Distinguishing graphs by their left and right homomorphism profiles

Delia Garijo [a], Andrew Goodall [b], Jaroslav Nešetřil [b]

[a] *Department of Applied Mathematics I, University of Seville, Seville, Spain*

[b] *Department of Applied Mathematics and Institute of Theoretical Computer Science (ITI), Charles University, Prague, Czech Republic*

A B S T R A C T

We introduce a new property of graphs called 'q-state Potts unique-ness' and relate it to chromatic and Tutte uniqueness, and also to 'chromatic–flow uniqueness', recently studied by Duan, Wu and Yu.

We establish for which edge-weighted graphs H homomor-phism functions from multigraphs G to H are specializations of the Tutte polynomial of G, in particular answering a question of Freed-man, Lovász and Schrijver. We also determine for which edge-weighted graphs H homomorphism functions from multigraphs G to H are specializations of the 'edge elimination polynomial' of Averbouch, Godlin and Makowsky and the 'induced subgraph poly-nomial' of Tittmann, Averbouch and Makowsky.

Unifying the study of these and related problems is the notion of the left and right homomorphism profiles of a graph.

1. Introduction

The question of whether a graph is uniquely determined by its characteristic polynomial (spectrum) or by its chromatic polynomial has received much attention. In recent years it has been conjectured that almost all graphs are determined by their chromatic polynomial [6] and that almost all graphs are determined by their characteristic polynomial [35]. On the other hand, for example, all trees have the same chromatic polynomial and almost every tree is cospectral with another tree [32]. Showing that a particular graph is determined by a given polynomial invariant often involves intricate arguments. Some insight may be found by a comparative study of the way in which related polynomial invariants determine graphs up to isomorphism. In this paper, which expands and develops the extended abstract [17], we show how graph homomorphisms might provide a fruitful theoretical basis for such a study.

E-mail addresses: dgarijo@us.es (D. Garijo), goodall.aj@gmail.com, goodall.aj@googlemail.com (A. Goodall), nesetril@kam.mff.cuni.cz (J. Nešetřil).

The chromatic polynomial is a specialization of the Tutte polynomial. Bollobás et al. [6] precede their conjecture on the chromatic polynomial by the weaker conjecture that almost all graphs are determined by their Tutte polynomial. Many families of graphs have been shown to be determined by the Tutte polynomial that are not determined by the chromatic polynomial, the first paper devoted to the subject being [11]. Another well-known specialization of the Tutte polynomial is the flow polynomial. The question of whether the flow polynomial determines a graph was only recently considered by Duan et al. [13]. These authors also explore when a graph is determined by its chromatic polynomial and flow polynomial jointly.

In Section 3, we use right homomorphism profiles to unify discussion of the question of when a graph is determined by a polynomial graph invariant such as the chromatic polynomial, flow polynomial or Tutte polynomial. Our first main result here is Theorem 11, showing that the 'colouring uniqueness' of [19] coincides with Tutte uniqueness. This prompts introducing the idea of 'q-state Potts uniqueness', which as far was we know has not yet been studied.

In Section 3.3, we establish by example some graph invariants that are not determined by the 2-state Potts partition function and also that '2-state Potts uniqueness' differs from Tutte uniqueness, chromatic–flow uniqueness, chromatic uniqueness, and q-state Potts uniqueness for $q \geq 3$. The 2-state Potts partition function is a specialization not only of the Tutte polynomial but also of the 'Ising polynomial' of Andrén and Markström [2]: a pair of Tutte equivalent or 'isomagnetic' graphs are also 2-state Potts equivalent.

In Section 3.4, we find graph invariants that are determined by the partition function of the q-state Potts model and use these to show that all the 'chromatic–flow unique' graphs of Duan et al. [13] are also 'q-state Potts unique'.

Section 4 of this paper concerns the question of when a homomorphism profile determines a given graph up to isomorphism, first considered by Lovász [26]. The case of left homomorphism profiles by cycles includes the question of graphs determined by their characteristic polynomial (Corollary 27). In this section, we consider the following type of problem for a graph parameter h: find a minimal set of graphs G for which the values $h(G)$ are sufficient to determine that h is a Tutte–Grothendieck invariant. The answer to this specific question is given by Theorem 53; to reach it we use left homomorphism profiles. Theorem 53 also includes an answer to a question of Freedman et al. [16, Example 3.3]. A similar question[1] for the trivariate generalization $\xi(G; x, y, z)$ of the Tutte polynomial of Averbouch, Godlin and Makowsky [3] is answered by Theorem 37 in Section 4.3. Likewise, in Section 4.4, we consider the similar question for the recently introduced 'induced subgraph polynomial' $Q(G; x, y)$ of Tittmann et al. [33].

In Section 5, we highlight some open problems.

2. Preliminaries

2.1. Homomorphism profiles

A homomorphism of a graph G to a graph H is a function $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. When G and H are multigraphs, i.e., they might have parallel edges and loops, a homomorphism of G to H is a pair of functions $f_V : V(G) \to V(H)$ and $f_E : E(G) \to E(H)$ with the property that if $e \in E(G)$ has endpoints u and v then $f_E(e)$ has endpoints $f_V(u)$ and $f_V(v)$. When G and H are simple, this corresponds to a homomorphism, as previously defined. In the case of multigraphs the function f_E maps parallel edges (resp., loops) in G to parallel edges (resp., loops) in H.

For a multigraph G and an edge-weighted graph H with symmetric adjacency matrix $A(H) = (a_{u,v})$, we define

$$\hom(G, H) = \sum_{f:V(G)\to V(H)} \prod_{\substack{e\in E(G) \\ \text{endpoints } i \text{ and } j}} a_{f(i),f(j)}.$$

[1] Raised by J. Makowsky at the 'Graph Limits, Homomorphisms and Structures' workshop, Hraniční zámeček, Czech Republic, January 2009.

When $a_{u,v} \in \mathbb{Z}_{\geq 0}$ indicates the multiplicity of edges joining u and v in a multigraph H, the quantity $\hom(G, H)$ is equal to the number of homomorphisms from G to H.

From now on in this paper we shall usually avoid using the longer word *multigraph* and allow the term *graph* to include the possibility of loops and parallel edges, unless explicitly stated otherwise by specifying the graph to be simple. On the other hand, in an *edge-weighted graph* we assume there are no parallel edges (but there are possibly loops of non-zero weight).

Definition 1. Let $\mathcal{G}$ be a family of graphs and $\mathcal{H}$ a family of edge-$\mathbb{C}$-weighted graphs. The *right $\mathcal{H}$-profile* of $G \in \mathcal{G}$ is the vector $(\hom(G, H) : H \in \mathcal{H})$. The *left $\mathcal{G}$-profile* of $H \in \mathcal{H}$ is the vector $(\hom(G, H) : G \in \mathcal{G})$.

We say that the right $\mathcal{H}$-profile *distinguishes* a pair of non-isomorphic graphs G and G' if $(\hom(G, H) : H \in \mathcal{H}) \neq (\hom(G', H) : H \in \mathcal{H})$. A graph G is *determined* by its right $\mathcal{H}$-profile if $G \cong G'$ whenever G' has the same right $\mathcal{H}$-profile as G; in other words, G is distinguished from other graphs by its right $\mathcal{H}$-profile. Similarly, the left $\mathcal{G}$-profile determines a graph H if $H \cong H'$ whenever H' has the same left $\mathcal{G}$-profile as H.

As usual, C_k, P_k and K_k denote the cycle, path and complete graph on k vertices, respectively. The graph P_1 is an isolated vertex, C_1 is a loop on one vertex, and C_2 consists of two parallel edges joining a pair of vertices.

Example 2. If $\mathcal{G} = \{P_1\} \cup \{C_k : k \in \mathbb{Z}_{>0}\}$ then the left $\mathcal{G}$-profiles of H and H' are the same if and only if H and H' are *cospectral*. (See Corollary 27.)

If $\mathcal{H} = \{K_q : q \in \mathbb{Z}_{>0}\}$ then G and G' have the same right $\mathcal{H}$-profile if and only if G and G' are *chromatically equivalent*.

2.2. Tutte–Grothendieck invariants

Let $G = (V, E)$ be a graph with $k(G)$ components, rank $r(G) = |V| - k(G)$ and nullity $n(G) = |E| - r(G)$. The graphs resulting by *deleting* and *contracting* an edge $e \in E$ are denoted by $G \setminus e$ and G/e, respectively. An edge e is a *bridge* in G if $r(G \setminus e) = r(G) - 1$ and a *loop* in G if $n(G/e) = n(G) - 1$. Call an edge *ordinary* if it is neither a bridge nor a loop of G.

Definition 3 (*[9]*)**.** A function F from (isomorphism classes of) graphs to $\mathbb{C}[\alpha, \beta, \gamma, x, y]$ is a *generalized Tutte–Grothendieck invariant* if it satisfies, for each graph $G = (V, E)$ and any edge $e \in E$,

$$F(G) = \begin{cases} \gamma^{|V|} & E = \emptyset, \\ xF(G/e) & e \text{ a bridge}, \\ yF(G \setminus e) & e \text{ a loop}, \\ \alpha F(G/e) + \beta F(G \setminus e) & e \text{ ordinary}. \end{cases} \tag{1}$$

For $A \subseteq E$, the subgraph (V, A) is obtained from G by deleting edges not in A. Given $G = (V, E)$, the rank of the spanning subgraph (V, A) is denoted by $r(A)$. A generalized Tutte–Grothendieck invariant is an evaluation of the *Tutte polynomial*, defined by

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}. \tag{2}$$

The coefficients of the Tutte polynomial are non-negative integers (see for example [4,5]), a fact while not evident from its definition in Eq. (2) is more readily seen in its alternative formulation as a Tutte–Grothendieck invariant with $\alpha = \beta = \gamma = 1$.

Theorem 4 (*[9]*)**.** *If F is a generalized Tutte–Grothendieck invariant satisfying (1) then*

$$F(G) = \gamma^{k(G)} \alpha^{r(G)} \beta^{n(G)} T\left(G; \frac{x}{\alpha}, \frac{y}{\beta}\right).$$

See [5] for how to interpret this evaluation when $\alpha = 0$ or $\beta = 0$.

Definition 5 (*See for example [36, Section 4.4]*)**.** The *q-state Potts partition function* $P(G) = P(G; q, y)$ (monochrome polynomial, bad colouring polynomial, coboundary polynomial) of a graph $G = (V, E)$ is defined by

$$P(G; q, y) = \sum_{\phi:V\to[q]} y^{|\{ij\in E:\phi(i)=\phi(j)\}|}.$$

It is easily verified that the q-state Potts partition function P satisfies

$$P(G) = \begin{cases} q^{|V|} & E = \emptyset, \\ (y+q-1)P(G/e) & e \text{ a bridge}, \\ yP(G\setminus e) & e \text{ a loop}, \\ (y-1)P(G/e) + P(G\setminus e) & e \text{ ordinary}. \end{cases}$$

By Theorem 4,

$$P(G; q, y) = q^{k(G)}(y-1)^{r(G)}T\left(G; \frac{y-1+q}{y-1}, y\right). \tag{3}$$

In particular, the *chromatic polynomial* $P(G; q)$ is given by

$$P(G; q) = q^{k(G)}(-1)^{r(G)}T(G; 1-q, 0).$$

2.3. Homomorphism functions and Tutte–Grothendieck invariants

In [19], a *local function* $h(G)$ is a function defined on graphs G with the property that

$$\frac{h(G)}{h(G/e)} = \begin{cases} \alpha & e \text{ ordinary}, \\ a & e \text{ a bridge}, \\ A & e \text{ a loop}, \end{cases} \tag{4}$$

and

$$\frac{h(G)}{h(G\setminus e)} = \begin{cases} \beta & e \text{ ordinary}, \\ b & e \text{ a bridge}, \\ B & e \text{ a loop}, \end{cases} \tag{5}$$

where $\alpha, a, A, \beta, b, B \in \mathbb{Q}\setminus\{0\}$ are constants (i.e., independent of G and e).

Proposition 6. *Suppose that h is a local function defined on all graphs G by Eqs. (4) and (5) and which is multiplicative over disjoint unions.*

Then $h(G) = \gamma^{k(G)}\alpha^{r(G)}\beta^{n(G)}$ for constants α, β, γ.

Proof. In Eqs. (4) and (5), we must have $A = B$, since $h(G) = Ah(G/e) = Bh(G\setminus e)$ and $G/e = G\setminus e$ for a loop e. Since both $h(K_3) = \alpha h(C_2) = \alpha\beta h(P_2)$ and $h(K_3) = \beta h(P_3) = \beta a h(P_2)$, it follows that $a = \alpha$. Similarly, $A = \beta$, since both $h(C_2) = \alpha h(C_1) = \alpha A h(P_1)$ and $h(C_2) = \beta h(P_2) = \beta a h(P_1)$. Suppose further that h is multiplicative over disjoint unions and $h(P_1) = \gamma$ for some non-zero constant γ. Then $a = \gamma b$, since $h(P_2) = ah(P_1) = bh(P_1 \cup P_1) = bh(P_1)^2$.

The function $h(G)$ must be determined by the recursion given by Eqs. (4) independently of the order in which the edges e are deleted and contracted from G. Hence a graph parameter $h(G)$ that is multiplicative over disjoint unions is local if and only if the following simplified versions of Eqs. (4) and (5) hold for some constants α, β, γ:

$$\frac{h(G)}{h(G/e)} = \begin{cases} \alpha & e \text{ not a loop}, \\ \beta & e \text{ a loop}; \end{cases} \qquad \frac{h(G)}{h(G\setminus e)} = \begin{cases} \beta & e \text{ not a bridge}, \\ \dfrac{\alpha}{\gamma} & e \text{ a bridge}. \end{cases} \tag{6}$$

These together say that

$$h(G) = \begin{cases} \gamma^{|V|} & E = \emptyset, \\ \alpha h(G/e) & e \text{ a bridge}, \\ \beta h(G\setminus e) & e \text{ a loop}, \\ \alpha h(G/e) = \beta h(G\setminus e) = \delta\alpha h(G/e) + (1-\delta)\beta h(G\setminus e) & e \text{ ordinary}, \end{cases}$$

where δ is arbitrary. By Theorem 4, this yields, for any δ,

$$h(G) = \gamma^{k(G)}(\delta\alpha)^{r(G)}[(1-\delta)\beta]^{n(G)}T\left(G; \tfrac{1}{\delta}, \tfrac{1}{1-\delta}\right)$$
$$= \gamma^{k(G)}\alpha^{r(G)}\beta^{n(G)},$$

with $\left(\frac{1}{\delta}-1\right)\left(\frac{1}{1-\delta}-1\right)=1$ and $T(G;x,y)=(x-1)^{r(E)}y^{|E|}$ when $(x-1)(y-1)=1$. This completes the proof. $\square$

For fixed H, the function $\hom(G,H)$ is multiplicative over disjoint unions, so the graph parameter $h(G)T(G;x,y)$ cannot be a homomorphism number if $h(G)$ is not multiplicative over disjoint unions. By Proposition 6, if $h(G)$ is a local function and $h(G)T(G;x,y)$ is a homomorphism number then $h(G)=\gamma^{k(G)}\alpha^{r(G)}\beta^{n(G)}$ for some constants α,β,γ.

Let $K_q^{a,b}$ denote the edge-$\mathbb{C}$-weighted complete graph on q vertices with loops attached at each vertex, having weight a on loops and weight b on non-loops. A multigraph can be regarded as an edge-$\mathbb{Z}_{\geq 0}$-weighted graph with edge weights indicating multiplicities.

Theorem 7 *([19]). For every connected graph H, the following statements are equivalent.*

(i) *There exist $x, y \in \mathbb{Q}$ and a local function h such that* $\hom(G,H)=h(G)T(G;x,y)$ *for every graph G.*
(ii) *There exist $a,b,q\in\mathbb{Z}_{\geq 0}$, $q\geq 1$, such that $H\cong K_q^{a,b}$.*

Allowing disconnected graphs H in Theorem 7, a generalized Tutte–Grothendieck invariant can arise from $\hom(G,H)$ only by taking H equal to the disjoint union of copies of one such connected graph $K_q^{a,b}$.

