])$ such that $\rho_{p,q}(f) \leq 1$ and $\rho_{p',q'}(g) \leq 1$ we consider the functions $F = R|f|$ and $G = R|g|$. Moreover, we define the following sequences of functions $f_k(\varphi) = F(\varphi, 2^{-k})$ and $g_k(\varphi) = G(\varphi, 2^{-k})$, $\varphi \in [0, 2\pi]$ and $k \in \mathbb{N} \cup \{0\}$. Notice that for all $x \in I_k = [2^{-k}, 2^{-k+1})$ we have that $f_{k-1}(\varphi) \leq F(\varphi, x) \leq f_k(\varphi)$ and $g_{k-1}(\varphi) \leq G(\varphi, x) \leq g_k(\varphi)$. Indeed, it follows

$$\begin{aligned} \sum_{k=1}^\infty f_{k-1}(\varphi) \chi_{I_k}(x) &\leq F(\varphi, x) \leq \sum_{k=1}^\infty f_k(\varphi) \chi_{I_k}(x), \\ \sum_{k=1}^\infty g_{k-1}(\varphi) \chi_{I_k}(x) &\leq G(\varphi, x) \leq \sum_{k=1}^\infty g_k(\varphi) \chi_{I_k}(x). \end{aligned}$$

Using Remark 4.7 and these inequalities, we obtain

$$\int_0^{2\pi} \left(\sum_{k=1}^\infty f_{k-1}^p(\varphi) 2^{-k} \right)^{q/p} d\varphi \leq \int_0^{2\pi} \left(\int_0^1 |F(\varphi, x)|^p dx \right)^{q/p} d\varphi \leq 2\pi C_p^q$$

and therefore

$$\int_0^{2\pi} \left(\sum_{k=0}^\infty f_k^p(\varphi) 2^{-k} \right)^{q/p} d\varphi \leq 2^{1+\frac{q}{p}} \pi C_p^q. \tag{4.3}$$

Following the same argument, we obtain the inequality for the sequence $\{g_k\}$.

Hence, by Lemma 4.6 we have

$$\begin{aligned} \int_0^{2\pi} \int_0^1 (T_H f) g \, dx \, d\theta &\leq \int_0^{2\pi} \int_0^{2\pi} F(\theta, |\varphi - \theta|) G(\varphi, |\varphi - \theta|) \, d\theta \, d\varphi \\ &\leq \int_0^{2\pi} \left(\sum_{k=1}^{\infty} f_k(\theta) \int_0^{2\pi} g_k(\varphi) \chi_{I_k}(|\theta - \varphi|) \, d\varphi \right) \, d\theta \\ &\leq \int_0^{2\pi} \sum_{k=1}^{\infty} f_k(\theta) 2^{2-k} \left(\frac{1}{2^{2-k}} \int_{\theta-2^{-k+1}}^{\theta+2^{-k+1}} g_k(\varphi) \, d\varphi \right) \, d\theta \\ &\leq \int_0^{2\pi} \sum_{k=1}^{\infty} f_k(\theta) 2^{2-k} M g_k(\theta) \, d\theta. \end{aligned}$$

Applying Hölder’s inequality it follows

$$\begin{aligned} \int_0^{2\pi} \int_0^1 (T_H) f(\theta, x) g(\theta, x) \, dx \, d\theta &\leq 4 \int_0^{2\pi} \left(\sum_{k=1}^{\infty} f_k^p(\theta) 2^{-k} \right)^{1/p} \left(\sum_{k=1}^{\infty} (M g_k)^{p'}(\theta) 2^{-k} \right)^{1/p'} \, d\theta \\ &\leq 4 \left(\int_0^{2\pi} \left(\sum_{k=1}^{\infty} f_k^p(\theta) 2^{-k} \right)^{q/p} \, d\theta \right)^{1/q} \left(\int_0^{2\pi} \left(\sum_{k=1}^{\infty} (M g_k)^{p'}(\theta) 2^{-k} \right)^{q'/p'} \, d\theta \right)^{1/q'}. \end{aligned}$$

Hence, by a classical result of Fefferman and Stein [9, Theorem 1, p.107], the inequality (4.3), and its version for $\{g_k\}$, we get

$$\begin{aligned} \int_0^{2\pi} \int_0^1 (T_H f(\theta, x)) g(\theta, x) \, dx \, d\theta \\ \leq 2^{2+\frac{1}{p}+\frac{1}{q}} \pi^{1/q} C_p A_{p',q'} \left(\int_0^{2\pi} \left(\sum_{k=1}^{\infty} g_k^p(\theta) 2^{-k} \right)^{q'/p'} \, d\theta \right)^{1/q'} \leq 16\pi C_p C_{p'} A_{p',q'}. \end{aligned}$$

Finally, we conclude the proof of the boundedness of the Bergman projection using the last inequality with [3, Theorem 1, p. 303]. □

An important consequence of this result is the following corollary about the dual of $RM(p, q)$ for $1 < p, q < \infty$.

