HYPERCYCLIC ALGEBRAS FOR D-MULTIPLES OF CONVOLUTION OPERATORS

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Abstract. It is shown in this short note the existence, for each nonzero member of the ideal of D-multiples of convolution operators acting on the space of entire functions, of a scalar multiple of it supporting a hypercyclic algebra.

1. Introduction

Let us consider the vector space $L(X)$ of all operators on X, that is, the family of all continuous linear self-mappings $T : X \to X$, where X is a Hausdorff topological vector space. An operator $T \in L(X)$ is said to be *hypercyclic* provided that there is some vector $x_0 \in X$ (called hypercyclic for T) such that the orbit $\{T^n x_0 : n \in \mathbb{N}\}$ of x_0 under T is dense in X ($\mathbb{N} := \{1, 2, \dots\}$). The set of hypercyclic vectors for T will be denoted by $HC(T)$. In this short paper, we are concerned with the *algebraic size* of $HC(T)$ when T runs over an important class of operators on the space of entire functions. For background on hypercyclic operators we refer the reader to the excellent books [5, 18]. An account of concepts and results about algebraic structures inside nonlinear sets can be found in [2].

It is well known that if X is an F-space (i.e., complete and metrizable, and hence Baire) and T is a hypercyclic operator then the set $HC(T)$ is *residual,* that is, it contains a dense G_δ subset of X; we can say that $HC(T)$ is topologically large. Furthermore, for any topological vector space and any hypercyclic $T \in L(X)$, the family $HC(T)$ is algebraically large; specifically, it contains, except for 0, a *dense* (even T-invariant) vector subspace of X (see [8, 14, 19, 26]). A number of criteria have been established for an operator $T \in L(X)$ to support a *closed infinite dimensional* vector subspace all of whose nonzero vectors are T-hypercyclic on an F-space X; however, not every hypercyclic operators supports such a subspace (see [5, Chapter 8] and [18, Chapter 10]).

If G is a nonempty open subset of $\mathbb{C}$, the symbol $H(G)$ will stand for the class of all holomorphic functions in G. We are mainly interested in operators defined on the space $H(\mathbb{C})$ of entire functions $\mathbb{C} \to \mathbb{C}$. It is well known that $H(\mathbb{C})$ becomes an F-space when endowed with the topology of uniform convergence on

2010 *Mathematics Subject Classification.* 30E10, 30H50, 30K20, 46E10, 47A16.
Key words and phrases. hypercyclic operator, convolution operator, derivative operator, convexity, algebrability.

compacta. An operator $T \in L(H(\mathbb{C}))$ is said to be a *convolution operator* provided that it commutes with translations, that is, $T \circ \tau_a = \tau_a \circ T$ for all $a \in \mathbb{C}$, where $\tau_a f := f(\cdot + a)$. Then $T \in L(H(\mathbb{C}))$ happens to be a convolution operator if and only if T commutes with the derivative operator $D : f \mapsto f'$, and if and only if it is an infinite order linear differential operator with constant coefficients $T = \Phi(D)$, where Φ is an entire function with finite exponential type, that is, there exist constants $A, B \in (0, +\infty)$ such that $|\Phi(z)| \leq Ae^{B|z|}$ for all $z \in \mathbb{C}$. For such a function $\Phi(z) = \sum_{n \geq 0} a_n z^n$ and $f \in H(\mathbb{C})$, we have $\Phi(D)f = \sum_{n=0}^{\infty} a_n f^{(n)}$. Godefroy and Shapiro [16] proved that every non-scalar convolution operator is hypercyclic, so covering the classical Birkhoff [13] and MacLane [21] results on hypercyclicity of the translation operator τ_a (take $\Phi(z) = e^{az}$, $a \neq 0$) and of D (take $\Phi(z) = z$), respectively. For every non-scalar convolution operator T, the set $HC(T)$ contains, except for 0, a closed infinite dimensional vector subspace of $\mathcal{H}(\mathbb{C})$: see [23–25] and also [2, Section 4.5] and [18, Section 10.1].

