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ON CERTAIN ANTI-INVARIANT SUBMANIFOLDS OF AN S-MANIFOLD

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Abstract: In this paper, the geometry of certain anti-invariant submanifolds of an S-manifold, ([1]), namely those which are normal to the structure vector fields, is studied.

0 – Introduction

Many authors have studied anti-invariant submanifolds of Kaehler and Sasakian manifolds, and obtained interesting results. The main ones can be found in [8].

On the other hand, for manifolds with an f-structure f, David E. Blair has introduced S-manifolds as the analogue of the Kaehler structure in the almost complex case and of the quasi-Sasakian structure in the almost contact case ([1]). Recently, different kinds of submanifolds of S-manifolds have been investigated ([4, 5]).

The purpose of the present paper is to study certain anti-invariant submanifolds of S-manifolds, namely those which are normal to the structure vector fields. The integral submanifolds of the distribution determined by $-f^2$ are examples of such submanifolds.

In section 1, we give a brief summary of basic formulas for submanifolds of Riemannian manifolds. In section 2, we investigate the integral submanifolds of the distribution determined by $-f^2$ in an S-manifold and in section 3 we study anti-invariant submanifolds of S-manifolds, which are normal to the structure vector fields. In the last section, we consider such submanifolds in S-space forms.

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1 – Preliminaries

Let N^n be an n-dimensional Riemannian manifold and M^m an m-dimensional submanifold of N^n. Let g be the metric tensor field on N^n as well as the metric induced on M^m. We denote by $\tilde{\nabla}$ the covariant differentiation in N^n and by ∇ the covariant differentiation in M^m determined by the induced metric in M^m. Let $T(N)$ (resp. $T(M)$) be the Lie algebra of vector fields in N^n (resp. in M^m) and $T(M)^\perp$ the set of all vector fields normal to M^m. The Gauss–Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \quad \tilde{\nabla}_X V = -A_V X + D_X V, \tag{1.1}$$

$X, Y \in T(M)$, $V \in T(M)^\perp$, where D is the connection in the normal bundle, σ is the second fundamental form of M^m and A_V the Weingarten endomorphism associated with V. A_V and σ are related by $g(A_V X, Y) = g(\sigma(X,Y), V)$.

We denote by $\tilde{R}$, R and R^D the curvature tensors associated with $\tilde{\nabla}$, ∇ and D respectively. The normal connection D is said to be flat if R^D vanishes identically.

The Gauss equation is given by

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) &+ g(\sigma(X,Z),\sigma(Y,W)) \\ &- g(\sigma(X,W),\sigma(Y,Z)), \qquad X,Y,Z,W \in T(M).\end{aligned} \tag{1.2}$$

Moreover, we have the following Ricci equation:

$$\begin{aligned}&\tilde{R}(X,Y,U,V) = R^D(X,Y,U,V) - g([A_U, A_V]X, Y), \\ &\qquad X,Y \in T(M), \quad U,V \in T(M)^\perp, \quad \text{where } [A_U, A_V] = A_U A_V - A_V A_U.\end{aligned} \tag{1.3}$$

Now, for $X, Y, Z \in T(M)$, the covariant derivative of the second fundamental form σ is defined as follows:

$$(\nabla'_X \sigma)(Y,Z) = D_X \sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{1.4}$$

The second fundamental form σ is said to be parallel if

$$\nabla' \sigma \equiv 0. \tag{1.5}$$

The mean curvature vector H, defined by $H = (\operatorname{trace} \sigma)/m$, is said to be parallel if $D_X H = 0$, for any $X \in T(M)$. On the other hand, M^m is said to be minimal if $H \equiv 0$. Furthermore, M^m is called a totally geodesic submanifold in N^n if $\sigma \equiv 0$. Finally, M^m is said to be totally umbilical if $\sigma(X,Y) = g(X,Y)H$, $X, Y \in T(M)$.