Corollary 4.8. *Let $1 < p, q < \infty$. Then $(RM(p, q))^* \cong RM(p', q')$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.*

Proof. One part of the proof follows immediately. Indeed, applying the Hölder's inequality, one has that the functional defined by

$$\lambda_g(f) = \int_{\mathbb{D}} f(z)\overline{g(z)} \, dA(z), \quad f \in RM(p, q), \quad g \in RM(p', q'),$$

where A is the normalized area measure on the unit disc \mathbb{D} , is bounded and $\|\lambda_g\|_{(RM(p,q))^*} \leq 2\rho_{p',q'}(g)$. Moreover g is unique, since if we assume that $\lambda_{g_1} = \lambda_{g_2}$ we have that $\lambda_{g_1}(z^n) = \lambda_{g_2}(z^n)$, for all $n \in \mathbb{N}$, and hence, $g_1 = g_2$ because $\lambda_g(z^n) = \frac{a_n}{n+1}$, where a_n is the n -th Taylor coefficient of g .

Now, let λ be a functional in $(RM(p, q))^*$. We have to show that there exists $g \in RM(p', q')$ such that

$$\lambda(f) = \int_{\mathbb{D}} f(z)\overline{g(z)} \, dA(z) \quad \text{for every } f \in RM(p, q).$$

Using the Hahn-Banach theorem, this functional can be extended to a certain $\Lambda \in (L^q(\mathbb{T}, L^p[0, 1]))^*$ such that $\|\lambda\|_{(RM(p,q))^*} = \|\Lambda\|_{(L^q(\mathbb{T}, L^p[0,1]))^*}$. Now, by means of [3, Theorem 1, p. 304] there is a function $h \in L^{q'}(\mathbb{T}, L^{p'}[0, 1])$ such that

$$\Lambda(f) = \int_{\mathbb{D}} f(z)\overline{h(z)} \, dA(z) \quad \text{for every } f \in L^q(\mathbb{T}, L^p[0, 1])$$

and $\|\Lambda\|_{(L^q(\mathbb{T}, L^p[0,1]))^*} = \|h\|_{L^{q'}(\mathbb{T}, L^{p'}[0,1])}$.

Let $g = T_K h$, where T_K is the Bergman projection, and notice that, using Theorem 4.3, $g \in RM(p', q')$. So, by Fubini's theorem we have, for $f \in RM(p, q)$,

$$\begin{aligned} \lambda(f) &= \Lambda(f) = \int_{\mathbb{D}} f(z)\overline{h(z)} \, dA(z) = \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{f(w)}{(1-z\bar{w})^2} \, dA(w)\overline{h(z)} \, dA(z) \\ &= \int_{\mathbb{D}} f(w) \int_{\mathbb{D}} \frac{\overline{h(z)}}{(1-z\bar{w})^2} \, dA(z) \, dm(w) = \int_{\mathbb{D}} f(w)\overline{T_K h(w)} \, dA(w) \\ &= \int_{\mathbb{D}} f(w)\overline{g(w)} \, dA(w). \end{aligned}$$

Also, we obtain that $\rho_{p,q}(g) \leq C\|h\|_{L^{q'}(\mathbb{T}, L^{p'}[0,1])} = C\|\lambda\|_{(RM(p,q))^*}$ by Theorem 4.3. \square

For the cases not covered by Theorem 4.3, its statement does not hold. In fact, we have

Theorem 4.9. *Let $1 \leq p, q \leq +\infty$. If $\max\{p, q\} = +\infty$ or $\min\{p, q\} = 1$, then the Bergman projection P does not send $L^q(\mathbb{T}, L^p[0, 1])$ into $RM(p, q)$.*

Before starting with the proof of the theorem, we state the following elementary lemma.

Lemma 4.10. *If $z, w \in \Omega := \{re^{i\theta} : 0 < \theta < 1/2, 0 < r < 1 - 2\theta\}$, then*

- (1) $|1 - z| \asymp 1 - |z|$,
- (2) $\left| \text{Arg} \left(\frac{1-z}{1-w} \right) \right| \leq \arctan \left(\frac{1}{2} \right) < \frac{\pi}{4}$,
- (3) $\text{Re} \left[\left(\frac{1-z}{1-w} \right)^2 \right] \geq \frac{3}{5} \left| \frac{1-z}{1-w} \right|^2$.