Therefore, many linear combinations of hypercyclic functions with respect to a convolution operator are still hypercyclic. But things seem to be more disappointing with respect to multiplication of hypercyclic functions. Let us give an overview on the short history of this topic. Firstly, we have that for every $a \neq 0$, every $f \in HC(\tau_a)$ and every $n \in \mathbb{N}$ with $n \geq 2$, the n-power $f^n \notin HC(\tau_a)$ (see [3, Cor. 2.4]). As a positive result, it was proved in [3, Th. 2.3] (see also [4]) the existence of a function $f \in H(\mathbb{C})$ –in fact, of a residual subset of them– such that $f^n \in HC(D)$ for all $n \in \mathbb{N}$. Going into this area in more depth, it has been recently shown in [7] the existence (for any prescribed nonconstant $\Phi \in H(\mathbb{C})$ of subexponential type, meaning that for given $\varepsilon > 0$ there is a constant $A = A(\varepsilon) \in (0, +\infty)$ such that $|\Phi(z)| \leq A\, e^{\varepsilon |z|}$ for all $z \in \mathbb{C}$) of an infinitely generated multiplicative *group* consisting, except for the constant function 1, of non-vanishing $\Phi(D)$-hypercyclic entire functions.

The mentioned result in [3, Th. 2.3] was extended by Bayart and Matheron who proved that there is even a residual set of functions in $H(\mathbb{C})$ generating a *hypercyclic algebra* for the derivative operator, that is, every non-null function in one of these algebras is D-hypercyclic [5, Th. 8.26]. Recall that the algebra generated by a function f is nothing but the set $\{Q \circ f : Q \text{ polynomial}, Q(0) = 0\}$. The existence of (one-generated) algebras of hypercyclic functions for D was also independently proved by Shkarin in [25] and extended –also with residuality of the set of generating functions– by Bès, Conejero and Papathanasiou in [9] to operators of the form $P(D)$, where P is a nonconstant polynomial with $P(0) = 0$.

Recently, important advances have been carried out by the last three mentioned authors in [10, 12], but let us first recall the notion of algebrability. If X is a (linear) algebra then a subset $A \subset X$ is called *algebrable* whenever $A \cup \{0\}$ contains a subalgebra that is not finitely generated, and provided that X is a Fréchet algebra, A is called *dense algebrable* if such infinitely generated subalgebra can be taken dense in X. For any $A \subset \mathbb{C}$ the symbol $\mathrm{conv}(A)$ denotes its convex hull. If $S \subset \mathbb{C}$ is an arc, then S is said to be *strictly convex* if for each pair of distinct $z, w \in S$ one has $S \cap \mathrm{conv}(\{z,w\}) = \{z,w\}$. As usual, the open unit disc $\{z : |z| < 1\}$ will be denoted by $\mathbb{D}$, and the space $H(\mathbb{C})^{\mathbb{N}}$ is assumed to be endowed with the product topology.

Theorem 1.1. *Let $\Phi \in H(\mathbb{C})$ be of finite exponential type.*

(a) ([10]) *Assume that the level set $\{z : |\Phi(z)| = 1\}$ contains a nontrivial, strictly convex compact arc Γ such that $\mathrm{conv}(\Gamma \cup \{0\}) \setminus (\Gamma \cup \{0\}) \subset \Phi^{-1}(\mathbb{D})$. Then the set of entire functions that generate a hypercyclic algebra for $\Phi(D)$ is residual in $H(\mathbb{C})$.*

(b) ([12]) *Assume that $|\Phi(0)| < 1$ and that the level set $\{z : |\Phi(z)| = 1\}$ contains a nontrivial, strictly convex compact arc Γ such that $\mathrm{conv}(\Gamma \cup \{0\}) \setminus \Gamma \subset \Phi^{-1}(\mathbb{D})$. Then the set of sequences $f = (f_j)_{j=1}^{\infty}$ that generate a dense hypercyclic algebra for $\Phi(D)$ that is not contained in a finitely generated hypercyclic algebra for $\Phi(D)$ is residual in $H(\mathbb{C})^{\mathbb{N}}$. In particular, the set $HC(\Phi(D))$ is dense algebrable.*

As applications of the last theorem, it is proved in [10] or [12] that, provided $a \neq 0$, $|b| \le 1$, $0 < c \le 1$, $|d| < 1$ and $k, n \in \mathbb{N}$, the operators $\cos(aD)$, $\sin(aD)$, $(aD^k + b)^n$, $D\,e^D$, $e^D - cI$ support hypercyclic algebras, while $\sin(aD)$, $\cos(D)$ and $aD^k + d$ support dense, infinitely generated hypercyclic algebras.