Remark 1. Compare [16, Example 3.3], where connection matrices are used to deduce that there is an edge-$\mathbb{R}$-weighted graph H such that $\hom(G,H)=(1-x)^{k(G)}(1-y)^{|V|}T(G;x,y)$ if and only if $(x-1)(y-1)=q$ for integers $q\geq 1$. (In fact, more is proved in [16], since H is also allowed to have positive real weights on its vertices.) This result and Proposition 6 give an alternative proof of Theorem 7.

For a minor-closed class of graphs $\mathcal{G}$, we define a function h on graphs to be *$\mathcal{G}$-local* if it is only required to satisfy Eqs. (4) and (5) for $G\in\mathcal{G}$. By the argument beginning the proof of Proposition 6, if a $\mathcal{G}$-local function h is also multiplicative over disjoint unions and $\mathcal{G}$ contains K_3 (i.e., some graph with a cycle of length at least three) then h satisfies the simplified recurrence (6). However, now it is not necessarily the case that $h(G)=\gamma^{k(G)}\alpha^{r(G)}\beta^{n(G)}$ for constants α,β,γ, since the recurrence (6) need not hold for graphs G outside the set $\mathcal{G}$.

From the proof of Theorem 2.7 in [19], it is straightforward to prove the following result, since the argument of the proof only uses graphs from the given set $\mathcal{G}$.

Theorem 8. *Let H be a connected graph and $\mathcal{G}=\{K_1^{k,0},K_2^{0,k},C_k,P_k : k\in\mathbb{Z}_{>0}\}$. The following statements are equivalent.*

(i) *There exist $x,y\in\mathbb{Q}$ and a $\mathcal{G}$-local function h such that* $\hom(G,H)=h(G)T(G;x,y)$ *for every graph $G\in\mathcal{G}$.*
(ii) *There exist $a,b,q\in\mathbb{Z}_{\geq 0}$, $q\geq 1$, such that $H\cong K_q^{a,b}$ and* $\hom(G,H)=h(G)T(G;x,y)$ *for every graph G.*

In Section 4.5, we prove a generalization of Theorem 8.

3. Right homomorphism profiles and '*q*-state Potts uniqueness'

3.1. *The Tutte polynomial and colouring uniqueness*

The problem of finding graphs determined by polynomial invariants has been studied for many polynomials (see [30] for a survey). A graph G is said to be *Tutte unique* if $T(G;x,y)=T(G';x,y)$ implies that $G\cong G'$, for every other graph G'. Tutte uniqueness has been studied for several families of graphs, such as complete multipartite graphs, wheels and hypercubes (see [11]). The following notion was motivated by the result of Theorem 7 above.

Definition 9 (*[19]*)**.** A finite graph G is *colouring unique* if $\hom(G, K_q^{y,1}) = \hom(G', K_q^{y,1})$ for all $q \geq 1$, $y \in \mathbb{Z}_{\geq 0}$ implies that $G \cong G'$ for every graph G'.

A graph G is colouring unique if and only if G is determined by its right $\{K_q^{y,1} : q \geq 1, y \in \mathbb{Z}_{\geq 0}\}$-profile. Whether a graph is determined by its right $\{K_q^{y,1} : q \in \mathbb{Z}_{>0}\}$-profile includes the question of chromatic uniqueness ($y = 0$) and flow uniqueness ($y = 1 - q$). Observe that chromatically unique graphs are colouring unique. Similarly, colouring unique graphs are Tutte unique. We now prove that the converse is also true.

Lemma 10 (*[1, Lemma 2.1]*)**.** *Let $f = f(x_1, x_2, \ldots, x_n)$ be a polynomial in n variables over an arbitrary field $\mathbb{F}$. Suppose that the degree of f as a polynomial in x_i is at most t_i for $1 \leq i \leq n$, and let $A_i \subseteq \mathbb{F}$ be a set of at least $t_i + 1$ distinct elements of $\mathbb{F}$. If $f(x_1, x_2, \ldots, x_n) = 0$ for all n-tuples $(x_1, x_2, \ldots, x_n) \in A_1 \times A_2 \times \cdots \times A_n$ then f is identically zero.*

Theorem 11. *Suppose that G, G' are graphs with $\max\{r(G), r(G')\} < r$ and $\max\{n(G), n(G')\} < s$. Let $A, B \subseteq \mathbb{C}$ be sets with $|A| = r$, $|B| = s$. Then $T(G; x, y) = T(G'; x, y)$ if and only if $T(G; u, v) = T(G'; u, v)$ for all $(u, v) \in A \times B$. In particular, G is Tutte unique if and only if G is colouring unique.*

Proof. If $T(G; x, y) = T(G'; x, y)$ identically then $T(G; u, v) = T(G'; u, v)$ for all $(u, v) \in \mathbb{C} \times \mathbb{C}$. Suppose now that $T(G; u, v) = T(G'; u, v)$ for all $(u, v) \in A \times B$. For all $(x, y) \in A \times B$ we have the following equality:

$$T(G; x, y) = \sum_{(u,v)\in A\times B} \prod_{\substack{(a,b)\in A\times B \\ a\neq u, b\neq v}} \frac{x-a}{u-a}\frac{y-b}{v-b} T(G; u, v). \tag{7}$$

A similar equality holds with G' in place of G.

By Lemma 10, it follows that equality (7) is a polynomial identity. Hence if $T(G'; u, v) = T(G; u, v)$ for all $(u, v) \in A \times B$ it follows that $T(G'; x, y) = T(G; x, y)$ identically too.

For the last part of the theorem, observe that the set $\left\{\left(\frac{y-1+q}{y-1}, y\right) : q \geq 1, y \in \mathbb{Z}_{\geq 0}\right\}$ contains arbitrarily large rectangles. In order to contain the rectangle $A \times B$ for given subsets $A, B \subseteq \mathbb{Z}_{\geq 0} \setminus \{0, 1\}$ with $|A| = r$, $|B| = s$, allow q to range over the set $\{(a-1)(b-1) : (a, b) \in A \times B\}$ and y to range over B.

Suppose that G is colouring unique, i.e., if $T(G'; u, v) = T(G; u, v)$ for all $(u, v) \in \{(\frac{y-q+1}{y-1}, y) : q \geq 1, y \in \mathbb{Z}_{\geq 0}\}$ then $G' \cong G$. By taking $r > \max\{r(G), r(G')\}$, $s > \max\{n(G), n(G')\}$, the equality $T(G'; u, v) = T(G; u, v)$ for all $(u, v) \in A \times B$ implies the identity $T(G'; x, y) = T(G; x, y)$. $\square$

3.2. The q-state Potts partition function

We recall from Definition 5 the q-state Potts partition function,

$$P(G; q, y) = \sum_{\phi: V(G)\to[q]} y^{|\{ij\in E(G):\phi(i)=\phi(j)\}|}, \tag{8}$$

and that it is the specialization of the Tutte polynomial given in Eq. (3) in Section 2.2. In particular, for $y = 0$ it is equal to the chromatic polynomial $P(G; q)$, and for $y = 1 - q$ it specializes to the *flow polynomial* $F(G; q)$.

The result of Theorem 11 prompts the question as to which Tutte polynomial invariants of G are not determined by the right $\mathcal{H}$-profile of G when $\mathcal{H}$ is a proper subset of $\{K_q^{y,1} : q \geq 1, y \in \mathbb{Z}_{\geq 0}\}$. The right $\{K_q^{0,1} : q \in \mathbb{Z}_{>0}\}$-profile of G gives by interpolation the chromatic polynomial of G, and the right $\{K_q^{1-q,1} : q \in \mathbb{Z}_{>0}\}$-profile of G gives the flow polynomial of G. The case where y is variable and q fixed gives a profile determining the q-state Potts partition function of G, since by Eq. (8) we see that $\hom(G, K_q^{y,1}) = P(G; q, y)$.

Definition 12. A graph is *q-state Potts unique* if it is determined up to isomorphism by its right $\{K_q^{y,1} : y \in \mathbb{Z}_{\geq 0}\}$-profile.

Our aim in this section is to begin an exploration of which graph invariants are determined by $P(G; q, y)$ (fixed q). In particular, we shall be interested in seeing how q-state Potts uniqueness relates to 'chromatic–flow uniqueness' explored by Duan et al. [13].

Fix an arbitrary orientation of $G = (V, E)$. Let $\delta : \mathbb{Z}_q^V \to \mathbb{Z}_q^E$ denote the coboundary map (taking the difference between the values at the head and tail of an edge) and $\partial : \mathbb{Z}_q^E \to \mathbb{Z}_q^V$ the boundary map (taking the net flow into a vertex from incoming and outgoing edges). The set $\mathrm{im}(\delta)$ of $\mathbb{Z}_q$-*tensions of* G has size $q^{r(G)}$ and the set $\ker(\partial)$ of $\mathbb{Z}_q$-*flows of* G has size $q^{n(G)}$. The weight enumerator of $\ker(\partial)$ (known also as the boundary polynomial or bad flow polynomial of G) is defined by

$$F(G; q, x) = \sum_{f \in \ker(\partial)} x^{|\{e \in E: f(e)=0\}|}.$$

For a lengthier exposition of the subject of the last paragraph and a full proof of the following lemma, see, for example, [21].

Lemma 13. *The q-state Potts partition function of G is given by*

$$P(G; q, y) = q^{k(G)} \sum_{f \in \ \mathrm{im}(\delta)} y^{|\{e \in E: f(e)=0\}|},$$

and

$$\begin{aligned} F(G; q, x) &= q^{-|V|}(x-1)^{|E|} P\left(G; q, \frac{x-1+q}{x-1}\right) \\ &= (x-1)^{n(G)} T\left(G; x, \frac{x-1+q}{x-1}\right). \end{aligned}$$

The 2-state Potts partition function is given by

$$\begin{aligned} P(G; 2, y) &= 2^{k(G)} \sum_{\text{cutsets} C} y^{|E|-|C|} \\ &= 2^{|V|-|E|}(y-1)^{|E|} F\left(G; 2, \frac{y+1}{y-1}\right), \end{aligned}$$

in which

$$F(G; 2, x) = \sum_{\text{Eulerian subgraphs } C} x^{|E|-|C|}.$$

Proof. The first equation uses the 1-to-$q^{k(G)}$ correspondence between $\mathbb{Z}_q$-tensions of G and vertex $\mathbb{Z}_q$-colourings of G. The second equation follows by MacWilliams duality and Eq. (3). When $q = 2$, $\ker(\partial)$ is the subspace of Eulerian subgraphs of G ($\mathbb{Z}_2$-flows) and $\mathrm{im}(\delta)$ the subspace of cutsets of G ($\mathbb{Z}_2$-tensions). $\square$

3.3. *Some 2-state Potts equivalent graphs*

To begin our discussion of q-state Potts uniqueness, we consider the case $q = 2$, i.e., the partition function of the Ising model $P(G; 2, y)$. We return to general q in Section 3.4.

A graph is *simple* if it has no 1-edge or 2-edge cycles, and *cosimple* if it has no 1-edge or 2-edge cutsets. As observed in [13, Corollary 2.5], whether G is simple can be detected given both $P(G; q)$ and $|E|$, but not whether G is cosimple. Similarly, whether G is cosimple can be detected by $F(G; q)$ and $|E|$ jointly, but not whether G is simple. On the other hand, since by Lemma 13 $P(G; 2, y)$ records both the number of 1-edge and 2-edge cycles and the number of 1-edge and 2-edge cutsets, the 2-state Potts partition function determines both whether G is simple or cosimple. (Similarly, in [13], it is shown that the chromatic and flow polynomial when taken together determine whether G is simple or cosimple.)

Fig. 1. Graphs with the same 2-state Potts partition function but different Tutte polynomials. (They also have different chromatic polynomials and different flow polynomials.) One is 2-connected, the other not.

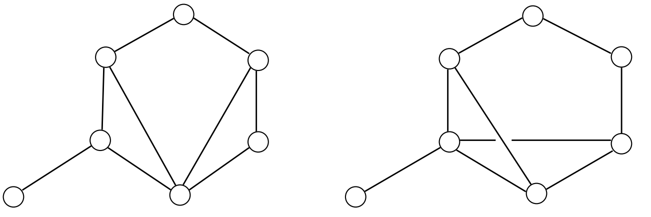

Fig. 2. Simple graphs with the same 2-state Potts partition function but different Tutte polynomials. They also have different chromatic polynomials.

The graphs in Fig. 1 have the same 2-state Potts partition function. Their q-state Potts model partition functions are respectively $(y^2+q-1)^3$ and $(y+q-1)^3+(y^3-1)(y^3+3(q-1)y+(q-1)(q-2))$. These are equal for $q \in \{1, 2\}$ but differ for $q \geq 3$. This example also shows that whether a graph is 2-connected cannot be determined by the 2-state Potts partition function. On the other hand, since $F(G; 2, 1) = 2^{|E|-|V|+k(G)}$ and $F(G; 2, x) = 2^{-|V|}(x-1)^{|E|}P\left(G; 2, \frac{x+1}{x-1}\right)$, we can decide whether G is connected given $P(G; 2, y)$, and more generally find $k(G)$.

An example of a pair of graphs that are both simple and share the same 2-state Potts partition function is given in Fig. 2 (taken from Fig. 3 in [2]). The value of the chromatic polynomial $P(G; q)$ for these two graphs differs by $q(q-1)^2(q-2)$. Hence the graphs in Fig. 2 have different q-state Potts model partition functions for $q \geq 3$. We have not yet found for any $q \geq 3$ an example of a pair of graphs with the same q-state Potts model partition function but different 2-state Potts model partition function.

Lemma 14. *Let $G = (V, E)$ be a connected graph. The 2-state Potts partition function $P(G; 2, y)$ determines the following graph parameters:*

(i) *$|V|$ and $|E|$;*
(ii) *for each $0 \leq k \leq |E|$ the number of Eulerian subgraphs of size k, in particular, the girth $g(G)$ of G and the number of cycles of this size; whether G is simple and, if so, the number of triangles;*
(iii) *for each $0 \leq k \leq |E|$ the number of cutsets of size k, in particular, the edge connectivity $\lambda(G)$ of G and whether G is cosimple;*
(iv) *whether G is bipartite and whether G is Eulerian.*

Proof. Part (i) follows since $P(G; 2, 1) = 2^{|V|}$ and the degree of $P(G; 2, y)$ in y is $|E|$. For (ii) and (iii), we use Lemma 13. The coefficient of $y^{|E|-k}$ in $2^{-1}P(G; 2, y)$ is equal to the number of cutsets of size k. The polynomial $F(G; 2, x)$ can be recovered from $P(G; 2, y)$ by setting $x = \frac{y+1}{y-1}$, and the coefficient of $x^{|E|-k}$ in $F(G; 2, x)$ is equal to the number of Eulerian subgraphs of size k. Given that G is simple (has no 1-edge or 2-edge cycles), a 3-edge Eulerian subgraph must be a triangle. An Eulerian subgraph of minimal size $g(G)$ is a cycle so the coefficient of $x^{|E|-g(G)}$ is equal to the number of cycles of size $g(G)$ in the graph of this girth. For part (iv), a graph G is bipartite if and only if $P(G; 2, 0) \neq 0$, and G is Eulerian if and only if $F(G; 2, 0) = (-1)^{|E|}2^{-|V|}P(G; 2, -1) \neq 0$. $\square$

A list of parameters similar to Lemma 14 that are determined by chromatic polynomial has been instrumental in proving chromatic uniqueness results.