2 – Integral submanifolds of the canonical distribution

Let (N^{2n+s}, g) be a $(2n+s)$-dimensional Riemannian manifold. N^{2n+s} is said to be an S-manifold, if there exists on N^{2n+s} an f-structure ([7]) of rank $2n$ and s global vector fields $\xi_1, \ldots, \xi_s$ (structure vector fields) such that ([1]):

i) If $\eta_1, \ldots, \eta_s$ are the dual 1-forms of $\xi_1, \ldots, \xi_s$, then,

$$f\,\xi_\alpha = 0\,, \quad \eta_\alpha \circ f = 0\,, \quad f^2 = -I + \sum \xi_\alpha \otimes \eta_\alpha\,, \qquad (2.1)$$
$$g(X,Y) = g(fX, fY) + \Phi(X,Y)\,,$$

for any $X, Y \in T(N)$, $\alpha = 1, \ldots, s$, where $\Phi(X,Y) = \sum \eta_\gamma(X)\,\eta_\gamma(Y)$.

ii) The f-structure is normal, that is,

$$[f,f] + 2\sum \xi_\alpha \otimes d\eta_\alpha = 0\,,$$

where $[f,f]$ is the Nijenhuis torsion of f.

iii) $\eta_1, \ldots, \eta_s$ satisfy

$$\eta_1 \wedge \ldots \wedge \eta_s \wedge (d\eta_\alpha)^n \neq 0 \quad \text{and} \quad d\eta_1 = \ldots = d\eta_s = F\,, \qquad (2.2)$$

for any α, where F is the fundamental 2-form defined by

$$F(X,Y) = g(X, fY)\,, \quad X, Y \in T(N)\,.$$

In the case $s = 1$, an S-manifold is a Sasakian manifold. For $s \geq 2$, examples of S-manifolds are given in [1, 2, 3]. Thus, the bundle space of a principal toroidal bundle over a Kaehler manifold with certain conditions is an S-manifold. In this way, a generalization of the Hopf fibration $\pi'\colon S^{2n+1} \to \mathbf{PC}^n$ is introduced as a canonical example of an S-manifold playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry.

The covariant differentiation $\tilde{\nabla}$ of N^{2n+s} satisfies ([1]),

$$\tilde{\nabla}_X \xi_\alpha = -fX\,, \qquad (2.3)$$

$$(\tilde{\nabla}_X f)Y = \sum_\alpha \Big\{ g(fX, fY)\xi_\alpha + \eta_\alpha(Y)\, f^2 X \Big\}\,, \qquad (2.4)$$

for any $X, Y \in T(N)$, $\alpha = 1, \ldots, s$.

When the curvature tensor of a $(2n+s)$-dimensional $(n > 1)$ S-manifold has the form ([6]),

$$\begin{aligned}
\tilde{R}(X,Y,Z,W) = \sum_{\alpha,\beta} \Big\{ & g(fX, fW)\,\eta_\alpha(Y)\,\eta_\beta(Z) - g(fX, fZ)\,\eta_\alpha(Y)\,\eta_\beta(W) + \\
& + g(fY, fZ)\,\eta_\alpha(X)\,\eta_\beta(W) - g(fY, fW)\,\eta_\alpha(X)\,\eta_\beta(Z) \Big\} \\
& + \frac{1}{4}(k+3s)\Big\{ g(fX, fW)\, g(fY, fZ) - g(fX, fZ)\, g(fY, fW) \Big\} \qquad (2.5) \\
& + \frac{1}{4}(k-s)\Big\{ F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W) \Big\}\,,
\end{aligned}$$

$X, Y, Z, W \in T(N)$, then the f-sectional curvature of N^{2n+s} is constant k and N^{2n+s} is called an S-space form.

Now, let

$$\mathcal{L}(p) = \left\{ X \in T_p(N),\ p \in N^{2n+s} \ /\ \eta_\alpha(X) = 0,\ \alpha = 1, ..., s \right\} .$$

Then $\mathcal{L}$ determines a distribution induced by $-f^2$. Regarding to the integral submanifolds of this distribution, we have:

Theorem 2.1. *Let N^{2n+s} be an S-manifold. Then, integral submanifolds of maximal dimension of the distribution $\mathcal{L}$ are of dimension n.*