Proof. The first identity follows immediately using the triangle inequality and the definition of the set Ω :

$$1 - r \leq |1 - re^{i\theta}| \leq \sqrt{(1 - r)^2 + \theta^2} \leq \sqrt{\frac{5}{4}}(1 - r).$$

To prove the second one it is enough to show that $\tan(\text{Arg}(1 - \bar{z})) \leq \frac{1}{2}$ for $z \in \Omega$, because we have that $\text{Arg}(1 - \bar{z}) \in (0, \arctan(1/2))$ and $\text{Arg}(1 - z) \in (-\arctan(1/2), 0)$. Clearly, one can see, for $re^{i\theta} \in \Omega$, that

$$\tan(\text{Arg}(1 - \bar{z})) = \frac{r \sin(\theta)}{1 - r \cos(\theta)} \leq \frac{(1 - 2\theta) \sin(\theta)}{1 - (1 - 2\theta) \cos(\theta)}.$$

To finish the proof of (2), we have to show that $\frac{(1-2\theta) \sin(\theta)}{1-(1-2\theta) \cos(\theta)} \leq \frac{1}{2}$. But this is clear because the function $f(\theta) = \frac{1}{2}(1 - (1 - 2\theta) \cos(\theta)) - (1 - 2\theta) \sin(\theta)$ for $\theta \in (0, \frac{1}{2})$ satisfies that $f(0) = 0$ and $f'(\theta) = 2\theta \cos(\theta) + \frac{1}{2}(5 - 2\theta) \sin(\theta) \geq 0$ for $\theta \in (0, \frac{1}{2})$.

The last inequality follows immediately from (2). \square

Proof of Theorem 4.9. The case $p = +\infty$. Let us recall that the Bergman projection P is a bounded operator from $L^\infty(\mathbb{D})$ onto the Bloch space \mathcal{B} (see [8, p. 47, Theorem 7] or [19, p. 102, Theorem 5.2]). Moreover, using lacunary sequences, it is possible to find functions in \mathcal{B} whose Taylor coefficients do not go to zero (see [2, Lemma 2.1]). Therefore, $\mathcal{B} \not\subseteq H^q, 1 \leq q \leq +\infty$. Thus, the Bergman projection P is not bounded from $L^q(\mathbb{T}, L^\infty[0, 1])$ to $RM(\infty, q) = H^q$.

The case $q = +\infty, 1 \leq p < +\infty$. We show that there exists a function $f \in L^\infty(\mathbb{T}, L^p[0, 1])$ such that

$$|P(f)(a)| \gtrsim (1 - a)^{-1/p}, \text{ for every } a \in \left(\frac{3}{4}, 1 \right),$$

so that $P(f) \notin RM(p, \infty)$. To prove this, take the set

$$\Omega = \{re^{i\theta} : 0 < \theta < 1/2, 0 < r < 1 - 2\theta\}.$$

Given $\alpha \in \mathbb{R}$, consider the function

$$f(re^{i\theta}) := \begin{cases} 0, & re^{i\theta} \notin \Omega, \\ \theta^\alpha K(1 - \theta, re^{-i\theta}), & re^{i\theta} \in \Omega, \end{cases}$$

where, as usual, K is the Bergman kernel. Taking $\alpha = 2 - \frac{1}{p} = 1 + \frac{1}{p'}$, we have $f \in L^\infty(\mathbb{T}, L^p[0, 1])$. Indeed, for $0 < \theta < 1/2$,

$$\begin{aligned} \int_0^1 |f(re^{i\theta})|^p dr &= \theta^{p\alpha} \int_0^{1-2\theta} |K(1 - \theta, re^{-i\theta})|^p dr \leq \theta^{p\alpha} \int_0^1 \frac{dr}{(1 - (1 - \theta)r)^{2p}} \\ &= \frac{\theta^{p\alpha}}{2p - 1} \frac{\theta^{1-2p} - 1}{1 - \theta} \leq \frac{2}{2p - 1} \theta^{p\alpha+1-2p} = \frac{2}{2p - 1} < +\infty. \end{aligned}$$

