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FIXED POINT APPROXIMATION METHODS FOR NONEXPANSIVE MAPPINGS: OPTIMIZATION PROBLEMS

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RESUMEN

Numerosos problemas en diferentes áreas de las matemáticas pueden ser reformulados como un problema de punto fijo de una aplicación no expansiva. Por ejemplo, dado un operador monótono en un espacio de Banach, es conocido que el operador resolvente asociado es una aplicación no expansiva cuyo conjunto de puntos fijos coinciden con el conjunto de ceros del operador monótono. Igualmente, el operador complementario de una aplicación no expansiva es monótono y su conjunto de ceros es el conjunto de puntos fijos de la aplicación. La conexión existente entre la teoría del punto fijo y la teoría de operadores monótonos permite establecer equivalencias entre un problema de punto fijo para aplicaciones no expansivas y otros problemas como por ejemplo la búsqueda de una solución de una desigualdad variacional, un minimizante de una función convexa o un punto de silla de un problema minimax.

Motivado por las anteriores aplicaciones, el estudio de métodos iterativos para la aproximación de puntos fijos de aplicaciones no expansivas en espacios de Banach ha adquirido una gran relevancia en los últimos años, especialmente en el caso particular de los espacios de Hilbert. Entre los algoritmos iterativos investigados hasta hoy, cabe destacar los siguientes:

- La iteración de Mann, cuyo esquema algorítmico consiste en definir la nueva iterada como la combinación convexa de la iterada anterior y su imagen por la aplicación no expansiva.

- La iteración de Halpern, cuya formula recursiva viene dada por la combinación convexa de un punto arbitrario y la imagen de la iterada anterior por la aplicación no expansiva.

Ambos algoritmos, en sus fórmulas implícitas y explícitas, han sido ampliamente estudiados y siguen siendo el objeto de investigación de muchos trabajos. A pesar del gran número de resultados en cuanto a la convergencia de estos algoritmos, aún existen importantes problemas abiertos referentes a las propiedades geométricas del espacio, hipótesis sobre la aplicación no expansiva u otros aspectos como el comportamiento asintótico de la sucesión de constantes de la combinación convexa.

Un importante campo de aplicación de estos métodos es el problema de aproximación de ceros de un operador monótono. Otro ejemplo es el uso de la teoría de aproximación del punto fijo para resolver problemas de viabilidad convexa como

es el caso del problema *multiple-sets split feasibility* que consiste en encontrar un punto perteneciente a la intersección de una familia finita de conjuntos cerrados y convexos cuya imagen por una aplicación lineal pertenece a la intersección de otra familia finita de conjuntos cerrados y convexos. Este problema constituye una vía de enfoque de problemas de otras disciplinas científicas como la reconstrucción de imágenes o la terapia con radiación de intensidad modulada.

Esta extensa teoría que tiene como objeto de estudio las aplicaciones no expansivas y los operadores monótonos ha sido desarrollada principalmente en el marco de los espacios de Banach. Recientemente algunos conceptos y técnicas propias de los espacios vectoriales lineales han sido extendidos al marco más general de los espacios métricos. En concreto, en variedades de Riemann, el estudio de métodos iterativos para resolver problemas de optimización, desigualdades variacionales y la búsqueda de ceros de operadores monótonos ha sido el centro de investigación de muchos trabajos. Especialmente en variedades de Hadamard, que son variedades con curvatura negativa que presentan muy buenas propiedades geométricas.

En esta tesis, se estudia los problemas que aparecen en la conexión entre las teorías de operadores monótonos y aplicaciones no expansivas tanto en espacios lineales como no lineales. El Capítulo 1 está dedicado a diferentes enfoques para aproximar puntos fijos de aplicaciones de tipo no expansivo definidas en espacios de Banach. En el Capítulo 2, se desarrolla una teoría de operadores monótonos y punto fijo de aplicaciones no expansivas en variedades de Hadamard, extendiendo resultados de la teoría clásica conocida en espacios de Hilbert.

Contents

Introduction

Many problems arising in different areas of mathematics, such as optimization, variational analysis and differential equations, can be modeled by the equation

$$x = Tx,$$

where T is a nonlinear operator defined on a metric space. The solutions to this equation are called fixed points of T. If T is a contraction defined on a complete metric space X, Banach contraction principle establishes that T has a unique fixed point and for any $x \in X$, the sequence of Picard iterates $\{T^n x\}$ strongly converges to the fixed point of T. However, if the mapping T is a nonexpansive mapping, that is,

$$d(T(x), T(y)) \leq d(x, y), \ \forall x, y \in X,$$

then we must assume additional conditions on T and/or the underlying space to ensure the existence of fixed points. Since the sixties, the study of the class of nonexpansive mappings is one of the major and most active research areas of nonlinear analysis. This is due to the connection with the geometry of Banach spaces along with the relevance of these mappings in the theory of monotone and accretive operators.

If we denote by X^* the dual space of a Banach space X, a set-valued operator $A : X \to 2^{X^*}$ with domain $\mathcal{D}(A)$ is said to be monotone if

$$\langle x^* - y^*, x - y \rangle \geq 0, \quad \forall x, y \in \mathcal{D}(A) \text{ and } x^* \in A(x), \ y^* \in A(y).$$

On the other hand, a set-valued operator $A : X \to 2^X$ is said to be *accretive* if

$$\langle x^* - y^*, j(x - y) \rangle \geq 0, \quad \forall \, x, y \in \mathcal{D}(A) \text{ and } x^* \in A(x), \ y^* \in A(y),$$

where $j(x-y) \in J(x-y)$ and J denotes the normalized duality mapping. One of the most relevant facts in the theory of monotone and accretive operators is that the two classes of operators coincide in the setting of Hilbert spaces; see [16]. The concepts of monotonicity and accretivity have turned out to be very powerful in diverse fields such as operator theory, numerical analysis, differentiability of convex functions and partial differential equations; see [88, 76, 27, 129]. In particular, one of the reasons is that the class of monotone operators is broad enough to cover subdifferentials of convex functions, which are operators of increasing importance in optimization theory. Recall that, given a function $f : X \to (\infty, -\infty]$, the subdifferential of f is the set-valued operator $\partial f : X \to 2^{X^*}$ defined by

$$\partial f(x_0) = \{x^* \in X^* : f(x) \geq f(x_0) + \langle x - x_0, x^* \rangle, \, \forall x \in X\},$$

for any $x_0 \in X$. Then, if f is lower semicontinuous and convex, ∂f is monotone; see [27]. This fact establishes an equivalence between convex minimization problems and the search for zeros of monotone operators. A zero of an operator A is a point $x \in X$ such that $0 \in A(x)$.

Regarding the problem of the existence of zeros it is essential the concept of maximal monotone operators. We say that $A : X \to 2^{X^*}$ is maximal monotone if is a monotone operator on X such that, for any $x_1 \in X$ and $y_1 \in X^*$, the inequality

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0, \quad \forall x_2 \in \mathcal{D}(A) \text{ and } y_2 \in A(x_2),$$

implies that $y_1 \in A(x_1)$. Likewise, its relationship with the notion of upper-semicontinuity constitutes a fruitful tool; see [14].

The relationship between the theory of monotone operators and the theory of nonexpansive mappings is basically determined by two facts: (1) if T is a nonexpansive mapping then the complementary operator $I - T$ is monotone and (2) the resolvent of a monotone operator A is nonexpansive. Moreover, in both cases the fixed point set of the nonexpansive mapping coincides with the set of zeros of the monotone operator.

The resolvent of a monotone operator in the setting of a Banach space was introduced by Brezis, Crandall and Pazy in [10]. They set up the fundamental properties of the resolvent, with special emphasis on the strong connection between its fixed points and the zeros of the monotone operator. From this starting point, the study of the asymptotic behavior of the resolvent operator has awakened the interest of many researchers. See, for instance, [20, 95, 56] and references therein. In the framework of a Hilbert space H, given a maximal monotone operator $A : H \to 2^H$, the resolvent of A of order $\lambda > 0$ is the single-valued mapping $J_\lambda : H \to H$ defined by

$$J_\lambda(x) = (I + \lambda A)^{-1}(x),$$

for any $x \in H$. It is straightforward to check that $A^{-1}(0) = \mathrm{Fix}(J_\lambda)$, where $\mathrm{Fix}(J_\lambda)$ denotes the fixed point set of J_λ. Moreover, the resolvent is not just nonexpansive but also firmly nonexpansive; that is,

$$\|J_\lambda(x) - J_\lambda(y)\|^2 \leq \langle x - y, J_\lambda(x) - J_\lambda(y)\rangle,$$

for all $x, y \in H$; see, for instance, [20]. Thus the problems of existence and approximation of zeros of maximal monotone operators can be formulated as the corresponding problems for fixed points of firmly nonexpansive mappings. It is this approach, applicable to other related problems as well, which makes firmly nonexpansive mappings an important tool in monotone operator and optimization theory.

In the interface between monotone operators and nonexpansive type mappings another class of nonlinear mappings appears, the so called pseudo-contractive mappings. Recall that a mapping $T : H \to 2^H$ is said to be pseudo-contractive if, for any $r > 0$,

$$\|x - y\| \leq \|(1 + r)(x - y) - r(u - v)\|, \ \forall\, x, y \in H, \ u \in T(x), \ v \in T(y).$$

This concept was introduced by Browder and Petryshyn, in [16], and they proved that a mapping T is pseudo-contractive if and only if the complementary operator $I - T$ is monotone. This means that the problem of solving an equation for monotone operators may be formulated as a fixed point problem of a pseudo-contractive mapping.

Concerning the fixed point approximation problem, we recall that the sequence of Picard iterations $\{T^n x\}$ strongly converges for contractions on complete metric spaces. However, if T is nonexpansive, even when it has a fixed point, this sequence $\{T^n x\}$ does not converge in general. For this reason, in the last decades, the development of feasible iterative methods for approximating fixed points of a nonexpansive mapping T has been of particular importance. For instance, [19, 29] constitute nice surveys about the asymptotic behavior of nonexpansive mappings in Hilbert and Banach spaces. It is worth mentioning two types of iterative procedures, Mann iteration and Halpern iteration. Both algorithms have been studied extensively and are still the focus of a host of research works.

Mann iteration, initially due to Mann [72], is the averaged algorithm defined by the recursive scheme

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,\ n \geq 0,$$

where x_0 is an arbitrary point in the domain of T and $\{\alpha_n\}$ is a sequence in $[0, 1]$. One of the classical results, due to Reich [94], states that if the underlying space is uniformly convex and has a Fréchet differentiable norm, T has fixed points and $\sum_n \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by Mann algorithm weakly converges to a fixed point of T. Moreover, a counterexample provided by Genel and Lindenstrauss ([45]) shows that in infinite-dimensional spaces Mann iteration cannot have strong convergence. References [52, 94, 44, 121] can be consulted for the convergence of Mann algorithm.

Halpern iteration, first presented in [50] by Halpern, is generated by the recursive formula

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T x_n,\ n \geq 0,$$

where x_0 and u are arbitrary points in the domain of T and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Unlike Mann iteration, Halpern algorithm can be proved to have strong convergence provided that the underlying space is smooth enough; see [65, 116, 95, 99, 120, 28, 108] and references therein. We should mention that it is still not clear whether Halpern algorithm converges if the underlying space does not have a smooth norm. We give more details about the convergence of both methods in Section 1.1.2.

One of the sources of the relevance of the iterative methods for approximating a fixed point of a nonexpansive mapping is its application in other fields. Besides the problem of zeros of a monotone operator, they can be applied to finding a solution to a variational inequality and a minimizer of a convex function, among other problems. For instance, in a Hilbert space H, given a monotone operator A defined on a closed convex subset C, the variational inequality problem associated to A, $\mathrm{VIP}(A, C)$, is formulated as finding a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0,\ \forall x \in C.$$

It can be readily proved that the VIP(A, C) is equivalent to the problem of finding a fixed point of the nonexpansive mapping

$$T = P_C(I - \lambda A),$$

where $\lambda > 0$ is an arbitrary real number, I is the identity mapping and P_C is the metric projection onto C. On the other hand, if $f : C \to \mathbb{R}$ is a differentiable convex function and we denote by A the gradient operator of f, then the VIP(A, C) is the optimality condition for the minimization problem

$$\min_{x \in C} f(x).$$

So an approach to approximating a solution of the VIP(A, C) or a minimizer of f is via a fixed point problem.

Another example of applications of the approximation fixed point theory is the multiple-sets split feasibility problem, MSSFP, introduced by Censor et al [24], which consists of finding a point belonging to a family of closed convex sets in one space such that its image under a linear transformation is contained in another family of closed convex sets in the image space. It serves as a model for inverse problems where constraints are imposed on the solutions in the domain of a linear transformation as well as in its range. In particular, the MSSFP arises in the field of intensity-modulated radiation therapy; see [22, 25] and references therein. Formally, in the setting of Hilbert spaces, the MSSFP is formulated as finding a point x^* satisfying

$$x^* \in C := \bigcap_{i=1}^{N} C_i \quad \text{and} \quad Ax^* \in Q := \bigcap_{j=1}^{M} Q_j,$$

where $N, M \geq 1$ are integers, $\{C_i\}_{i=1}^N$ and $\{Q_j\}_{j=1}^M$ are closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. It generalizes the convex feasibility problem (cf. [31]) and the two-sets split feasibility problem (cf. [23, 21, 130, 124]).

Regarding the problem of finding a zero of a monotone operator A, in the case of Hilbert spaces, Rockafellar [103], inspired by Moreau and Martinet [78, 75], defined the proximal point algorithm for monotone operators by means of the iterative scheme

$$0 \in A(x_{n+1}) + \lambda_n(x_{n+1} - x_n), \; n \geq 0,$$

where $\{\lambda_n\}$ is a sequence of real positive numbers and x_0 is an initial point. In the study of its convergence, the key tool is the asymptotic behavior of the resolvent of the monotone operator A.

This extensive theory dealing with nonexpansive mappings and monotone operators has mainly been developed in the framework of Banach spaces. Out of the setting of linear vector spaces, some concepts and techniques have been extended to other metric spaces. In particular, in the setting of Riemannian manifolds, relevant advances have been made in this direction. The study of optimization methods to solve minimization problems on Riemannian manifolds has been the subject of many works. It has offered a new way to solve non-convex constrained minimization problems in Euclidean spaces by means of convex problems on Riemannian manifolds; see [36, 41, 42]. A generalization of the convex minimization problem is the variational inequality problem. In the study of this problem several classes of monotone vector fields have been introduced (see [81, 82, 36] for single-valued vector fields and [35] for point-to-set vector fields) and convergence properties of iterative methods to solve them have been presented (see, for instance, [41]).

Riemannian manifolds constitute a broad and fruitful framework for the development of different fields. However, most of the extended methods previously mentioned require the Riemannian manifold to have nonpositive sectional curvature. This is an important property which is enjoyed by a large class of Riemannian manifolds and it is strong enough to imply tight topological restrictions and rigidity phenomena. More precisely, Hadamard manifolds, which are complete simply connected and finite dimensional Riemannian manifolds of nonpositive sectional curvature, have become a suitable setting for diverse disciplines. A Hadamard manifold is an example of hyperbolic space and geodesic space, more precisely, a Busemann nonpositive curvature (NPC) space and a CAT(0) space; see [59, 54, 113].

This thesis studies the problems that arise in the interface between the fixed point theory for nonexpansive type mappings and the theory of monotone operators in linear and nonlinear settings.

In Chapter 1, we present different approaches to approximate solutions of fixed point problems for nonexpansive type mappings in the setting of Banach spaces. In particular, in Section 1.2, we study the behavior of an approximating curve and its discretized iteration for finding a common solution to a fixed point problem for a nonexpansive mapping T and the variational inequality

$$\langle (I-\psi)q, J(x-q)\rangle \geq 0,\ \forall x \in \mathrm{Fix}(T),$$

associated to a contraction ψ, where J is the normalized duality mapping and $\mathrm{Fix}(T)$ is the fixed point set. As a consequence we obtain a hybrid steepest descent method, first studied by Yamada [117], which extends the viscosity approximation method; see [79, 123, 109].

Section 1.3 is devoted to the discussion of a general perturbation technique for approximating a fixed point of a nonexpansive mapping involving a sequence of nonexpansive mappings which converges in some sense to the original mapping. It consists of a Halpern type algorithm which has strong convergence under suitable conditions. Previous perturbation techniques are due to Yang and Zhao [130, 128], and Xu [124], for Mann type algorithms. To end Chapter 1, we give some applications of the previous methods to solve other related problems such as variational inequalities, accretive inclusions and convex feasibility problems. In particular, we study diverse iterative approaches for solving the multiple-sets split feasibility problem.

In Chapter 2, we present some contributions to the approximation fixed point theory and monotonicity theory in the setting of a Hadamard manifold. Section 2.2 is devoted to introducing the concept of monotonicity for set-valued vector fields, establishing a relationship with the notions of maximal monotonicity and upper semicontinuity. We define the class of accretive vector fields and prove that it coincides with the class of monotone vector fields, as happens in Hilbert spaces. These equivalences will be the key in Section 2.4 to prove the existence of singularities (zeros) of a maximal strongly monotone vector field. Then the concept of resolvent, previously defined implicitly on Hilbert manifold by Iwamiya and Okochi [53], will be given and proved to be well-defined. We also analyze the asymptotic behavior of the resolvent by using the notion and properties of the Yosida regularization; we then obtain some existence results of singularities under boundary conditions. Regarding the approximation of singularities, in Section 2.5, we provide a proximal point algorithm for maximal monotone vector fields which coincides with the one introduced by Rockafellar [103] in the framework of Hilbert spaces.

Concerning the fixed point theory in the setting of Hadamard manifolds, in Section 2.3, we introduce the notion of firmly nonexpansive mappings consistent with the definition given by Goebel and Reich [48] on the Hilbert ball. The properties of this class of mappings establish a strong relation between monotone vector fields and firmly nonexpansive mappings by means of the resolvent. By using the concept of complementary vector fields (cf. [80]) we establish a connection between monotonicity and the class of pseudo-contractive operators introduced by Reich and Shafrir [101] in the more general setting of hyperbolic spaces. In Section 2.6, we study the convergence of different algorithms for nonexpansive type mappings. In particular, we prove the convergence of Picard iteration for firmly nonexpansive mappings, define and study Mann and Halpern iterations for nonexpansive mappings and present a viscosity approximation method. In order to illustrate the application of these methods, we provide some numerical examples for Mann and Halpern iterations. Finally, Section 2.7 focuses on how to apply the previous results to different problems: minimization problems, minimax problems, variational inequalities and equilibrium problems.

Chapter 1

Iterative methods in Banach spaces

In this chapter we will focus on the linear case when X is a real Banach space. We first provide the theoretical framework in which the problems we deal with are formulated, including some knowledge of geometry of Banach spaces as well as some iterative methods for approximating fixed points of nonexpansive type mappings. In Section 1.2 we study the behavior of an approximating curve and its discretized iteration for finding a common solution to a fixed point problem and a variational inequality. As a consequence we obtain a hybrid steepest descent method which extends the viscosity approximation method. We dedicate Section 1.3 to the discussion of general perturbation techniques for approximating a fixed point of a nonexpansive mapping, involving a sequence of nonexpansive mappings which converges in some sense to the original mapping. Finally we analyze the applications of the previous methods to solve other related problems such as variational inequalities, accretive inclusions and convex feasibility problems. In particular, we propose different approaches to approximate a solution to the multiple-sets split feasibility problem.