With the aim of contributing to shedding light on this research line, and taking advantage of Theorem 1.1, we prove in this short note that every D-multiple of a nonzero convolution operator possesses some scalar multiple supporting a hypercyclic algebra.

2. D-multiples of convolution operators and hypercyclic algebras

Geometrical properties of analytic functions play a central role in the results of [10, 12]. We recall in the next lemma a convexity property due to R. Nevanlinna, which can be found, for instance, in [17, Theorem 2.33].

Lemma 2.1. *If $f \in H(\mathbb{D})$ is one-to-one and satisfies $f'(0) = 1$ and $f(0) = 0$ then the set $f(r\,\mathbb{D})$ is a convex domain for every $r \in (0, 2 - \sqrt{3})$.*

Let us denote by $L_{\rm cvl}(H(\mathbb{C}))$ the ring of all convolution operators on $H(\mathbb{C})$, and by $DL_{\rm cvl}(H(\mathbb{C}))$ the ideal generated by the differentiation operator D, that is,

$$DL_{\rm cvl}(H(\mathbb{C})) := \{DS : S \in L_{\rm cvl}(H(\mathbb{C}))\}.$$

Next, we state and prove our result. We denote by $\mathbb{T}$ the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

Theorem 2.2. *Assume that $T \in DL_{\rm cvl}(H(\mathbb{C})) \setminus \{0\}$. Then there is $\rho > 0$ such that, for every $\lambda \in \mathbb{C}$ with $|\lambda| > \rho$, the set $HC(\lambda T)$ is dense algebrable.*

Proof. We are going to apply Theorem 1.1. By assumption, there is a convolution operator S such that $T = DS$. Therefore, there exists a nonzero entire function $\Psi(z)$ with finite exponential type such that $S = \Psi(D)$. If $\Phi(z) := z\Psi(z)$, then we get $T = \Phi(D)$, Φ has finite exponential type, $\Phi \neq 0$ and $\Phi(0) = 0$. Let $N \in \mathbb{N}$ be the order of the zero of Φ at the origin or, that is the same, $N = \min\{n \in \mathbb{N} : \Phi^{(n)}(0) \neq 0\}$. According to the local representation theorem (see, e.g., [1, Section 3.3]), there exist $\alpha > 0$ as well as a one-to-one function $\varphi : \alpha\,\mathbb{D} \to \mathbb{C}$ such that $\varphi^N = \Phi$ on $\alpha\,\mathbb{D}$. Note that $\varphi(0) = 0$.

Consider the inverse mapping $\psi : \varphi(\alpha\,\mathbb{D}) \to \alpha\,\mathbb{D}$ of φ. Since the image $\varphi(\alpha\,\mathbb{D})$ is an open subset of $\mathbb{C}$ containing 0, there exists $\beta > 0$ such that $\beta\,\mathbb{D} \subset \varphi(\alpha\,\mathbb{D})$. Then $\psi \in H(\beta\,\mathbb{D})$, $\psi(0) = 0$, $\varphi \circ \psi(z) = z$ $(z \in \beta\,\mathbb{D})$ and ψ is one-to-one on $\beta\,\mathbb{D}$. In particular, $\psi'(0) \neq 0$. Define the function $\gamma : \mathbb{D} \to \mathbb{C}$ as

$$\gamma(z) := \frac{\psi(\beta z)}{\beta\,\psi'(0)}.$$

Note that $\gamma \in H(\mathbb{D})$, $\gamma(0) = 0$, $\gamma'(0) = 1$ and γ is one-to-one on $\mathbb{D}$. According to Lemma 2.1, the set $\gamma(r\,\mathbb{D})$ is a convex domain for each $r \in (0, 2 - \sqrt{3})$. But this is equivalent to the fact that $\kappa(P) > 0$, κ denoting the signed curvature at the points P of the curve $\Gamma_r := \gamma(r\mathbb{T})$, where $\mathbb{T}$ is oriented counterclockwise (see [17, Remark on p. 166]). If $P = (x(t), y(t))$ $(t \in \mathbb{R})$ is a (2π-periodic) smooth parametrization of Γ_r (obtained, for instance, via composition of γ with $(r\cos t, r\sin t)$), then