Proof: Since N^{2n+s} is an S-manifold, we have (2.2) and hence $\eta_1 \wedge (d\eta_1)^n \neq 0$. Moreover, it is easy to show that $(d\eta_1)^{n+1} = 0$ and then, from Darboux's Theorem, for any $p \in N^{2n+s}$, there exists a neighbourhood $V(p)$ and functions f^1, $x^1, ..., x^n$, $y^1, ..., y^n$ in $V(p)$ such that

$$\eta_1 = df^1 - \sum_{j=1}^{n} y^j \, dx^j \ .$$

Hence

$$d\eta_1 = \sum_{j=1}^{n} dx^j \wedge dy^j$$

and

$$(d\eta_1)^n = n! \, (-1)^{\frac{1}{2}n(n-1)} \, dx^1 \wedge ... \wedge dx^n \wedge dy^1 \wedge ... \wedge dy^n \ .$$

Now, since $d\eta_1 = ... = d\eta_s$, $\eta_\alpha - \eta_1$ is a closed 1-form, for any α. From Poincaré's Lemma, $W(p) \subset V(p)$ exists such that $\eta_\alpha - \eta_1$ is exact within $W(p)$. Hence, we can find differentiable functions $g^1, g^2, ..., g^s$ on $W(p)$ such that $\eta_\alpha = \eta_1 + dg^\alpha$. Then, taking $z^1 = f^1$, $z^\alpha = f^1 + g^\alpha$, $\alpha = 2, ..., s$ and $U = W(p)$, we have a local chart around p, $(U; x^1, ..., x^n, y^1, ..., y^n, z^1, ..., z^s)$ such that in U,

$$d\eta_\alpha = dz^\alpha - \sum_{j=1}^{n} y^j \, dx^j$$

for any α. Then, for a point p with coordinates

$$\left(x_0^1, ..., x_0^n, y_0^1, ..., y_0^n, z_0^1, ..., z_0^s \right)$$

in the coordinate neighbourhood,

$$x^1 = x_0^1, \ \ ..., \ \ x^n = x_0^n , \qquad z^1 = z_0^1, \ \ ..., \ \ z^s = z_0^s ,$$

defines an n-dimensional integral submanifold of $\mathcal{L}$ in the neighbourhood and a maximal integral submanifold containing this coordinate slice is an integral submanifold of $\mathcal{L}$ in N^{2n+s}.

On the other hand, if M^m is an m-dimensional integral submanifold of $\mathcal{L}$ and $m > n$, let $X_1, ..., X_m$ be m linearly independent local vector fields tangent to M^m. Extend them to a basis by $X_{m+1}, ..., X_{2n}, \xi_1, ..., \xi_s$. For $\alpha = 1, ..., s$, $i, j = 1, ..., m$, we have $\eta_\alpha(X_i) = 0 = d\eta_\alpha(X_i, X_j)$. Thus, since $m > n$,

$$\left(\eta_1 \wedge ... \wedge \eta_s \wedge (d\eta_\alpha)^n\right) \left(X_1, ..., X_{2n}, \xi_1, ..., \xi_s\right) = 0 ,$$

which is a contradiction. ∎

In the proof of the above theorem, we have seen that if X, Y are vector fields tangent to an integral submanifold of $\mathcal{L}$, then $\eta_\alpha(X) = \eta_\alpha(Y) = d\eta_\alpha(X, Y) = 0$, for any α. Conversely, we prove:

Proposition 2.2. *Let M^m be a submanifold of an S-manifold N^{2n+s}. Then, M^m is an integral submanifold of the distribution $\mathcal{L}$ if and only if η_α and $d\eta_\alpha$ restricted to M^m vanish, for any α. Moreover, if M^m is normal to the structure vector fields, then M^m is an integral submanifold of the distribution $\mathcal{L}$ if and only if fX is normal to M^m in N^{2n+s}, for any vector field X in $\mathcal{L}$.*

Proof: If $X, Y \in T(M)$ are such that $\eta_\alpha(X) = \eta_\alpha(Y) = 0$ and $d\eta_\alpha(X, Y) = 0$, for any α, then $0 = -\frac{1}{2}\eta_\alpha([X, Y])$. Hence $[X, Y]$ belongs to $\mathcal{L}$ and M^m is an integral submanifold of $\mathcal{L}$.