1.1 Theoretical Framework

We provide the theoretical framework necessary to develop and discuss the subject we will focus on in the setting of Banach spaces. Likewise we introduce some iterative process well-known in the literature for approximating fixed points of nonexpansive mappings within this framework. This brief introduction will be very useful to motivate and understand the iterative methods that we present in the following sections.

Throughout this chapter X is a real Banach space with norm $\|\cdot\|$. The dual space of X will be denoted by X^*, which is a Banach space itself endowed with the norm

$$\|x^*\|_* := \sup\{|\langle x, x^*\rangle| : x \in X,\ \|x\| \leq 1\},\ \ x^* \in X^*,$$

where we write $\langle x, x^*\rangle = x^*(x)$ for the application of an element $x^* \in X^*$ on an element $x \in X$. We will omit the index whenever it becomes clear from the context which norm is meant. In some particular cases we will work in the setting of a Hilbert space denoted by H. By $x_n \to x$, $x_n \rightharpoonup x$ or $x_n \overset{*}{\rightharpoonup} x$, we denote the strong, weak or weak* convergence of $\{x_n\}$ to x, respectively. The extended real line will be denoted by $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

1.1.1 Geometry of Banach spaces

This section gathers some basic definitions and geometrical properties of Banach spaces which can be found in [27]. We also survey basic results of convex analysis and connect them to characterize those spaces in which some iteration methods are defined and proved to converge to solutions to fixed point problems for certain nonlinear mappings.

Elements of Convex Analysis

We briefly introduce some elements of convex analysis which can be found in many texts on the general theory of convex analysis. See, for instance, [90, 40, 9].

Definition 1.1.1. A function $f : X \to \overline{\mathbb{R}}$ is said to be

- *proper* if its effective domain, $\mathcal{D}(f) = \{x \in X : f(x) < \infty\}$, is nonempty;

- *convex* if
$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$
for all $\lambda \in (0,1)$ and $x, y \in \mathcal{D}(f)$;

- *lower semicontinuous* at $x_0 \in \mathcal{D}(f)$ if
$$f(x_0) \le \liminf_{x \to x_0} f(x).$$
We say that f is lower semicontinuous on $\mathcal{D}(f)$ if it is so at every $x_0 \in \mathcal{D}(f)$;

- *Gâteaux differentiable* at $x_0 \in \mathcal{D}(f)$ if there exists an element $f'(x_0) \in X^*$ such that
$$\lim_{t \to 0} \frac{f(x_0 + ty) - f(x_0)}{t} = \langle y, f'(x_0) \rangle, \ \ \forall y \in \mathcal{D}(f);$$

- *Fréchet differentiable* at $x_0 \in \mathcal{D}(f)$ if it is Gâteaux differentiable and
$$\lim_{t \to 0} \sup_{\|y\|=1} \left| \frac{f(x_0 + ty) - f(x_0)}{t} - \langle y, f'(x_0) \rangle \right| = 0;$$

- *subdifferentiable* at $x_0 \in \mathcal{D}(f)$ if there exists a functional $x^* \in X^*$, called *subgradient* of f at x_0, such that
$$f(x) \ge f(x_0) + \langle x - x_0, x^* \rangle, \ \ \forall x \in X.$$
The set of all subgradients of f at x_0
$$\partial f(x_0) = \{x^* \in X^* : f(x) \ge f(x_0) + \langle x - x_0, x^* \rangle, \forall x \in X\}$$
is called the *subdifferential* of f at x_0.

Proposition 1.1.2. *(Cioranescu* [27]*) Let $f : X \to \overline{\mathbb{R}}$ be a proper convex lower semicontinuous function. Then*

(i) *the function f is subdifferentiable on int$\mathcal{D}(f)$, the interior of $\mathcal{D}(f)$;*

(ii) *the function f is Gâteaux differentiable at $x \in$ int$\mathcal{D}(f)$ if and only if it has a unique subgradient $\partial f(x) = \nabla f(x)$ called the* gradient *of f.*

Classes of Banach spaces

Definition 1.1.3. A Banach space X

- is *smooth* or has a *Gâteaux differentiable* norm if
$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{1.1.1}$$
exists for each $x, y \in S_X = \{v \in X : \|v\| = 1\}$, the unit sphere of X;

- has a *uniformly Gâteaux differentiable* norm if for each $y \in S_X$ the limit (1.1.1) is uniformly attained for $x \in S_X$;

- is *uniformly smooth* if the limit (1.1.1) is attained uniformly for any $x, y \in S_X$;

- is *uniformly convex* if the *modulus of convexity*, $\delta : [0,2] \longrightarrow [0,1]$ defined by
$$\delta(\epsilon) := \inf\left\{1-\left\|\frac{x+y}{2}\right\| : \|x\|\le 1,\ \|y\|\le 1,\ \|x-y\|\ge\epsilon\right\},$$
satisfies that $\delta(\epsilon) > 0$ for all $\epsilon > 0$.

Duality Mappings

Recall that a *gauge* is a continuous strictly increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(0) = 0$ and $\lim_{t\to\infty}\phi(t) = \infty$. Associated with a gauge ϕ the *duality mapping* is the mapping $J_\phi : X \to 2^{X^*}$ given by
$$J_\phi(x) = \left\{j_\phi(x) \in X^* : \langle x, j_\phi(x)\rangle = \|j_\phi(x)\|\,\|x\|,\ \phi(\|x\|) = \|j_\phi(x)\|\right\} \tag{1.1.2}$$
It is easily seen that $J_\phi(x)$ is nonempty for each $x \in X$ and $J_\phi(\cdot)$ is odd (cf. [27]).

If the gauge ϕ is given by $\phi(t) = t$ for all $t \in \mathbb{R}^+$, then the corresponding duality mapping is called the *normalized duality mapping*, and is denoted by J. Hence the normalized duality mapping J is defined by
$$J(x) = \{j(x) \in X^* : \langle x, j(x)\rangle = \|x\|^2 = \|j(x)\|^2\}. \tag{1.1.3}$$
We can use another way to describe duality mappings. Given a gauge ϕ, we define
$$\Phi(t) = \int_0^t \phi(s)ds.$$

Then it can be proved that Φ is convex and, for any $x \in X$,

$$J_\phi(x) = \partial\Phi(\|x\|).$$

Thus we have the following subdifferential inequality: for any $x, y \in X$,

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, j_\phi(x+y)\rangle, \quad j_\phi(x+y) \in J_\phi(x+y). \tag{1.1.4}$$

For the normalized duality mapping J, the subdifferential inequality (1.1.4) turns into

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad j(x+y) \in J(x+y). \tag{1.1.5}$$

The relation between the normalized duality mapping J and the general duality mapping J_ϕ is easily seen to be

$$J_\phi(x) = \frac{\phi(\|x\|)}{\|x\|}J(x), \quad x \neq 0, \ x \in X. \tag{1.1.6}$$

The following result gathers the relation between the geometric properties of the classes of Banach spaces and the features of the normalized duality mapping.

Proposition 1.1.4. *(Cioranescu [27]) Let X be a Banach space. Given any gauge ϕ, the following assertions hold.*

(i) *The space X is smooth if and only if the duality mapping J_ϕ is single-valued.*

(ii) *The space X is uniformly smooth if and only if the duality mapping J_ϕ is single-valued and norm-to-norm uniformly continuous on bounded sets of X.*

(iii) *If the space X has a uniformly Gâteaux differentiable norm then J_ϕ is norm-to-weak* uniformly continuous on bounded sets of X.*

Following Browder [15], we say that a Banach space X has a *weakly continuous duality mapping* if there exists a gauge ϕ such that J_ϕ is single-valued and weak-to-weak* sequentially continuous; that is,

$$\text{if } \{x_n\} \subset X, \ x_n \rightharpoonup x, \ \text{then } J_\phi(x_n) \overset{*}{\rightharpoonup} J_\phi(x).$$

It is known that the space l^p, for $1 < p < \infty$, has a weakly continuous duality mapping with gauge $\phi(t) = t^{p-1}$. The following result constitutes an important property satisfied by this class of spaces. See [27] and [98] for more details on duality mappings.

Theorem 1.1.5. *(Lim-Xu [66]) Suppose that X has a weakly continuous duality mapping J_ϕ associated with a gauge ϕ and that $\{x_n\}$ is a sequence converging weakly to x. Then*

$$\limsup_{n\to\infty} \Phi(\|x_n - z\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|z - x\|)$$

for all $z \in X$. In particular, X satisfies Opial's property; that is,

$$x_n \rightharpoonup x \quad \Longrightarrow \quad \limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - z\| \tag{1.1.7}$$

for all $z \in X$, $z \neq x$.

Nonlinear mappings

The following definition contains the nonlinear mappings we are working with and that will appear throughout the entire chapter.

Definition 1.1.6. Let $C \subseteq X$ be a nonempty set. We say that a mapping $T : C \to X$ is

- *L-Lipschitz* if there exists a constant $L > 0$ such that for all $x, y \in C$,

$$\|Tx - Ty\| \leq L\|x - y\|;$$

- a *α-contraction* if it is Lipschitz with constant $\alpha < 1$;
- *nonexpansive* if it is Lipschitz with constant 1; that is, for all $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\|;$$

- *α-averaged* if there exists $\alpha \in (0, 1)$ such that

$$T = (1 - \alpha)I + \alpha S,$$

  for some nonexpansive mapping S;

- firmly nonexpansive if for all $x, y \in C$ there exists $j(Tx - Ty) \in J(Tx - Ty)$ such that
$$\|Tx - Ty\|^2 \leq \langle x - y, j(Tx - Ty)\rangle,$$
where J is the normalized duality map; equivalently, if for all $x, u \in C$ the function $\theta : [0,1] \to [0,\infty]$ defined by
$$\theta(t) = d((1-t)x + tTx, (1-t)y + tTy), \tag{1.1.8}$$
is nonincreasing (see [48]).

We will denote the fixed point set of T as

$$\mathrm{Fix}\,(T) := \{x \in X : x = Tx\} \tag{1.1.9}$$

In the particular case of a Hilbert space, these nonlinear mappings satisfy some properties which are crucial for the methods developed in Section 1.4 and can be deduced from the definitions.

Proposition 1.1.7. *Let H be a Hilbert space. Given a mapping T defined on H, the following assertions hold.*

(i) *If there exist an averaged mapping S, a nonexpansive mapping V and $\alpha \in (0,1)$ such that $T = (1-\alpha)S + \alpha V$, then T is averaged.*

(ii) *A mapping T is firmly nonexpansive if $T = (I + V)/2$ for some nonexpansive mapping V. This means that any firmly nonexpansive mapping is $1/2$-averaged.*

(iii) *A mapping T is firmly nonexpansive if and only if the complementary operator $I - T$ is firmly nonexpansive.*

(iv) *The composition of finitely many averaged mappings is averaged; that is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composition $T_1 \cdots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composition T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

(iv) The weighted sum of finitely many averaged mappings is averaged; that is, if each of the mappings $\{T_i\}_{i=1}^N$ is α_i-averaged, with $\{\alpha_i\}$ real numbers in $(0,1)$, and $\{\lambda_i\}$ is a sequence of real numbers in $(0,1]$ such that $\sum_{i=1}^N \lambda_i = 1$, then $\sum_{i=1}^N \lambda_i T_i$ is an α-averaged mapping, where $\alpha = \max\{\alpha_i : 1 \leq i \leq N\}$.

(v) *If the mappings* $\{T_i\}_{i=1}^N$ *are averaged and have a nonempty common fixed point, then*

$$\bigcap_{i=1}^{N} Fix(T_i) = Fix(T_1 \cdots T_N) = Fix\left(\sum_{i=1}^{N} \lambda_i T_i\right),$$

where $\{\lambda_i\}_{i=1}^N$ *is a set of real numbers in* $(0,1]$ *satisfying* $\sum_{i=1}^N \lambda_i = 1$.

The metric projection on Hilbert spaces

Let C be a nonempty closed convex subset of a Hilbert space H. The (metric or nearest point) *projection* onto C is the mapping $P_C : H \to C$ which assigns to each $x \in H$ the unique point $P_C x$ in C with the property

$$\|x - P_C x\| = \min\{\|x - y\| : y \in C\}. \tag{1.1.10}$$

Projections are characterized as follows (see, for example, [5]).

Proposition 1.1.8. *Given* $x \in H$ *and* $z \in C$. *Then* $z = P_C x$ *if and only if*

$$\langle x - z, y - z\rangle \le 0, \text{ for all } y \in C. \tag{1.1.11}$$

As consequence we have that

(i) $\|P_C x - P_C y\|^2 \le \langle x - y, P_C x - P_C y\rangle$ *for all* $x, y \in H$*; that is, the projection is firmly nonexpansive;*

(ii) $\|x - P_C x\|^2 \le \|x - y\|^2 - \|y - P_C x\|^2$ *for all* $x \in H$ *and* $y \in C$*;*

(iii) *If* C *is a closed subspace, then* P_C *coincides with the orthogonal projection from* H *onto* C*; that is, for* $x \in H$, $x - P_C x$ *is orthogonal to* C *(i.e.,* $\langle x - P_C x, y\rangle = 0$ *for* $y \in C$*).*

If C is a closed convex subset with a particulary simple structure, then the projection P_C has a closed form expression as described below.

1. If $C = \{x \in H : \|x - u\| \le r\}$ is a closed ball centered at $u \in H$ with radius $r > 0$, then

$$P_C x = \begin{cases} u + r\frac{(x-u)}{\|x-u\|}, & \text{if } x \notin C \\ x, & \text{if } x \in C \end{cases}$$

2. If $C = [a, b]$ is a closed rectangle in $\mathbb{R}^n$, where $a = (a_1, a_2, \cdots, a_n)^T$ and $b = (b_1, b_2, \cdots, b_n)^T$, then, for $1 \leq i \leq n$, $P_C x$ has the i^{th} coordinate given by

$$(P_C x)_i = \begin{cases} a_i, & \text{if } x_i < a_i, \\ x_i, & \text{if } x_i \in [a_i, b_i], \\ b_i, & \text{if } x_i > b_i. \end{cases}$$

3. If $C = \{y \in H : \langle a, y \rangle = \alpha\}$ is a hyperplane, with $a \neq 0$ and $\alpha \in \mathbb{R}$, then

$$P_C x = x - \frac{\langle a, x \rangle - \alpha}{\|a\|^2} a.$$

4. If $C = \{y \in H : \langle a, y \rangle \leq \alpha\}$ is a closed halfspace, with $a \neq 0$ and $\alpha \in \mathbb{R}$, then

$$P_C x = \begin{cases} x - \frac{\langle a, x \rangle - \alpha}{\|a\|^2} a, & \text{if } \langle a, x \rangle > \alpha \\ x, & \text{if } \langle a, x \rangle \leq \alpha. \end{cases}$$

5. If C is the range of an $m \times n$ matrix A with full column rank, then

$$P_C x = A(A^* A)^{-1} A^* x$$

   where A^* is the adjoint of A.

Nonexpansive retraction

Our interest in nonexpansive retractions focusses on the generalization of two results in the fixed point theory. Firstly a linear one in reflexive Banach spaces, the fact that the convergence of the means defines a sunny projection on the fixed point sets (Theorem 1.1.30). On the other hand, the firm nonexpansivity of metric projections on the fixed point sets, a nonlinear result in Hilbert spaces, extended in some sense to smooth Banach spaces (Lemma 1.1.9).

Given a subset K of C and a mapping $T : C \to K$. Recall that T is a *retraction* of C onto K if $Tx = x$ for all $x \in K$. We say that T is *sunny* if for each $x \in C$ and $t \in [0, 1]$, we have

$$T(tx + (1 - t)Tx) = Tx,$$

whenever $tx + (1-t)Tx \in C$.

Although metric projections can be well-defined in any strictly convex reflexive Banach space, they are no longer nonexpansive in general. In fact, it is known (cf. [88]) that if nearest point projections are nonexpansive whenever they exist for closed convex subsets C of a Banach space X with dimension at least three, then X must be a Hilbert space. Moreover, it is also known [93] that if every closed and convex subset of a Banach space X with dimension at least three is a nonexpansive retract of X, then X is necessarily a Hilbert space.

The following result characterizes sunny nonexpansive retractions on a smooth Banach space.

Lemma 1.1.9. *(Bruck* [18]*, Reich* [92]*, Goebel-Reich* [48]*) Let X be a smooth Banach space and let $C \supseteq K$ be two nonempty closed convex subsets of X. Assume that $Q : C \to K$ is a retraction from C onto K. Then the following three statements are equivalent.*

(a) *Q is sunny and nonexpansive.*

(b) $\|Qx - Qy\|^2 \le \langle x - y, J(Qx - Qy)\rangle$ *for all* $x, y \in C$.

(c) $\langle x - Qx, J(y - Qx)\rangle \le 0$ *for all* $x \in C$ *and* $y \in K$.

Consequently, there is at most one sunny nonexpansive retraction from C onto K. Note that in terms of the duality mapping J_ϕ, (b) *and* (c) *can be re-expressed as*

(b′) $\|Qx - Qy\|\phi(\|Qx - Qy\|) \le \langle x - y, J_\phi(Qx - Qy)\rangle$ *for all* $x, y \in C$.

(c′) $\langle x - Qx, J_\phi(y - Qx)\rangle \le 0$ *for all* $x \in C$ *and* $y \in K$.

Note that when X is a Hilbert space the unique sunny nonexpansive retraction from C to K is the metric projection onto K since (c) turns into its characterization inequality, and from (b) we deduce its firm nonexpansivity.

The first result regarding the existence of sunny nonexpansive retractions on the fixed point set of a nonexpansive mapping is due to Bruck.

Theorem 1.1.10. *(Bruck* [18]*) If X is strictly convex and uniformly smooth and if $T : C \to C$ is a nonexpansive mapping having a nonempty fixed point set $Fix(T)$, then there exists a sunny nonexpansive retraction of C onto $Fix(T)$.*

In a more general setting within the framework of smooth Banach spaces, Reich [95] and O'Hara-Pillay-Xu [86] provided constructive proof for the existence of the sunny nonexpansive retraction from C onto $\mathrm{Fix}\,(T)$, as it will be stated in Theorem 1.1.30.

Demiclosedness principle

A fundamental result in the theory of nonexpansive mappings is Browder's demiclosedness principle.

Definition 1.1.11. A mapping $T : C \to X$ is said to be *demiclosed* (at y) if the conditions that $\{x_n\}$ converges weakly to x and that $\{Tx_n\}$ converges strongly to y imply that $x \in C$ and $Tx = y$. Moreover, we say that X satisfies the *demiclosedness principle* if for any closed convex subset C of X and any nonexpansive mapping $T : C \to X$, the mapping $I - T$ is demiclosed.

The demiclosedness principle plays an important role in the theory of nonexpansive mappings (and other classes of nonlinear mappings as well). It is an interesting problem to identify those Banach spaces which satisfy the demiclosedness principle for nonexpansive mappings. The following theorem provides a partial answer to the problem.

Theorem 1.1.12. *(Browder* [12]*, Goebel-Kirk* [47]*) The demiclosedness principle for nonexpansive mappings holds in a Banach space which is either uniformly convex or satisfies Opial's property* (1.1.7)*.*

Accretive and monotone operators

The concepts of monotonicity and accretivity constitute a valuable tool in studying important operators which appear in different areas.

Definition 1.1.13. Let $A : X \to 2^X$ be a set-valued operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ in X. The operator A is said to be

- *accretive* if for each $x, y \in \mathcal{D}(A)$ and any $u \in A(x)$, $v \in A(y)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y)\rangle \geq 0,$$

where J is the normalized duality map.

- *m-accretive* if it is accretive and $\mathcal{R}(I + A) = X$.