$$\kappa(P) = \frac{x'(t)y''(t) - y'(t)x''(t)}{((x'(t))^2 + (y'(t))^2)^{3/2}},$$

which is a continuous function of P. Therefore, a segment joining two points P, Q in Γ_r can intersect this curve only at P, Q (otherwise κ would be ≤ 0 at some point of Γ_r). In other words, each curve Γ_r is strictly convex.

But neither dilations nor rotations alter the strict convexity of an arc as well as the convexity of a domain, so that all curves $\psi(\beta\,r\mathbb{T})$ are strictly convex and all domains $\psi(\beta\,r\mathbb{D})$ $(0 < r < 2 - \sqrt{3})$ are convex. Equivalently, all curves $\Psi_r := \psi(r\,\mathbb{T})$ are strictly convex and all domains $\psi(r\,\mathbb{D})$ $(0 < r < (2 - \sqrt{3})\beta)$ are convex. Define

$$\rho := ((2 - \sqrt{3})\beta)^{-N}.$$

Fix $\lambda \in \mathbb{C}$ with $|\lambda| > \rho$ and set $\Phi_\lambda := \lambda\,\Phi$. Note that $\lambda T = \Phi_\lambda(D)$ and $|\Phi_\lambda(0)| = 0 < 1$. Consider the level set

$$\mathcal{L} := \{z \in \mathbb{C} : |\Phi_\lambda(z)| = 1\}$$

and the curve $\Gamma := \Psi_{|\lambda|^{-1/N}}$. This curve is a strictly convex compact arc, and if $z \in \Gamma$ then $|\varphi(z)| = |\lambda|^{-1/N}$, hence $|\Phi_\lambda(z)| = |\lambda| \cdot |\varphi(z)|^N = |\lambda| \cdot |\lambda|^{-1} = 1$. Therefore $z \in \mathcal{L}$, so $\Gamma \subset \mathcal{L}$. Denote, as usual, by ∂A and $\overline{A}$ the boundary and the closure, respectively, of a set $A \subset \mathbb{C}$. From the maximum modulus principle, we get $\Gamma = \partial\psi(|\lambda|^{-1/N}\mathbb{D})$ and

$$\overline{\psi(|\lambda|^{-1/N}\mathbb{D})} = \Gamma \cup \psi(|\lambda|^{-1/N}\mathbb{D}) \subset \{z \in \mathbb{C} : |\varphi(z)| \leq |\lambda|^{-1/N}\}.$$

Finally, take $z_0 \in \mathrm{conv}(\Gamma \cup \{0\}) \setminus \Gamma$. Since $\overline{\psi(|\lambda|^{-1/N}\mathbb{D})}$ is a convex set containing $\Gamma \cup \{0\}$, we get $|\varphi(z_0)| \leq |\lambda|^{-1/N}$. But $z_0 \notin \Gamma$, so $|\varphi(z_0)| < |\lambda|^{-1/N}$ and $|\Phi_\lambda(z_0)| = |\lambda| \cdot |\varphi(z_0)|^N < |\lambda| \cdot |\lambda|^{-1} = 1$. Consequently, we obtain $\mathrm{conv}(\Gamma \cup \{0\}) \setminus \Gamma \subset (\Phi_\lambda)^{-1}(\mathbb{D})$, and an application of Theorem 1.1(b) yields the desired conclusion. □

Remarks 2.3. 1. Recall that an operator T on a Hausdorff topological space X is said to be *supercyclic* whenever there is a vector $x_0 \in X$ –called supercyclic for T– whose projective orbit $\{\lambda T^n x_0 : n \in \mathbb{N}\}$ is dense in X. The set of supercyclic vectors for T is denoted by $SC(T)$. Observe that Theorem 2.2 also shows, in particular, that $SC(T)$ is dense algebrable for every $T \in DL_{\mathrm{cvl}}(H(\mathbb{C})) \setminus \{0\}$.