Now let us see that this function f satisfies that $|P(f)(a)| \gtrsim (1 - a)^{-1/p}$ for every $a \in (\frac{3}{4}, 1)$. We have that the Bergman projection of the function f , for $a \in (0, 1)$, is

$$\begin{aligned} P(f)(a) &= \int_0^{1/2} \theta^\alpha \left(\int_0^{1-2\theta} \frac{rdr}{(1 - are^{-i\theta})^2(1 - (1 - \theta)re^{i\theta})^2} \right) \frac{d\theta}{\pi} \\ &= \int_0^{1/2} \theta^\alpha \left(\int_0^{1-2\theta} \left(\frac{1 - (1 - \theta)re^{-i\theta}}{1 - are^{-i\theta}} \right)^2 \frac{rdr}{|1 - (1 - \theta)re^{-i\theta}|^4} \right) \frac{d\theta}{\pi}. \end{aligned}$$

By Lemma 4.10 (applying first (3) and then (1)), we obtain

$$\begin{aligned} |P(f)(a)| &\geq \operatorname{Re} [P(f)(a)] \geq \frac{3}{5} \int_0^{1/2} \theta^\alpha \left(\int_0^{1-2\theta} \left| \frac{1 - (1 - \theta)re^{i\theta}}{1 - are^{i\theta}} \right|^2 \frac{rdr}{|1 - (1 - \theta)re^{-i\theta}|^4} \right) \frac{d\theta}{\pi} \\ &\succ \int_0^{1/2} \theta^\alpha \left(\int_0^{1-2\theta} \frac{rdr}{(1 - ar)^2(1 - (1 - \theta)r)^2} \right) d\theta \\ &\geq \int_0^{1-a} \theta^\alpha \left(\int_0^{1-2\theta} \frac{rdr}{(1 - ar)^4} \right) d\theta \\ &\geq \frac{1}{12a} \int_0^{1-a} \theta^\alpha ((1 - a(1 - 2\theta))^{-3} - (1 - a/4)^{-3}) d\theta. \end{aligned}$$

Using that $\theta < 1 - a$ and $3/4 \leq a < 1$ we deduce $(1 - a(1 - 2\theta))^{-3} - (1 - a/4)^{-3} \geq (1 - a(1 - 2\theta))^{-3}/2$ and $1 - a(1 - 2\theta) < 3(1 - a)$. Hence

$$\begin{aligned} \frac{1}{12a} \int_0^{1-a} \theta^\alpha ((1 - a(1 - 2\theta))^{-3} - (1 - a/4)^{-3}) d\theta &\geq \frac{1}{24a} \int_0^{1-a} \theta^\alpha (1 - a(1 - 2\theta))^{-3} d\theta \\ &\geq \frac{1}{24a} \frac{1}{27} \frac{1}{(1 - a)^3} \int_0^{1-a} \theta^\alpha d\theta \\ &\asymp (1 - a)^{\alpha-2} = (1 - a)^{-1/p}. \end{aligned}$$

Thus, for $a > 3/4$, we have $|P(f)(a)| \gtrsim (1 - a)^{-1/p}$ and the function $P(f)$ does not belong to $RM(p, \infty)$.

The remaining cases. For the remaining cases, we use the fact that if the Bergman projection $P : L^{q'}(\mathbb{T}, L^{p'}[0, 1]) \rightarrow RM(p', q')$ is bounded then $P : L^q(\mathbb{T}, L^p[0, 1]) \rightarrow RM(p, q)$ is bounded since, for $f \in L^q(\mathbb{T}, L^p[0, 1])$,

$$\begin{aligned} \rho_{p,q}(P(f)) \asymp \rho_{p,q}(rP(f)) &\asymp \sup_{g \in B_{L^{q'}(\mathbb{T}, L^{p'}[0, 1])}} \left| \int_{\mathbb{D}} P(f)(w) \overline{g(w)} dA(w) \right| \\ &= \sup_{g \in B_{L^{q'}(\mathbb{T}, L^{p'}[0, 1])}} \left| \int_{\mathbb{D}} f(w) \overline{P(g)(w)} dA(w) \right| \\ &\leq \rho_{p,q}(f) \sup_{g \in B_{L^{q'}(\mathbb{T}, L^{p'}[0, 1])}} \rho_{p',q'}(P(g)) \leq C\rho_{p,q}(f), \end{aligned}$$

where C is the norm of the operator $P : L^{q'}(\mathbb{T}, L^{p'}[0, 1]) \rightarrow RM(p', q')$ and, as usual, $B_{L^{q'}(\mathbb{T}, L^{p'}[0, 1])}$ denotes the unit ball of $L^{q'}(\mathbb{T}, L^{p'}[0, 1])$. \square

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