Definition 1.1.14. Let $A : X \to 2^{X^*}$ be a set-valued operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ in X^*. The operator A is said to be

- *monotone* if for each $x, y \in \mathcal{D}(A)$ and any $u \in A(x)$, $v \in A(y)$,

$$\langle u - v, x - y \rangle \geq 0; \tag{1.1.12}$$

- *strictly monotone* if for each $x, y \in \mathcal{D}(A)$ and any $u \in A(x)$, $v \in A(y)$, the strict inequality of (1.1.12) holds;

- *strongly monotone* if there exists a constant $\eta > 0$ such that for each $x, y \in \mathcal{D}(A)$ and any $u \in A(x)$, $v \in A(y)$,

$$\langle u - v, x - y \rangle \geq \eta \|x - y\|^2; \tag{1.1.13}$$

- *maximal monotone* if it is a monotone operator which is not proper contained in any other monotone operator on X; in other words, for any $x \in X$ and $u \in X^*$, the inequality

$$\langle u - v, x - y \rangle \geq 0, \quad \forall y \in \mathcal{D}(A) \text{ and } v \in A(y), \tag{1.1.14}$$

  implies that $u \in A(x)$;

- *inverse strongly monotone* (ism) if there exists a constant $\nu > 0$ such that for all $x, y \in \mathcal{D}(A)$ and any $u \in A(x)$, $v \in A(y)$,

$$\langle x - y, u - v \rangle \geq \nu \|u - v\|^2.$$

Remark 1.1.15. Note that when the underlying space is Hilbert, the normalized duality mapping is the identity operator and then the notions of accretive and monotone operator coincide.

Definition 1.1.16. Let $A : X \to 2^X$ be a set-valued operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ in X. Given $\lambda > 0$, the *resolvent* of order λ of A is the set-valued mapping $J_\lambda : X \to 2^X$ defined as

$$J_\lambda = (I + \lambda A)^{-1}. \tag{1.1.15}$$

Recall the important Minty Theorem for m-accretive operators on Hilbert spaces enclosed in the following theorem.

Theorem 1.1.17. *(Minty* [76]*, Bruck-Reich* [20]*, Goebel-Reich* [48]*) Let X be a Banach space and $A : X \to 2^X$ an accretive operator. Then the resolvent J_λ is single-valued and firmly nonexpansive, for any $\lambda > 0$. Furthermore, if A is defined on a Hilbert space H, A is m-accretive if and only if $\mathcal{D}(J_\lambda) = H$, for any $\lambda > 0$.*

Some of the problems we will discuss in Section 1.4 are enunciated on a Hilbert space. So next we recall some properties satisfied in this setting with the aim of studying these problems. In particular, the following proposition gathers some results on the relationship between nonexpansive mappings and monotone operators.

Proposition 1.1.18. *(Byrne and Xu-Kim* [21, 127]*) Let $T : H \to H$ be a mapping.*

(i) *If $T : H \to H$ is a ρ-contraction then the complement $I - T$ is $(1 - \rho)$-strongly monotone.*

(ii) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.*

(iii) *If T is $\nu-$ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.*

(iv) *The mapping T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Regarding the properties of the subdifferential of a convex function, the following results are extremely useful.

Theorem 1.1.19. *(Minty* [76]*, Moreau* [78]*) Let $f : H \to \overline{\mathbb{R}}$ be a proper lower semicontinuous convex function. Then its subdifferential ∂f is a maximal monotone operator.*

Lemma 1.1.20. *(Baillon-Haddad* [2]*) If $f : H \to \mathbb{R}$ is a differentiable convex function with a L-Lipschitz gradient ∇f, then ∇f is $\frac{1}{L}$-ism.*

Technical Lemmas about convergent sequences

Lemma 1.1.21. *(Xu* [120]*) Let* $\{\alpha_n\} \subset (0,1)$ *be a sequence satisfying* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*. Let* $\{a_n\} \subset \mathbb{R}^+$ *be a sequence such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n, \text{ where } \limsup_{n\to\infty} b_n \leq 0.$$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.1.22. *(Maingé* [70]*) Let* $\{a_n\}$, $\{c_n\} \subset \mathbb{R}^+$, $\{\alpha_n\} \subset (0,1)$ *and* $\{b_n\} \subset \mathbb{R}$ *be sequences such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + b_n + c_n, \quad \text{for all } n \geq 0.$$

Assume $\sum_{n=0}^{\infty} c_n < \infty$*. Then the following results hold:*

(a) *If* $b_n \leq \alpha_n C$ *where* $C \geq 0$*, then* $\{a_n\}$ *is a bounded sequence.*

(b) *If* $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\limsup_{n\to\infty} b_n/\alpha_n \leq 0$*, then* $\lim_{n\to\infty} a_n = 0$.

Definition 1.1.23. Let X be a complete metric space and $C \subseteq X$ be a nonempty subset. A sequence $\{x_n\} \subset X$ is called *Fejér monotone* with respect to C if

$$d(x_{n+1}, y) \leq d(x_n, y)$$

for all $y \in C$ and $n \geq 0$.

Lemma 1.1.24. *(Browder* [15]*, Ferreira-Oliveira* [42]*) Let* X *be a complete metric space and* $C \subseteq X$ *a nonempty subset. If* $\{x_n\}$ *is Fejér monotone with respect to* C*, then* $\{x_n\}$ *is bounded. Furthermore, if a cluster point* x *of* $\{x_n\}$ *belongs to* C *then* $\{x_n\}$ *converges strongly to* x*. In the particular case of a Hilbert space, given the set of all weakly cluster points of* $\{x_n\}$,

$$\omega_w(x_n) = \{x : \exists\, x_{n_j} \rightharpoonup x\},$$

$\{x_n\}$ *converges weakly to a point* $x \in C$ *if and only if* $\omega_w(x_n) \subseteq C$.

Lemma 1.1.25. *(Suzuki* [108]*) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\gamma_n\} \subset [0,1]$ be a sequence with $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$. Assume that $x_{n+1} = \gamma_n y_n + (1-\gamma_n)x_n$, for all $n \ge 0$ and*

$$\limsup_{n\to\infty} \Big(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \Big) \le 0.$$

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.1.26. *(Reich* [96]*) Let $\{x_n\}$ be a bounded sequence contained in a separable subset D of a Banach space X. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that*

$$\lim_{k\to\infty} \|x_{n_k} - y\|$$

exists for all $y \in D$.

Lemma 1.1.27. *(Reich* [96]*) Let D be a closed convex subset a real Banach space X with a uniformly Gâteaux differentiable norm, and let $\{x_n\}$ be a sequence in D such that*

$$f(y) := \lim_{n\to\infty} \|x_n - y\|$$

exists for all $y \in D$. Given a gauge ϕ, if the function f attains its minimum over D at u, then

$$\limsup_{n\to\infty} \langle y - u, J_\phi(x_n - u)\rangle \le 0$$

for all $y \in D$.

Remark 1.1.28. The proof of Lemma 1.1.27 appears in [96] for the particular case of the normalized duality mapping, but it is readily extended to the case of a generalized duality mapping J_ϕ thanks to the properties of the gauge ϕ.

1.1.2 Iterative algorithms for nonexpansive mappings

Let T be a self-mapping defined on a closed convex subset C of a Banach space X. We know since 1922 that if T is a contraction defined on a complete metric space X, the Banach contraction principle sets up that, for any $x \in X$, Picard iteration $\{T^n x\}$ converges strongly to the unique fixed point of T. If the mapping T is nonexpansive we must assume additional conditions to ensure the existence of fixed points of T

and, even when a fixed point exists, the sequences of iterates in general do not converge to a fixed point. In the particular case when T is firmly nonexpansive, Picard iteration does converge assuming the existence of a fixed point (see, for instance, [48]). The study of iterative methods for approximating fixed points of a nonexpansive mapping T has yielded a host of works in the last decades. The most relevant progresses are mainly based on two types of iterative algorithms: Mann and Halpern iterations.

Mann iteration is essentially an averaged algorithm which generates a sequence recursively

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.1.16}$$

where the initial guess $x_0 \in C$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$.

Halpern iteration generates a sequence via the recursive formula

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0 \tag{1.1.17}$$

where the initial guess $x_0 \in C$ and anchor $u \in C$ are arbitrary (but fixed) and the sequence $\{\alpha_n\}$ is contained in $[0, 1]$.

Both iterations have extensively been studied for decades. The following subsections are devoted to present some background about both iterative algorithms. We first recall certain convergence results for Mann iteration and both implicit and explicit schemes of Halpern iteration. Then diverse approaches regarding these methods will be presented.

Mann iteration

Mann iteration (1.1.16) was first implicitly introduced by Mann, in [72], in a simpler way. Then Krasnosel'skij [61] studied the iterative scheme (1.1.16) in the particular case when $\alpha_n = \lambda$. He provided a result which assures the weak convergence of Picard iteration for averaged mappings in a Hilbert space. Ishikawa, in [52], proved that if $0 < a \leq \alpha_n < 1$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$, which implies the convergence of $\{x_n\}$ to a fixed point of T if the range of T lies in a compact subset of X. Therefore, we can let α be variable and still obtain weak convergence (see, for instance, [32]). Moreover the weak convergence of the Mann iterative algorithm remains true in a more general class of Banach spaces as the following theorem claims.

Theorem 1.1.29. *(Reich* [94]*) Let $C \subseteq H$ be a closed convex set of a uniformly convex Banach space with Fréchet differentiable norm and $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. If the sequence $\{\alpha_n\}_{n\geq 0} \subset (0,1)$ satisfies $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = +\infty$, then, for any $x_0 \in C$, the sequence $\{x_n\}$ defined by the Mann iteration converges weakly to a fixed point of T.*

It is worth mentioning other works regarding the convergence of Mann iteration such as [52, 44, 121].

Halpern implicit iteration

An iterative approach for solving the problem of approximating a fixed point of a mapping T, which may have multiple solutions, is to replace it by a family of perturbed problems admitting a unique solution, and then to get a particular original solution as the limit of these perturbed solutions as the perturbation vanishes. For example, given a closed convex set $C \subseteq H$, $T : C \to C$, $u \in C$ and $t \in (0,1)$, Browder [13, 14] studied the approximating curve $\{x_t\}$ defined by

$$x_t = tu + (1-t)Tx_t; \tag{1.1.18}$$

that is, x_t is the unique fixed point of the contraction $tu + (1-t)T$. He proved that if the underlying space H is Hilbert, $\{x_t\}$ converges strongly as $t \to 0$ to the fixed point of T closest to u. This result has been generalized and extended to a more general class of Banach spaces as it is summarized below.

Theorem 1.1.30. *The net $\{x_t\}$ generated by the implicit algorithm* (1.1.18) *converges strongly as $t \to 0$ to a fixed point of T and the mapping $Q : C \to Fix(T)$ given by*

$$Q(u) := \lim_{t\to 0} x_t \tag{1.1.19}$$

defines the sunny nonexpansive retraction from C onto $Fix(T)$ under either one of the following assumptions:

(i) *The underlying space X is a Hilbert space (Browder [15]);*

(ii) *The underlying space X is uniformly smooth (Reich [95]);*

(iii) *The underlying space X is reflexive, uniformly Gâteaux differentiable and has the fixed point property for nonexpansive mappings (Reich [95], Takahashi-Ueda [111]);*

(iv) *The underlying space X is reflexive and has a weakly continuous duality mapping (Reich [92], O'Hara-Pillay-Xu [86]).*

Those Banach spaces where the approximating curve $\{x_t\}$ strongly converges are said to have *Reich's property* since Reich was the first to show that all uniformly smooth Banach spaces have this property.

Halpern explicit iteration

Halpern was the first in introducing the explicit iterative algorithm (1.1.17) for finding a fixed point of a nonexpansive mapping $T : C \to C$ with $\mathrm{Fix}\,(T) \neq \emptyset$, where C is a closed convex subset of a Hilbert space H. This iterative method is now commonly known as *Halpern iteration* although Halpern initially considered the case where C is the unit closed ball and $u = 0$. He proved that $\{x_n\}$ converges strongly to the fixed point of T which is closest to u (i.e., $P_{\mathrm{Fix}\,(T)}u$) essentially when $\alpha_n = n^{-a}$ with $a \in (0, 1)$. He also showed that the following two conditions

(1) $\lim_{n\to\infty} \alpha_n = 0$, and

(2) $\sum_{n=0}^{\infty} \alpha_n = \infty$

are necessary for the convergence of the sequence $\{x_n\}$ to a fixed point of the mapping T. For example, taking $T(x) = x$ we see that condition (1) is necessary, and for condition (2) it suffices to consider $T(x) = -x$. Ten years later, Lions [65] improved Halpern's result by proving the strong convergence of $\{x_n\}$ to $P_{\mathrm{Fix}\,(T)}u$ if the sequence $\{\alpha_n\}$ satisfies conditions (1), (2) and

(3) $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0$.

Both Halpern's and Lion's conditions on the sequence $\{\alpha_n\}$ exclude the natural choice of $\alpha_n = \frac{1}{n}$. Then, to avoid this problem, Wittmann [116], in 1992, obtained strong convergence to $P_{\mathrm{Fix}\,(T)}u$ of $\{x_n\}$ (still in a Hilbert space H), by replacing condition (3) with the following more general one

(3') $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Conditions (3) and (3') are not comparable. For instance, if $\{\alpha_n\}$ is given by $\alpha_{2n} = (n+1)^{-\frac{1}{4}}$ and $\alpha_{2n+1} = (n+1)^{-\frac{1}{4}} + (n+1)^{-1}$, then (3) holds while (3') does not hold. Note that if the sequence is decreasing condition (3') is a consequence of conditions (1) and (2), so in this particular case conditions (1) and (2) are necessary and sufficient.

In 1994, Reich [99] proved the strong convergence of the algorithm (1.1.17) with the two necessary and decreasing conditions on the parameters in the case when X is uniformly smooth with a weakly continuous duality mapping.

In 2002, Xu [120] succeeded in improving the previous results twofold. First weakening condition (3) by removing the square from the denominator so that the natural choice of $\alpha_n = \frac{1}{n}$ is included:

(3*) $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n} = 0$.

Secondly, he proved strong convergence in the framework of uniformly smooth Banach spaces.

Note that conditions (3') and (3*) are independent in general. For example, the sequence defined by $\alpha_{2n} = \frac{1}{\sqrt{n}}$ and $\alpha_{2n+1} = \frac{1}{\sqrt{n-1}}$ satisfies (3') but fails to satisfy (3*). However, if there exists the limit of $\frac{\alpha_{n-1}}{\alpha_n}$, with α_n verifying conditions (1) and (2), it is easy to check that (3*) holds when (3') does.

Recently O'Hara, Pillay and Xu [86] extended the proof of Xu [120] to a more general class of Banach spaces and improved the approach of Shimizu and Takahashi [106] by showing that the use of the Banach limit can be avoided. The following theorem gathers the previous results.

Theorem 1.1.31. *Let X be either a uniformly smooth Banach space or a reflexive Banach space having a weakly continuous duality mapping J_ϕ. Assume that $\{\alpha_n\} \subset [0,1]$ satisfies conditions (1), (2) and (3') or (3*). Then the sequence $\{x_n\}$ generated by scheme* (1.1.17) *converges strongly to $Q(u)$, where Q is the unique sunny nonexpansive retraction from C onto $Fix(T)$.*

The uniform smoothness assumption can be weakened to the hypothesis that the norm of X is uniformly Gâteaux differentiable and each nonempty closed bounded convex subset C of X has the fixed point property for nonexpansive mappings, i.e., every nonexpansive self-mapping of C has a fixed point.

The existence of a sunny nonexpansive retraction from C to $\mathrm{Fix}\,(T)$, given by Theorem 1.1.30, is an important tool in the proof of the previous theorem. However, while in the uniformly smooth setting the explicit definition of the sunny retraction is crucial (cf. [121]), only the existence of such retraction is needed when X has a weakly continuous duality mapping (cf. [86]).

It is straightforward to see that the proof of Theorem 1.1.31 also works if condition (3') or (3^*) is replaced by the condition of Cho et al. [26] below:

($3^\diamond$) $|\alpha_{n+1} - \alpha_n| \leq \circ(\alpha_{n+1}) + \sigma_n$ where $\sum_{n=1}^{\infty} \sigma_n < \infty$.

Condition ($3^\diamond$) seems weaker than condition (3^*). However, there are no essential differences.

Since conditions (1) and (2) are necessary for Halpern iteration (1.1.17) to converge in norm for all nonexpansive mappings T, a natural question is whether they are also sufficient for strong convergence of Halpern iteration (1.1.17). There are some cases where the answer is affirmative but in general it has been proved to be negative in [110] (see also the following subsection).

Theorem 1.1.32. *(Xu [122]) Let X be a smooth Banach space, C a closed convex subset of X, and $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let ϕ be a gauge and J_ϕ its associated duality map. Assume conditions* (1) *and* (2)*. Then the sequence $\{x_n\}$ generated by Halpern iteration* (1.1.17) *converges strongly to $z \in Fix(T)$ if and only if the following condition holds:*

$$\limsup_{n\to\infty} \langle u - z, J_\phi(x_n - z)\rangle \leq 0. \tag{1.1.20}$$

In particular, under conditions (1) *and* (2)*, we have that $\{x_n\}$ converges strongly to z if either of the following two conditions is satisfied:*

(i) *X is uniformly smooth and J_ϕ is weakly continuous, and $\{x_n\}$ is weakly asymptotically regular (i.e., $x_{n+1} - x_n \rightharpoonup 0$);*

(ii) *X is uniformly smooth and $\{x_n\}$ is asymptotically regular (i.e., $x_{n+1} - x_n \to 0$ in norm).*

Averaged mappings

We have seen in the previous subsection that much effort has been devoted to weaken the third condition on the sequence $\{\alpha_n\}$ and the geometric properties of the Banach space. What are the sufficient and necessary conditions concerning $\{\alpha_n\}$ is still an open problem. However there exist some partial answers. Xu [120] proved that if we replace Tx_n in the scheme (1.1.17) with the mean

$$T_n x_n = (1/n) \sum_{k=0}^{n-1} T^k x_n,$$

then we do have strong convergence under conditions (1) and (2). The main contribution, independently due to Chidume-Chidume [28] and Suzuki [108], is that conditions (1) and (2) are indeed sufficient for the strong convergence of Halpern iteration (1.1.17) if T is an *averaged* mapping, a term coined in [3].

If we require that the net $\{x_t\}$ of solutions of the implicit equation (1.1.18) converge in norm, then the uniform smoothness of X can be weakened to uniform Gâteaux differentiability.

Theorem 1.1.33. *(Chidume-Chidume* [28]*, Suzuki* [108] *) Let X be a Banach space whose norm is uniformly Gâteaux differentiable, $C \subseteq X$ a closed convex subset and $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the explicit scheme*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(\lambda T x_n + (1 - \lambda)x_n), \quad n \geq 0, \tag{1.1.21}$$

where $u \in C$, $\lambda \in (0, 1)$ and the sequence $\{\alpha_n\} \subset [0, 1]$ satisfies (1) *and* (2)*. Assume that $\{x_t\}$ defined as in* (1.1.18) *converges strongly to $z \in Fix(T)$ as $t \to 0$. Then $\{x_n\}$ converges strongly to z.*

In particular, the following result holds true.

Corollary 1.1.34. *Let X be a uniformly smooth Banach space, $C \subseteq X$ a closed convex subset and $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let $\{\alpha_n\}$ satisfy conditions* (1) *and* (2) *and let $\lambda \in (0, 1)$. Then the sequence $\{x_n\}$ defined by (1.1.21) converges strongly to $Q(u)$, where Q is the unique sunny nonexpansive retraction from C onto $Fix(T)$.*

Conditions (1) and (2) are also necessary for the convergence of algorithm (1.1.21), as it is showed in [108] with an example.