The second statement follows immediately from the first, since $d\eta_\alpha(X, Y) = F(X, Y) = -g(fX, Y)$, for any α. ∎

We notice that we have not used the hypothesis of normality about f. So, the results are valid for f-manifolds with complemented frames $\eta_1, ..., \eta_s$ and fundamental 2-form $F = d\eta_\alpha$, $\alpha = 1, ..., s$ (see [1] for the definition).

3 – Anti-invariant submanifolds of an S-manifold, normal to the structure vector fields

In this section, we assume that N^{2n+s} is an $(2n + s)$-dimensional S-manifold and M^m is an m-dimensional submanifold immersed in N^{2n+s} and normal to the structure vector fields $\xi_1, ..., \xi_s$. Then, M^m is said to be anti-invariant in N^{2n+s} if $fT_p(M) \subseteq T_p(M)^\perp$ for each $p \in M^m$. We notice that integral submanifolds of the distribution $\mathcal{L}$ are anti-invariant submanifolds of N^{2n+s} normal to the structure vector fields by virtue of Proposition 2.2.

Since $fT_p(M) \subseteq T_p(M)^\perp$ at each point p of M^m, we have the decomposition of $T_p(M)^\perp$ into the direct sum

$$T_p(M)^\perp = fT_p(M) \oplus \nu_p(M) ,$$

where $\nu_p(M)$ is the orthogonal complement of $fT_p(M)$ in the normal space $T_p(M)^\perp$. Note that $f\nu_p(M) \subseteq \nu_p(M)$ and $\xi_\alpha \in \nu_p$, for any α.

For any vector field V in the normal bundle $T(M)^\perp$, put

$$fV = tV + nV , \tag{3.1}$$

where tV (resp. nV) is the tangential (resp. normal) part of fV. Then t is a tangent-bundle valued 1-form on the normal bundle and n is an endomorphism of the normal bundle. It is easy to show that if n does not vanish, it defines an f-structure in the normal bundle. Using (1.1), we have, comparing the normal parts, from (2.4) and (3.1)

$$(D_X n)V = D_X nV - nD_X V = -\sigma(X, tV) - fA_V X , \tag{3.2}$$

for any $X \in T(M)$, $V \in T(M)^\perp$. If $D_X n = 0$ for all X, then the f-structure n in the normal bundle is said to be parallel. Now, for later use, we prove:

Lemma 3.1. *Let M^m be an anti-invariant submanifold of an S-manifold normal to the structure vector fields. Then:*

i) *$A_{\xi_\alpha} = 0$ for any α;*

ii) *$A_{fX}Y = A_{fY}X$, $X, Y \in T(M)$.*

Proof: Since $\xi_\alpha \in T(M)^\perp$, from (1.1) and (2.3), we have:

$$-fX = \tilde{\nabla}_X \xi_\alpha = -A_{\xi_\alpha} X + D_X \xi_\alpha \ .$$

But, $fX \in T(M)^\perp$ because M^m is anti-invariant. Then, i) holds.

On the other hand, from (1.1) and (2.4), we have

$$A_{fX}Y = -\tilde{\nabla}_Y fX + D_Y fX = -\sum_\alpha g(fX, fY)\,\xi_\alpha - f\nabla_Y X - f\sigma(X,Y) + D_Y fX ,$$

and since σ is a symmetric tensor field, ii) holds. ∎

Proposition 3.2. *Let M^m be an anti-invariant submanifold of an S-manifold normal to the structure vector fields. Then:*

i) *If $\nabla'\sigma \equiv 0$, then $A_{fX} = 0$, for any $X \in T(M)$.*

ii) *If $Dn \equiv 0$, then $A_W = 0$, for any $W \in \nu$.*

Consequently, if i) and ii) hold simultaneously, then M^m is totally geodesic.