2. Part(b) of Theorem 1.1 tells us that, in fact, there are –in the topological sense– many algebraically independent sequences of functions generating algebras contained, except for 0, in the sets $HC(\lambda T)$ of the conclusion of our theorem.

3. Note that the number ρ in Theorem 2.2 is obtained constructively in the proof. Specifically, we can take $\rho = ((2-\sqrt{3})\beta)^{-N}$, where β is the radius of a disc with center at 0 where the inverse of a local holomorphic branch of the Nth-root of Φ is defined, and N is the valence of Φ at the origin. This allows us to exhibit explicit examples. For instance, the operator

$$T : f \in H(\mathbb{C}) \longmapsto \mu f' + f'(\cdot + 1) \in H(\mathbb{C}),$$

where $|\mu| \geq 3e$, supports hypercyclic algebras. In order to prove this, we first notice that $T = \Phi(D)$ with $\Phi(z) = \mu z + ze^z$. Then $\Phi(0) = 0$. Also, $\Phi'(0) = \mu + 1 \neq 0$, so that $N = 1$. If z_1, z_2 are distinct points of $\mathbb{D}$ and L is the segment joint z_1 to z_2, then by assuming $\Phi(z_1) = \Phi(z_2)$ we get

$$\begin{aligned}\mu|z_1 - z_2| > 2e|z_1 - z_2| &= \int_L 2e\,|dz| \geq \int_L |e^z + ze^z|\,|dz| = \int_L |\Phi'(z) - \mu|\,|dz| \\ &\geq \left|\int_L (\Phi'(z) - \mu)\,dz\right| = |(\Phi(z_2) - \mu z_2) - (\Phi(z_1) - \mu z_1)| = \mu|z_1 - z_2|,\end{aligned}$$

which is absurd. Hence $\Phi(z_1) \neq \Phi(z_2)$ and Φ is one-to-one on $\alpha\,\mathbb{D}$, where $\alpha = 1$. Given $w \in \mathbb{C}$ with $|w| < 2.10 + \sqrt{3}$, we have for all $z \in \mathbb{T}$ that

$$\begin{aligned}|(\Phi(z) - w) - (\mu z - w)| = |\Phi(z) - \mu z| = |ze^z| < e^{|z|} = e \\ < 3e - (2.10 + \sqrt{3}) < |\mu| - |w| \leq |\mu z - w|.\end{aligned}$$

According to Rouché's theorem (see, e.g., [1]), and taking into account that the value w is reached exactly once by μz in the geometric interior $\mathbb{D}$ of the Jordan curve $\mathbb{T}$ (namely, at $z = w/\mu$), we deduce that w is reached exactly once by Φ in

$\mathbb{D}$, so that we can take as β, ρ the respective quantities

$$\beta = 2.10 + \sqrt{3}, \;\; \rho = ((2-\sqrt{3})(2.10+\sqrt{3}))^{-1} < ((2-\sqrt{3})(2+\sqrt{3}))^{-1} = 1.$$

The selection $\lambda := 1$ in Theorem 2.2 ensures the existence of a dense infinitely generated algebra contained in $HC(T)$.

In the next and final remark, we will make use of the following assertion, known as Hadamard's factorization theorem, that can be found in [20, pp. 68–70].

Lemma 2.4. *Assume that $f \in H(\mathbb{C}) \setminus \{0\}$ has finite exponential growth order ρ. Let $\Pi(z)$ be the canonical product formed with the zeros of f. Then there exists a polynomial Q of degree not greater than ρ such that $f(z) = e^{Q(z)}\Pi(z)$ for all $z \in \mathbb{C}$.*