Remark 1.1.35. It is not hard to see that the conclusions in Theorem 1.1.33 and Corollary 1.1.34 remain true if the parameter λ in the definition of x_{n+1} is replaced with λ_n satisfying

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1.$$

Viscosity approximation method

Given a nonexpansive self-mapping T on a closed convex subset C, a real number $t \in (0,1]$ and a contraction ψ on C, define the mapping $T_t : C \to C$ by

$$T_t x = t\psi(x) + (1-t)Tx, \quad x \in C.$$

It is easily seen that T_t is a contraction; hence T_t has a unique fixed point which is denoted by x_t. That is, x_t is the unique solution to the fixed point equation

$$x_t = t\psi(x_t) + (1-t)Tx_t, \quad t \in (0,1]. \tag{1.1.22}$$

The explicit iterative discretization of (1.1.22) is

$$x_{n+1} = \alpha_n \psi(x_n) + (1-t)Tx_n, \quad n \geq 0, \tag{1.1.23}$$

where $\{\alpha_n\} \subset [0,1]$. Note that these two iterative processes (1.1.22) and (1.1.23) have Browder and Halpern iterations as special cases by taking $\psi(x) = u \in C$ for any $x \in C$.

The viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [79] in the framework of a Hilbert space. The convergence of the implicit (1.1.22) and explicit (1.1.23) algorithms has been the subject of many papers because under suitable conditions these iterations converge strongly to the unique solution $q \in \mathrm{Fix}\,(T)$ of the variational inequality

$$\langle (I-\psi)q, J(x-q)\rangle \geq 0 \ \ \forall x \in \mathrm{Fix}\,(T), \tag{1.1.24}$$

where J is the normalized duality mapping. This fact allows us to apply this method to convex optimization, linear programming and monotone inclusions. See [123] and references therein for convergence results regarding viscosity approximation methods.

1.2 Alternative iterative methods

Let us focus on the problem of finding a fixed point of the nonexpansive mapping $T : C \to C$ with $\mathrm{Fix}\,(T) \neq \emptyset$. Recall the implicit iterative method defined by the approximating curve (1.1.18), presented by Browder, which indeed converges to a fixed point of T. In this direction, Combettes and Hirstoaga [34] introduced a new type of approximating curve for fixed point problems in the setting of Hilbert spaces. This curve whose iterative scheme is a more general version of the implicit formula

$$x_t = T(tu + (1-t)x_t), \tag{1.2.1}$$

was proved to converge to the best approximation to u from $\mathrm{Fix}\,(T)$. Now let us recover from Section 1.1.2 the viscosity method of approximating the particular fixed point of T which is the unique solution of certain variational inequality. As a generalization of the implicit formula (1.2.1), and motivated by the viscosity iterative algorithm (1.1.22), given a contraction $\psi : C \to C$, we define the approximation curve

$$x_t = T(t\psi(x_t) + (1-t)x_t), \tag{1.2.2}$$

that is, for any $t \in (0, 1]$, x_t is the unique fixed point of the contraction $T_t = T(t\psi + (1-t)I)$, which constitutes a hybrid method of the ones mentioned above.

From the explicit discretized iteration

$$x_{n+1} = T(\alpha_n\psi(x_n) + (1-\alpha_n)x_n), \tag{1.2.3}$$

with the sequence $\{\alpha_n\} \subset [0, 1]$, we can obtain the so-called hybrid steepest descent method

$$x_{n+1} = Tx_n - \alpha_n g(Tx_n). \tag{1.2.4}$$

This latter procedure was suggested by Yamada [117] as an extension of the viscosity approximation method for solving the variational inequality $\mathrm{VIP}(g, \mathrm{Fix}\,(T))$

$$\langle g(p), x - p\rangle \geq 0 \quad \forall x \in \mathrm{Fix}\,(T), \tag{1.2.5}$$

in the case when g is strongly monotone and Lipschitz continuous (see also [118, 71]). However, we can get the convergence of the algorithm (1.2.4) just requiring $I - \mu g$ to be a contraction for some $\mu > 0$, which is satisfied in the particular case when g is strongly monotone and Lipschitz continuous.

In this chapter, we aim to analyze the behavior of these alternative iterative methods motivated mainly by the purpose of solving variational inequality problems and finding zeros of accretive operators as we will show in Section 1.4.

First of all, note that a simple manipulation shows that the classical implicit viscosity iteration (1.1.22) and the approximating curve (1.2.2) just defined are equivalent in the sense that the convergence of one of them implies the convergence of the other. Indeed, if $x_t = T(t\psi(x_t) + (1-t)x_t)$, let us denote

$$y_t := t\psi(x_t) + (1-t)x_t, \tag{1.2.6}$$

which implies $x_t = T(y_t)$. Therefore, the curve (1.2.6) turns into

$$y_t = t\psi \circ T(y_t) + (1-t)T(y_t), \tag{1.2.7}$$

which is actually the implicit viscosity iteration (1.1.22) for the contraction $\psi \circ T$. Thus the curve $\{y_t\}$ converges to the fixed point of T which satisfies the variational inequality (1.1.24) and so does the curve $\{x_t\}$ by the nonexpansivity of T. Conversely, it would be analogous. The same reasoning works for the explicit case. Then, since the viscosity method was proved to converge in uniformly smooth Banach spaces, this proves the convergence of these alternative methods in this setting.

Our contribution is the convergence of the alternative methods in the framework of reflexive Banach spaces with weakly continuous duality mapping, and provide a different proof in uniformly smooth Banach spaces avoiding the use of Banach limits as it was done in [123] for the viscosity method.

1.2.1 Implicit algorithm

In this section we prove the convergence of the implicit algorithm (1.2.2) in two different frameworks. The result is gathered in the following theorem.

Theorem 1.2.1. *Let X be either a reflexive Banach space with a weakly continuous duality mapping J_ϕ or a uniformly smooth Banach space, C a nonempty closed convex subset of X, $T : C \to C$ a nonexpansive mapping with fixed point set $Fix(T) \neq \emptyset$ and $\psi : C \to C$ a ρ-contraction. Then the approximating curve $\{x_t\} \subset C$ defined by*

$$x_t = T(t\psi(x_t) + (1-t)x_t), \;\; t \in (0, 1] \tag{1.2.8}$$

converges strongly, as $t \to 0$, to the unique solution $q \in Fix(T)$ of the inequality

$$\langle (\psi - I)q, J_\phi(x - q)\rangle \le 0, \ \ \forall x \in Fix(T); \tag{1.2.9}$$

that is, q is the unique fixed point of the contraction $Q \circ \psi$, where Q is the unique sunny nonexpansive retraction from C to $Fix(T)$.

Proof. First of all, note that if $Q : C \to \mathrm{Fix}\,(T)$ is the unique sunny nonexpansive retraction whose existence is assured by Theorem 1.1.30, by the characterization Lemma 1.1.9, $q \in C$ is the unique fixed point of $Q \circ \psi$ if and only if $q \in \mathrm{Fix}\,(T)$ satisfies the variational inequality (1.2.9). Throughout the proof, when X is uniformly smooth, the duality mapping is J_ϕ for any gauge ϕ.

We observe that we may assume that C is separable. To see this, consider the set K defined by

$$\begin{aligned} &K_0 := \{q\}, \\ &K_{n+1} := co(K_n \cup T(K_n) \cup \psi(K_n)), \\ &K := \overline{\textstyle\bigcup_n K_n}. \end{aligned}$$

Then $K \subseteq C$ is a nonempty convex closed and separable set. Moreover K is invariant under T, ψ and, therefore, $T_t = T(t\psi + (1-t)I)$. Then $\{x_t\} \subset K$ and we may replace C with K.

We will prove that $\{x_t\}$ converges, as $t \to 0$, to the point $q \in \mathrm{Fix}\,(T)$ which is the unique solution of the inequality (1.2.9).

The sequence $\{x_t\}$ is bounded. Indeed, given $p \in \mathrm{Fix}\,(T)$,

$$\begin{aligned} \|x_t - p\| &= \|T(t\psi(x_t) + (1-t)x_t) - Tp\| \\ &\le \|t(\psi(x_t) - \psi(p)) + (1-t)(x_t - p) + t(\psi(p) - p)\| \\ &\le (t\rho + (1-t))\|x_t - p\| + t\|\psi(p) - p\|. \end{aligned}$$

Then, for any $t \in (0, 1]$,

$$\|x_t - p\| \le \frac{1}{1-\rho}\|\psi(p) - p\|.$$

Take an arbitrary sequence $\{t_n\} \subset (0, 1]$ such that $t_n \to 0$, as $n \to 0$, and denote $x_n = x_{t_n}$ for any $n \ge 0$. Let $\Gamma := \limsup_{n\to\infty}\langle \psi(q) - q, J_\phi(x_n - q)\rangle$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty}\langle \psi(q) - q, J(x_{n_k} - q)\rangle = \Gamma.$$

Since $\{x_{n_k}\}$ is bounded, by Lemma 1.1.26, there exists a subsequence, which also will be denoted by $\{x_{n_k}\}$ for the sake of simplicity, satisfying that

$$g(x) := \lim_{k\to\infty} \|x_{n_k} - x\|$$

exists for all $x \in C$.

We define the set

$$A := \{z \in C : g(z) = \min_{x\in C} g(x)\}$$

and note that A is a nonempty bounded, closed and convex set since g is a continuous convex function and $\lim_{\|x\|\to\infty} g(x) = \infty$. Moreover,

$$\|x_{n_k} - Tz\| \le t_{n_k}\|\psi(x_{n_k}) - x_{n_k}\| + \|x_{n_k} - z\|,$$

for any $z \in C$. Since the sequence $\{t_{n_k}\}$ converges to 0 as $k \to \infty$, we deduce that $g(Tz) \le g(z)$ for any $z \in C$. Then $T(A) \subseteq A$, in other words, T maps A into itself.

Since A is a nonempty bounded, closed and convex subset of either a reflexive Banach space with a weakly continuous duality mapping J_ϕ or a uniformly smooth Banach space, it has the fixed point property for nonexpansive mappings (see [47]), that is $\mathrm{Fix}\,(T) \cap A \ne \emptyset$.

If X is reflexive with weakly continuous duality mapping J_ϕ, we can assume that $\{x_{n_k}\}$ has been chosen to be weakly convergent to a point $\bar{x}$. Since X satisfies Opial's property, we have $A = \{\bar{x}\}$. Then, since $\bar{x} \in \mathrm{Fix}\,(T)$, we obtain by inequality (1.2.9) that

$$\Gamma = \langle \psi(q) - q, J_\phi(\bar{x} - q)\rangle \le 0.$$

If X is uniformly smooth, let $\bar{x} \in \mathrm{Fix}\,(T) \cap A$. Then $\bar{x}$ minimize g over C and, since the norm is uniformly Gâteaux differentiable, by Lemma 1.1.27,

$$\limsup_{k\to\infty} \langle x - \bar{x}, J_\phi(x_{n_k} - \bar{x})\rangle \le 0 \tag{1.2.10}$$

holds, for all $x \in C$, and in particular for $x = \psi(\bar{x})$.

We shall show that $\{x_{n_k}\}$ converges strongly to $\bar{x}$. Denote

$$\delta_k := \langle \psi(\bar{x}) - \bar{x}, J_\phi(t_{n_k}(\psi(x_{n_k}) - x_{n_k}) + (x_{n_k} - \bar{x})) - J_\phi(x_{n_k} - \bar{x})\rangle.$$

Since J is norm-to-norm uniformly continuous on bounded sets, then $\lim_{k\to\infty}\delta_k = 0$. Moreover, since Φ is convex and nondecreasing, from the subdifferential inequality (1.1.4) we deduce that

$$\begin{aligned}\Phi(\|x_{n_k}-\bar{x}\|) &\leq \Phi(\|t_{n_k}(\psi(x_{n_k})-\psi(\bar{x}))+(1-t_{n_k})(x_{n_k}-\bar{x})+t_{n_k}(\psi(\bar{x})-\bar{x})\|)\\ &\leq \Phi(\|t_{n_k}(\psi(x_{n_k})-\psi(\bar{x}))+(1-t_{n_k})(x_{n_k}-\bar{x})\|)+t_{n_k}\delta_k\\ &\quad +t_{n_k}\langle\psi(\bar{x})-\bar{x}, J_\phi(x_{n_k}-\bar{x})\rangle\\ &\leq (1-(1-\rho)t_{n_k})\Phi(\|x_{n_k}-\bar{x}\|)+t_{n_k}\delta_k\\ &\quad +t_{n_k}\langle\psi(\bar{x})-\bar{x}, J_\phi(x_{n_k}-\bar{x})\rangle.\end{aligned} \tag{1.2.11}$$

From (1.2.11) and by (1.2.10), we obtain

$$\begin{aligned}\lim_{k\to\infty}\Phi(\|x_{n_k}-\bar{x}\|) &\leq \limsup_{k\to\infty}\frac{1}{1-\rho}\Big(\delta_k+\langle\psi(\bar{x})-\bar{x}, J_\phi(x_{n_k}-\bar{x})\rangle\Big)\\ &\leq 0.\end{aligned}$$

Bearing in mind that Φ is positive and $\Phi(0)=0$, this implies that $\lim_k x_{n_k}=\bar{x}$. Since $\bar{x}$ is a fixed point of T, we also have

$$\Gamma = \lim_{k\to\infty}\langle\psi(q)-q, J_\phi(x_{n_k}-q)\rangle = \langle\psi(q)-q, J_\phi(\bar{x}-q)\rangle \leq 0.$$

By applying (1.2.11) to $\{x_n\}$ and q, since $\Gamma\leq 0$ in both cases, we obtain

$$\lim_{n\to\infty} x_n = q$$

as required. □

1.2.2 Explicit algorithm

We next analyze the explicit (1.2.3) and hybrid steepest descent (1.2.4) algorithms in the setting of Banach spaces, whose convergence results generalize in some sense or constitute a different approach for proving the previously stated results by Combettes and Hirstoaga [34], Xu [123], Yamada [117], and Xu and Kim [127].

The following lemma collects some properties of the iteration (1.2.3) in the setting of normed spaces.

Lemma 1.2.2. *Let X be a normed space and $\{x_n\}$ be the sequence defined by the explicit algorithm* (1.2.3)*.*

(1) *For all $n \geq 0$,*

$$\|\psi(x_n) - x_n\| \leq (1+\rho)\|x_n - x_0\| + \|\psi(x_0) - x_0\|, \tag{1.2.12}$$
$$\|x_n - Tx_n\| \leq \|x_{n+1} - x_n\| + \alpha_n\|\psi(x_n) - x_n\|. \tag{1.2.13}$$

(2) *For all $n \geq 1$,*

$$\begin{aligned} \|x_{n+1} - x_n\| \leq\ & (1-(1-\rho)\alpha_n)\|x_n - x_{n-1}\| \\ & + |\alpha_n - \alpha_{n-1}|\,\|\psi(x_{n-1}) - x_{n-1}\|. \end{aligned} \tag{1.2.14}$$

(3) *If $Fix(T) \neq \emptyset$, then $\{x_n\}$ is bounded for every $x_0 \in C$.*

Proof.

(1) Let $n \geq 0$. Thanks to the definition of the sequence $\{x_n\}$ and since T is nonexpansive and ψ a contraction, the following inequalities hold.

$$\begin{aligned} \|\psi(x_n) - x_n\| &\leq \|\psi(x_n) - \psi(x_0)\| + \|\psi(x_0) - x_0\| + \|x_0 - x_n\| \\ &\leq (1+\rho)\|x_n - x_0\| + \|\psi(x_0) - x_0\|. \\ \|x_n - Tx_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - Tx_n\| \\ &\leq \|x_{n+1} - x_n\| + \|\alpha_n\psi(x_n) + (1-\alpha_n)x_n - x_n\| \\ &= \|x_{n+1} - x_n\| + \alpha_n\|\psi(x_n) - x_n\|. \end{aligned}$$

(2) For any $n \geq 1$, it follows that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|T(\alpha_n \psi(x_n) + (1-\alpha_n)x_n) \\
&\quad - T(\alpha_{n-1}\psi(x_{n-1}) + (1-\alpha_{n-1})x_{n-1})\| \\
&\leq \|\alpha_n \psi(x_n) + (1-\alpha_n)x_n - \alpha_{n-1}\psi(x_{n-1}) \\
&\quad - (1-\alpha_{n-1})x_{n-1}\| \\
&= \|\alpha_n(\psi(x_n) - \psi(x_{n-1})) + (1-\alpha_n)(x_n - x_{n-1}) \\
&\quad + (\alpha_n - \alpha_{n-1})(\psi(x_{n-1}) - x_{n-1})\| \\
&\leq \alpha_n \rho \|x_n - x_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \, \|\psi(x_{n-1}) - x_{n-1}\| \\
&= (1-(1-\rho)\alpha_n)\|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \, \|\psi(x_{n-1}) - x_{n-1}\|.
\end{aligned}$$

(3) Let p be a fixed point of T.

$$\begin{aligned}
\|x_{n+1} - p\| &= \|T(\alpha_n \psi(x_n) + (1-\alpha_n)x_n) - Tp\| \\
&\leq \|\alpha_n \psi(x_n) + (1-\alpha_n)x_n - p\| \\
&= \|\alpha_n(\psi(x_n) - \psi(p)) + (1-\alpha_n)(x_n - p) + \alpha_n(\psi(p) - p)\| \\
&\leq \alpha_n \rho \|x_n - p\| + (1-\alpha_n)\|x_n - p\| + \alpha_n \|\psi(p) - p\| \\
&= (1-(1-\rho)\alpha_n)\|x_n - p\| + (1-\rho)\alpha_n \frac{\|\psi(p) - p\|}{1-\rho} \\
&\leq \max\left\{\|x_n - p\|, \frac{\|\psi(p) - p\|}{1-\rho}\right\}.
\end{aligned}$$

By induction, we obtain that for all $n \geq 0$,

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\psi(p) - p\|}{1-\rho}\right\},$$

thus $\{x_n\}$ is bounded.

□

Theorem 1.2.3. *Let X be either a reflexive Banach space with a weakly continuous duality mapping J_ϕ or a uniformly smooth Banach space, C a nonempty closed convex subset of X, $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$, $\psi : C \to C$ a ρ-contraction and $\{\alpha_n\}$ a sequence in $[0,1]$ satisfying*

(H1) $\lim_{n\to\infty} \alpha_n = 0$

(H2) $\sum_{n=1}^{\infty} \alpha_n = \infty$

(H3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \dfrac{\alpha_n}{\alpha_{n+1}} = 1$.

Then, for any $x_0 \in C$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = T(\alpha_n \psi(x_n) + (1-\alpha_n)x_n) \tag{1.2.15}$$

converges strongly to the unique solution $q \in Fix(T)$ of the inequality

$$\langle (\psi - I)q, J_\phi(x-q)\rangle \leq 0, \quad \forall x \in Fix(T); \tag{1.2.16}$$

that is, q is the unique fixed point of the contraction $Q \circ \psi$, where Q is the unique sunny nonexpansive retraction from C to $Fix(T)$.

Proof. As we justified in the proof of the previous theorem if $Q : C \to \mathrm{Fix}(T)$ is the unique sunny nonexpansive retraction whose existence is assured by Theorem 1.1.30, by the characterization Lemma 1.1.9, $q \in C$ is the unique fixed point of $Q \circ \psi$ if and only if $q \in \mathrm{Fix}(T)$ satisfies the variational inequality (1.2.9). Throughout the proof, when X is uniformly smooth, the duality mapping is J_ϕ for any gauge ϕ.