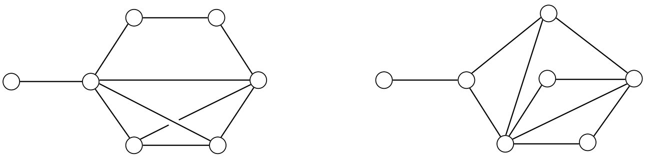

Fig. 3. Graphs with different chromatic number and different clique number but the same 2-state Potts partition function.

Lemma 15 (*[24]*)**.** *Let $G = (V, E)$ be a connected simple graph. The chromatic polynomial $P(G; q)$ determines the following graph parameters:*

(i) *$|V|$ and $|E|$;*
(ii) *whether G is 2-connected;*
(iii) *the number of triangles, and $|\{$chordless 4-cycles$\}| - 2|\{$4-cliques$\}|$;*
(iv) *the girth $g(G)$ and the number of cycles of this size;*
(v) *the chromatic number $\chi(G)$.*

Duan et al. [13] provide an analogous list of flow polynomial invariants that suffices to prove their uniqueness results.

Lemma 16 (*[13, Theorem 3.1]*)**.** *Let $G = (V, E)$ be a connected cosimple graph. The flow polynomial $F(G; q)$ determines the following graph parameters:*

(i) *$|V|$ and $|E|$;*
(ii) *whether G is 2-connected;*
(iii) *the edge connectivity $\lambda(G)$ and the number of bonds of this size;*
(iv) *the flow number $\phi(G)$.*

The corresponding list of parameters determined by the Tutte polynomial (see for example [30, Lemma 3.9]) that has been used to prove Tutte uniqueness results [29,11,18] starts with the union of the lists given in Lemmas 15 and 16. An important addition is that $T(G; x, y)$ determines the number of cliques of any given size in G, in particular the clique number $\omega(G)$. Further, $T(G; x, y)$ determines the number of 4-cycles and 5-cycles of G, and amongst the 4-cycles the number that have exactly one chord. The latter refines the knowledge obtained from the chromatic polynomial concerning 4-cycles, namely the quantity in Lemma 15(iii). Read and Whitehead [31] showed that the Tutte polynomial cannot only tell whether G is simple or cosimple but for each $0 \leq k \leq |E|$ also gives the number of edges of multiplicity k and the number of "chains" (maximal class of edges in series) of length k. The pair of graphs in Fig. 1 show that the 2-state Potts model can do no better than detect whether a graph is simple.

The chromatic number $\chi(G)$ is not determined by $P(G; 2, y)$ (except for deciding whether $\chi(G) = 2$), and the flow number $\phi(G)$ is not determined by $P(G; 2, y)$ (except for deciding whether $\phi(G) = 2$). Also, the clique number $\omega(G)$ is not determined by $P(G; 2, y)$ (except for deciding whether $\omega(G) = 2$). The graphs in Fig. 3 (which are the same as those of Fig. 4 in [2]) have different chromatic numbers and different clique numbers but the same 2-state Potts partition function.

3.4. Examples of q-state Potts unique graphs

The q-state Potts partition function for $q \geq 3$ contains a lot of the information that can be obtained from the 2-state Potts partition function (Lemma 14).

Lemma 17. *The q-state Potts partition function $P(G; q, y)$ determines the following parameters of a graph G:*

(i) *$|V(G)|$, $|E(G)|$, $k(G)$;*
(ii) *the girth $g(G)$ and the number of cycles of this length; if $g(G) \geq 3$ (i.e., G is simple) then whether G is bipartite;*
(iii) *the edge connectivity $\lambda(G)$ and the number of cutsets of this size; if $\lambda(G) \geq 3$ (i.e., G is cosimple) then whether G is Eulerian;*

(iv) *the number of vertex q-colourings with given number of monochromatic edges, and the number of* $\mathbb{Z}_q$*-flows of given size support;*
(v) *whether* $\chi(G) = q$ *and whether* $\phi(G) = q$*.*

Proof. (i) $P(G; q, 1) = q^{|V|}$, the degree of $P(G; q, y)$ as a polynomial in y is equal to $|E|$, and $k(G)$ can be obtained from $F(G; q, 1) = q^{|E|-|V|+k(G)}$. (ii) The support of a $\mathbb{Z}_q$-flow f is defined by $S(f) = \{e \in E : f(e) \neq 0\}$. A $\mathbb{Z}_q$-flow f has *minimal support* if, for all $\mathbb{Z}_q$-flows f' such that $S(f') \subseteq S(f)$, either $f' = 0$ or $f' = f$. A result of Tutte [34] for integer-valued flows implies that if f is a $\mathbb{Z}_q$-flow of minimal support then $S(f)$ is a cycle (2-regular subgraph) on which f takes values $\pm a$ for some non-zero constant a. Moreover, any $\mathbb{Z}_q$-flow is a linear combination of such flows of minimal support. Hence, if the girth is g, for each cycle of this length there are $(q-1)\mathbb{Z}_q$-flows supported on this cycle and so the coefficient of $y^{|E|-g}$ in $P(G; q, y)$ is equal to the number of cycles of length g multiplied by $(q-1)$. Part (iii) is dual to part (ii): the minimal supports of $\mathbb{Z}_q$-tensions of G are bonds of G. Part (iv) follows by Definition 5 of the q-state Potts partition function and by Lemma 13. Part (v) is a consequence of part (iv). □

We have no examples of graphs with the same q-state Potts partition function that do not also have the same Tutte polynomial. This mirrors the fact that Duan et al. in [13] report not having found a pair of chromatic–flow equivalent graphs that are distinguished by the Tutte polynomial. In the light of this it is difficult to say whether for example there are graphs G, H with $P(G; q_1, y) \neq P(H; q_1, y)$ but $P(G; q_2, y) = P(H; q_2; y)$ when $q_2 > 2$.

A graph G is said to be *super-edge-connected* if every minimum edge cut of G is a set of edges incident with some vertex (a 'vertex cutset'). The following is a useful result for proving that certain regular graphs are q-state Potts unique.

Lemma 18. *Let* $q \geq 2, r \geq 3$*, and suppose G is an r-regular super-edge-connected graph. If* $P(H; q, y) = P(G; q, y)$ *then H is also r-regular and super-edge-connected.*

Proof. The proof is by simple adaptation of the proof of [13, Lemma 3.3]. □

Proposition 19. K_4, K_5 *and* $K_{3,3,}$ *are q-state Potts unique for* $q \geq 2$*.*

Proof. We adapt the proof of Theorem 3.4 in [13]. Any graph q-state Potts equivalent to K_4 has 4 vertices and is 3-regular super-edge-connected. By Lemma 18 this forces it to be isomorphic to K_4. A graph q-state Potts equivalent to K_5 must have 5 vertices, 10 edges, girth 3, and must be 4-regular and super-edge-4-connected. The only possibility is a graph isomorphic to K_5.

If $P(G; q, y) = P(K_{3,3}; q, y)$ then G is simple and cosimple, has girth 4, and it is super-edge-connected and 3-regular. By Lemma 18 G is also 3-regular and super-edge-connected. Let $C = v_1v_2v_3v_4$ be a 4-cycle of G and v_5 and v_6 the other two vertices. If $v_5v_6 \notin E(G)$ then v_5 is incident with three vertices of C thus producing a triangle, in contradiction to $g(G) = 4$. Hence $v_5v_6 \in E(G)$, and vertex v_5 is joined to two non-adjacent vertices of C, say v_1 and v_3. Similarly, v_6 is adjacent to v_2 and v_4. This makes $G \cong K_{3,3}$. □

Proposition 20. *Cycles are q-state Potts unique for* $q \geq 2$*.*

Proof. From $F(C_n; q, x) = x^n + (q-1)$ we know that if $F(H; q, x) = F(C_n; q, x)$ then H has n vertices and n edges and that the minimum support q-flows in fact have support the whole edge set of H. Since minimum supports of q-flows are cycles, it follows that H is a cycle. □

The wheel W_5, consisting of one central vertex joined to five other vertices on a cycle, is an example of a graph that is neither chromatically unique nor flow unique. On the other hand, it is chromatic–flow unique [13, Proposition 4.2].

Proposition 21. *The wheel* W_5 *is q-state Potts unique for* $q \geq 2$*.*

Proof. By Lemma 17, if $P(G; q, y) = P(W_5; q, y)$ then G is both simple and cosimple and has the same number of vertices, edges and triangles as W_5. Also, $\lambda(G) = 3 = \lambda(W_5)$, and so each vertex of G has degree at least 3. Hence the degree sequence of G is either 333 335 or 333 344. Following the same argument as in the proof of Proposition 4.2 in [13], we can prove that the sequence has to be 333 335, and so G is isomorphic to W_5. □

The graph $L_k = C_k \times K_2$ is called a *ladder*. The *Möbius ladder* M_k is formed by joining every pair of opposite vertices in C_{2k}. The square of the k-cycle C_k^2 is obtained by adding all the edges between vertices distance 2 apart. If k is odd then $\chi(L_k) = 3$ and $\chi(M_k) = 2$. If k is even then $\chi(L_k) = 2$ and $\chi(M_k) = 3$. In [11] the graphs L_k, M_k, C_k^2 are shown to be Tutte unique; in [13], the stronger result is shown that they are determined by the chromatic and flow polynomial jointly; here, they are deduced to be also q-state Potts unique.

Theorem 22. *The ladders L_k, $k \geq 3$, are q-state Potts unique for $q \geq 2$.*

Proof. L_3 is determined by having 6 vertices, 9 edges, girth 3, 2 triangles and 3 cycles of length 4 – all these parameters are determined by the q-state Potts partition function. The proof of Theorem 4.4 in [13] that establishes the chromatic–flow uniqueness of L_k for $k \geq 6$ in fact only requires $k \geq 4$. The parameters determined by the chromatic and flow polynomial of a graph G required in this proof are also determined by the q-state Potts partition function, with the exception of the chromatic number $\chi(G)$. But all that is needed is to determine whether $\chi(G) = 2$ or not, and this can be detected by $P(G; q, y)$. (Although in the proof of Theorem 4.4 in [13] the 2-connectivity of G is one of the parameters listed as relevant to the proof, in fact it is not actually used.) $\square$

The small cases $M_2 \cong K_4$ and $M_3 \cong K_{3,3}$ of Möbius ladders are q-state Potts unique (Proposition 19). It is again the case that the proofs in [13] use parameters that occur amongst those determined by the q-state Potts partition function listed in Lemma 17. Hence we have the following.

Theorem 23. *The Möbius ladders M_k, $k \geq 2$, are q-state Potts unique for $q \geq 2$.*

For squares of cycles, $C_3^2 \cong K_3, C_4^2 \cong K_4$ and $C_5^2 \cong K_5$ have already been seen to be q-state Potts unique. Duan et al. [13] in the proof of their Theorem 4.6 that C_k^2 is chromatic–flow unique just consider $k \geq 10$, since C_k^2 is known to be chromatically unique for $k \leq 9$. Their proof in fact only requires $k \geq 6$, and again uses parameters determined by the q-state Potts model. (As similarly noted in the proof of Theorem 22, 2-connectedness, although listed as relevant in [13, proof of Theorem 4.6], is not actually used.)

Theorem 24. *The squares of cycles C_k^2, $k \geq 3$, are q-state Potts unique for $q \geq 2$.*

Let $\theta(a_1, \ldots, a_s)$ denote the *s-bridge graph* consisting of s internally disjoint paths of lengths $a_1, \ldots, a_s$. A 3-bridge graph is commonly known as a theta graph. The flow polynomial $F(G; q)$ does not distinguish any pair of s-bridge graphs. On the other hand, some s-bridge graphs are chromatically unique; those that are not have the same chromatic polynomial as a non-isomorphic graph which in all cases discovered so far is not another s-bridge graph. See, for example, [25].

A *chain* in a graph is a maximal class of edges that lie in series. Read and Whitehead [31] show that the Tutte polynomial determines the chain lengths of a graph (and dually the number of edges of given multiplicity). They then deduce that s-bridge graphs are Tutte unique.

The 4-bridge graph $\theta(1, 1, 1, 3)$ has the same 2-state Potts partition function as three 2-cycles vertex-glued together in a path (see Fig. 1). This indicates that at least for $q = 2$ we cannot say that if H has the same q-state Potts partition function as a s-bridge graph then H must also be a s-bridge graph.

Since the cycle space of an s-bridge graph has dimension $s - 1$, independent of the number of edges, it is more convenient to work with $F(G; q, x)$ than $P(G; q, y)$ when trying to establish whether a pair of s-bridge graphs are distinguished from each other by the q-state Potts partition function. If $F(G; q, x) = F(H; q, x)$ for connected graphs G, H then $P(G; q, y) = P(H; q, y)$.

Theorem 25. *Theta graphs are q-state Potts unique for $q \geq 2$.*

Proof. A simple calculation gives

$$F(\theta(a_1, a_2, a_3); q, x) = x^{a_1+a_2+a_3} + (q-1)(x^{a_1} + x^{a_2} + x^{a_3}) + (q-1)(q-2).$$

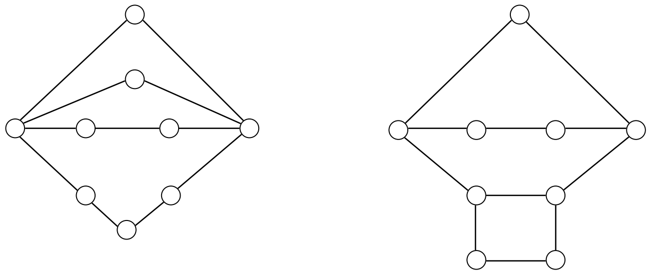

Fig. 4. Graphs with the same chromatic polynomial but different 2-state Potts partition functions.

Hence $F(\theta(a_1, a_2, a_3); q, x) = F(\theta(b_1, b_2, b_3); q, x)$ if and only if $x^{a_1} + x^{a_2} + x^{a_3} = x^{b_1} + x^{b_2} + x^{b_3}$, i.e., $\{a_1, a_2, a_3\} = \{b_1, b_2, b_3\}$. Hence theta graphs are distinguished from each other by the q-state Potts partition function.

If $F(H; q, x) = F(\theta(a_1, a_2, a_3); q, x)$ then H is connected, and has $a_1+a_2+a_3$ edges and cycle space dimension 2. Therefore H has two cycles C_1, C_2, and a third cycle or union of two edge-disjoint cycles $C_3 = C_1 \triangle C_2$. The intersection $C_1 \cap C_2$ is a path, possibly just a single vertex, since C_1 and C_2 cannot meet in disjoint paths without making the dimension of the cycle space greater than 2.

If C_1, C_2 share no edges then the subgraph $C_1 \cup C_2$ supports a $\mathbb{Z}_q$-flow and has size $|C_1| + |C_2|$. But the supports of the three non-zero $\mathbb{Z}_q$-flows of H are subgraphs of sizes $a_1 + a_2$, $a_2 + a_3$, $a_3 + a_1$, and no two of these integers is the sum of the third. Hence the cycles C_1, C_2 meet in a path with at least one edge. The result is a generalized theta graph. $\square$

It turns out that $F(\theta(a_1, a_2, a_3, a_4); q, x) = F(\theta(b_1, b_2, b_3, b_4); q, x)$ if and only if $a_1+a_2+a_3+a_4 = b_1 + b_2 + b_3 + b_4$, and we have the polynomial identity (in $\mathbb{Z}[x]$)

$$\sum_{i \neq j} x^{a_i+a_j} + (q-2)\sum_i x^{a_i} = \sum_{i \neq j} x^{b_i+b_j} + (q-2)\sum_i x^{b_i}. \tag{9}$$

For $q = 2$ this identity holds if and only if the multiset $\{a_i + a_j : i \neq j\}$ is equal to the multiset $\{b_i + b_j : i \neq j\}$. Hence the multibridge graph $\theta(a_1, a_2, a_3, 2k - a_1 - a_2 - a_3)$ has the same 2-state Potts partition function as $\theta(k-a_1, k-a_2, k-a_3, a_1+a_2+a_3-k)$, where $0 < a_i < k$, $a_1+a_2+a_3 > k$ and $k \geq 2$. For example, $P(\theta(1, 3, 4, 4); 2, y) = P(\theta(2, 2, 3, 5); 2, y)$.