Remark 2.5. Suppose that $T \in L_{\rm cvl}(H(\mathbb{C}))$ is invertible. Then there is an operator $S \in L(H(\mathbb{C}))$ such that $ST = I = TS$, where I is the identity on $H(\mathbb{C})$. Since T is a convolution operator, one has $T\tau_{-a} = \tau_{-a}T$ for all $a \in \mathbb{C}$. Taking inverses, we get $\tau_a S = S\tau_a$ for all $a \in \mathbb{C}$, so that S is also a convolution operator. Hence there are $\Phi, \Psi \in H(\mathbb{C})$ with finite exponential type such that $T = \Phi(D)$ and $S = \Psi(D)$. If by $\mathbf{1}$ we represent the constant function $\mathbf{1}(z) = 1$, we obtain:

$$\mathbf{1}(D) = I = \Phi(D)\Psi(D) = (\Phi \cdot \Psi)(D).$$

But the (linear) mapping $\Phi \in \{\text{entire functions with finite exponential type}\} \mapsto \Phi(D) \in L_{\rm cvl}(H(\mathbb{C}))$ is one-to-one because $\Phi(D) \neq 0$ as soon as $\Phi \neq 0$: indeed, according to the Malgrange–Ehrenpreis theorem (see [6, 15, 22]), any nonzero convolution operator is even surjective. Therefore, $\Phi(z) \cdot \Psi(z) = 1$ for all $z \in \mathbb{C}$ and so Φ is an entire function without zeros having exponential growth order $\rho \leq 1$. It follows from Lemma 2.4 that $\Phi(z) = e^{az+b}$ for some scalars $a, b \in \mathbb{C}$. Consequently, we have

$$T = \Phi(D) = e^{aD+b} = e^b \cdot e^{aD} = e^b \cdot \tau_a,$$

that is, T is a nonzero scalar multiple of a translation operator (conversely, if $T = M\tau_a$ with $M \neq 0$ and $a \in \mathbb{C}$ then T is invertible: indeed, it possesses inverse $M^{-1}\tau_{-a}$). Now, assume that $T \in L_{\rm cvl}(H(\mathbb{C}))$ is invertible (so that $T = M\tau_a$ as above), that $f \in H(\mathbb{C})$ and that $(n_k) \subset \mathbb{N}$ is a strictly increasing sequence such that $f_k(z) \to z$ $(k \to \infty)$ compactly in $\mathbb{C}$, where

$$f_k(z) := M^{n_k}(f(z + n_k a))^2.$$

Now, we proceed similarly to [3] (in fact, the following is a special case of [10, Theorem 2.1], but we keep the proof for the sake of completeness). Since f cannot be identically 0, any f_k is not identically 0 either, so that we can find a closed Jordan curve J surrounding 0 such that $f_k(z) \neq 0$ $(z \in J,\ k \in \mathbb{N})$. Now, we invoke the argument principle (see, e.g., [1, p. 152]) and the fact that the identity $z \mapsto z$ vanishes only at 0 to deduce that $N_k \to 1$ $(k \to \infty)$, where by N_k we have denoted the number of zeros of f_k inside J, counting multiplicities. But, clearly, each N_k is an even integer, hence (N_k) cannot tend to 1, a contradiction. Hence $f^2 \notin HC(T)$ for any $f \in H(\mathbb{C})$. In other words, T cannot support hypercyclic algebras.

In view of these considerations, of Theorem 2.2, and of the fact that T is invertible if and only if T is not in any ideal $(D - cI)L_{\rm cvl}(H(\mathbb{C}))$ $(c \in \mathbb{C})$, we want to conclude this paper by expressing what is our *conjecture:* A nonzero operator $T \in L_{\rm cvl}(H(\mathbb{C}))$ supports hypercyclic algebras if and only if T is *not* invertible; if this is the case, then $HC(T)$ is dense algebrable. In particular, our conjecture claims for an affirmative answer to Question 3(ii) in [11] whether the only hypercyclic convolution operators on $H(\mathbb{C})$ not supporting hypercyclic algebras must be scalar multiples of translations.

Acknowledgements. The authors are grateful to the referee for helpful comments and suggestions. The authors have been supported by the Plan Andaluz de Investigación de la Junta de Andalucía FQM-127 Grant P08-FQM-03543 and by MEC Grant MTM2015-65242-C2-1-P.

References

[1] L. V. Ahlfors, *Complex Analysis*, 3rd ed., McGraw-Hill, London, 1979.