Since T has fixed points, by Lemma 1.2.2 (3) we have that $\{x_n\}$ is bounded, and therefore so are $\{T(x_n)\}$ and $\{\psi(x_n)\}$. The fact that $\{x_n\}$ is asymptotically regular is a consequence of Lemma 1.2.2 (2). Indeed, by hypothesis we have that $\sum_{n=1}^{\infty}(1-\rho)\alpha_{n-1} = \infty$ and either $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ or

$$\limsup_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = \lim_{n\to\infty} \left| 1 - \frac{\alpha_{n-1}}{\alpha_n} \right| = 0. \tag{1.2.17}$$

Then inequality (1.2.14)

$$\|x_{n+1} - x_n\| \leq (1-(1-\rho)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\, \|\psi(x_{n-1}) - x_{n-1}\|$$

allows us to use Lemmas 1.1.21 and 1.1.22(b) to deduce that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{1.2.18}$$

Then, by using inequality (1.2.13) and hypothesis (H1) we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| \leq \lim_{n\to\infty} \|x_{n+1} - x_n\| + \lim_{n\to\infty} \alpha_n \|\psi(x_n) - x_n\| = 0. \tag{1.2.19}$$

Distinguishing both cases according to the underlying space we will see now that

$$\limsup_{n\to\infty} \langle \psi(q) - q, J_\phi(x_n - q)\rangle \leq 0, \tag{1.2.20}$$

where ϕ will be the identity function, that is, $J_\phi = J$ will be the normalized duality mapping in the case of a uniformly smooth Banach space.

Assume first that X is a reflexive Banach space with weakly continuous duality mapping J_ϕ. Take a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\limsup_{n\to\infty} \langle \psi(q) - q, J_\phi(x_n - q)\rangle = \lim_{k\to\infty} \langle \psi(q) - q, J_\phi(x_{n_k} - q)\rangle.$$

Since X is reflexive and $\{x_n\}$ bounded, we may assume that $x_{n_k} \rightharpoonup \bar{x}$. From Theorem 1.1.5 we know that X satisfies Opial's property, therefore Demiclosedness principle for nonexpansive mappings holds (see Theorem 1.1.12). Since $\{(I-T)x_n\}$ converges to 0 from (1.2.19), this implies that $\bar{x} \in \mathrm{Fix}(T)$. Then by inequality (1.2.16) and the weak-to-weak* uniform continuity of J_ϕ,

$$\begin{aligned}\limsup_{n\to\infty} \langle \psi(q) - q, J_\phi(x_n - q)\rangle &= \lim_{k\to\infty} \langle \psi(q) - q, J_\phi(x_{n_k} - q)\rangle \\ &= \langle \psi(q) - q, J_\phi(\bar{x} - q)\rangle \\ &\leq 0.\end{aligned}$$

If X is uniformly smooth we proceed as follows. Let $\{\beta_k\}$ be a null sequence in $(0,1)$ (i.e., $\{\beta_k\} \to 0$, as $k \to \infty$) and define $\{y_k\}$ by

$$y_k := T(\beta_k \psi(y_k) + (1 - \beta_k) y_k).$$

We have proved in Theorem 1.2.1 that $\{y_k\}$ converges strongly to q. For any $n, k \geq 0$ define

$$\delta_{n,k} := \|x_n - Tx_n\|^2 + 2\|x_n - Tx_n\| \|y_k - Tx_n\|$$

and

$$\epsilon_k := \sup_{n\geq 0}\{\|\psi(y_k)-x_n\|\|J(\beta_k(\psi(y_k)-x_n)+(1-\beta_k)(y_k-x_n))-J(y_k-x_n)\|\}.$$

For any fixed $k\in\mathbb{N}$, by (1.2.19), $\lim_{n\to\infty}\delta_{n,k}=0$. Moreover $\lim_{k\to\infty}\epsilon_k=0$ because of the uniform continuity of J over bounded sets. By using inequality (1.1.5) and the nonexpansivity of T we obtain

$$\begin{aligned}
\|y_k-x_n\|^2 &\leq (\|Tx_n-x_n\|+\|y_k-Tx_n\|)^2\\
&= \|x_n-Tx_n\|^2+2\|x_n-Tx_n\|\|y_k-Tx_n\|+\|y_k-Tx_n\|^2\\
&\leq \delta_{n,k}+\|(1-\beta_k)(y_k-x_n)+\beta_k(\psi(y_k)-x_n)\|^2\\
&\leq \delta_{n,k}+(1-\beta_k)^2\|y_k-x_n\|^2\\
&\quad +2\beta_k\langle\psi(y_k)-x_n,J(\beta_k(\psi(y_k)-x_n)+(1-\beta_k)(y_k-x_n))\rangle\\
&\leq \delta_{n,k}+(1-\beta_k)^2\|y_k-x_n\|^2+2\beta_k\langle\psi(y_k)-x_n,J(y_k-x_n)\rangle\\
&\qquad +2\beta_k\epsilon_k\\
&= \delta_{n,k}+(1-\beta_k)^2\|y_k-x_n\|^2+2\beta_k\langle y_k-x_n,J(y_k-x_n)\rangle\\
&\quad +2\beta_k\langle\psi(y_k)-y_k,J(y_k-x_n))\rangle)+2\beta_k\epsilon_k\\
&= \delta_{n,k}+((1-\beta_k)^2+2\beta_k)\|y_k-x_n\|^2+2\beta_k\epsilon_k\\
&\quad 2\beta_k\langle\psi(y_k)-y_k,J(y_k-x_n)\rangle
\end{aligned}$$

Then we deduce that

$$\langle\psi(y_k)-y_k,J(x_n-y_k)\rangle\leq\frac{1}{2}\,(\frac{\delta_{n,k}}{\beta_k}+\beta_k\|y_k-x_n\|^2+2\epsilon_k),$$

and therefore

$$\limsup_{n\to\infty}\langle\psi(y_k)-y_k,J(x_n-y_k)\rangle\leq\frac{\beta_k}{2}\limsup_{n\to\infty}\|y_k-x_n\|^2+\epsilon_k. \tag{1.2.21}$$

On the other hand

$$\begin{aligned}
\langle\psi(q)-q,J(x_n-q)\rangle &= \langle\psi(q)-q,J(x_n-q)-J(x_n-y_k)\rangle\\
&\quad +\langle(\psi(q)-q)-(\psi(y_k)-y_k),J(x_n-y_k)\rangle\\
&\quad +\langle\psi(y_k)-y_k,J(x_n-y_k)\rangle.
\end{aligned} \tag{1.2.22}$$

Note that

$$\lim_{k\to\infty}(\sup_{n\geq 0}\{\langle\psi(q)-q, J(x_n-q)-J(x_n-y_k)\rangle\})=0 \tag{1.2.23}$$

because J is norm to norm uniformly continuous on bounded sets. By using (1.2.21), (1.2.23) and passing first to $\limsup_{n\to\infty}$ and then to $\lim_{k\to\infty}$, from (1.2.22) we obtain

$$\limsup_{n\to\infty}\langle\psi(q)-q, J(x_n-q)\rangle\leq 0. \tag{1.2.24}$$

Finally we prove that $\{x_n\}$ converges strongly to q. Set

$$\eta_n := \|\psi(q)-q\|\,\|J_\phi(\alpha_n\psi(x_n)+(1-\alpha_n)x_n-q)-J_\phi(x_n-q)\|.$$

Hypothesis (H1) implies that $\eta_n\to 0$, as $n\to\infty$. By using the nonexpansivity of T, subdifferential inequality 1.1.4, the convexity of Φ and the fact that ψ is a contraction we obtain that

$$\begin{aligned}
\Phi(\|x_{n+1}-q\|) &\leq \Phi(\|\alpha_n\psi(x_n)+(1-\alpha_n)x_n-q\|)\\
&= \Phi(\|\alpha_n(\psi(x_n)-\psi(q))+(1-\alpha_n)(x_n-q)+\alpha_n(\psi(q)-q)\|)\\
&\leq \Phi(\|\alpha_n(\psi(x_n)-\psi(q))+(1-\alpha_n)(x_n-q)\|)\\
&\quad +\alpha_n\langle\psi(q)-q, J_\phi(\alpha_n\psi(x_n)+(1-\alpha_n)x_n-q)\rangle\\
&\leq \Phi(\|\alpha_n(\psi(x_n)-\psi(q))+(1-\alpha_n)(x_n-q)\|)\\
&\quad +\alpha_n(\langle\psi(q)-q, J_\phi(x_n-q)\rangle+\eta_n)\\
&\leq (1-(1-\rho)\alpha_n)\Phi(\|x_n-q\|)+\alpha_n b_n,
\end{aligned}$$

where $b_n=\langle\psi(q)-q, J_\phi(x_n-q)\rangle+\eta_n$. If X is a reflexive Banach space with a weakly continuous duality mapping J_ϕ, thanks to (1.2.20) we deduce that $\limsup_{n\to\infty} b_n\leq 0$. Thus from Lemma 1.1.21 we obtain that $\{x_n\}$ converges strongly to q, since Φ is positive and $\Phi(0)=0$. In the case of a uniformly smooth Banach space the previous inequalities for the normalized duality mapping J, with $\Phi(t)=t^2/2$, turn into

$$\|x_{n+1}-q\|^2\leq(1-(1-\rho)\alpha_n)\|x_n-q\|^2+2\alpha_n b_n,$$

where $\limsup_{n\to\infty} b_n\leq 0$ since (1.2.24) holds. Thus the result follows from Lemma 1.1.21 as well.

□

Corollary 1.2.4. *Let X be either a reflexive Banach space with a weakly continuous duality mapping J_ϕ or a uniformly smooth Banach space, C a nonempty closed convex subset of X, $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$ and $g : C \to C$ a mapping such that $I - \mu g$ is a contraction for some $\mu > 0$. Assume that $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying hypotheses* (H1)-(H3) *in Theorem* 1.2.3*. Then the sequence $\{x_n\}$ defined by the iterative scheme*

$$x_{n+1} = Tx_n - \alpha_n g(Tx_n), \tag{1.2.25}$$

converges strongly to the unique solution $q \in Fix(T)$ of the inequality problem

$$\langle g(q), J_\phi(x - q)\rangle \geq 0, \ \forall x \in Fix(T). \tag{1.2.26}$$

Proof. Consider the sequence $\{z_n\}$ defined by $z_n = Tx_n$, for any $n \geq 0$. Then

$$\begin{aligned} z_{n+1} &= T(Tx_n - \alpha_n g(Tx_n)) \\ &= T(z_n - \frac{\alpha_n}{\mu}\mu g(z_n)) \\ &= T(\alpha_n'(I - \mu g)z_n + (1 - \alpha_n')z_n), \end{aligned}$$

where $\alpha_n' = \frac{\alpha_n}{\mu}$ for all $n \geq 0$, so the sequence $\{\alpha_n'\}$ satisfies hypotheses (H1)-(H3). Since $\psi := I - \mu g$ is a contraction, Theorem 1.2.3 implies the strong convergence of $\{z_n\}$ to the unique solution $q \in \mathrm{Fix}\,(T)$ of the inequality problem

$$\langle (I - \psi)q, J_\phi(x - q)\rangle \geq 0, \ \forall x \in \mathrm{Fix}\,(T),$$

which is equivalent to (1.2.26). Therefore, from the iteration scheme (1.2.25) we deduce that the sequence $\{x_n\}$ converges strongly to q. □

In Section 1.4.2 we will show how to apply this previous result for approximating solutions to variational inequality problems and therefore to minimization problems.

Remark 1.2.5. It is easily seen that the conclusions in Theorems 1.2.1, 1.2.3 and Corollary 1.2.4 remain true if the uniform smoothness assumption of X is replaced with the following two conditions:

(a) X has a uniformly Gâteaux differentiable norm;

(b) X has Reich's property (see Theorem 1.1.30).

1.3 Perturbation techniques

In this section we present a new approach for the problem of finding a fixed point of a nonexpansive mapping $T : C \to C$ with $\mathrm{Fix}\,(T) \neq \emptyset$. In some applications, as we will see in Section 1.4.3, the involved mapping T is often the projection P_C onto the closed convex set C in a Hilbert space H. The complexity in the computations of the projection P_C may bring difficulties, due to the fact that projections may lack to have a closed form (unless C is as simple as a closed ball or a half-space). There exist some perturbation techniques to avoid this inconvenience with the implementation. These methods consist of considering a sequence $\{C_n\}$ of closed convex subsets of H, instead of the original set C, in the hope that the projections P_{C_n} are relatively easy to compute and converge in some sense to the projection P_C.

The first appearance of this idea is due to Yang and Zhao, in [130, 128], who proved the convergence of a Mann type algorithm in a finite-dimensional Hilbert space assuming the convergence in the sense of Mosco of $\{C_n\}$ to C. They mainly applied their results to approximate solutions to the split feasibility problem (see Section 1.4.3). Motivated by this approach, Xu, in [124], studied a more general algorithm which generates a sequence $\{x_n\}$ according to the recursive formula

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n x_n,\ n \geq 0, \tag{1.3.1}$$

where $\{T_n\}$ is a sequence of nonexpansive mappings defined on a Banach space X tending to the mapping T in some sense. Then, under assumptions

(i) $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n D_\rho(T_n, T) < \infty$, for every $\rho > 0$,

where $D_\rho(T_n, T) = \sup\{\|T_n x - Tx\| : \|x\| \leq \rho\}$, he proved that in the setting of a uniformly convex Banach space X having a Fréchet differentiable norm the sequence $\{x_n\}$ weakly converges to a fixed point of T. It is worth mentioning that Xu's result contains Theorem 2 by Reich in [94].

Aiming to get strong convergence to a fixed point of T by means of this technique, we present a Halpern type iterative scheme, considering a sequence $\{T_n\}$ of nonexpansive self-mappings of C which are viewed as perturbations and will be assumed to converge in some sense to the originally given mapping T. Our iterative algorithm is then defined to generate a sequence $\{x_n\}$ according to the recursive formula

$$x_{n+1} = \alpha_{n+1} u + (1 - \alpha_{n+1}) S_{n+1} x_n, \ n \geq 0, \tag{1.3.2}$$

where $x_0, u \in C$ are arbitrary points, $\{\alpha_n\}$ is a sequence in $[0, 1]$ and, for each integer $n \geq 0$, S_n is the averaged mapping defined by $S_n = (1 - \lambda) I + \lambda T_n$, with $\lambda \in (0, 1)$ fixed. Then, under certain conditions on the sequence $\{T_n\}$, we can prove the strong convergence of $\{x_n\}$ to a fixed point of T in the setting of either reflexive Banach spaces having a weakly continuous duality mapping J_ϕ or uniformly smooth Banach spaces.

Theorem 1.3.1. *Let X be a reflexive Banach space having a weakly continuous duality mapping J_ϕ, C a nonempty closed convex subset of X, $T : C \to C$ a nonexpansive mapping such that $Fix(T) \neq \emptyset$, $\{T_n\}$ a sequence of nonexpansive self-mappings defined on C and $\{\alpha_n\}$ a sequence in $[0, 1]$. Assume that the following conditions are satisfied.*

(i) $\lim_{n\to\infty} \|T_n y_n - T y_n\| = 0, \ \forall \{y_n\} \subset C$ *bounded;*

(ii) $\sum_{n=0}^{\infty} \|T_n p - Tp\| < \infty, \ \forall p \in Fix(T)$*;*

(H1) $\lim_{n\to\infty} \alpha_n = 0$*;*

(H2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ generated by the algorithm (1.3.2) *converges strongly to $Q(u)$, where Q is the unique sunny nonexpansive retraction from C onto $Fix(T)$.*

Proof. First of all, note that by Theorem 1.1.30 the unique sunny nonexpansive retraction $Q : C \to \text{Fix}(T)$ is given by $Q(u) = \lim_{t\to 0} z_t$ where $z_t = tu + (1 - t) T z_t$ for each $t \in (0, 1)$.

We prove the theorem in the following steps.

Step 1. $\{x_n\}$ is bounded.
Let $p \in \text{Fix}(T)$. By the nonexpansivity of S_{n+1},

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_{n+1}\|u - p\| + (1 - \alpha_{n+1})\|S_{n+1} x_n - p\| \\
&\leq \alpha_{n+1}\|u - p\| + (1 - \alpha_{n+1})\Big(\|x_n - p\| + \|S_{n+1} p - p\|\Big) \\
&\leq (1 - \alpha_{n+1})\|x_n - p\| + \alpha_{n+1}\|u - p\| + \lambda\|T_{n+1} p - Tp\|
\end{aligned}$$

where $\sum_{n=0}^{\infty} \|T_n p - Tp\| < \infty$ by (ii). Thus, by Lemma 1.1.22(a), we obtain that $\{x_n\}$ is a bounded sequence.

Moreover, since

$$\begin{aligned} \|S_{n+1}x_n - p\| &\leq \lambda\|T_{n+1}x_n - p\| + (1-\lambda)\|x_n - p\| \\ &\leq \lambda\Big(\|x_n - p\| + \|T_{n+1}p - Tp\|\Big) + (1-\lambda)\|x_n - p\| \\ &= \|x_n - p\| + \lambda\|T_{n+1}p - Tp\|, \end{aligned}$$

where $\{\|x_n - p\|\}$ is bounded and $\{\|T_{n+1}p - Tp\|\}$ tends to 0, we conclude that $\{S_{n+1}x_n\}$ and $\{T_{n+1}x_n\}$ are bounded.