Proof: From i) of Lemma 3.1 and (1.4) we get $g(D_X\sigma(Y,Z),\xi_\alpha)=0$, for any $X,Y,Z\in T(M)$ and any α. Now, from (1.1), (2.3) and i) of Lemma 3.1 again, we obtain

$$\begin{aligned} 0 = g\left(D_X\sigma(Y,Z),\xi_\alpha\right) &= g\left(\tilde{\nabla}_X\sigma(Y,Z),\xi_\alpha\right) \\ &= -g\left(\tilde{\nabla}_X\xi_\alpha,\sigma(Y,Z)\right) = g\left(fX,\sigma(Y,Z)\right) = g(A_{fX}Y,Z)\ , \end{aligned}$$

which gives i). Now, if $X\in T(M)$ and $W\in\nu$, from (3.2), we have $0=(D_Xn)W=-fA_WX$, and then $0=f^2A_WX=-A_WX$. So, ii) holds. ∎

Now, we choose a local field of orthonormal frames

$$\left\{E_1,\dots,E_m,E_{m+1},\dots,E_n,\ E_1^*=fE_1,\ \dots,\ E_n^*=fE_n,\ \xi_1,\dots,\xi_s\right\}$$

on N^{2n+s}, in such a way that, restricted to M^m, $\{E_1,\dots,E_m\}$ are tangent to M^m. We use the conventions that the ranges of indices are, respectively:

$$i,j,k=1,\dots,m\,,\qquad a,b=m+1,\dots,n,1^*,\dots,n^*\ .$$

Moreover, to simplify the notation, we write A_a instead of A_{E_a}. Then, we have:

Proposition 3.3. *Let M^m be an anti-invariant submanifold of an S-manifold normal to the structure vector fields. If $Dn\equiv 0$ and $DH\equiv 0$, then M^m is minimal.*

Proof: Using (1.1) and (2.3), we get

$$\begin{aligned} g\Big(\sum_{i=1}^{n}\sigma(E_i,E_i),fX\Big) &= -g\Big(\sum_{i=1}^{n}\sigma(E_i,E_i),\tilde{\nabla}_X\xi_\alpha\Big) \\ &= g\Big(D_X\Big(\sum_{i=1}^{n}\sigma(E_i,E_i)\Big),\xi_\alpha\Big)=0\ , \end{aligned}$$

for any $X\in T(M)$ and any α, because H is parallel, where we have taken account of Lemma 3.1, i). Now, from Proposition 3.2, ii), $A_W=0$, for any $W\in\nu$. Thus, $g(H,W)=0$ and $H=0$. ∎

Theorem 3.4. *Let M^m be an anti-invariant submanifold of an S-manifold normal to the structure vector fields. If $Dn\equiv 0$ and $R^D\equiv 0$, then M^m is of constant curvature s.*

Proof: From (1.2), ii) of Lemma 3.1 and ii) of Proposition 3.2, we have, for any $X, Y, Z, W \in T(M)$,

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) + \sum_{i=1}^{m}\Big\{g\big(\sigma(X,Z), fE_i\big)\, g\big(\sigma(Y,W), fE_i\big) \\ &\quad - g\big(\sigma(X,W), fE_i\big)\, g\big(\sigma(Y,Z), fE_i\big)\Big\} \\ &= R(X,Y,Z,W) + g(A_{fX}Z, A_{fY}W) - g(A_{fY}Z, A_{fX}W)\ .\end{aligned}$$

Comparing with (1.3), we obtain

$$\begin{aligned}(3.3)\qquad & R^D(fX,fY,Z,W) - \tilde{R}(fX,fY,Z,W) = \\ & = R(X,Y,Z,W) - \tilde{R}(X,Y,Z,W)\,, \qquad X,Y,Z,W \in T(M)\ .\end{aligned}$$

Now, since the submanifold is anti-invariant (and then, $F(X,Y) = g(X, fY) = 0$, for any $X, Y \in T(M)$), Lemma 2.2 in [1] gives:

$$\tilde{R}(fX,fY,Z,W) = \tilde{R}(X,Y,Z,W) - s\Big\{g(X,W)\,g(Y,Z) - g(X,Z)\,g(Y,W)\Big\}\ .$$

Consequently, regarding to (3.3) we get

$$\begin{aligned}(3.4)\qquad R^D(fX,fY,Z,W) &= R(X,Y,Z,W) \\ &\quad - s\Big\{g(X,W)\,g(Y,Z) - g(X,Z)\,g(Y,W)\Big\}\ .\end{aligned}$$