Chen et al. [10] show that $\theta(a_1, a_2, a_3, a_4)$ is chromatically unique except when $\{a_1, a_2, a_3, a_4\} = \{2, a, a + 1, a + 2\}$, $a \geq 2$, in which case the graph obtained by edge-gluing $\theta(3, a, a + 1)$ to C_{a+2} has the same chromatic polynomial. (The case $a = 2$ is illustrated in Fig. 4.) The Eulerian subgraphs of $\theta(2, a, a + 1, a + 2)$ have sizes $0, a + 2, a + 3, a + 4, 2a + 1, 2a + 2, 2a + 3, 3a + 5$, whereas the Eulerian subgraphs of $\theta(3, a, a + 1)$ edge-glued with C_{a+2} have sizes $0, a + 2, a + 3, a + 4, 2a + 1, 2a + 3, 2a + 4, 3a + 3$. Hence these graphs are distinguished by the 2-state Potts partition function but not by the chromatic polynomial. On the other hand, we have just seen that the graphs $\theta(a_1, a_2, a_3, 2k - a_1 - a_2 - a_3)$ are not 2-state Potts unique but are chromatically unique when $\{a_1, a_2, a_3\} \neq \{2, a, a + 1\}$, $2 \leq a \leq k - 2$, $2a \geq k - 2$.

For $q \geq 3$, we have no examples of $\theta(a_1, a_2, a_3, a_4)$ such that there is non-isomorphic G with $F(G; q, x) = F(\theta(a_1, a_2, a_3, a_4); q, x)$. Such a G will not be a multibridge graph, since by Eq. (9) it is not difficult to deduce that non-isomorphic 4-bridge graphs are distinguished from each other by the q-state Potts partition function when $q \geq 3$.

4. Left homomorphism profiles

Lovász [26] proved that if $\mathcal{G}$ is the set of all graphs then every graph H is determined by its left $\mathcal{G}$-profile. Likewise, if $\mathcal{H}$ is the set of all graphs then every graph G is determined by its right $\mathcal{H}$-profile. Dvořák [14] showed that every graph is still determined by its left $\mathcal{G}$-profile if $\mathcal{G}$ is the set of all 2-degenerated graphs. A graph H is *k-degenerated* if each subgraph of H contains a vertex of degree at most k. Every graph with tree-width k is k-degenerated. 1-degenerated graphs are precisely forests, but there are 2-degenerated graphs with arbitrary tree-width; the complete graph with each edge subdivided by two new vertices is 2-degenerate.

Lovász [27] later extended his result for left $\mathcal{G}$-profiles to the case where the graph H on the right is allowed to have real edge and vertex weights (with the definition of isomorphism suitably extended and, when there are vertex weights, the removal of 'twin vertices').

By Theorem 8, the graph $K_q^{a,b}$ for $a, b \in \mathbb{Z}_{\geq 0}$ is determined by its left $\{K_1^{k,0}, K_2^{0,k}, C_k, P_k : k \in \mathbb{Z}_{>0}\}$-profile amongst all edge-$\mathbb{Z}_{\geq 0}$-weighted graphs. In Section 4.1, we prove that each Potts model graph $K_q^{a,b}$ is determined by its left $\{C_k, K_{1,k} : k \in \mathbb{Z}_{>0}\}$-profile amongst all edge-$\mathbb{C}$-weighted graphs. In Section 4.5, we also find that a Potts model graph is determined by its left $\{K_1^{k,0}, K_2^{0,k} : k \in \mathbb{Z}_{>0}\}$-profile.

4.1. Cycles and stars

A k-walk in a graph is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k$, where $e_{i+1} = v_i v_{i+1}$ for $0 \leq i \leq k-1$. A k-walk is closed if $v_0 = v_k$. A 0-walk is just a vertex, and is always closed. A 1-walk is a walk from a vertex to an adjacent vertex. A closed 1-walk is a loop.

Lemma 26. *Let H be an edge-weighted graph with adjacency matrix A. Then*

$$\hom(C_k, H) = \operatorname{tr}(A^k).$$

Proof. The matrix A^k has (i, j) entry the sum of edge-weighted k-walks from i to j, as can be proved by induction. (The weight of a walk is the product of its edge weights, with multiplicities counted for repeated edges.) A closed k-walk corresponds to a homomorphic image of C_k. The diagonal entries of A^k then together sum to $\hom(C_k, H)$. $\square$

Corollary 27. *Let H be an edge-weighted graph H on q vertices with adjacency matrix A. Then the left $\{C_k : 1 \leq k \leq q\}$-profile of H determines the spectrum of A.*

Proof. If A has eigenvalues $\lambda_1, \ldots, \lambda_q$ then $\operatorname{tr}(A^k) = \sum_i \lambda_i^k$. In particular, $\operatorname{tr}(A^0) = q$ gives the number q of vertices of H, i.e., the size of A. Given these power sums for $1 \leq k \leq q$, Newton's relations yield the elementary symmetric polynomials in the λ_i, and hence the λ_i are uniquely determined (as the roots of the characteristic polynomial of A). $\square$

Corollary 28. *The left $\{P_1\} \cup \{C_k : k \in \mathbb{Z}_{>0}\}$-profile of an edge-weighted graph H determines the spectrum of its adjacency matrix.*

Proof. Use $\hom(P_1, H)$ to determine the number of vertices of H, and by Corollary 27 the left $\{C_k : k \in \mathbb{Z}_{>0}\}$-profile determines its spectrum. $\square$

Restricting attention to simple unweighted undirected graphs, graphs determined by their spectrum include K_n, $K_{n,n}$ and C_n. It is conjectured that almost all graphs are determined by their spectrum [35]. On the other hand, almost all trees are not determined by their spectrum, and there are many constructions of cospectral non-isomorphic graphs.

Lemma 29. *Let H be an edge-weighted graph on q vertices with adjacency matrix A, and let* $\mathbf{1}$ *denote the $q \times 1$ all-1 vector. Then the left $\{K_{1,k} : 1 \leq k \leq q\}$-profile of H determines the vector $A\mathbf{1}$.*

Proof. The homomorphic image of $K_{1,k}$ is a multiset of k edges incident with a common vertex. If H has adjacency matrix $A = (a_{u,v})_{u,v \in [q]}$ then

$$\hom(K_{1,k}, H) = \sum_{v \in [q]} \left(\sum_{u \in [q]} a_{u,v} \right)^k,$$

by taking all possible choices of a multiset of k edges incident with common vertex v as the image of $K_{1,k}$. By taking $k = 1, \ldots, q$, we can determine the column sums $\sum_{u \in [q]} a_{u,v}$ of A, i.e., the vector $\mathbf{1}^\top A$. Since A is symmetric, this also gives the row sums and the vector $A\mathbf{1}$. $\square$

Corollary 30. *The left* $\{K_{1,k} : k \in \mathbb{Z}_{>0}\}$*-profile of an edge-weighted graph* H *with adjacency matrix* A *determines whether* $\mathbf{1}$ *is an eigenvector of* A *and, if so, its associated eigenvalue.*

Proof. Use $K_{1,0} = P_1$ to determine the size of A, and the result follows by Lemma 29. $\square$

We now show that the graph $K_q^{a,b}$ for complex weights a, b on loops and non-loops is, like K_q, determined by its spectrum and the fact that it has eigenvector $\mathbf{1}$ with eigenvalue $a + (q - 1)b$.

Lemma 31. *Suppose that* A *is a symmetric matrix over* $\mathbb{C}$ *and that* I, J *are the* $q \times q$ *identity and all-1 matrices respectively. If* A *is cospectral with* $(a - b)I + bJ$ *and the all-1 vector* $\mathbf{1}$ *is an eigenvector of* A *with eigenvalue* $a + (q - 1)b$ *then* $A = (a - b)I + bJ$*.*

Proof. Take A to be a $q \times q$ real symmetric matrix and $A^\top = A$; we shall deduce the result when A has complex entries at the end of the proof.

The eigenvalues of $(a-b)I + bJ$ are $a + (q-1)b$ (eigenvector $\mathbf{1}$) and $a - b$ (with multiplicity $q - 1$).

Let C be a real orthogonal matrix such that $C((a - b)I + bJ)C^\top = \mathrm{diag}(a + (q - 1)b, a - b, a - b, \ldots, a - b)$. (The columns of C form an orthonormal real basis for the eigenvectors.) By hypothesis there is a real orthogonal matrix D such that $DAD^\top = \mathrm{diag}(a + (q - 1)b, a - b, a - b, \ldots, a - b)$. Moreover, we may assume that the first row of D is equal to $q^{-\frac{1}{2}}\mathbf{1}^\top$, the all-1 vector scaled by $q^{-\frac{1}{2}}$. (The columns of D comprise an orthonormal basis of right eigenvectors of A and its rows an orthonormal basis of left eigenvectors. We are given that $\mathbf{1}$ is an eigenvector. Since A is symmetric, its left eigenvectors are the transposes of its right eigenvectors. By Gram–Schmidt orthonormalization, the matrix D can be built so that its rows form a real orthonormal basis for the left eigenvectors of J, starting with $q^{-\frac{1}{2}}\mathbf{1}^\top$ as the first row.)

Hence

$$A = D^\top C((a - b)I + bJ)C^\top D = (a - b)I + bD^\top CJC^\top D.$$

It remains to prove that $D^\top CJC^\top D = J$. First, $CJC^\top = \mathrm{diag}(q, 0, 0, \ldots, 0)$, since $C((a - b)I + bJ)C^\top = \mathrm{diag}(a + (q - 1)b, a - b, a - b, \ldots, a - b)$. Second, since the first column of $D^\top$ is equal to $q^{-\frac{1}{2}}\mathbf{1}$, we have

$$D^\top \mathrm{diag}(q, 0, 0, \ldots, 0) = q^{\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \cdots & & \cdots & \cdots & \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

so that, since D has first row $q^{-\frac{1}{2}}\mathbf{1}^\top$,

$$D^\top \mathrm{diag}(q, 0, 0, \ldots, 0)D = J.$$

This is what we wished to prove, yielding $A = (a - b)I + bJ$.

To deduce the result for A over $\mathbb{C}$, let $\overline{A}$ denote the complex conjugate of A. Then $A + \overline{A}$ is a real symmetric matrix with eigenvalues $a + \bar{a} + (q - 1)(b + \bar{b})$ (eigenvector $\mathbf{1}$) and $a + \bar{a} - b - \bar{b}$ (multiplicity $q - 1$). By the previous argument we obtain $A + \overline{A} = (a + \bar{a} - b - \bar{b})I + (b + \bar{b})J$. Similarly, the purely imaginary symmetric matrix $A - \overline{A}$ is equal to $(a - \bar{a} - b + \bar{b})I + (b - \bar{b})J$. Therefore $A = (a - b)I + bJ$. $\square$

Corollary 32. *The Potts model graph* $K_q^{a,b}$ *is determined by its left* $\{C_k, K_{1,k} : 0 \le k \le q\}$*-profile.*

Proof. The adjacency matrix of $K_q^{a,b}$ is $(a - b)I + bJ$. Suppose that H has the same left $\{C_k, K_{1,k} : 0 \le k \le q\}$-profile as $K_q^{a,b}$. By Corollary 27, the left $\{P_1\} \cup \{C_k : 1 \le k \le q\}$-profile determines the spectrum of H. By Lemma 29, the left $\{K_{1,k} : 0 \le k \le q\}$-profile determines that H has eigenvector $\mathbf{1}$ with eigenvalue $a + (q - 1)b$. The result now follows by Lemma 31. $\square$

Thus we can determine whether an edge-weighted graph H is a Potts model graph from its left $\{C_k, K_{1,k} : k \in \mathbb{Z}_{>0}\}$-profile.

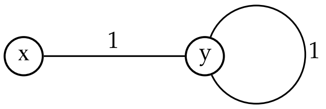

Fig. 5. Vertex- and edge-weighted graph H such that $\hom(G, H) = I(G; x, y)$.

4.2. The independence polynomial

NOTATION. To avoid too much clutter, weights on non-loop edges will always be assumed to be equal to 1 unless otherwise indicated, i.e., $K_q^y = K_q^{y,1}$ in the earlier Potts model graph notation. The coclique on p vertices (p independent vertices) is denoted by $\overline{K}_p$, and if loops of weight y are attached to each vertex the resulting graph is denoted by $\overline{K}_p^y$.

Dohmen et al. [12] introduce a bivariate generalization of the chromatic polynomial of G which, for $p, q \in \mathbb{Z}_{\geq 0}$ and q-set Q containing a p-set P of 'proper colours', counts the number of vertex Q-colourings of G with the property that all monochrome edges have endpoint colours belonging to $Q \setminus P$. By definition, this quantity is equal to $\hom(G, H)$, where H is the weighted complete graph with edges uv weight 1 for $u \neq v$, loops uu weight 1 for $q - p$ vertices (namely $Q \setminus P$) and loops weight 0 for the remaining p vertices (P). This graph is the *join* of a p-clique K_p and a $(q-p)$-clique with loops of weight 1 attached to each vertex, i.e., the graph $K_p + K_{q-p}^1$. In particular, taking $p = 1$ and interpolating over $q \in \mathbb{Z}_{>0}$ determines [12, Corollary 2] the *independence polynomial* of G of Gutman and Harary [22], defined by

$$I(G; x) = \sum_{\text{independent sets } U \subseteq V} x^{|U|}.$$

(A set of vertices U is independent in G if the induced subgraph on U is a coclique.) A bivariate version of the independence polynomial is

$$I(G; x, y) = \sum_{\text{independent sets } U \subseteq V} x^{|U|} y^{|V \setminus U|} = y^{|V|} I(G; x/y).$$

Twin vertices in an edge-weighted graph H on vertex set $[q]$ with adjacency matrix $A(H) = (a_{u,v})_{u,v \in [q]}$ are vertices s, t with the property that $a_{s,v} = a_{t,v}$ for all $v \in [q]$. In particular, $a_{s,s} = a_{s,t} = a_{t,t}$. (In a similar way, parallel edges in a multigraph are defined as edges with the same incidence relation to all other edges.) The induced subgraph on a set of twin vertices must either form a clique with loops on each vertex in which all edges have the same weight, or form a coclique (independent set of vertices).

Twin vertices s, t can be 'reduced' to a single vertex $u \notin [q]$ that is given a weight equal to the sum of the weights of s and t; the adjacency matrix of the graph on $[q] \cup \{u\} \setminus \{s, t\}$ thus obtained has $a_{u,v} = a_{s,v} = a_{t,v}$ for all $v \in [q]$. Conversely, in a vertex-weighted graph, a vertex of positive integer weight p can be 'split' into a set of twin vertices whose vertex weights sum to p. Much as an edge-$\mathbb{Z}_{\geq 0}$-weighted graph can be transformed into an unweighted multigraph by 'splitting' an edge of weight p into p parallel edges, so a vertex-$\mathbb{Z}_{>0}$-weighted and edge-weighted graph can be transformed into an edge-weighted graph by splitting vertices of weight p into p twin unweighted vertices.

Proposition 33. *The independence polynomial of G is determined the right $\{\overline{K}_{q-p} + K_p^1 : q \in \mathbb{Z}_{>0}, q > p\}$-profile of G, where $p \geq 1$.*

Proof. The graph $\overline{K}_{q-p} + K_p^1$ equal to the join of the p-clique with loops of weight 1 on each vertex and the $(q-p)$-coclique with no loops is obtained from the vertex- and edge-weighted graph in Fig. 5 with $x = q - p$ and $y = p$ by splitting each of these two weighted vertices into sets of unweighted twin vertices. Thus

$$\hom(G, \overline{K}_{q-p} + K_p^1) = \sum_{\text{independent sets } U \subseteq V} (q-p)^{|U|} p^{|V \setminus U|} = I(G; q-p, p). \quad \square$$

Table 1
Evaluations of graph polynomials from counting homomorphisms.