Step 2. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

We can write

$$\begin{aligned} x_{n+1} &= \alpha_{n+1}u + (1-\alpha_{n+1})\big(\lambda T_{n+1}x_n + (1-\lambda)x_n\big) \\ &= \alpha_{n+1}u + (1-\alpha_{n+1})\lambda T_{n+1}x_n + (1-\alpha_{n+1})(1-\lambda)x_n \\ &= \gamma_n y_n + (1-\gamma_n)x_n, \end{aligned}$$

where, for $n \geq 0$,

$$\gamma_n = \alpha_{n+1} + (1-\alpha_{n+1})\lambda$$

and

$$y_n = \frac{\alpha_{n+1}u + (1-\alpha_{n+1})\lambda T_{n+1}x_n}{\gamma_n} = \frac{\alpha_{n+1}}{\gamma_n}u + \left(1 - \frac{\alpha_{n+1}}{\gamma_n}\right)T_{n+1}x_n.$$

Note that the sequence $\{y_n\}$ is bounded because it is the convex combination of two bounded sequences, and the sequence $\{\gamma_n\}$ satisfies

$$\lim_{n\to\infty} \gamma_n = \lambda \in (0,1). \tag{1.3.3}$$

We estimate

$$\begin{aligned} \|y_{n+1}-y_n\| &\leq \Big|\frac{\alpha_{n+2}}{\gamma_{n+1}}-\frac{\alpha_{n+1}}{\gamma_n}\Big|\|u\|+\Big|\frac{1-\alpha_{n+2}}{\gamma_{n+1}}-\frac{1-\alpha_{n+1}}{\gamma_n}\Big|\|T_{n+1}x_n\|+ \\ &\quad +\frac{1-\alpha_{n+2}}{\gamma_{n+1}}\lambda\|T_{n+2}x_{n+1}-T_{n+1}x_n\| \\ &\leq \Big|\frac{\alpha_{n+2}}{\gamma_{n+1}}-\frac{\alpha_{n+1}}{\gamma_n}\Big|\|u\|+\Big|\frac{1-\alpha_{n+2}}{\gamma_{n+1}}-\frac{1-\alpha_{n+1}}{\gamma_n}\Big|\|T_{n+1}x_n\|+ \\ &\quad +\frac{1-\alpha_{n+2}}{\gamma_{n+1}}\lambda\Big(\|x_{n+1}-x_n\|+\|T_{n+2}x_n-T_{n+1}x_n\|\Big). \end{aligned}$$

We denote

$$\begin{aligned} \beta_n &= \Big|\frac{\alpha_{n+2}}{\gamma_{n+1}}-\frac{\alpha_{n+1}}{\gamma_n}\Big|\|u\|+\Big|\frac{1-\alpha_{n+2}}{\gamma_{n+1}}-\frac{1-\alpha_{n+1}}{\gamma_n}\Big|\|T_{n+1}x_n\|+ \\ &\quad +\frac{1-\alpha_{n+2}}{\gamma_{n+1}}\lambda\|T_{n+2}x_n-T_{n+1}x_n\|. \end{aligned}$$

By (H1), (1.3.3), (i) and the fact that $\{T_{n+1}x_n\}$ is bounded, we obtain

$$\lim_{n\to\infty}\beta_n=0.$$

Then it follows that

$$\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\leq\beta_n+\Big(\frac{1-\alpha_{n+2}}{\gamma_{n+1}}\lambda-1\Big)\|x_{n+1}-x_n\|,$$

and, since $\{x_n\}$ is bounded, we obtain that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

Now, by Lemma 1.1.25, we deduce that $\lim_{n\to\infty}\|y_n-x_n\|=0$, which implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\gamma_n\|y_n-x_n\|=0.$$

Step 3. $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

Indeed, we can write

$$\|x_n - Tx_n\| \;\leq\; \|x_n - x_{n+1}\| + \|x_{n+1} - T_{n+1}x_n\| + \|T_{n+1}x_n - Tx_n\|$$

where

$$\begin{aligned}\|x_{n+1} - T_{n+1}x_n\| &\leq \alpha_{n+1}\|u - S_{n+1}x_n\| + \|S_{n+1}x_n - T_{n+1}x_n\| \\ &= \alpha_{n+1}\|u - S_{n+1}x_n\| + (1-\lambda)\|x_n - T_{n+1}x_n\| \\ &\leq \alpha_{n+1}\|u - S_{n+1}x_n\| + \\ &\quad +(1-\lambda)\Big(\|x_n - Tx_n\| + \|T_{n+1}x_n - Tx_n\|\Big).\end{aligned}$$

Then

$$\|x_n - Tx_n\| \;\leq\; \frac{1}{\lambda}\Big(\alpha_{n+1}\|u - S_{n+1}x_n\| + (2-\lambda)\|T_{n+1}x_n - Tx_n\| + \|x_{n+1} - x_n\|\Big).$$

Therefore, by step 2, (i), (H1) and the boundedness of $\{S_{n+1}x_n\}$,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Step 4. $\limsup_{n\to\infty}\langle u - Q(u), J_\phi(x_n - Q(u))\rangle \leq 0$, where Q the unique sunny nonexpansive retraction from C onto $\mathrm{Fix}\,(T)$.

Take a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\limsup_{n\to\infty}\langle u - Q(u), J_\phi(x_n - Q(u))\rangle = \lim_{k\to\infty}\langle u - Q(u), J_\phi(x_{n_k} - Q(u))\rangle.$$

Since X is reflexive and $\{x_n\}$ bounded, we may assume that $x_{n_k} \rightharpoonup x$.

Step 3 combined with the demiclosedness principle (Theorem 1.1.12) implies that $x \in \mathrm{Fix}\,(T)$. Then by Lemma 1.1.9(c′)

$$\begin{aligned}\limsup_{n\to\infty}\langle u - Q(u), J_\phi(x_n - Q(u))\rangle &= \lim_{k\to\infty}\langle u - Q(u), J_\phi(x_{n_k} - Q(u))\rangle \\ &= \langle u - Q(u), J_\phi(x - Q(u))\rangle \\ &\leq 0.\end{aligned}$$

Step 5. $x_n \to Q(u)$.

Using the subdifferential inequality (1.1.4) and the nonexpansivity of S_n we obtain

$$\begin{aligned}
\Phi(\|x_n - Q(u)\|) &= \Phi(\|\alpha_n(u - Q(u)) + (1-\alpha_n)(S_n x_{n-1} - Q(u))\|) \\
&\leq \Phi((1-\alpha_n)\|S_n x_{n-1} - Q(u)\|) + \alpha_n\langle u - Q(u), J_\phi(x_n - Q(u))\rangle \\
&\leq (1-\alpha_n)\Phi(\|x_{n-1} - Q(u)\| + \|S_n Q(u) - Q(u)\|) + \\
&\quad +\alpha_n\langle u - Q(u), J_\phi(x_n - Q(u))\rangle \\
&\leq (1-\alpha_n)\Phi(\|x_{n-1} - Q(u)\|) \\
&\quad +\Phi(\|x_{n-1} - Q(u)\| + \|S_n Q(u) - Q(u)\|) - \Phi(\|x_{n-1} - Q(u)\|) \\
&\quad +\alpha_n\langle u - Q(u), J_\phi(x_n - Q(u))\rangle \\
&\leq (1-\alpha_n)\Phi(\|x_{n-1} - Q(u)\|) + \phi(a)\|S_n Q(u) - Q(u)\| \\
&\quad +\alpha_n\langle u - Q(u), J_\phi(x_n - Q(u))\rangle \\
&\leq (1-\alpha_n)\Phi(\|x_{n-1} - Q(u)\|) \\
&\quad +\alpha_n\langle u - Q(u), J_\phi(x_n - Q(u))\rangle + \lambda\phi(a)\|T_n Q(u) - Q(u)\|,
\end{aligned}$$

where a is such that

$$a = \sup_{n\geq 1}\{\|S_n Q(u) - Q(u)\| + \|x_{n-1} - Q(u)\|\} < \infty.$$

Therefore, by using Lemma 1.1.22(b), Step 4 and condition (ii), we conclude that $x_n \to Q(u)$.

□

Remark 1.3.2. Condition (i) in Theorem 1.3.1 equivalently says that the sequence $\{T_n\}$ converges to T uniformly over any bounded subset of C; that is,

$$\lim_{n\to\infty} \sup\{\|T_n x - Tx\| : x \in D\} = 0,$$

where D is any given bounded subset of C. This condition is satisfied if we take T_n as the average of the identity I and T, that is,

$$T_n = \beta_n I + (1-\beta_n)T, \tag{1.3.4}$$

where $\{\beta_n\}$ is a positive null sequence in $(0,1)$. For this choice of $\{T_n\}$, condition (ii) is fulfilled automatically.

We next prove a convergence result for the iterative algorithm (1.3.2) in the setting of uniformly smooth Banach spaces. This setting looks more natural than the setting of Banach spaces which have a weakly continuous duality mapping since the former setting includes both l^p and L^p spaces for $1 < p < \infty$, while the latter excludes L^p for $1 < p < \infty$, $p \neq 2$.

Theorem 1.3.3. *Let X be a uniformly smooth Banach space, C a closed convex subset of X, $T : C \to C$ a nonexpansive mapping such that $Fix(T) \neq \emptyset$, $\{T_n\}$ a sequence of nonexpansive self-mappings defined on C and $\{\alpha_n\}$ a sequence in $[0,1]$. Assume that the following conditions are satisfied.*

(i) $\lim_{n\to\infty} \|T_n y_n - T y_n\| = 0, \ \forall \{y_n\} \subset C$ *bounded;*

(ii) $\sum_{n=0}^{\infty} \|T_n p - Tp\| < \infty \ \forall p \in Fix(T)$*;*

(H1) $\lim_{n\to\infty} \alpha_n = 0$*;*

(H2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ generated by the algorithm (1.3.2) converges strongly to $Q(u)$, where Q is the unique sunny nonexpansive retraction from C onto $Fix(T)$.

Proof. As in the proof of Theorem 1.3.1, we divide this proof into the following steps:

Step 1. $\{x_n\}$ is bounded,

Step 2. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$,

Step 3. $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$,

Step 4. $\limsup_{n\to\infty} \langle u - Q(u), J(x_n - Q(u)) \rangle \leq 0$,

Step 5. $x_n \to Q(u)$ in norm.

Step 1. - **Step 3.** can be similarly proved as in the proof of Theorem 1.3.1. The proof of **Step 4.** is however different since we do not have weak continuity of the duality map, so instead we will tactically utilize the uniform smoothness of X.

We proceed as follows. Recall that z_t satisfies the equation (1.1.18) which leads to the identity $z_t - x_n = t(u - x_n) + (1-t)(Tz_t - x_n)$. Using (1.1.5), we derive that

$$\begin{aligned} \|z_t - x_n\|^2 &\leq (1-t)^2\|Tz_t - x_n\|^2 + 2t\langle u - x_n, J(z_t - x_n)\rangle \\ &\leq (1-t)^2\big(\|z_t - x_n\| + \|Tx_n - x_n\|\big)^2 + \\ &\quad +2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2 \\ &\leq (1+t^2)\|z_t - x_n\|^2 + 2t\langle u - z_t, J(z_t - x_n)\rangle + \\ &\quad +(1-t)^2\|Tx_n - x_n\|\big(\|Tx_n - x_n\| + \|z_t - x_n\|\big). \end{aligned}$$

Therefore, setting

$$\beta_n = \|Tx_n - x_n\|\big(\|Tx_n - x_n\| + \|z_t - x_n\|\big),$$

we get

$$\langle u - z_t, J(x_n - z_t)\rangle \leq \frac{t}{2}\|z_t - x_n\| + \frac{1}{2t}\beta_n. \tag{1.3.5}$$

Besides, by Step 3 and the fact that $\{x_n\}$ and $\{z_t\}$ are bounded, we get

$$\beta_n \to 0 \quad \text{as } n \to \infty \text{ (uniformly in } t \in (0,1)),$$

and

$$\exists\, L > 0 \text{ such that } \|z_t - x_n\| \leq L \;\; \forall t \in (0,1), \forall n \geq 0.$$

It follows from (1.3.5) that

$$\limsup_{n\to\infty}\langle u - z_t, J(x_n - z_t)\rangle \leq \frac{t}{2}L. \tag{1.3.6}$$

Since the normalized duality mapping J is norm-to-norm uniformly continuous on bounded sets of X, we have $J(x_n - z_t) \to J(x_n - Q(u))$ as $t \to 0^+$ uniformly for all $n \geq 0$. This suffices for us to derive from (1.3.6) that

$$\limsup_{n\to\infty}\langle u - Q(u), J(x_n - Q(u))\rangle \leq 0. \tag{1.3.7}$$

Finally we prove **Step 5.**; that is, $x_n \to Q(u)$. Using the subdifferential inequality (1.1.5) and the nonexpansivity of S_n we obtain

$$\begin{aligned}
\|x_n - Q(u)\|^2 &= \|\alpha_n(u - Q(u)) + (1-\alpha_n)(S_n x_{n-1} - Q(u))\|^2 \\
&\leq (1-\alpha_n)\|S_n x_{n-1} - Q(u)\|^2 + 2\alpha_n\langle u - Q(u), J(x_n - Q(u))\rangle \\
&\leq (1-\alpha_n)\Big(\|x_{n-1} - Q(u)\| + \|S_n Q(u) - Q(u)\|\Big)^2 \\
&\quad +2\alpha_n\langle u - Q(u), J(x_n - Q(u))\rangle \\
&= (1-\alpha_n)\|x_{n-1} - Q(u)\|^2 + 2\alpha_n\langle u - Q(u), J(x_n - Q(u))\rangle + \\
&\quad +\|S_n Q(u) - Q(u)\|\Big(\|S_n Q(u) - Q(u)\| + 2\|x_{n-1} - Q(u)\|\Big) \\
&\leq (1-\alpha_n)\|x_{n-1} - Q(u)\|^2 + 2\alpha_n\langle u - Q(u), J(x_n - Q(u))\rangle \\
&\quad +\lambda M\|T_n Q(u) - Q(u)\|,
\end{aligned}$$

where M is such that

$$\sup_{n\geq 0}\{\|S_n Q(u) - Q(u)\| + 2\|x_{n-1} - Q(u)\|\} \leq M,$$

which exists by the boundedness of $\{x_n\}$ and (i). Then, using Lemma 1.1.22(b), by step 4 and (ii) we get $x_n \to Q(u)$.

□

Remark 1.3.4. It is easily seen that the conclusion of Theorem 1.3.3 remains true if the uniform smoothness assumption of X is replaced with the two weaker conditions

(a) X is uniformly Gâteaux differentiable

(b) X has Reich's property.

Note that if the fixed point set of T is contained in the common fixed point set of the mappings $\{T_n\}$, then condition (ii) is trivially satisfied and both Theorems 1.3.1 and 1.3.3 can be rewritten as follows.

Theorem 1.3.5. *Let X be either a reflexive Banach space having a weakly continuous duality mapping J_ϕ or a uniformly smooth Banach space. Let C be a nonempty closed convex subset of X. Let $T : C \to C$ be a nonexpansive mapping and assume that $\{T_n\}$ is a sequence of nonexpansive self-mappings defined on C such that*

(i) $\lim_{n\to\infty} \|T_n y_n - Ty_n\| = 0,\ \forall\{y_n\} \subset C$ *bounded,*

(iii) $\bigcap_{n\geq 0} Fix(T_n) \supseteq Fix(T) \neq \emptyset$.

Let $\{\alpha_n\} \subset [0,1]$ *satisfy* (H1) *and* (H2) *in Theorem* 1.3.1. *Then the sequence* $\{x_n\}$ *generated by the algorithm* (1.3.2) *converges strongly to* $Q(u)$*, where* Q *is the unique sunny nonexpansive retraction from* C *onto* $Fix(T)$.

As a consequence of Theorem 1.3.1 and Theorem 1.3.3 we deduce the following convergence result. It will be applied to approximate a solution to the convex feasibility problems studied in Section 1.4.

Corollary 1.3.6. *Let* X *be either a reflexive Banach space having a weakly continuous duality mapping* J_ϕ *or a uniformly smooth Banach space and* $C \subseteq X$ *a nonempty closed convex set. Assume that* $T : C \to C$ *is a nonexpansive self-mapping such that* $Fix(T) \neq \emptyset$ *and* $\{T_n\}$ *is a sequence of nonexpansive self-mappings defined on* C *satisfying*

$$\sum_{n=0}^{\infty} D_\rho(T_n, T) < \infty,\ \forall \rho > 0, \tag{1.3.8}$$

where

$$D_\rho(T_n, T) = \sup\{\|T_n x - Tx\| : x \in C,\ \|x\| \leq \rho\}.$$

Let $\{\alpha_n\} \subset [0,1]$ *satisfy* (H1) *and* (H2) *in Theorem 1.3.1. Then the sequence* $\{x_n\}$ *generated by the algorithm* (1.3.2) *converges strongly to* $Q(u)$*, where* Q *is the unique sunny nonexpansive retraction from* C *onto* $Fix(T)$.

Proof. It suffices to prove that (1.3.8) implies conditions (i) and (ii) in Theorem 1.3.1.

(i) Let $\{y_n\}$ be a bounded sequence. Then there exists a constant $\rho > 0$ such that $\|y_n\| \leq \rho$, $\forall n \geq 0$. Therefore

$$\lim_{n\to\infty} \|T_n y_n - Ty_n\| \leq \lim_{n\to\infty} \sup_{\|x\|\leq\rho} \|T_n x - Tx\| = \lim_{n\to\infty} D_\rho(T_n, T) = 0.$$

(ii) Obviously, for every $p \in \mathrm{Fix}\,(T)$,

$$\sum_{n=0}^{\infty} \|T_n p - Tp\| \leq \sum_{n=0}^{\infty} \sup_{\|x\|\leq\|p\|} \|T_n x - Tx\| = \sum_{n=0}^{\infty} D_{\|p\|}(T_n, T) < \infty.$$

□

1.4 Applications

1.4.1 Zeros of accretive operators

Let X be a real Banach space and $A : X \to 2^X$ a set-valued m-accretive operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ in X. The problem of finding a zero of A in $\mathcal{D}(A)$ consist of

$$\text{finding } z \in \mathcal{D}(A) \text{ such that } 0 \in Az. \tag{1.4.1}$$

When $C = \overline{\mathcal{D}(A)}$ is convex, this problem has extensively been investigated due to its many applications in solving related problems; for instance, minimization problems, variational inequality problems and nonlinear evolution equations. Let us denote the zero set of A by

$$A^{-1}(0) = \{z \in \mathcal{D}(A) : 0 \in Az\}.$$

By Theorem 1.1.17, we know that the resolvent of A is a firmly nonexpansive mapping from X to $\mathcal{D}(A)$. It is straightforward to see that $A^{-1}(0)$ coincides with the fixed point set of J_λ, for any $\lambda > 0$. Therefore an interesting approach for solving the problem of finding a zero of A is via iterative methods for approximating a fixed point of nonexpansive mappings.

As a consequence of the convergence of the implicit iterative scheme (1.2.8) presented in Section 1.2, we obtain Reich's result (cf. [95]) for approximating zeros of accretive operators in uniformly smooth Banach spaces. However, the following theorem in the setting of reflexive Banach spaces with weakly continuous duality mapping constitutes a new approach.

Theorem 1.4.1. *Let A be an m-accretive operator in a reflexive Banach space X with a weakly continuous duality mapping J_ϕ. Then, for each $x \in X$, the sequence $\{J_\lambda(x)\}$ converges strongly, as $\lambda \to \infty$, to the unique zero of A, $q \in A^{-1}(0)$, which satisfies the variational inequality*

$$\langle x - q, J_\phi(y - q)\rangle \leq 0 \ \ \forall y \in A^{-1}(0). \tag{1.4.2}$$

Proof. Given $x \in X$ we consider the approximating curve $\{x_t\}$ such that $x_t = J_{1/t}x$, for any $t \in (0, 1)$. By definition of the resolvent of A, we obtain the following equivalence:

$$\begin{aligned} x_t = (I + \frac{1}{t}A)^{-1}x \quad &\Leftrightarrow \quad x \in x_t + \frac{1}{t}Ax_t \\ &\Leftrightarrow \quad t(x - x_t) \in Ax_t \\ &\Leftrightarrow \quad x_t + t(x - x_t) \in (I + A)x_t \\ &\Leftrightarrow \quad x_t = (I + A)^{-1}(x_t + t(x - x_t)) \\ &\Leftrightarrow \quad x_t = T(tf(x_t) + (1 - t)x_t), \end{aligned}$$

where $T = (I + A)^{-1}$ is the nonexpansive resolvent of order 1, and $f = x$ is a constant mapping which is a contraction. Therefore, Theorem 1.2.1 implies the strong convergence of $\{x_t\}$, as $t \to 0$, to the unique solution to the inequality (1.2.9); in other words, $\{J_\lambda x\}$ strongly converges, as $\lambda \to \infty$, to the unique zero $q \in A^{-1}(0)$ which is solution to the inequality (1.4.2). □

Remark 1.4.2. If we define the mapping $Q : X \to A^{-1}(0)$ such that, for any $x \in X$,

$$Qx = \lim_{\lambda \to \infty} J_\lambda x,$$

then, since Qx satisfies the inequality (1.4.2), by Lemma 1.1.9 we can claim that Q is the unique sunny nonexpansive retraction from X to $A^{-1}(0)$.