[2] R. M. Aron, L. Bernal-González, D. Pellegrino, and J. B. Seoane-Sepúlveda, *Lineability: The search for linearity in Mathematics*, Monographs and Research Notes in Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2016.

[3] R. M. Aron, J. A. Conejero, A. Peris, and J. B. Seoane-Sepúlveda, *Powers of hypercyclic functions for some classical hypercyclic operators*, Integr. Equ. Oper. Theory **58** (2007), no. 4, 591–596.

[4] R. M. Aron, J. A Conejero, A. Peris, and J. B. Seoane-Sepulveda, *Sums and products of bad functions*, Contemp. Math. **435** (2007), 47–52.

[5] F. Bayart and E. Matheron, *Dynamics of Linear Operators*, Cambridge Tracts in Mathematics, Cambridge University Press, 2009.

[6] C. A. Berenstein and R. Gay, *Complex analysis and special topics in harmonic analysis*, Springer, New York, 1995.

[7] L. Bernal-González, J. A. Conejero, G. Costakis, and J. A. Seoane-Sepúlveda, *Multiplicative structures of hypercyclic functions for MacLane's operator*, J. Operator Theory (2017), in press, available at arXiv1710.10413v1 [mathFA].

[8] J. P. Bès, *Invariant manifolds of hypercyclic vectors for the real scalar case*, Proc. Amer. Math. Soc. **127** (1999), no. 6, 1801–1804.

[9] J. Bès, J. A. Conejero, and D. Papathanasiou, *Convolution operators supporting hypercyclic algebras*, J. Math. Anal. Appl. **445** (2017), no. 2, 1232–1238.

[10] ———, *Hypercyclic algebras for convolution and composition operators*, Preprint (2017), available at arXiv:1706.08022v1 [math.FA].

[11] ———, *Hypercyclic algebras for convolution and composition operators*, Preprint (2018), available at arXiv:1706.08022v2 [math.FA].

[12] J. Bès and D. Papathanasiou, *Algebrable sets of hypercyclic vectors for convolution operators*, Preprint (2017), available at arXiv:1706.08651v2 [math.FA].

[13] G. D. Birkhoff, *Démonstration d'un théorème élémentaire sur les fonctions entières*, C. R. Acad. Sci. Paris **189** (1929), 473–475.

[14] P. S. Bourdon, *Invariant manifolds of hypercyclic operators*, Proc. Amer. Math. Soc. **118** (1993), no. 3, 845–847.
[15] L. Ehrenpreis, *Mean periodic functions I*, Amer. J. Math. **77** (1955), 293–328.
[16] G. Godefroy and J. H. Shapiro, *Operators with dense, invariant, cyclic vectors manifolds*, J. Funct. Anal. **98** (1991), no. 2, 229–269.
[17] M. O. González, *Complex Analysis. Selected Topics*, Marcel Dekker, New York-Basel-Hong Kong, 1992.
[18] K.-G. Grosse-Erdmann and A. Peris, *Linear Chaos*, Springer, London, 2011.
[19] D. Herrero, *Limits of hypercyclic and supercyclic operators*, J. Funct. Anal. **99** (1991), no. 1, 179–190.
[20] A. S. B. Holland, *Introduction to the theory of entire functions*, Academic Press, New York and London, 1973.
[21] G. R. MacLane, *Sequences of derivatives and normal families*, J. Anal. Math. **2** (1952), no. 1, 72–87.
[22] B. Malgrange, *Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution*, Ann. Institut Fourier (Grenoble) **6** (1955/1956), 271–355.
[23] Q. Menet, *Hypercyclic subspaces and weighted shifts*, Adv. Math. **255** (2014), 305–337.
[24] H. Petersson, *Hypercyclic subspaces for Fréchet spaces operators*, J. Math. Anal. Appl. **319** (2006), no. 2, 764–782.
[25] S. Shkarin, *On the set of hypercyclic vectors for the differentiation operator*, Israel J. Math. **180** (2010), no. 1, 271–283.
[26] J. Wengenroth, *Hypercyclic operators on nonlocally convex spaces*, Proc. Amer. Math. Soc. **131** (2003), no. 6, 1759–1761.

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