H	$\hom(G, H)$
K_q	Chromatic polynomial, $P(G; q)$
K_q^y	q-state Potts partition function, $P(G; q, y)$
$\overline{K}_q^y$	$q^{k(G)}y^{\lvert E\rvert}$
$K_1^1 + \overline{K}_{q-1}$	Independence polynomial, $I(G; q-1)$
$K_{q-1}^1 + \overline{K}_1$	$(q-1)^{\lvert V\rvert} I(G; (q-1)^{-1})$
$K_p^1 + \overline{K}_{q-p}$	Bivariate independence polynomial, $I(G; p, q-p)$
$K_p^1 + \overline{K}_{q-p}^y$	[$p = 1, y = 1$ is the Widom–Rowlinson model]
$\overline{K}_p^1 + K_{q-p}^y$	
$K_{q-p}^1 + K_p$	Dohmen–Pönitz–Tittmann polynomial [12] at (p, q)
$K_{q-p}^1 + K_p^y$	Averbouch–Godlin–Makowsky polynomial [3] at $(q, y-1, (p-q)(y-1))$
$\overline{K}_{q-p}^1 + \overline{K}_p^y$	

The graph parameter $\hom(G, K_{q-p}^1 + K_p)$ when $p = 1$ is equal to the independence polynomial of G. As remarked above, taking $1 \le p \le q$ gives evaluations of the Dohmen–Pönitz–Tittmann polynomial. This in turn is the case $y = 0$ of the graph parameter

$$\hom(G, K_{q-p}^1 + K_p^y) = \sum_{U\subseteq V} (q-p)^{|U|} P(G-U; p, y),$$

shown in Theorem 34 to be an evaluation of the Averbouch–Godlin–Makowsky polynomial.

Recall that $\hom(G, K_p^1 + \overline{K}_{q-p}) = I(G; q-p, p)$. Putting loops of weight y on each of the $q-p$ vertices in the $(q-p)$-coclique gives $K_p^1 + \overline{K}_{q-p}^y$. For a graph G,

$$\begin{aligned}\hom(G, K_p^1 + \overline{K}_{q-p}^y) &= \sum_{U\subseteq V(G)} p^{|U|}(q-p)^{k(G-U)} y^{|E(G-U)|}\\ &= \sum_{\text{induced subgraphs } H \text{ of } G} p^{|V(G)|-|V(H)|}(q-p)^{k(H)} y^{|E(H)|}.\end{aligned}$$

When $p = 1 = y$, $\hom(G, K_1^1 + \overline{K}_{q-1}^1)$ is the partition function of the Widom–Rowlinson model [37] (see also [15]).

Table 1 summarizes the polynomial graph invariants obtained by counting homomorphisms to cliques or cocliques with loops of constant weight attached to their vertices, and the various joins of these graphs. For the blank entries in this table there is obviously a state sum formula for $\hom(G, H)$ similar to that given above for $\hom(G, K_p^1 + \overline{K}_{q-p}^y)$, but we could not find a well-known interpretation for it.

4.3. The Averbouch–Godlin–Makowsky polynomial

Averbouch, Godlin and Makowsky [3] introduce their polynomial $\xi(G; x, y, z)$ as a simultaneous trivariate generalization of the Tutte polynomial and matching polynomial. The polynomial $\xi(G; x, y, z)$ includes the polynomial of Dohmen et al. [12] as the specialization $\xi(G; q, -1, q-p)$. The q-state Potts partition function $P(G; q, y)$ is the specialization $\xi(G; q, y-1, 0)$.

For edge $e = uv$, $G \dagger e$ denotes the induced graph $G - \{u, v\}$. The polynomial $\xi(G)$ is defined by the following confluent recurrence relation [3]:

$$\begin{aligned}&\xi(P_0; x, y, z) = 1, \qquad \xi(P_1; x, y, z) = x,\\ &\xi(G \oplus H; x, y, z) = \xi(G; x, y, z)\xi(H; x, y, z),\\ &\xi(G; x, y, z) = \xi(G\setminus e) + y\,\xi(G/e; x, y, z) + z\,\xi(G\dagger e; x, y, z).\end{aligned} \tag{10}$$

(The boundary conditions are for the empty graph P_0 and a single isolated vertex P_1.)

Theorem 34. *For all graphs G,*

$$\hom(G, K_{q-p}^1 + K_p^y) = \xi(G; q, y-1, (p-q)(y-1)). \tag{11}$$

Proof. We check that the recurrence (10) is satisfied by $\hom(G, K_{q-p}^1 + K_p^y)$ with the appropriate values of the three arguments of $\xi(G)$. Eq. (11) holds trivially for $G = P_0, P_1$. Also, both left- and right-hand sides of Eq. (11) are multiplicative over disjoint unions. It remains to prove that $\hom(G, K_{q-p}^1 + K_p^y)$ satisfies the recurrence relation (10).

Let Q be a set of size q and $P \subseteq Q$ size p. Let $G = (V, E)$ and $e = uv \in E$. Partition the range of summation in

$$\hom(G, K_{q-p}^1 + K_p^y) = \sum_{\phi:V\to Q} y^{|\{ij\in E:\phi(i)=\phi(j)\in P\}|} \tag{12}$$

into three classes according as $\phi(u) \neq \phi(v)$, $\phi(u) = \phi(v) \notin P$ or $\phi(u) = \phi(v) \in P$. For short, let us write here $h(G)$ for the function $\hom(G, K_{q-p}^1 + K_p^y)$. Restricting the summation (12) to each of these classes separately,

$$\sum_{\substack{\phi:V\to Q\\ \phi(u)\neq\phi(v)}} y^{|\{ij\in E:\phi(i)=\phi(j)\in P\}|} = h(G \setminus e) - h(G/e),$$

$$\sum_{\substack{\phi:V\to Q\\ \phi(u)=\phi(v)\notin P}} y^{|\{ij\in E:\phi(i)=\phi(j)\in P\}|} = (q-p)h(G \dagger e),$$

and

$$\sum_{\substack{\phi:V\to Q\\ \phi(u)=\phi(v)\in P}} y^{|\{ij\in E:\phi(i)=\phi(j)\in P\}|} = yh(G/e) - y(q-p)h(G \dagger e).$$

(If $\phi(u) = \phi(v) \notin P$ then all contributions to the weight of the vertex colouring ϕ from edges incident with e are 1, so the weight of ϕ is the same as the weight of ϕ restricted to $V \setminus \{u, v\}$. There are $q-p$ choices for the colour $\phi(u) = \phi(v) \notin P$ on the endpoints of e.) Hence

$$\begin{aligned} h(G) &= h(G \setminus e) - h(G/e) + (q-p)h(G \dagger e) + yh(G/e) - y(q-p)h(G \dagger e) \\ &= h(G \setminus e) + (y-1)h(G/e) + (p-q)(y-1)h(G \dagger e). \end{aligned}$$

Thus $h(G)$ is the evaluation of $\xi(G)$ at the point $(q, y-1, (p-q)(y-1))$. $\square$

We now prepare to prove a converse to Theorem 34. As in [19], the key lemma is the following elementary result.

Lemma 35. *Let $u_1, \ldots, u_r$ be distinct non-zero complex numbers and $\ell \in \mathbb{Z}_{>0}$. Suppose that, for $\ell \leq k \leq \ell + r - 1$,*

$$c_1u_1^k + c_2u_2^k + \cdots + c_ru_r^k = 0.$$

Then $c_1 = c_2 = \cdots = c_r = 0$.

Lemma 36. *Let H be an edge-$\mathbb{C}$-weighted graph on q vertices and $x, y, z \in \mathbb{C}$. If $\hom(G, H) = \xi(G; x, y, z)$ for all $G \in \{K_1^{k,0} : k \in \mathbb{Z}_{>0}\}$ then there is an integer p, $0 \leq p \leq q$, such that*

(i) *$x = q$ and $z = (p-q)y$,*
(ii) *the weights on p loops of H are equal to $1 + y$ and the weights on the remaining $q - p$ loops of H are equal to 1.*

Proof. The graph $K_1^{0,0}$ is an isolated single vertex. We have $\xi(K_1^{0,0}; x, y, z) = x$ and $\hom(K_1^{0,0}, H) = |V(H)| = q$.

Let H on vertex set $[q]$ have adjacency matrix $A = (a_{u,v})$. Let $\ell_k = \xi(K_1^{k,0}; q, y, z)$. Using the recurrence relation (10) for $\xi(G)$, for $k \geq 1$ we have $\ell_k = (1+y)\ell_{k-1} + z$. With boundary condition

$\ell_0 = q$, it follows that $\ell_k = \xi(K_1^{k,0}; q, y, z) = (q + zy^{-1})(1 + y)^k - zy^{-1}$ when $y \neq 0$ and $\xi(K_1^{k,0}; q, 0, z) = q + kz$ when $y = 0$. Assume first that $y \neq 0$. By hypothesis,

$$\hom(K_1^{k,0}, H) = \sum_{v \in [q]} a_{v,v}^k = (q + zy^{-1})(1 + y)^k - zy^{-1}.$$

By Lemma 35, with $u_1, \ldots, u_r$ taking the distinct non-zero values amongst $\{a_{v,v} : v \in [q]\} \cup \{1+y, 1\}$, it follows that $zy^{-1} \in \mathbb{Z}$ and, setting $p = q + zy^{-1}$, that $|\{v \in [q] : a_{v,v} = 1 + y\}| = p$ and $|\{v \in [q] : a_{v,v} = 1\}| = q - p$. The statement of the lemma results.

When $y = 0$, $\ell_k = q + kz$, and since this is also equal to $\sum a_{v,v}^k$ it follows that $z = 0$ and $a_{v,v} = 1$ for each $v \in [q]$. So in this case the statement of the lemma holds with $p = q$ (or $p = 0$). □

We reach our desired converse to Theorem 34.

Theorem 37. *Let H be an edge-$\mathbb{C}$-weighted graph on q vertices and $x, y, z \in \mathbb{C}$. If $\hom(G, H) = \xi(G; x, y, z)$ for all $G \in \{K_1^{k,0}, K_2^{0,k} : k \in \mathbb{Z}_{>0}\}$ then there is an integer p, $0 \le p \le q$, such that*

(i) $x = q \in \mathbb{Z}_{>0}$, $z = (p - q)y$,

(ii) $H \cong K_{q-p}^1 + K_p^{1+y}$.

Proof. By Lemma 36, we have $x = q$, $z = (p-q)y$, and p loops of H with weight $1+y$ and $q-p$ loops with weight 1. It remains to determine the weights on the non-loop edges of H.

Let $m_k = \xi(K_2^{0,k}; q, y, (p-q)y)$. Using the recurrence relation (10), $m_k = m_{k-1} + y\ell_{k-1} + (p-q)y$, with boundary value $m_0 = q^2$. Since $\ell_k = (1 + y)\ell_{k-1} + (p - q)y$, $\ell_0 = q$, we have $m_k - m_{k-1} = \ell_k - \ell_{k-1}$, from which we obtain $m_k = \ell_k + q^2 - q = p(1+y)^k + q^2 - p$. By hypothesis, we also have

$$\hom(K_2^{0,k}, H) = \sum_{(u,v) \in [q] \times [q]} a_{u,v}^k = p(1 + y)^k + q^2 - p.$$

Lemma 35 implies that $|\{(u, v) \in [q] \times [q] : a_{u,v} = 1+y\}| = p$ and $|\{(u, v) \in [q] \times [q] : a_{u,v} = 1\}| = q^2 - p$. The result follows. (The case $y = 0$ is trivial, corresponding to all edges, loops and non-loops, of weight 1.) □

Remark 2. Averbouch et al. [3, Theorem 6] take a vertex- and edge-weighted graph H on vertex set $[q]$ having p vertices of weight -1 with attached loops of weight 1, and the remaining vertices of weight 1 with loops of weight $1 + y$. They show that $\hom(G, H) = \xi(G; q - 2p, y, py)$; indeed, a simple adaptation of the proof of Theorem 34 can be used to demonstrate this result. (In the edge elimination reduction $G \dagger e$, if the endpoints of e are the same colour and not in P then each vertex has weight -1 and all incident edges weight 1, so there is no overall weight change upon extracting the edge e.)

This raises the question as to what are possible choices for H when it is allowed to have both vertex and edge weights, i.e., what is the analogue of Theorem 37 in this situation?

4.4. The Tittmann–Averbouch–Makowsky polynomial

In order to study the polynomial $Q(G; x, y)$ of Tittmann et al. [33], we will find it convenient to first define homomorphisms between coloured graphs.

A *k-coloured* graph (G, κ) is a (possibly weighted) graph G together with a function $\kappa : V(G) \to [k]$ assigning a colour $\kappa(v)$ to each vertex v. (The colouring κ is not necessarily proper.) A *colour-preserving* homomorphism from (G, κ) to (H, κ_0) is a homomorphism $f : V(G) \to V(H)$ with the property that $\kappa_0(f(v)) = \kappa(v)$ for each $v \in V(G)$.

Given a multigraph G, edge-weighted graph H with edges ij of weight β_{ij}, and k-colourings $\kappa : V(G) \to [k]$, $\kappa_0 : V(H) \to [k]$, define

$$\hom_c((G, \kappa), (H, \kappa_0)) = \sum_{\substack{f : V(G) \to V(H) \\ \kappa_0 f = \kappa}} \prod_{uv \in E(G)} \beta_{f(u)f(v)},$$

which for a multigraph H ($\beta_{ij} \in \mathbb{Z}_{\ge 0}$) is equal to the number of colour-preserving homomorphisms $(G, \kappa) \to (H, \kappa_0)$.

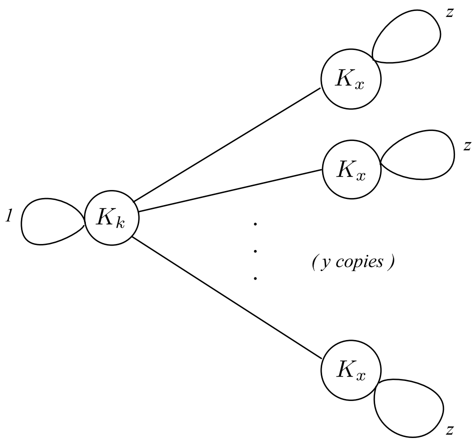

Fig. 6. The graph $H_{k,x,y,z}$. This can be viewed as an ornamented star $K_{1,y}$: the central vertex is replaced by k twin vertices forming a clique with loops on each vertex, all edges having weight 1. The y pendant vertices are replaced by x-cliques, with loops of weight z attached to each vertex. The single lines joining cliques each stand for kx edges joining all pairs of vertices between K_k^1 and K_x^z. All edges are of weight 1 unless otherwise indicated. The case $z = 1$ is the graph of [33, Figure 6], for which $\hom(G, H_{k,x,y,1}) = k^{|V(G)|}Q(G; x/k, y)$. The case $y = 1$ gives – after renaming parameters – the graph in Fig. 7.

Given a colour-preserving homomorphism $f : G \to H$ and a fixed colouring $\kappa_0 : V(H) \to [k]$, the condition $f^{-1}(\kappa_0^{-1}(c)) = \kappa^{-1}(c)$ for each $c \in [k]$ partitions the set of all homomorphisms $f : G \to H$ according to the colouring κ. Hence, for any fixed $\kappa_0 : V(H) \to [k]$,

$$\sum_{\kappa:V(G)\to[k]} \hom_c((G, \kappa), (H, \kappa_0)) = \hom(G, H). \tag{13}$$

(Cf. Eq. (2) in [28].)