In [38], the authors studied a different iterative method for m-accretive operators in a uniformly smooth Banach space with a weakly continuous duality mapping. They proved the strong convergence of the Halpern type algorithm

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)J_{r_n}x_n, \; n \geq 0, \tag{1.4.3}$$

where $\{\alpha_n\}$ satisfies conditions (1) and (2) in Section 1.1.2, and $\{r_n\} \subset (0, +\infty)$ is such that

$$\lim_{n \to \infty} r_n = \infty.$$

The inspiration for this method is Rockafellar's proximal point algorithm, formulated by

$$x_{n+1} = J_{r_n}x_n, \; n \geq 0,$$

for maximal monotone operators in Hilbert spaces [103]. Early results on the proximal point algorithm in Banach spaces (which take into account computational errors) can be found in [20, 85]. More recent results were obtained in [6].

Xu, in [125], presented an improvement of the previous result by removing either the uniform smoothness of X or the assumption of a weakly continuous duality mapping.

Using Theorem 1.3.5, we get strong convergence of a modified algorithm under better conditions. To this end, we need the following lemma.

Lemma 1.4.3. *(Miyadera [77]) Let $A : \mathcal{D}(A) \to 2^X$ be an accretive operator. If $x \in \mathcal{D}(J_\lambda)$, then, for any $\lambda > 0$ and any $\mu > 0$,*

$$\frac{\mu}{\lambda}x + \big(1 - \frac{\mu}{\lambda}\big)J_\lambda x \in \mathcal{D}(J_\mu)$$

and

$$J_\lambda x = J_\mu\big(\frac{\mu}{\lambda}x + \big(1 - \frac{\mu}{\lambda}\big)J_\lambda x\big).$$

Theorem 1.4.4. *Let X be either a reflexive Banach space having a weakly continuous duality mapping J_ϕ or a uniformly smooth Banach space. Let A be an m-accretive operator with $A^{-1}(0) \neq \emptyset$. If $\{x_n\}$ is the sequence generated by the algorithm*

$$x_{n+1} = \alpha_{n+1}u + (1 - \alpha_{n+1})((1 - \lambda)x_n + \lambda J_{r_{n+1}}x_n),\ n \geq 0, \tag{1.4.4}$$

where $\lambda > 0$, the sequence $\{\alpha_n\}$ satisfies conditions (H1) *and* (H2) *in Theorem* 1.3.1 *and*

$$\lim_{n\to\infty} r_n = r \in (0, +\infty),$$

then $\{x_n\}$ converges strongly to $x \in A^{-1}(0)$.

Proof. Since A is m-accretive, by Theorem 1.1.17 we have that the resolvent J_λ is firmly nonexpansive for any $\lambda > 0$, in particular, nonexpansive. Then define, for any $n > 0$, $T_n = J_{r_n}$ and $T = J_r$. Since $\mathrm{Fix}\,(J_{r_n}) = \mathrm{Fix}\,(J_r)$ for all $n \geq 0$, we have

$$\bigcap_{n\geq 0} \mathrm{Fix}\,(T_n) = \mathrm{Fix}\,(T) = A^{-1}(0) \neq \emptyset.$$

We next verify that condition (i) of Theorem 1.3.5 holds. Indeed, given $\{y_n\}$ bounded, since J_{r_n} is nonexpansive, we can find a constant $\rho > 0$ such that

$$\sup_{n\geq 0}\{\|y_n\| + \|J_{r_n}y_n\|\} \leq \rho.$$

By using Lemma 1.4.3 and the nonexpansivity of the resolvent, we obtain

$$\begin{aligned}
\|T_n y_n - T y_n\| &= \|J_{r_n}y_n - J_r y_n\| \\
&= \|J_{r_n}y_n - J_{r_n}\big(\frac{r_n}{r}y_n + \big(1-\frac{r_n}{r}\big)J_r y_n\big)\| \\
&\leq \|y_n - \big(\frac{r_n}{r}y_n + \big(1-\frac{r_n}{r}\big)J_r y_n\big)\| \\
&\leq (1-\frac{r_n}{r})\|y_n\| + (1-\frac{r_n}{r})\|J_r y_n\| \\
&\leq (1-\frac{r_n}{r})\rho.
\end{aligned}$$

Since $r_n \to r$, as $n \to 0$, we obtain

$$\lim_{n\to\infty}\|T_n y_n - T y_n\| = 0.$$

Therefore we can now apply Theorem 1.3.5 to prove the strong convergence of $\{x_n\}$. □

Remark 1.4.5. The previous result remains true in a reflexive Banach space with a uniformly Gâteaux differentiable norm and Reich's property.

Kamimura and Takahashi, in [55], provided a perturbed version with errors of the iteration (1.4.3) for a maximal monotone operator A in a Hilbert space H with $\mathcal{D}(A) = H$. They proved the strong convergence of the sequence defined by the algorithm

$$x_{n+1} = \alpha_n u + (1-\alpha_n)y_n,\ n \geq 0,$$

where $\|y_n - J_{r_n}x_n\| \leq e_n$ with $\sum_{n\geq 0} e_n < \infty$, and $r_n \to \infty$. In this regard, by Theorems 1.3.1 and 1.3.3 we have the following result.

Theorem 1.4.6. *Let X, A, λ, $\{\alpha_n\}$ and $\{r_n\}$ be as in Theorem* 1.4.4. *Let the sequence $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} = \alpha_{n+1}u + (1-\alpha_{n+1})\big((1-\lambda)x_n + \lambda T_{n+1}x_n\big),\ n \geq 0, \tag{1.4.5}$$

where for each $n \geq 0$, T_n is given by

$$T_n = J_{r_n} + e_n$$

and the sequence of errors $\{e_n\}$ satisfies the condition

$$\sum_{n \geq 0} \|e_n\| < \infty.$$

Then $\{x_n\}$ converges strongly to a point of $A^{-1}(0)$.

Proof. Let $T = J_r = (I + rA)^{-1}$.

We need to prove that conditions (i) and (ii) in Theorem 1.3.1 hold.

(i) Given a bounded sequence $\{y_n\}$, we have

$$\|T_n y_n - T y_n\| = \|J_{r_n} y_n + e_n - J_r y_n\| \leq \|J_{r_n} y_n - J_r y_n\| + \|e_n\|.$$

In Theorem 1.4.4 we proved that $\lim_{n\to\infty} \|J_{r_n} x_n - J_r x_n\| = 0$. This, together with the hypothesis $\lim_{n\to\infty} \|e_n\| = 0$, implies that

$$\lim_{n\to\infty} \|T_n y_n - T y_n\| = 0.$$

(ii) Let $p \in \mathrm{Fix}\,(T)$. Since $\mathrm{Fix}\,(J_{r_n}) = \mathrm{Fix}\,(J_r) = \mathrm{Fix}\,(T)$, for all $n \geq 0$,

$$\|T_n p - Tp\| = \|J_{r_n} p + e_n - J_r p\| = \|e_n\|.$$

Hence

$$\sum_{n=0}^{\infty} \|T_n p - Tp\| = \sum_{n=0}^{\infty} \|e_n\| < \infty.$$

Therefore, the strong convergence of the sequence $\{x_n\}$ generated by (1.4.5) is an immediate consequence of Theorems 1.3.1 and 1.3.3. □

In a Hilbert space H, the concept of maximal monotone operators coincides with that of m-accretive operators. If we consider the problem of minimizing a proper lower semicontinuous convex function $f : H \to \overline{\mathbb{R}}$, it is known that

$$z \in \arg\min_{x \in H} f(x) \Leftrightarrow 0 \in \partial f(z),$$

and that $T = \partial f$ is a maximal monotone operator. Then the iteration scheme (1.4.4) is a method for minimizing f.

1.4.2 Variational inequality problems

In this section let us assume that the underlying space H is a Hilbert space and $C \subseteq H$ is a closed convex set. Given a monotone operator $A : H \to H$, the variational inequality problem VIP (A, C) consists of finding $p \in C$ such that

$$\langle Ap, p - x \rangle \leq 0, \ \ \forall x \in C. \tag{1.4.6}$$

If $f : H \to \overline{\mathbb{R}}$ is a lower semicontinuous convex function, a necessary and sufficient condition for the constrained convex minimization problem

$$\min_{x \in C} f(x), \tag{1.4.7}$$

is the VIP (A, C), where the operator A is the subdifferential of f, ∂f, which is a maximal monotone operator by Theorem 1.1.19. This means that solving the minimization problem (1.4.7) is equivalent to finding a solution of a variational inequality. Thus, in order to solve a broad range of convexly constrained nonlinear inverse problems in real Hilbert space, Yamada [117] presented an hybrid steepest descent method for approximating solutions to the variational inequality problem VIP$(g, \mathrm{Fix}\,(T))$, for an operator g and the fixed point set of a nonexpansive mapping $T : C \to C$. In particular, he proved that, when g is strongly monotone and Lipschitz continuous, the sequence $\{x_n\}$ defined by the algorithm

$$x_{n+1} = Tx_n - \alpha_n g(Tx_n), \quad n \geq 0, \tag{1.4.8}$$

converges strongly to the fixed point of T, q, which is the unique solution to the inequality

$$\langle g(q), x - q \rangle \geq 0 \ \ \forall x \in \mathrm{Fix}\,(T). \tag{1.4.9}$$

As a consequence of Corollary 1.2.4 we get a more general result.

Theorem 1.4.7. *Let $T : C \to C$ be a nonexpansive mapping with $Fix\,(T) \neq \emptyset$ and $g : C \to C$ a mapping such that $I - \mu g$ is a contraction for some $\mu > 0$. Assume that $\{\alpha_n\}$ is a sequence in $[0, 1]$ satisfying hypotheses*

(H1) $\lim_{n\to\infty} \alpha_n = 0$*;*

(H2) $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(H3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \dfrac{\alpha_n}{\alpha_{n+1}} = 1.$

Then the sequence $\{x_n\}$ *defined by the iterative scheme* (1.4.8) *converges strongly to the unique solution* $q \in Fix(T)$ *of the VIP* $(g, Fix(T))$ (1.4.9).

Indeed this theorem improves Yamada's result since for any Lipschitz and strongly monotone operator g there exists $\mu > 0$ such that $I - \mu g$ is a contraction as it is stated in the following lemma.

Lemma 1.4.8. *Let* $A : H \to H$ *be a single-valued* L*-Lipschitz and* η*-strongly monotone operator and* ψ *a* ρ*-contraction. Then,*

(i) *for any* $0 < \mu < 2\eta/L^2$*, the mapping* $I - \mu A$ *is an* α*-contraction with* $\alpha = \sqrt{1 - \mu(2\eta - \mu L^2)}$*;*

(ii) *for any* $\gamma < \eta/\rho$*, the mapping* $A - \gamma\psi$ *is* R*-Lipschitz and* δ*-strongly monotone with* $R = L + \gamma\rho$ *and* $\delta = \eta - \gamma\rho$*.*

Proof.

(i) By applying the L-Lipschitz continuity and η-strong monotonicity of A we obtain

$$\begin{aligned} \|(I - \mu B)x - (I - \mu B)y\|^2 &= \|x - y\|^2 + \mu^2\|Bx - By\|^2 \\ &\quad -\mu\langle x - y, Bx - By\rangle \\ &\leq \|x - y\|^2 + \mu^2 L^2\|x - y\|^2 - 2\mu\eta\|x - y\|^2 \\ &= (1 - \mu(2\eta - \mu L^2))\|x - y\|^2. \end{aligned}$$

Then, for any $0 < \mu < 2\eta/L^2$, the mapping $I - \mu A$ is a contraction with constant $\sqrt{1 - \mu(2\eta - \mu L^2)}$.

(ii) Since A is L-Lipschitz and ψ is a ρ-contraction,

$$\|(A - \gamma\psi)x - (A - \gamma\psi)y\| \leq \|Ax - Ay\| + \gamma\|\psi x - \psi y\| \leq (L + \gamma\rho)\|x - y\|,$$

that is, $A-\gamma\psi$ is Lipschitz with constant $R = L+\gamma\rho$. The strong monotonicity of $A-\gamma\psi$ is consequence of the strong monotonicity of A as it is showed as follow.

$$\begin{aligned}
\langle (A-\gamma\psi)x-(A-\gamma\psi)y, x-y\rangle &= \langle Ax-Ay, x-y\rangle - \gamma\langle \psi x-\psi y, x-y\rangle \\
&\geq \eta\|x-y\|^2 - \gamma\|\psi x-\psi y\|\|x-y\| \\
&\geq (\eta-\gamma\rho)\|x-y\|^2,
\end{aligned}$$

where $\delta = \eta-\gamma\rho > 0$.

□

Let us consider now the following particular variational inequality problem. Let $T : H \to H$ be a nonexpansive mapping with $\mathrm{Fix}\,(T) \neq \emptyset$, $\psi : H \to H$ be a contraction and A be a Lipschitz self-operator on H which is strongly monotone. Then the VIP $(A-\gamma\psi, \mathrm{Fix}\,(T))$

$$\langle (A-\gamma\psi)q, q-x\rangle \leq 0, \ \ \forall x \in \mathrm{Fix}\,(T), \tag{1.4.10}$$

where $\gamma > 0$, is the optimality condition for the minimization problem

$$\min_{x\in \mathrm{Fix}\,(T)} f(x) - h(x)$$

where f is a differentiable function with subdifferential $\partial f = A$ and h is a potential function for $\gamma\psi$ (i.e. $h'(x) = \gamma\psi(x)$ for $x \in H$). Marino and Xu [73] presented an iterative method to solve the VIP $(A-\gamma\psi, \mathrm{Fix}\,(T))$ for a linear bounded operator. Lemma 1.4.8 and Theorem 1.4.7 allow us to apply the hybrid steepest descent method (1.4.8) to solve such variational inequality dispensing with the linear condition on the operator A.

Theorem 1.4.9. *Let T be a nonexpansive mapping with nonempty fixed point set $Fix\,(T)$, A an L-Lipschitz and η-strongly monotone operator and ψ a ρ-contraction on a Hilbert space. Then, for any $0 < \gamma < \eta/\rho$, the sequence $\{x_n\}$ defined by the iterative scheme*

$$x_{n+1} = Tx_n - \alpha_n(A-\gamma\psi)Tx_n,$$

where $\{\alpha_n\} \subset [0,1]$ satisfies hypotheses (H1)-(H3) *in Theorem* 1.4.7*, converges strongly to the unique solution to the variational inequality* (1.4.10)*.*

Proof. Note that, for any $0 < \gamma < \eta/\rho$, Lemma 1.4.8 implies that there exists $\mu > 0$ such that $I - \mu g$ is a contraction, where $g = A - \gamma\psi$. Then, by Theorem 1.4.7 we obtain the strong convergence of the sequence $\{x_n\}$ to the unique solution to the variational inequality (1.4.10).

□

1.4.3 Multiple-set split feasibility problem

The intensity-modulated radiation therapy (IMRT) is an advanced mode of high-precision radiotherapy that utilizes computer-controlled linear accelerators to deliver precise radiation doses to a malignant tumor or specific areas within the tumor. Two problems are pertinent to this medical treatment. The first one is to calculate the radiation dose absorbed in the irradiated tissue based on a given distribution of beamlet intensities. The second one is the inverse problem of the first one, that is to find a distribution of radiation intensities (radiation intensity map) deliverable by all beamlets which would result in a clinically acceptable dose distribution; that is, the dose to each tissue should be within the desired upper and lower bounds which are prescribed based on medical diagnosis, knowledge and experience. The latter which has received a great deal of attention recently can mathematically be formulated as a multiple-sets split feasibility problem (MSSFP); see [24, 22, 25] and references therein. Our aim is to provide a theoretical background of algorithmic developments and convergence results for iteratively solving the MSSFP by means of optimization and fixed point approaches. Different methods and aspects of the problem, such as random iterations, minimum-norm solutions or perturbation techniques, will be investigated.

For the sake of generality we consider the multiple-sets split feasibility problem (MSSFP) in general Hilbert spaces (not necessarily finite-dimensional). Thus the MSSFP is formulated as finding a point x^* with the property

$$x^* \in C := \bigcap_{i=1}^{N} C_i \quad \text{and} \quad Ax^* \in Q := \bigcap_{j=1}^{M} Q_j \tag{1.4.11}$$

where $N, M \geq 1$ are integers, $\{C_i\}_{i=1}^N$ and $\{Q_j\}_{j=1}^M$ are closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. This problem was first investigated by Censor et al. in order to model the inverse problem of IMRT; see [22, 24, 25].

The case where $N = M = 1$, called split feasibility problem (SFP), was introduced by Censor and Elfving [23], modeling phase retrieval and other image restoration problems, and further studied by many researchers; see, for instance, [21, 130, 124].

From now on, we assume that MSSFP is consistent, i.e., it is solvable, and S denotes its solution set, otherwise it will be pointed out.

The Gradient-Projection Method

Consider the convex minimization problem

$$\min_{x \in C} f(x) \tag{1.4.12}$$

where C is a closed convex subset of a Hilbert space H and $f : C \to \mathbb{R}$ is a differentiable convex function, with gradient ∇f. The convexity of f implies the monotonicity of ∇f.

For this smooth convex minimization problem, as we mention in the previous Section 1.4.2, a necessary and sufficient condition so that a point $x^* \in C$ is an optimal solution to (1.4.12) is the following variational inequality

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad x \in C. \tag{1.4.13}$$

That is, the VIP $(\nabla f, C)$ is equivalent to the minimization problem (1.4.12).

If we consider the mapping $T = P_C(I - \gamma \nabla f)$ for some $\gamma > 0$, from the inequality (1.1.11) which characterizes the projection P_C, we deduce that x^* is a fixed point of T if and only if

$$\langle x - x^*, x^* - (x^* - \gamma \nabla f(x^*)) \rangle = \gamma \langle x - x^*, \nabla f(x^*) \rangle \geq 0.$$

Then, likewise the VIP $(\nabla f, C)$ (1.4.13) is equivalent to the fixed point problem for T.

Lemma 1.4.10. *Let $f : C \to \mathbb{R}$ be a differentiable convex function such that its gradient ∇f is an L-Lipschitz mapping. Then, for any $0 < \gamma < 2/L$, the mapping $T = P_C(I - \gamma \nabla f)$ is $(2 + \gamma L)/4$-averaged.*

Proof. If ∇f is L-Lipschitz, Lemma 1.1.20 implies that ∇f is $1/L$-ism and from Proposition 1.1.18 we deduce that $I - \gamma\nabla f$ is $\gamma L/2$-averaged. Thus, since the projection P_C is firmly-nonexpansive, from Proposition 1.1.7 we obtain that the composition is averaged. In fact, the mapping T is $(2 + \gamma L)/4$-averaged. □

This lemma allows us to use the fixed point iterative methods for averaged mappings to approximate a minimizer of the function f. In particular, from this fact it was born the *gradient-projection method* (GPM) which generates a sequence $\{x_n\}$ via the iterative algorithm

$$x_{n+1} = P_C(I - \gamma_n \nabla f)x_n \tag{1.4.14}$$

where the initial guess $x_0 \in C$ is arbitrarily chosen and $\{\gamma_n\}$ is a sequence of positive stepsizes.

Theorem 1.4.11. *Assume that the minimization problem* (1.4.7) *has a solution. If the gradient ∇f of f is L-Lipschitz and the sequence $\{\gamma_n\}$ satisfies the property*

$$0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < \frac{2}{L},$$

then the sequence $\{x_n\}$ generated by the GPM (1.4.14) *converges weakly to a minimizer of the function f.*

For the proof of this theorem, the reader can consult [89] for the case of finite-dimensional Hilbert spaces and [126] for the general case of infinite-dimensional Hilbert spaces.