This and the hypothesis of flat normal connection, prove our assertion. ■

Proposition 3.5. *Let M^n be an anti-invariant submanifold of an S-manifold N^{2n+s} normal to the structure vector fields. Then, $R^D \equiv 0$ if and only if M^n is of constant curvature s.*

Proof: Since ξ_α are Killing vector fields ([1]), from (2.4) we easily prove $\tilde{R}(X,\xi_\alpha)Y = -(\tilde{\nabla}_X f)Y$, for any $X, Y \in T(N)$ and any α. Now, if $V \in T(M)^\perp$, $X, Y \in T(M)$, from (1.3) and i) of Lemma 3.1, we have $R^D(V,\xi_\alpha,X,Y) = \tilde{R}(V,\xi_\alpha,X,Y) = 0$.

On the other hand, as $\dim(M) = n$, then $\nu = \langle \xi_1, \dots, \xi_s\rangle$. Now, from (1.2) and (1.3) we obtain (3.4) as in the proof of the Theorem 3.4. ■

If the Weingarten endomorphisms are commutative, we can choose a local field of orthonormal frames for which all A_a are simultaneously diagonal. Then, if we put $\sigma_{ij}^a = g(\sigma(E_i,E_j),E_a)$, we have $\sigma_{ij}^a = 0$, when $i \neq j$. But, from Lemma 3.1, ii), we see that $\sigma_{ij}^{k^*} = \sigma_{jk}^{i^*} = \sigma_{ik}^{j^*}$ and so, $\sigma_{ij}^{k^*} = 0$, unless $i = j = k$.

On the other hand, if the f-structure n in the normal bundle is parallel, from Proposition 3.2, ii), we can easily prove that the Weingarten endomorphisms are commutative if and only if we can choose an orthonormal frame for which $\sigma_{ij}^{k^*} = 0$, unless $i = j = k$.

Proposition 3.6. *Let M^m be an anti-invariant submanifold of an S-manifold normal to the structure vector fields. If $DH \equiv 0$, $Dn \equiv 0$ and if the Weingarten endomorphisms are commutative, then M^m is totally geodesic.*

Proof: By virtue of Proposition 3.3, M^m is minimal. As the Weingarten endomorphisms are commutative, we have $A_{i^*} = 0$, $i = 1, ..., m$. But by Proposition 3.2, ii), $A_W = 0$, for any $W \in \nu$. Hence, the Weingarten endomorphisms are all zero. ∎

Proposition 3.7. *Let M^m $(m \geq 2)$ be an anti-invariant submanifold of an S-manifold normal to the structure vector fields. If $Dn \equiv 0$ and M^m is totally umbilical, then M^m is totally geodesic.*

Proof: We have from Proposition 3.2, $A_W = 0$, for any $W \in \nu$. Now, since M^m is totally umbilical:

$$
\begin{aligned}
g(A_{fX}E_i, E_j) &= g\Big(\sigma(E_i, E_j), fX\Big) = g(\delta_{ij}H, fX) \\
&= \frac{1}{m}\,\delta_{ij} \sum_k g\Big(\sigma(E_k, E_k), fX\Big) \qquad (3.5) \\
&= \frac{1}{m}\,\delta_{ij}\,\mathrm{trace}(A_{fX})\,, \qquad X \in T(M)\,.
\end{aligned}
$$

Thus, A_{fX} is diagonal, for any $X \in T(M)$. Consequently, the Weingarten endomorphisms are commutative and we can choose a new local field of orthonormal frames for which $\sigma_{ij}^{k^*} = 0$ unless $i = j = k$. But, from (3.5), $\sigma_{ij}^{k^*} = \frac{1}{m}\delta_{ij}\,\mathrm{trace}(A_{k^*})$. Since $m \geq 2$, then $\mathrm{trace}(A_{k^*}) = 0$ for any k so that, $A_V = 0$, for any $V \in T(M)^\perp$. ∎