In [33], Tittmann et al. define the 'induced subgraph polynomial' of a graph $G = (V, E)$ as follows:

$$Q(G; x, y) = \sum_{U\subseteq V} x^{|U|}y^{c(G[U])},$$

where $G[U]$ is the induced subgraph on U and $c(G[U])$ the number of its connected components. They show that $Q(G; x, y)$ for $x \in \mathbb{R}$ and $y \in \mathbb{Z}_{\geq 0}$ can be viewed as a partition function by counting graph homomorphisms to a vertex- and edge-weighted graph, but leave it as an open problem whether there are other points (x, y) for which $Q(G; x, y)$ is equal to a homomorphism number. Our main result towards which we work in this section is Theorem 47, which answers this question for homomorphisms to graphs with positive real vertex weights and complex edge weights.

Let $H_{k,x,y}$ be the graph formed by taking K_k^1 as the centre of a star with y vertices of degree 1 each replaced by a copy of K_x^1 (the graph $Star_y$ of [33, Figure 6] with vertices of weight x replaced by x twin vertices forming K_x^1 and with the root replaced by k twin vertices forming K_k^1). This is the graph $H_{k,x,y,z}$ illustrated in Fig. 6 above with $z = 1$.

By Theorem 10 in [33], $\hom(G, H_{1,x,y})$ is equal to the polynomial $Q(G; x, y)$. It is not difficult to see that, more generally, $\hom(G, H_{k,x,y}) = k^{|V(G)|}Q(G; x/k, y)$.

Let $\kappa_0 : V(H_{k,x,y}) \to [k] \cup \{0\} \to$ be a colouring which restricted to K_k^1 is an injection $V(K_k^1) \to [k]$ and which colours all the other xy vertices in the looped cliques K_x^1 with the colour 0. Then, for any colouring $\kappa : V(G) \to [k] \cup \{0\}$,

$$\hom_c((G, \kappa), (H_{k,x,y}, \kappa_0)) = \hom(G[\kappa^{-1}(0)], yK_x^1)$$

(where yK_x^1 denotes y disjoint copies of K_x^1), since each colour in $[k]$ occurs just once in $(H_{k,x,y}, \kappa)$ and on the looped clique K_k^1, so there is precisely one colour-preserving homomorphism from $G - \kappa^{-1}(0)$

to $H_{k,x,y}$, and this has weight 1. Setting $U = \kappa^{-1}(0)$, this gives

$$\begin{aligned}\hom_c((G,\kappa),(H_{k,x,y},\kappa_0)) &= y^{c(G[U])}\prod_{1\le i\le c(G[U])}\hom(G_i,K_x^1)\\ &= y^{c(G[U])}x^{|U|},\end{aligned}$$

where $G_1,\ldots,G_{c(G[U])}$ are the connected components of $G[U]$, containing altogether $|U|$ vertices. By Eq. (13),

$$\begin{aligned}\hom(G,H_{k,x,y}) &= \sum_{\kappa:V(G)\to[k]\cup\{0\}} x^{|\kappa^{-1}(0)|}y^{c(G[\kappa^{-1}(0)])}\\ &= \sum_{U\subseteq V(G)} x^{|U|}y^{c(G[U])}k^{|V(G)|-|U|}\\ &= k^{|V(G)|}Q(G;x/k,y).\end{aligned}$$

By the same argument, if we take $H_{k,x,y,z}$ to be the graph in Fig. 6 with loop weights z instead of 1 on each of the cliques K_x, then

$$\begin{aligned}\hom(G,H_{k,x,y,z}) &= \sum_{U\subseteq V(G)} y^{c(G[U])}k^{|V(G)|-|U|}\prod_{1\le i\le c(G[U])}\hom(G_i,K_x^z)\\ &= \sum_{U\subseteq V(G)} k^{|V(G)|-|U|}y^{c(G[U])}\prod_i P(G_i;x,z)\\ &= \sum_{U\subseteq V(G)} k^{|V(G)|-|U|}y^{c(G[U])}(z-1)^{r(G[U])}x^{c(G[U])}\prod_i T\left(G_i;\frac{z-1+x}{z-1},z\right)\\ &= k^{|V(G)|}\sum_{U\subseteq V(G)}\left(\frac{x}{k}\right)^{|U|}\left(\frac{xy}{z-1}\right)^{c(G[U])}\prod_{1\le i\le c(G[U])}T\left(G_i;\frac{z-1+x}{z-1},z\right).\end{aligned}$$

This polynomial in k,x,y,z includes the Averbouch–Godlin–Makowsky polynomial $\xi(G;x,y,z)$ and the Tittmann–Averbouch–Makowsky polynomial $Q(G;x,y)$ as specializations. (See Figs. 6 and 7.) We now return to the latter and settle the question of when an evaluation of $Q(G;x,y)$ is equal to a homomorphism number $\hom(G,H)$. We do this first for H with positive integer vertex weights and complex edge weights, and then deduce the result for H with positive real vertex weights and complex edge weights. We require a number of lemmas to obtain our first result in Theorem 44.

Note that $Q(G;x,y)$ does not distinguish parallel edges, nor do loops contribute anything, so that $Q(G;x,y) = Q(G';x,y)$, where G' is the simple graph obtained from G by removing all but one edge in each parallel class and removing any loops. Recall that $K_1^{k,0}$ denotes the graph consisting of k loops on a single vertex and $K_2^{0,k}$ two vertices joined by k parallel edges.

Lemma 38. *For $k \in \mathbb{Z}_{>0}$, $Q(K_1^{k,0};x,y) = xy+1$ and $Q(K_2^{0,k};x,y) = x^2y+2xy+1$.*

Proof. By direct calculation, $Q(K_1^{k,0};x,y) = Q(K_1;x,y) = xy+1$ and $Q(K_2^{0,k};x,y) = Q(K_2;x,y) = x^2y+xy+1$. $\square$

Lemma 39. *For $k \in \mathbb{Z}_{>0}$,*

$$\begin{aligned}Q(K_{1,k};x,y) &= (xy+1)^k + xy(x+1)^k,\\ Q(P_k;x,y) &= \frac{1-x+a}{2a}\left(\frac{x+1+a}{2}\right)^{k+1} - \frac{1-x-a}{2a}\left(\frac{x+1-a}{2}\right)^{k+1},\end{aligned}$$

where $a = \sqrt{(x-1)^2+4xy}$.

Proof. The first identity is given in [33, Corollary 19]. The second identity is the one given after Proposition 16 in [33], just written differently. $\square$

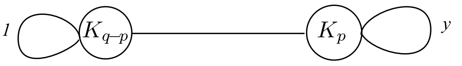

Fig. 7. The graph $K_{q-p}^1 + K_p^y$. The line between the cliques stands for the $(q-p)p$ edges joining them. This graph is a special case of the ornamented star of Fig. 6. An evaluation of the Averbouch–Godlin–Makowsky polynomial is given by $\hom(G, K_{q-p} + K_p^y) = \xi(G; q, y-1, (p-q)(y-1))$.

Lemma 40 *(See e.g. [20, Ch. 8, Ex. 20]).* *If*

$$A = \begin{pmatrix} a & \mathbf{b}^\top \\ \mathbf{b} & B \end{pmatrix}$$

then

$$\begin{aligned} \frac{\det(tI - A)}{\det(tI - B)} &= t - a - \mathbf{b}^\top (tI - B)^{-1}\mathbf{b} \\ &= t - a - \sum_{\theta \in \mathrm{ev}(B)} \frac{\mathbf{b}^\top E_\theta \mathbf{b}}{t - \theta}, \end{aligned}$$

where $\mathrm{ev}(B)$ *denotes the set of distinct eigenvalues of* B *and* E_θ *is the orthogonal projection onto the eigenspace of vectors with eigenvalue* θ.

We write $\phi_A(t) = \det(tI - A)$ for the characteristic polynomial of A (of the graph whose adjacency matrix is A).

Corollary 41. *Let* $B = I_y \otimes J_x$ *and*

$$A = \begin{pmatrix} 1 & \mathbf{1}^\top \\ \mathbf{1} & B \end{pmatrix}.$$

Then

$$\frac{\phi_A(t)}{\phi_B(t)} = \frac{t^2 - (x+1)t - x(y-1)}{t - x},$$

so that, writing $a = \sqrt{(x-1)^2 + 4xy}$, *the eigenvalues of* A *are* $\frac{1}{2}(x+1+a)$ *(with multiplicity* 1*),* $\frac{1}{2}(x+1-a)$ *(multiplicity* 1*),* x *(multiplicity* $y-1$*) and* 0 *(multiplicity* $xy - y$*).*

Proof. By Lemma 40,

$$\frac{\phi_A(t)}{\phi_B(t)} = t - 1 - \frac{\mathbf{1}^\top E_x \mathbf{1}}{t - x} - \frac{\mathbf{1}^\top E_0 \mathbf{1}}{t},$$

where $E_x = (xy)^{-1} J_{xy}$, $E_0 = I_{xy} - E_x$, and $\mathbf{1}^\top E_x \mathbf{1} = xy$, $\mathbf{1}^\top E_0 \mathbf{1} = 0$. This gives

$$\begin{aligned} \phi_A(t) &= \left(t - 1 - \frac{xy}{t - x}\right)\phi_B(t) \\ &= [t^2 - (x+1)t - x(y-1)](t-x)^{y-1} t^{xy-y}, \end{aligned}$$

with the matrix $B = I_y \otimes J_x$ having characteristic polynomial $\phi_B(t) = (t-x)^y t^{xy-y}$. $\square$

Lemma 42. *Suppose that* H *is a graph such that*

1. *there is an apex vertex* v_0 *attached to all the other vertices,*
2. *there are* $a \geq 0$ *loops on* v_0*, and*
3. $H - v_0$ *is a spectrally unique* x*-regular graph with adjacency matrix* B.

Then H *is determined up to isomorphism to be the graph with adjacency matrix*

$$A = \begin{pmatrix} a & \mathbf{1}^\top \\ \mathbf{1} & B \end{pmatrix}.$$

Proof. Since $H - v_0$ is x-regular, B has eigenvector $\mathbf{1}$ with eigenvalue x, and any other eigenvector of B with eigenvalue θ different to x is orthogonal to $\mathbf{1}$. If E_θ is the projection onto the θ-eigenspace of B then

$$\mathbf{1}^\top E_\theta \mathbf{1} = \begin{cases} x & \theta = x, \\ 0 & \theta \neq x. \end{cases}$$

By Lemma 40,

$$\phi_A(t) = \phi_B(t)[t - a - \frac{xt}{t - x}].$$

Hence if $H - v_0$ is uniquely determined by its characteristic polynomial $\phi_B(t)$ then H is also determined by its characteristic polynomial $\phi_A(t)$. $\square$

Lemma 43. *For $k \in \mathbb{Z}_{>0}$,*

$$Q(C_k; x, y) = \left(\frac{x + 1 + a}{2}\right)^k + \left(\frac{x + 1 - a}{2}\right)^k + (y - 1)x^k,$$

where $a^2 = (x - 1)^2 + 4xy$.

Proof. By [33, Theorem 10] and Lemma 26, $Q(C_k; x, y) = \hom(C_k, H_{1,x,y}) = \mathrm{tr}(A^k)$, where A is the adjacency matrix of $H_{1,x,y}$. The eigenvalues of the matrix A are calculated in Corollary 41. $\square$

We are now ready to prove our first main result about for which points (x, y) the evaluation $Q(G; x, y)$ is equal to a homomorphism number $\hom(G, H)$, and which graphs H yield these evaluations.

Theorem 44. *Let (H, α, β) be a weighted graph, where $\alpha : V(H) \to \mathbb{Z}_{>0}$ and $\beta : E(H) \to \mathbb{C}$. Then $\hom(G, H) = Q(G; x, y)$ for some point (x, y) if and only if $x, y \in \mathbb{Z}_{\geq 0}$ and $H \cong H_{1,x,y}$ up to twin vertices.*

Proof. Note that $H_{1,x,0}$ is the empty graph (equivalently, all its vertices have weight 0) and $H_{1,0,y} = K_1^1$; we have $Q(G; 0, y) = 1 = \hom(G, K_1^1)$ and $Q(G; x, 0) = 0$. Henceforth, we assume that x and y are non-zero.

The condition "up to twin vertices" is needed because a vertex of H with weight a can be split into twin vertices whose vertex weights sum to a without affecting $\hom(G, H)$. Upon splitting vertices with positive integer weight a into a unweighted twin vertices (i.e., weight 1), we just need to prove the statement for an edge-weighted graph (H, β).

Suppose that $\hom(G, H) = Q(G; x, y)$ for some non-zero $x, y \in \mathbb{C}$. Since $Q(P_1; x, y) = xy + 1$, we have $|V(H)| = xy + 1$.

Since $\hom(K_1^{k,0}, H) = \sum_{v \in V(H)} \beta_{v,v}^k = xy + 1$, it follows that $\beta_{v,v} = 1$ for each v.

For each $k \in \mathbb{Z}_{>0}$,

$$\hom(K_2^{0,k}, H) = \sum_{(u,v) \in V(H) \times V(H)} \beta_{u,v}^k = x^2y + 2xy + 1. \tag{14}$$

This forces $\beta_{u,v} \in \{0, 1\}$ for each $(u, v) \in V(H) \times V(H)$. There are $(xy + 1)^2$ pairs $(u, v) \in V(H) \times V(H)$ and $x^2y + 2xy + 1 = (xy + 1)^2 - x^2y(y - 1)$. We have seen that the $xy + 1$ loops each have weight 1. There are thus $x^2 \binom{y}{2}$ non-edges uv (weight $\beta_{u,v} = 0$), and the remaining $\binom{x+1}{2} y$ non-loop edges uv also have weight $\beta_{u,v} = 1$.

By Lemma 39,

$$\begin{aligned} \hom(K_{1,k}, H) &= \sum_{v \in V(H)} \left(\sum_{u \in V(H)} \beta_{u,v} \right)^k \\ &= 1 \cdot (xy + 1)^k + xy(x + 1)^k, \end{aligned}$$

which implies that there is a single apex vertex v_0 for which $\sum_{u\in V(H)}\beta_{u,v_0} = xy+1$, and $\sum_{u\in V(H)}\beta_{u,v_i} = x+1$ for the remaining xy vertices $v_1, v_2, \dots, v_{xy}$.

By Lemma 43,

$$\hom(C_k, H) = \sum_{\theta\in \mathrm{ev}(A)} \theta^k = \left(\frac{x+1+a}{2}\right)^k + \left(\frac{x+1-a}{2}\right)^k + (y-1)x^k,$$

where the sum ranges over eigenvalues θ of the adjacency matrix $A = (\beta_{u,v})$ of H, taken with multiplicity. Hence the eigenvalues of A are 0 (multiplicity $xy - y$), x (multiplicity $y-1$), and one eigenvalue is equal to $\frac{1}{2}(x+1+a)$, and one equal to $\frac{1}{2}(x+1-a)$.

We have thus seen that, by taking G in the family $\{K_1^{k,0}, K_2^{0,k}, K_{1,k}, C_k : k \in \mathbb{Z}_{>0}\}$, the homomorphism numbers $\hom(G,H)$ determine that the adjacency matrix of A is a (0, 1)-matrix (so H is an unweighted graph) of size $xy+1$ and rank $y+1$, each entry in the diagonal equal to 1, one vertex v_0 indexing a row and column all of whose entries are equal to 1, and each of the other rows and columns containing $x+1$ non-zero entries. Finally, the spectrum of H coincides with that of $H_{1,x,y}$.

The complete graph K_x^1 with a loop on each of its vertices is uniquely determined by its characteristic polynomial $t^{x-1}(t-x)$ amongst unweighted graphs. The disjoint union of y copies of this graph, which has adjacency matrix $I_y \otimes J_x$, is then also uniquely determined by its characteristic polynomial $t^{xy-y}(t-x)^y$ amongst unweighted graphs (see for example [35, Proposition 5]). Since H has an apex vertex of degree $xy+1$ whose deletion leaves an x-regular graph, and since its spectrum determines that its adjacency matrix A satisfies $\phi_A(t) = \phi_{I_y\otimes J_x}(t)\left(t-1-\frac{xt}{t-x}\right)$, Lemma 42 implies that H must be isomorphic to $H_{1,x,y}$. □

Theorem 44 can be extended from $\alpha : V(H) \to \mathbb{Z}_{>0}$ to $\alpha : V(H) \to \mathbb{R}_{>0}$ as follows.

Lemma 45. *If (H, α, β) is a weighted graph with $\alpha : V(H) \to \mathbb{Z}_{>0}$ and $\beta : E(H) \to \mathbb{C}$ such that $\hom(G,H) = k^{|V(G)|}Q(G; x/k, y)$ then $H \cong H_{k,x,y}$ up to twin vertices.*

Proof. The proof is by simple adaptation of proof of Theorem 44. □

Corollary 46. *If (H, α, β) is a weighted graph with $\alpha : V(H) \to \mathbb{Q}_{>0}$ and $\beta : E(H) \to \mathbb{C}$ then $\hom(G,H) = Q(G; x, y)$ for some point (x, y) if and only if $H \cong H_{1,x,y}$ up to twin vertices.*

Proof. Suppose that the least common multiple of the denominators of the vertex weights is equal to k. Multiply through the vertex weights of H by k to obtain a vertex-$\mathbb{Z}_{>0}$-weighted graph H'. Then $\hom(G,H) = Q(G; x, y)$ if and only if $\hom(G,H') = k^{|V(G)|}Q(G; x, y)$, and by Lemma 45 the latter holds if and only if $H' \cong H_{k,kx,y}$ up to twin vertices. Then $H \cong H_{1,x,y}$ up to twin vertices. □

Theorem 47. *If (H, α, β) is a weighted graph with $\alpha : V(H) \to \mathbb{R}_{>0}$ and $\beta : E(H) \to \mathbb{C}$ then $\hom(G,H) = Q(G; x, y)$ for some point (x, y) if and only if $H \cong H_{1,x,y}$ up to twin vertices.*

Proof. Let $((H_n, \alpha_n, \beta) : n = 1, 2, \dots)$ be a sequence of vertex- and edge-weighted graphs where the functions $\alpha_n : V(H_n) \to \mathbb{Q}_>0$ have the property that $\alpha_n(v) \to \alpha(v)$ for each $v \in V(H) = V(H_n)$, i.e., (α_n) converges pointwise to α. By continuity of $Q(G; x, y)$ in x and $\hom(G,H_n) \to \hom(G,H) = Q(G; x, y)$, there must be a sequence of reals (x_n) convergent to x such that $\hom(G,H_n) = Q(G; x_n, y)$. By Corollary 46, $H_n \cong H_{1,x_n,y}$ up to twin vertices, and taking the limit as $n \to \infty$ we get $H \cong H_{1,x,y}$ up to twin vertices. □

Now, let $H = (H, \alpha, \beta)$ be a vertex- and edge weighted graph, with $\alpha : V(H) \to \mathbb{C}$ (and as usual $\beta : V(H) \to \mathbb{C}$).

We do not have any examples of (H, α, β) where α takes negative values and for which $\hom(G,H) = Q(G; x, y)$. On the other hand, as noted in Remark 2 above, Averbouch et al. do obtain such an example for their "edge elimination polynomial" $\xi(G; x, y, z)$.

Tittmann et al. [33] pose a slightly weaker problem than determining whether there is a vertex- and edge-weighted graph H not isomorphic up to twins with $H_{1,x,y}$ such that $\hom(G,H) = Q(G; x, y)$:

what they ask is whether there is a point (x, y) with $y \notin \mathbb{Z}_{\geq 0}$ for which $\hom(G, H) = Q(G; x, y)$ for some H. To this question we do have an answer.

Lemma 48. *If* $\hom(C_k, H) = Q(C_k; x, y)$ *for* $k \in \mathbb{Z}_{>0}$ *then the eigenvalues of the matrix* $C = ((\alpha_u\alpha_v)^{\frac{1}{2}}\beta_{u,v})_{u,v\in V(H)}$ *are* x *(*$y-1$ *times),* $(x+1+a)/2$ *(once)* $(x+1-a)/2$ *(once) and* 0 *(*$|V(H)|-y-1$ *times), where* $a^2 = (x-1)^2 + 4xy$.

Proof. We have

$$\begin{aligned}
\hom(C_k, H) &= \sum_{v_0,v_1,\ldots,v_{k-1},v_k=v_0} \alpha_{v_0}\alpha_{v_1}\cdots\alpha_{v_{k-1}}\beta_{v_0,v_1}\beta_{v_1,v_2}\cdots\beta_{v_{k-1},v_0} \\
&= \sum_{v_0,v_1,\ldots,v_{k-1},v_k=v_0} \prod_{0\leq i\leq k-1} (\alpha_{v_i}\alpha_{v_{i+1}})^{\frac{1}{2}}\beta_{v_i,v_{i+1}} \\
&= \mathrm{tr}(C^k) \\
&= \sum_{\theta\in \mathrm{ev}(C)} \theta^k = \left(\frac{x+1+a}{2}\right)^k + \left(\frac{x+1-a}{2}\right)^k + (y-1)x^k,
\end{aligned}$$

where $\mathrm{ev}(C)$ denotes the multiset of eigenvalues of C. □

Corollary 49. *If* $Q(G; x, y) = \hom(G, H)$ *for some vertex- and edge-weighted graph* H *then* $y \in \mathbb{Z}_{>0}$ *or* $y = 0$ *and* H *has vertices all of weight* 0.

4.5. 'Local Tutte–Grothendieck invariants'

A '$\mathcal{G}$-local Tutte–Grothendieck invariant' is just required to satisfy the generalized Tutte–Grothendieck recurrence relations for a minor-closed class of graphs $\mathcal{G}$, not necessarily all graphs. This includes the case of parameters of the form $h(G)T(G; x, y)$ for a $\mathcal{G}$-local function h, since these satisfy a generalized Tutte–Grothendieck recurrence relation for $G \in \mathcal{G}$. Our main result in this section is Theorem 53, which proves that if a homomorphism number is a generalized Tutte–Grothendieck invariant locally on cycles, paths, multiple edges and multiple loops then it is in fact a Tutte–Grothendieck invariant on all multigraphs, and hence is of the form $\hom(G, K_q^{a,b})$, where $K_q^{a,b}$ is a Potts model graph.

Tutte–Grothendieck invariants have recurrence relations that vary depending on whether an edge is a bridge, loop or ordinary. However, it turns out that if a generalized Tutte–Grothendieck invariant is also equal to $\hom(G, H)$ for a connected edge-weighted graph H then in fact there is no dependence on edge type.[2] A familiar example is the evaluation of the chromatic polynomial $\hom(G, K_q)$. Note however that, by multiplying by a suitable local function, any generalized Tutte–Grothendieck invariant can be made to satisfy a contraction–deletion recurrence that is independent of edge type. For example, the flow polynomial $F(G; q)$ satisfies

$$F(G) = \begin{cases} F(G/e) - F(G\setminus e) & e \text{ ordinary,} \\ 0 & e \text{ a bridge,} \\ (q-1)F(G\setminus e) & e \text{ a loop,} \\ 1 & E = \emptyset. \end{cases}$$

Here, $h(G) = q^{|V|}F(G; q)$ satisfies $h(G) = qh(G/e) - h(G\setminus e)$ for all edges e of G.

Lemma 50. *Let* H *be an edge-*$\mathbb{C}$*-weighted graph on* q *vertices and* $\alpha, \beta, x, y \in \mathbb{C}$. *Suppose that for* $G \in \{K_1^{k,0}, K_2^{0,k} : k \in \mathbb{Z}_{>0}\}$ *the function* $\hom(G, H) = h(G)$ *satisfies the equations for a generalized*

[2] Averbouch et al. [3, p.4, n.2] remark that the recurrence relation for the polynomial $\xi(G)$ does not depend on the type of edge being contracted/deleted/eliminated, and that it may be interesting to explore the possibility of dependence on edge type. Theorem 37 would in fact incorporate this more general situation, due to the fact that for connected H homomorphism functions $\hom(G, H)$ do not satisfy recurrences dependent on edge type.

Tutte–Grothendieck invariant, i.e.,

$$h(G) = \begin{cases} \alpha h(G/e) + \beta h(G \setminus e) & e \text{ ordinary,} \\ xh(G/e) & e \text{ a bridge,} \\ yh(G \setminus e) & e \text{ a loop,} \\ q & G = K_1^{0,0}, \\ q^2 & G = K_2^{0,0}. \end{cases}$$

Then when $y \neq \beta$ the graph H has q loops on each of its vertices, each loop of weight y, $\frac{q(\alpha+\beta-y)}{y-\beta}$ ordinary edges of weight y, $\frac{q(\alpha+q\beta-x)}{\beta}$ ordinary edges of weight 0 (zero) and the remaining ordinary edges of weight β.

If $y = \beta$ then $\alpha = 0$, $x = y$, $H = \overline{K}_q^y$ and $\hom(G, H) = q^{k(G)}y^{|E|}$.

Proof. Let H have adjacency matrix $A = (a_{u,v})_{u,v\in[q]}$. Then, for each $k \in \mathbb{Z}_{>0}$,

$$h(K_1^{k,0}) = \hom(K_1^{k,0}, H) = \sum_{v\in[q]} a_{v,v}^k,$$

and also

$$h(K_1^{k,0}) = qy^k.$$

By Lemma 35, this implies that $a_{v,v} = y$ for each $v \in [q]$. For each $k \in \mathbb{Z}_{>0}$, we also have

$$h(K_2^{0,k}) = \hom(K_2^{0,k}, H) = \sum_{(u,v)\in[q]\times[q]} a_{u,v}^k,$$

and, by the recurrence relations, for $k \geq 2$,

$$h(K_2^{0,k}) = \alpha h(K_1^{k-1,0}) + \beta h(K_2^{0,k-1}),$$

with boundary condition $h(K_2^{0,1}) = qx$. Writing $m_k = h(K_2^{0,k})$, for $k \geq 2$,

$$m_k = \beta m_{k-1} + \alpha q y^{k-1}, \qquad m_1 = qx.$$

Set $M(t) = \sum_{k\geq 1} m_k t^{k-1}$. Then

$$M(t) - qx = \beta t M(t) + \frac{\alpha q y}{1-yt},$$

whence, for $y \neq \beta$,

$$\begin{aligned} M(t) &= \frac{qx}{1-\beta t} + \frac{\alpha q y}{(1-\beta t)(1-yt)}, \\ &= \frac{qx}{1-\beta t} - \frac{\alpha\beta q y}{(y-\beta)(1-\beta t)} + \frac{\alpha q y^2}{(y-\beta)(1-yt)}. \end{aligned}$$

This gives, for $k \geq 1$,

$$m_k = \left(qx - \frac{\alpha q y}{y-\beta}\right)\beta^{k-1} + \frac{\alpha q}{y-\beta} y^k. \tag{15}$$

For $y = \beta$, we obtain $m_k = qxy^{k-1} + \alpha q(k-1)y^{k-1}$. This implies that $\alpha = 0$ and also implies the situation described in the last statement of the lemma.

We return to the case $y \neq \beta$. Write $N(y) = |\{(u, v) \in [q] \times [q] : a_{u,v} = y\}|$ and $N(\beta) = |\{(u, v) \in [q] \times [q] : a_{u,v} = \beta\}|$. By Lemma 35, Eq. (15) implies that

$$N(y) = \frac{\alpha q}{y-\beta}$$

and

$$N(\beta) = \frac{qx(y-\beta) - \alpha qy}{\beta(y-\beta)}.$$

The first statement of the lemma now follows. $\square$

Compare the following proposition to Theorem 37, where graphs amongst $\{K_1^{k,0}, K_2^{0,k} : k \in \mathbb{Z}_{>0}\}$ were sufficient to determine that $H \cong K_{q-p}^1 + K_p^{1+y}$ when $\hom(G, H)$ is an evaluation of the Averbouch–Godlin–Makowsky polynomial $\xi(G)$.

Proposition 51. *Let $\mathcal{G} = \{K_1^{k,0}, K_2^{0,k} : k \in \mathbb{Z}_{>0}\}$. Suppose that for $G \in \mathcal{G}$ the graph parameter $h(G)$ satisfies the equation $h(G) = \alpha h(G/e) + \beta h(G \setminus e)$ for a generalized Tutte–Grothendieck invariant that is independent of whether the edge e is ordinary, a bridge or a loop. Suppose further that $h(G) = \hom(G, H)$ for some edge-$\mathbb{C}$-weighted graph H. Then $H \cong K_q^{\alpha+\beta,\beta}$. In particular, a Potts model graph is determined by its left $\{K_1^{k,0}, K_2^{0,k} : k \in \mathbb{Z}_{>0}\}$-profile.*

Proof. In the proof of Lemma 50, note that $\frac{\alpha q}{y-\beta}$ is the integer $N(y) \geq |\{v \in [q] : a_{v,v} = y\}|$, which is equal to q if and only if $y = \alpha + \beta$. The equation $y = \alpha + \beta$ holds if the relation $h(G) = \alpha h(G/e) + \beta h(G \setminus e)$ coincides with the relation $h(G) = yh(G \setminus e)$ when $G = K_1^{1,0}$ is a loop. Also, by the equations for $N(y)$ and $N(\beta)$ given at the end of the proof of Lemma 50, we find that $N(y) + N(\beta) = \frac{q(x-\alpha)}{\beta}$. It follows that $N(y) + N(\beta) = q^2$ (i.e., $a_{u,v} \neq 0$ for all $u, v \in [q]$) if and only if $x - \alpha = q\beta$. The equation $x = \alpha + q\beta$ holds if the relation $h(G) = \alpha h(G/e) + \beta h(G \setminus e)$ coincides with the relation $h(G) = xh(G/e)$ when $G = K_2^{0,1}$ is a bridge. Given then that $y = \alpha + \beta$ and $x = \alpha + q\beta$, the assertion of Lemma 50 is that $H \cong K_q^{\alpha+\beta,\beta}$.

For the final statement of the proposition, the recurrence for the generalized Tutte–Grothendieck invariant $h(G) = \hom(G, K_q^{\alpha+\beta,\beta})$ is independent of edge type. By Lemma 50, we conclude that if $\hom(G, H) = \hom(G, K_q^{\alpha+\beta,\beta})$ for $G \in \{K_1^{k,0}, K_2^{0,k} : k \in \mathbb{Z}_{>0}\}$ then $H \cong K_q^{\alpha+\beta,\beta}$. $\square$

Note that Lemma 50 provides examples of edge-weighted graphs H that are not Potts model graphs but are graphs for which $\hom(G, H)$ satisfies the equations for a generalized Tutte–Grothendieck invariant for $G \in \{K_1^{k,0}, K_2^{0,k} : k \in \mathbb{Z}_{>0}\}$. This occurs when the recurrence equations depend on edge type. We require some more profile information about H in order to eliminate these other possibilities for H.

Lemma 52. *Let H be an edge-$\mathbb{C}$-weighted graph on q vertices and $\alpha, \beta, x, y \in \mathbb{C}$. Suppose that for $G \in \{P_k, C_k : k \in \mathbb{Z}_{>0}\}$ the function $\hom(G, H) = h(G)$ satisfies the equations for a generalized Tutte–Grothendieck invariant, i.e.,*

$$h(G) = \begin{cases} \alpha h(G/e) + \beta h(G \setminus e) & e \text{ ordinary}, \\ xh(G/e) & e \text{ a bridge}, \\ yh(G \setminus e) & e \text{ a loop}, \\ q & G = P_1. \end{cases}$$

Then the adjacency matrix of H has eigenvalues x (with multiplicity $\frac{q\beta}{x-\alpha}$) and α (with multiplicity $\frac{qy(x-\alpha)-q\beta x}{\alpha(x-\alpha)}$). Furthermore, $y = \alpha + \beta$.