4 – Case of constant f-sectional curvature

In this section, let $N^{2n+s}(k)$ be an S-space form of constant f-sectional curvature k and let M^m be an anti-invariant submanifold of $N^{2n+s}(k)$ normal to the structure vector fields. From (2.5) and the Gauss equation (1.2), we obtain, for the curvature tensor R, the Ricci tensor S and the scalar curvature ρ of M^m:

$$\begin{aligned} R(X,Y,Z,W) = &\frac{1}{4}(k+3s)\Big\{g(X,W)\,g(Y,Z) - g(X,Z)\,g(Y,W)\Big\} \\ &+ g\Big(\sigma(X,W),\sigma(Y,Z)\Big) - g\Big(\sigma(X,Z),\sigma(Y,W)\Big) \ , \end{aligned} \tag{4.1}$$

$$\begin{aligned} S(X,Y) = &\frac{1}{4}\,(m-1)\,(k+3s)\,g(X,Y) \\ &+ \sum_i \Big\{ g\Big(\sigma(E_i,E_i),\sigma(X,Y)\Big) - g\Big(\sigma(X,E_i),\sigma(Y,E_i)\Big)\Big\}, \end{aligned} \tag{4.2}$$

$$\rho = \frac{1}{4}\,m\,(m-1)\,(k+3s) + \sum_{i,j}\Big\{ g\Big(\sigma(E_i,E_i),\sigma(E_j,E_j)\Big) - \|\sigma(E_i,E_j)\|^2\Big\}, \tag{4.3}$$

for any $X,Y,Z,W \in T(M)$. From Lemma 3.1, i), we get

$$\begin{aligned} \sum_i g\Big(\sigma(X,E_i),\sigma(Y,E_i)\Big) &= \sum_{i,a} g\Big(\sigma(X,E_i),E_a\Big)\,g\Big(\sigma(Y,E_i),E_a\Big) \\ &= \sum_a g(A_aX, A_aY) \ , \end{aligned} \tag{4.4}$$

for any $X,Y \in T(M)$. Now, suppose that M^m is minimal. Then we have

$$S(X,Y) = \frac{1}{4}\,(m-1)\,(k+3s)\,g(X,Y) - \sum_a g(A_aX,A_aY) \ , \tag{4.5}$$

$$\rho = \frac{1}{4}\,m\,(m-1)\,(k+3s) - \|\sigma\|^2 \ , \tag{4.6}$$

for any $X,Y \in T(M)$. From these formulas and (4.1), we easily prove:

Proposition 4.1. *Let M^m be an anti-invariant submanifold of an S-space form $N^{2n+s}(k)$ normal to the structure vector fields.*

i) *If M^m is minimal, then for the Ricci tensor S and the scalar curvature ρ of M^m:*

i.1) $S - \frac{1}{4}\,(m-1)\,(k+3s)\,g$ *is a negative semi-definite symmetric tensor;*

i.2) $\rho \leq \frac{1}{4}\,m\,(m-1)\,(k+3s)$.

ii) *If M^m is totally geodesic, then M^m is of constant curvature $\frac{1}{4}\,(k+3s)$.*

Now, from Proposition 3.7, we get:

Corollary 4.2. *Let M^m $(m \geq 2)$ be an anti-invariant submanifold of an S-space form $N^{2n+s}(k)$ normal to the structure vector fields. If $Dn \equiv 0$ and M^m is totally umbilical, then M^m is of constant curvature $\frac{1}{4}\,(k+3s)$.*

Proposition 4.3. *Let M^m be an anti-invariant submanifold of an S-space form $N^{2n+s}(k)$ normal to the structure vector fields such that $Dn \equiv 0$. Then M^m is of constant curvature $\frac{1}{4}(k+3s)$ if and only if the Weingarten endomorphisms are commutative.*

Proof: It follows easily from Proposition 3.2, ii), and (4.1). ∎

Finally, as a consequence of the above propositions, we have:

Theorem 4.4. *Let M^m be a minimal anti-invariant submanifold of an S-space form $N^{2n+s}(k)$ normal to the structure vector fields. Then, the following assertions are equivalent:*

i) *M^m is totally geodesic.*

ii) *M^m is of constant curvature $\frac{1}{4}(k+3s)$.*

iii) $S = \frac{1}{4}(m-1)(k+3s)\,g$.

iv) $\rho = \frac{1}{4}m(m-1)(k+3s)$.

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