Proof. Note that $h(P_k) = qx^{k-1}$. Writing $h(C_k) = c_k$, we have, for $k \geq 2$,

$$c_k = \alpha c_{k-1} + \beta q x^{k-1},$$

with boundary condition $c_1 = qy$. This is the same equation as for $m_k = h(K_2^{0,k})$ in the proof of Lemma 50 with the roles of α, β and x, y reversed. (The cycle C_k is the planar dual of the multiple edge $K_2^{0,k}$.) Hence,

$$c_k = \left(qy - \frac{q\beta x}{x-\alpha}\right)\alpha^{k-1} + \frac{q\beta}{x-\alpha}x^k,$$

and by Lemma 35 the desired conclusion now follows, since $h(C_k)$ is the sum of the kth powers of the eigenvalues of the adjacency matrix A of H.

Since A has q eigenvalues, counting multiplicities, we must have

$$q = \frac{q\beta}{x-\alpha} + \frac{qy(x-\alpha) - q\beta x}{\alpha(x-\alpha)},$$

whence $y = \alpha + \beta$. □

We reach our main theorem of this section.

Theorem 53. *Let H be a connected edge-$\mathbb{C}$-weighted graph and $\alpha, \beta, x, y \in \mathbb{C}$. Suppose that for $G \in \{K_1^{k,0}, K_2^{0,k}, P_k, C_k : k \in \mathbb{Z}_{>0}\}$ the function* $\hom(G, H) = h(G)$ *satisfies the equations for a generalized Tutte–Grothendieck invariant, i.e.,*

$$h(G) = \begin{cases} \alpha h(G/e) + \beta h(G \setminus e) & e \text{ ordinary}, \\ xh(G/e) & e \text{ a bridge}, \\ yh(G \setminus e) & e \text{ a loop}, \\ q^2 & G = K_2^{0,0}, \\ q & G = P_1 = K_1^{0,0}. \end{cases}$$

Then $x = \alpha + q\beta$, $y = \alpha + \beta$, and H is isomorphic to $K_q^{y,\beta}$.

Proof. H has q vertices by $h(P_1) = q$. By Lemma 52, the hypotheses imply that $y = \alpha + \beta$ and that the adjacency matrix $A = (a_{u,v})_{u,v\in[q]}$ of H has two distinct eigenvalues x and α, and thus satisfies $A^2 - (x+\alpha)A + \alpha x I = 0$. Since H is connected, this implies that every pair of distinct vertices are connected by an edge of non-zero weight. By Lemma 50, H has q loops of weight y but no ordinary edges of weight y, and the ordinary edges all must have weight β, since we have just seen that none of them can have zero weight. That $x = \alpha + q\beta$ also follows from this lemma. □

Remark 3. The *coloured Tutte polynomial* of an edge-coloured graph defined by Bollobás and Riordan [7] satisfies a contraction–deletion recurrence whose coefficients depend on edge colour as well as edge type. A *coloured homomorphism* between edge-coloured graphs G and H maps G to H so that edges of G are mapped to edges of the same colour in H. (We have already met in Section 4.4 colour-preserving homomorphisms between vertex-coloured graphs.) It is not difficult to see that if an evaluation of the coloured Tutte polynomial is a coloured homomorphism number $\hom(G, H)$ then the subgraph of H comprising edges of a single colour c is a q_c-state Potts model graph $K_{q_c}^{\alpha_c,\beta_c}$, the loop and non-loop weights α_c and β_c and number of vertices q_c all possibly varying with c. There may be additional vertices of H that are not incident with any edge of colour c. That H must take this form can be established by testing $\hom(G, H)$ for coloured versions of the graphs G of Theorem 53. Conversely, given an edge-weighted graph H, each of whose monochromatic subgraphs is a Potts model graph, the homomorphism function $\hom(G, H)$ satisfies a contraction–deletion recurrence of a form that implies it is an evaluation of the coloured Tutte polynomial.

5. Some open problems

The problem of determining a graph by its left or right profile has been studied in various contexts, leading to interesting notions of left and right convergence (see [8] for a survey) and homomorphism dualities (see for example [23]). Here we have seen how Tutte uniqueness corresponds to being determined by the right $\{K_q^{y,1} : q \geq 1,\ y \in \mathbb{Z}_{\geq 0}\}$-profile (Theorem 11). Moreover, graphs belonging to $\{K_q^{y,1} : q \geq 1,\ y \in \mathbb{Z}_{\geq 0}\}$ are each determined by their left profile by cycles and stars (Corollary 32) and by their left profile by the duals of these graphs, i.e., multiple edges joining two vertices and multiple loops on a single vertex (Proposition 51). In this paper, we have introduced the notion of 'q-state Potts uniqueness', the property of a graph being determined by its right $\{K_q^{y,1} : y \in \mathbb{Z}_{\geq 0}\}$-profile. There is a significant overlap between 'q-state Potts uniqueness' and 'chromatic–flow uniqueness', introduced by Duan et al. [13]. Some questions immediately raised concerning this topic are as follows.

Problem 54. Find further examples of s-bridge graphs for $s \geq 4$ that are not 2-state Potts unique. Is $\theta(a_1, \ldots, a_s) q$-state Potts unique for $q \geq 3, s \geq 4$?

Problem 55. Locally grid graphs are Tutte unique [29,18]: are they q-state Potts unique? Are they chromatic–flow unique?

Problem 56. Given $3 \leq q < q'$, is q-state Potts uniqueness different to q'-state Potts uniqueness? For $q \geq 3$, do there exist chromatic–flow unique graphs that are not q-state Potts unique, or vice versa?

In Section 4, we encountered a problem of the following form.

> Fix a set $\mathcal{H}$ of edge-weighted graphs. Find $\mathcal{G}$ (as small as possible) so that any $H \in \mathcal{H}$ is distinguished by its left $\mathcal{G}$-profile from all other non-isomorphic edge-weighted graphs.

When $\mathcal{H}$ is the set of Potts model graphs $\{K_q^{a,b} : q \in \mathbb{Z}_{>0}, a, b \in \mathbb{C}\}$, we found in Section 4 that we can take $\mathcal{G} = \{C_k, K_{1,k} : k \in \mathbb{Z}_{>0}\}$ or $\mathcal{G} = \{K_1^{k,0}, K_2^{0,k} : k \in \mathbb{Z}_{>0}\}$. In fact, it is easy to see that we can let k range over any infinite subset of $\mathbb{Z}_{>0}$ in either of these families while still preserving the property that the left $\mathcal{G}$-profile determines each graph in $\mathcal{H}$.

A concrete question in this area is as follows.

Problem 57. Is there a set $\mathcal{H}$ of edge-weighted graphs for which there is an infinite set $\mathcal{G}$ with the property that each $H \in \mathcal{H}$ is determined by its left $\mathcal{G}$-profile, but is not determined in this way by any proper subset of $\mathcal{G}$?

Note that if $\mathcal{G} = \{P_1\} \cup \{C_k : k \in \mathbb{Z}_{>0}\}$ and $\mathcal{H}$ is the set of spectrally unique edge-weighted graphs then each $H \in \mathcal{H}$ is determined by its left $\mathcal{G}$-profile, but not by its $\mathcal{G} - \{P_1\}$-profile. There could be any number of isolated vertices added to H, and its left $\{C_k : k \in \mathbb{Z}_{>0}\}$-profile would be unchanged. On the other hand, any subset of $\mathcal{G}$ containing P_1 and an infinite number of cycles still determines graphs in $\mathcal{H}$.

Problem 57 has an analogous formulation for right profiles: is there a set $\mathcal{G}$ of multigraphs with the property that there is an infinite set $\mathcal{H}$ for which the right $\mathcal{H}$-profile determines each $G \in \mathcal{G}$ but the same is not true for any proper subset of $\mathcal{H}$?

In Section 3, we considered a problem of the following form.

> Fix a set $\mathcal{H}$ of edge-weighted graphs. Find $\mathcal{G}$ (as large as possible) so that any $G \in \mathcal{G}$ is distinguished by its right $\mathcal{H}$-profile from all other non-isomorphic multigraphs.

For example, when $\mathcal{H} = \{K_q^{y,1} : q \geq 1, y \in \mathbb{Z}_{\geq 0}\}$ we have seen that $\mathcal{G}$ is the set of Tutte unique multigraphs. When $\mathcal{H} = \{K_q : q \in \mathbb{Z}_{>0}\}$, we restrict attention to simple graphs, in which case $\mathcal{G}$ comprises the set of chromatically unique graphs. When $\mathcal{H} = \{K_q^{q-1,-1} : q \in \mathbb{Z}_{>0}\}$, we restrict attention to cosimple graphs, in which case $\mathcal{G}$ comprises the set of flow unique graphs. A fundamental problem here is to determine whether $\mathcal{G}$ comprises almost all graphs, in the sense that the proportion of graphs on n vertices belonging to $\mathcal{G}$ is asymptotically equal to 1 as $n \to \infty$. An example where an answer to this question is known is the case $\mathcal{H} = \{\overline{K}_{q-1} + K_1^{1,1} : q \in \mathbb{Z}_{>0}\}$: by Proposition 33, the right $\mathcal{H}$-profile of G here determines the independence polynomial of G, and Noy [30] shows that the proportion of graphs on n vertices that are determined by their independence polynomial is asymptotically 0.

Acknowledgement

The first author's research was supported by projects O.R.I. MTM2008-05866-C03-01, PAI FQM-0164 and PAI P06-FQM-01649 The second author's research was partially supported by the hosting departments while visiting the first author in December 2008 and January 2010 and the third author in January 2009 and autumn 2009, and by AEOLUS. The third author's research was supported by ITI 1M0545, and the Centre for Discrete Mathematics, Theoretical Computer Science and Applications (DIMATIA).

References

[1] N. Alon, M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (1992) 125–134.
[2] D. Andrén, K. Markström, The bivariate Ising polynomial of a graph, Discrete Appl. Math. 157 (11) (2009) 2515–2524.
[3] I. Averbouch, B. Godlin, J. Makowsky, An extension of the bivariate chromatic polynomial, European J. Combin. 31 (2010) 1–17.
[4] N. Biggs, Algebraic Graph Theory, 2nd ed., Cambridge Univ. Press, Cambridge, 1994.
[5] B. Bollobás, Modern Graph Theory, in: Grad. Texts in Math, Springer, New York, 1998.
[6] B. Bollobás, L. Pebody, O. Riordan, Contraction-deletion invariants for graphs, J. Combin. Theory Ser. B 80 (2000) 320–345.
[7] B. Bollobás, O. Riordan, A Tutte polynomial for coloured graphs, Combin. Probab. Comput. 8 (1999) 45–94.
[8] C. Borgs, J. Chayes, L. Lovász, V. Sós, K. Vesztergombi, Counting graph homomorphisms, in: M. Klazar, J. Kratochvíl, M. Loebl, J. Matoušek, R. Thomas, P. Valtr (Eds.), Topics in Discrete Mathematics, Springer, New York, 2007, pp. 315–371.
[9] T. Brylawski, J. Oxley, The Tutte polynomial and its applications, in: N. White (Ed.), Matroid Applications, in: Encyclopedia Math. Appl., vol. 40, Cambridge Univ. Press, Cambridge, 1992, pp. 123–225.
[10] X.E. Chen, X.W. Bao, K.Z. Ouyang, Chromaticity of the graph $\theta(a, b, c, d)$, J. Shanxi Normal Univ. 20 (1992) 75–79 (in Chinese) English summary.
[11] A. de Mier, M. Noy, On graphs determined by their Tutte polynomial, Graphs Combin. 20 (2004) 105–119.
[12] K. Dohmen, A. Pönitz, P. Tittmann, A new two-variable generalization of the chromatic polynomial, Discrete Math. Theor. Comput. Sci. 6 (2003) 69–90.
[13] Y. Duan, H. Wu, Q. Yu, On chromatic and flow polynomial unique graphs, Discrete Appl. Math. 156 (12) (2008) 2300–2309.
[14] Z. Dvořák, On recognizing graphs by numbers of homomorphisms, Tech. Rep. 2006-287, ITI, Charles University, Prague, 2006.
[15] M. Dyer, C. Greenhill, The complexity of counting graph homomorphisms, Random Structures Algorithms 17 (2000) 260–289.
[16] M. Freedman, L. Lovász, A. Schrijver, Reflection positivity, rank connectivity, and homomorphisms of graphs, J. Amer. Math. Soc. 20 (2007) 37–51.
[17] D. Garijo, A. Goodall, J. Nešetřil, Graph homomorphisms, the Tutte polynomial and q-state Potts uniqueness, Electron. Notes Discrete Math. 34 (2009) 231–236.
[18] D. Garijo, A. Márquez, M. Revuelta, Tutte uniqueness of locally grid graphs, Ars Combin. 92 (2009) 377–396.
[19] D. Garijo, J. Nešetřil, M. Revuelta, Homomorphisms and polynomial invariants of graphs, European J. Combin. 30 (2009) 1659–1675.
[20] C. Godsil, G. Royle, Algebraic Graph Theory, in: Grad. Texts in Math., Springer, New York, 2001.
[21] A. Goodall, Fourier analysis on finite Abelian groups: some graphical applications, in: G. Grimmett, C. McDiarmid (Eds.), Combinatorics, Complexity, Chance: A Tribute to Dominic Welsh, in: Oxford Lecture Ser. Math. Appl., vol. 34, Oxford Univ. Press, Oxford, 2007, pp. 103–129. (Chapter 7).
[22] I. Gutman, F. Harary, Generalizations of the matching polynomial, Util. Math. 24 (1983) 97–106.
[23] P. Hell, J. Nešetřil, Graphs and Homomorphisms, in: Oxford Lecture Ser. Math. Appl., 28, Oxford Univ. Press, Oxford, 2004.
[24] K. Koh, K. Teo, The search for chromatically unique graphs, Graphs Combin. 6 (1990) 259–285.
[25] K. Koh, K. Teo, The search for chromatically unique graphs—II, Discrete Math. 172 (1997) 59–78.
[26] L. Lovász, Operations with structures, Acta Math. Hungar. 18 (1967) 321–328.
[27] L. Lovász, The rank of connection matrices and the dimension of graph algebras, European J. Combin. 27 (2006) 962–970.
[28] L. Lovász, A. Schrijver, Dual graph homomorphism functions, J. Combin. Theory Ser. A 117 (2010) 216–222.
[29] A. Márquez, A. de Mier, M. Noy, M.P. Revuelta, Locally grid graphs: classification and Tutte uniqueness, Discrete Math. 266 (1–3) (2003) 327–352.
[30] M. Noy, Graphs determined by polynomial invariants, Theoret. Comput. Sci. 307 (2) (2003) 365–384.
[31] R. Read, E.W. Whitehead Jr., A note on chain lengths and the Tutte polynomial, Discrete Math. 308 (10) (2008) 1826–1829.
[32] A. Schwenk, Almost all trees are cospectral, in: F. Harary (Ed.), New Directions in the Theory of Graphs, Academic Press, New York, 1973, pp. 275–307.
[33] P. Tittmann, I. Averbouch, J. Makowsky, The enumeration of vertex induced subgraphs with respect to the number of components, 2009. arXiv:0812.4147 [math.CO].
[34] W.T. Tutte, A class of Abelian groups, Canad. J. Math. 8 (1956) 13–28.
[35] E.R. van Dam, W.H. Haemers, Which graphs are determined by their spectrum? Linear Algebra Appl. 373 (2003) 241–272.
[36] D. Welsh, Complexity: Knots, Colourings and Counting, in: London Math. Soc. Lect. Notes Ser., Cambridge Univ. Press, Cambridge, 1993.
[37] B. Widom, J.S. Rowlinson, New model for the study of liquid–vapour phase transition, J. Chem. Phys. 52 (1970) 1670–1684.