Optimization Approach

The MSSFP consists of finding a point x^* satisfying two properties:

(i) the distance $d(x^*, C_i) = 0$ for all $i = 1, \cdots, N$;

(ii) the distance $d(Ax^*, Q_j) = 0$ for all $j = 1, \cdots, M$.

This motivated Censor et al. [24] to consider the minimization problem (1.4.12) for the proximity function

$$\begin{aligned} f(x) &= \frac{1}{2}\sum_{i=1}^{N}\alpha_i d^2(x, C_i) + \frac{1}{2}\sum_{j=1}^{M}\beta_j d^2(Ax, Q_j) \\ &= \frac{1}{2}\sum_{i=1}^{N}\alpha_i \|x - P_{C_i}x\|^2 + \frac{1}{2}\sum_{j=1}^{M}\beta_j \|Ax - P_{Q_j}Ax\|^2, \end{aligned} \tag{1.4.15}$$

where $\{\alpha_i\}$ and $\{\beta_j\}$ are positive real numbers, and P_{C_i} and P_{Q_j} are the metric projections onto C_i and Q_j, respectively.

It is evident that x^* is a solution to the MSSFP (1.4.11) if and only if $f(x^*) = 0$; that is, if x^* is a minimizer of f over H_1, since $f(x) \geq 0$ for all $x \in H_1$.

The proximity function f is convex and differentiable with gradient

$$\nabla f(x) = \sum_{i=1}^{N}\alpha_i (I - P_{C_i})x + \sum_{j=1}^{M}\beta_j A^*(I - P_{Q_j})Ax, \tag{1.4.16}$$

where A^* is the adjoint of A. See [24] for details.

Since, for every closed convex subset K of a Hilbert space, $I - P_K$ is nonexpansive, we get that the gradient $\nabla f(x)$ is L-Lipschitz continuous with constant

$$L = \sum_{i=1}^{N}\alpha_i + \sum_{j=1}^{M}\beta_j \|A\|^2. \tag{1.4.17}$$

Therefore, we can use the gradient-projection method (1.4.14) to solve the constraint minimization problem

$$\min_{\Omega} f(x) \tag{1.4.18}$$

where Ω is a closed convex subset of H_1 whose intersection with the solution set of the MSSFP is nonempty, and get a solution of the so-called *constrained multiple-sets split feasibility problem* (CMSSFP)

$$x^* \in \Omega \text{ such that } x^* \text{ solves (1.4.11).} \tag{1.4.19}$$

No matter we are dealing with either the MSSFP or the CMSSFP, S will also denote the solution set.

Theorem 1.4.12. *Define a sequence* $\{x_n\}$ *by the gradient-projection algorithm as follows*

$$x_{n+1} = P_\Omega \left(x_n - \gamma_n \left(\sum_{i=1}^{N} \alpha_i (I - P_{C_i}) x_n + \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j}) A x_n \right) \right), \quad (1.4.20)$$

where the sequence $\{\gamma_n\}$ *of stepsizes is such that*

$$0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < \frac{2}{L} \quad (1.4.21)$$

with L *given by* (1.4.17)*. Then* $\{x_n\}$ *converges weakly to a solution to the CMSSFP* (1.4.19)*.*

Proof. Since the algorithm (1.4.20) can be rewritten as

$$x_{n+1} = P_\Omega(x_n - \gamma_n \nabla f(x_n)), \quad (1.4.22)$$

where the mapping ∇f is L-Lipschitz, condition (1.4.21) allows us to apply Theorem 1.4.11 to conclude that the sequence $\{x_n\}$ converges weakly to a minimizer of f over Ω which is a solution to the CMSSFP. □

Remark 1.4.13. Censor et al. [24] considered the algorithm (1.4.20) in the case of constant stepsizes, $\gamma_n = \gamma$ for all $n \ge 0$.

Remark 1.4.14. Recall that the solution set $S = \text{Fix}\,(T)$ where $T : C \to C$ is the mapping $T = P_C(I - \gamma\nabla f)$. If ∇f is an L-Lipschitz mapping and $0 < \gamma < 2/L$, then T is nonexpansive by Lemma 1.4.10. Therefore, since the fixed point set of a nonexpansive mapping defined on a closed convex set is closed and convex, so is S.

Fixed Point Approach

For the special case where $N = M = 1$, that is the split feasibility problem (SFP), it is known that x^* is a solution of the SFP if and only if x^* solves the fixed point equation

$$x^* = P_{C_1}(I - \gamma A^*(I - P_{Q_1})A)x^* \quad (1.4.23)$$

where $\gamma > 0$ is any parameter.

It occurs to us that the MSSFP (1.4.11) is equivalent to a common fixed point problem of finitely many nonexpansive mappings, as we show below.

We decompose the MSSFP into the following N subproblems. For any $1 \leq i \leq N$, we want to find

$$x_i^* \in C_i \quad \text{and} \quad Ax_i^* \in Q := \bigcap_{j=1}^{M} Q_j. \tag{1.4.24}$$

In order to solve these subproblems, we define the function $g : H_1 \to \mathbb{R}^+$ by

$$g(x) = \frac{1}{2} \sum_{j=1}^{M} \beta_j \|Ax - P_{Q_j} Ax\|^2 \tag{1.4.25}$$

with $\beta_j > 0$ for all $1 \leq j \leq M$. Note that this function is a particular case of the proximity function (1.4.15). Thus it is convex and differentiable with gradient

$$\nabla g(x) = \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})Ax, \tag{1.4.26}$$

which is L'-Lipschitz continuous with constant

$$L' = \sum_{j=1}^{M} \beta_j \|A\|^2. \tag{1.4.27}$$

Then, if we define the mapping $T_i : H_1 \to H_1$ by

$$T_i = P_{C_i}(I - \gamma_i \nabla g) = P_{C_i}\left(I - \gamma_i \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})A\right), \tag{1.4.28}$$

for each $1 \leq i \leq N$, the solution set S_i of the subproblem (1.4.24) coincides with $\mathrm{Fix}\,(T_i)$, and therefore the solution set S of the MSSFP coincides with the common fixed point set of the mappings T_i's; that is,

$$S = \bigcap_{i=1}^{N} \mathrm{Fix}\,(T_i).$$

Therefore algorithms for finding common fixed points of a finite family of nonexpansive mappings may apply to solve the MSSFP. In particular, from Lemma 1.4.10 we deduce that, if $0 < \gamma_i < 2/L'$, T_i is $(2+\gamma_i L')/4$-averaged, for any $1 \leq i \leq N$. Therefore, by Proposition 1.1.7 we have that the composition and the weighted sum of the finitely family $\{T_i\}_{1\leq i\leq N}$ are averaged mappings. Moreover,

$$S = \mathrm{Fix}\,(T_N \cdots T_2 T_1) = \mathrm{Fix}\left(\sum_{i=1}^{N} \lambda_i T_i\right), \tag{1.4.29}$$

where $\{\lambda_i\}_{i=1}^N$ is a set of real numbers in $(0,1]$ satisfying $\sum_{i=1}^N \lambda_i = 1$. Thus the well-known weak convergence of Mann iteration (1.1.16) implies the validity of some simple iterative methods. For instance,

(i) the composition iteration

$$x_{n+1} = T_N \cdots T_2 T_1 x_n; \tag{1.4.30}$$

(ii) the parallel iteration

$$x_{n+1} = \sum_{i=1}^{N} \lambda_i T_i x_n = \sum_{i=1}^{N} \lambda_i P_{C_i}\left(I - \gamma_i \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})A\right) x_n, \tag{1.4.31}$$

where $\{\lambda_i\}_{i=1}^N$ is a set of real numbers in $(0,1]$ satisfying $\sum_{i=1}^N \lambda_i = 1$;

(iii) the cyclic iteration

$$x_{n+1} = T_{[n+1]} x_n = P_{C_{[n+1]}}\left(I - \gamma_{n+1} \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})A\right) x_n, \tag{1.4.32}$$

where, for each $n \geq 0$, $T_{[n]} = T_{n \bmod N}$ with the mod function taking values in $\{1, 2, \cdots, N\}$.

Theorem 1.4.15. *(Xu* [124]*) The sequence $\{x_n\}$ generated by any one of the algorithms* (1.4.30)-(1.4.32) *converges weakly to a solution to the MSSFP.*

Approach by random iterations

Let T_i be defined as in (1.4.28) for any $1 \leq i \leq N$. We assume that $0 \leq \gamma_i \leq 2/L'$ so that each T_i is an averaged mapping by Lemma 1.4.10. Then we know that the solution set S of the MSSFP is the common fixed point set of these mappings, i.e.,

$$S = \bigcap_{i=1}^{N} \mathrm{Fix}\,(T_i).$$

In previous subsections we have developed some iterative algorithms that converge weakly to a solution to the MSSFP in the deterministic sense that each mapping T_i is repeated regularly. The purpose of this section is to present a random iteration process in which each T_i may occur irregularly; the only requirement is that each T_i has to occur infinitely many times in the full process of iterations.

Let $r : \mathbb{N} \to \{1, 2, \cdots, N\}$ be a mapping from the set $\mathbb{N}$ of positive integers to the index set $\{1, 2, \cdots, N\}$ such that it assumes each value infinitely often. Define a sequence $\{x_n\}$ by the random iterative algorithm

$$x_{n+1} = T_{r(n)} x_n, \tag{1.4.33}$$

where $x_0 \in H_1$ is arbitrary. The convergence of this random iteration algorithm is given below. Related convergence of random products of nonexpansive mappings can be found in literature, for instance, [112, 39].

Theorem 1.4.16. *If H_1 is finite-dimensional, then the sequence $\{x_n\}$ generated by the random iteration algorithm* (1.4.33) *converges to a solution to the MSSFP.*

Proof. Let $p \in S$ the solution set of the MSSFP. Since p is a common fixed point of the mappings T_i's, which are nonexpansive, we get

$$\|x_{n+1} - p\| = \|T_{r(n)} x_n - p\| \leq \|x_n - p\|.$$

Hence, $\{x_n\}$ is Fejér monotone with respect to S; therefore, by Lemma 1.1.24, $\{x_n\}$ is bounded. Put

$$a = \lim_{n \to \infty} \|x_n - p\|.$$

Since H_1 is finite-dimensional, $\{x_n\}$ contains convergent subsequences. Let $\{x_{n_j}\}$ be such a convergent subsequence with limit $\hat{x}$, that is, $\hat{x}$ is a cluster point of $\{x_n\}$. Since the pool of mappings is finite, we may further assume, without loss of generality, that

$$T_{r(n_j)} = T_r, \quad \text{for some } 1 \le r \le N.$$

It follows that

$$x_{n_j+1} = T_{r(n_j)} x_{n_j} = T_r x_{n_j} \to T_r \hat{x}.$$

Hence

$$\lim_{n\to\infty} \|x_n - p\| = \lim_{j\to\infty} \|x_{n_j+1} - p\| = \|T_r \hat{x} - p\|.$$

On the other hand, we also have

$$\lim_{n\to\infty} \|x_n - p\| = \lim_{j\to\infty} \|x_{n_j} - p\| = \|\hat{x} - p\|.$$

So we must have

$$\|T_r \hat{x} - p\| = \|\hat{x} - p\|. \tag{1.4.34}$$

Since T_r is averaged, $T_r = (1-\alpha)I + \alpha V$, where $\alpha \in (0,1)$ and V is nonexpansive. Noting $\mathrm{Fix}\,(T_r) = \mathrm{Fix}\,(V)$ and (1.4.34), we obtain

$$\begin{aligned}
\|\hat{x} - p\|^2 &= \|T_r \hat{x} - p\|^2 \\
&= \|(1-\alpha)(\hat{x} - p) + \alpha(V\hat{x} - p)\|^2 \\
&= (1-\alpha)\|\hat{x} - p\|^2 + \alpha\|V\hat{x} - p\|^2 - \alpha(1-\alpha)\|\hat{x} - V\hat{x}\|^2 \\
&\le \|\hat{x} - p\|^2 - \alpha(1-\alpha)\|\hat{x} - V\hat{x}\|^2.
\end{aligned}$$

This implies that

$$\alpha(1-\alpha)\|\hat{x} - V\hat{x}\|^2 \le 0 \quad \Rightarrow \quad \hat{x} = V\hat{x}.$$

Therefore, $\hat{x} \in \mathrm{Fix}\,(T_r)$.

We next show that $\hat{x}$ is indeed a common fixed point of the mappings T_i's, that is, $\hat{x} \in S$. We reason by contradiction. Suppose on the contrary that $\hat{x}$ is not a common fixed point. Then, after renumbering the mappings T_i's, we may assume that there is an integer $2 \le k \le N$ such that

$$T_i \hat{x} = \hat{x} \text{ for } i < k, \quad T_i \hat{x} \ne \hat{x} \text{ for } i \ge k. \tag{1.4.35}$$

For each j, since $\{r(n)\}$ takes each value of the index set $\{1, 2, \cdots, N\}$ infinitely often, there is a minimal $m_j \geq n_j$ such that $r(l) < k$ for any $n_j \leq l \leq m_j - 1$ and $r(m_j) = k$. Then, by the nonexpansivity of each T_i, it follows that

$$\begin{aligned} \|x_{m_j} - \hat{x}\| &= \|T_{r(m_j-1)}x_{m_j-1} - \hat{x}\| \\ &\leq \|x_{m_j-1} - \hat{x}\| \leq \cdots \\ &\leq \|x_{n_j} - \hat{x}\| \to 0. \end{aligned}$$

Hence

$$x_{m_j} \to \hat{x}.$$

Without loss of generality (extracting a further subsequence if necessary), we may assume that

$$r(m_j) = k \text{ for all } j.$$

Hence, $x_{m_j+1} = T_{r(m_j)x_{m_j}} = T_k x_{m_j} \to T_k\hat{x}$. It follows that

$$a = \lim_{n\to\infty} \|x_n - p\| = \lim_{j\to\infty} \|x_{m_j+1} - p\| = \|T_k\hat{x} - p\|.$$

In the meanwhile, we have

$$a = \lim_{j\to\infty} \|x_{m_j} - p\| = \|\hat{x} - p\|.$$

Therefore,

$$\|T_k\hat{x} - p\| = \|\hat{x} - p\|.$$

Following the same reasoning used for (1.4.34) we deduce that $T_k\hat{x} = \hat{x}$ which contradicts (1.4.35).

Since $\hat{x}$ is an arbitrary cluster point, by Lemma 1.1.24 we get that $\{x_n\}$ converges to a solution to the MSSFP. □

Perturbation Techniques

Consider the consistent CMSSFP (1.4.19) with nonempty solution set S. As we mentioned in Section 1.3, the projection P_C, where C is a closed convex subset of H, may bring difficulties in computing it, unless C has a simple form (e.g., a closed ball or a half-space). Therefore some perturbed methods in order to avoid this inconvenience are presented.

We first establish a convergence result using approximate sets defined by means of the subdifferential, when $\{C_i\}$, $\{Q_j\}$ and Ω are level sets of convex functionals. See [43, 119, 25] for other works on this approach.

Then, for each $i \in \{1, 2, \cdots, N\}$ and $j \in \{1, 2, \cdots, M\}$, we consider

$$C_i = \{x \in H_1 : c_i(x) \leq 0\}, \quad Q_j = \{y \in H_2 : q_j(y) \leq 0\},$$

$$\Omega = \{x \in H_1 : \omega(x) \leq 0\},$$

where $c_i, \omega : H_1 \to \mathbb{R}$ and $q_j : H_2 \to \mathbb{R}$ are convex functions.

We iteratively define a sequence $\{x_n\}$ as follows. The initial $x_0 \in H_1$ is arbitrary; once x_n has been defined, we define the $(n+1)^{th}$ iterate x_{n+1} by

$$x_{n+1} = P_{\Omega_n}\left(x_n - \gamma_n\left(\sum_{i=1}^{N} \alpha_i (I - P_{C_i^n})x_n + \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j^n})Ax_n\right)\right) \quad (1.4.36)$$

where

$$\Omega_n = \{x \in H_1 : \omega(x_n) + \langle \zeta_n, x - x_n\rangle \leq 0\}, \quad (\zeta_n \in \partial\omega(x_n)), \quad (1.4.37)$$

$$C_i^n = \{x \in H_1 : c_i(x_n) + \langle \xi_i^n, x - x_n\rangle \leq 0\}, \quad (\xi_i^n \in \partial c_i(x_n)), \quad (1.4.38)$$

$$Q_j^n = \{y \in H_2 : q_j(Ax_n) + \langle \eta_j^n, y - Ax_n\rangle \leq 0\} \quad (\eta_j^n \in \partial q_j(Ax_n)), \quad (1.4.39)$$

and $\{\alpha_i\}$ and $\{\beta_j\}$ are families of positive real numbers.

Lemma 1.4.17. *Let $f : H \to \mathbb{R}$ be a convex function which is bounded on bounded sets. (Note that this condition is automatically satisfied if H is finite-dimensional.) Suppose $\{x_n\}$ is a bounded sequence in H and $\{x_n^*\}$ is another sequence in H such that $x_n^* \in \partial f(x_n)$ for each $n \geq 0$. Then $\{x_n^*\}$ is bounded.*

Proof. The subdifferential inequality implies that

$$f(x_n + y) - f(x_n) \geq \langle x_n^*, y\rangle \tag{1.4.40}$$

for all $y \in H$. Let

$$M = \sup\{|f(x_n + y) - f(x_n)| : n \geq 1, \|y\| \leq 1\}.$$

Then $M < \infty$ by assumption. It follows from (1.4.40) that $\|x_n^*\| \leq M$ for all $n \geq 0$. □

Theorem 1.4.18. *Assume that each of the functions $\{c_i\}_{i=1}^N$ and ω, and $\{q_j\}_{j=1}^M$ satisfies the property: it is bounded on every bounded subset of H_1 and H_2, respectively. (Note that this condition is automatically satisfied in a finite-dimensional Hilbert space.) Then the sequence $\{x_n\}$ generated by the algorithm* (1.4.36) *converges weakly to a solution to the CMSSFP, provided that the sequence $\{\gamma_n\}$ satisfies*

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \frac{2}{L}, \tag{1.4.41}$$

where the constant L is given by (1.4.17).

Proof. Given $n \geq 1$, if we consider the differentiable function

$$f_n(x) := \frac{1}{2}\sum_{i=1}^N \alpha_i \|x - P_{C_i^n} x\|^2 + \frac{1}{2}\sum_{j=1}^M \beta_j \|Ax - P_{Q_j^n} Ax\|^2,$$

with gradient

$$V_n = \nabla f_n = \sum_{i=1}^N \alpha_i (I - P_{C_i^n}) + \sum_{j=1}^M \beta_j A^*(I - P_{Q_j^n})A, \tag{1.4.42}$$

then x_{n+1} can be rewritten as

$$x_{n+1} = P_{\Omega_n}(x_n - \gamma_n V_n x_n). \tag{1.4.43}$$

Hence V_n is L-Lipschitz continuous with L given by (1.4.17) and, therefore, $P_{\Omega_n}(I - \gamma_n V_n)$ is nonexpansive (averaged indeed). It is easily seen that, for each integer $n \geq 1$,

- $\Omega \subseteq \Omega_n$;
- $C_i \subseteq C_i^n$ for $1 \leq i \leq N$;
- $Q_j \subseteq Q_j^n$ for $1 \leq j \leq M$.

Indeed, if $x \in \Omega$ (i.e., $\omega(x) \leq 0$), using the subdifferential inequality, we get

$$\omega(x_n) + \langle \xi_n, x - x_n \rangle \leq \omega(x) \leq 0.$$

This shows that $x \in \Omega_n$. The other two inclusion relations are similarly proved.

Since any $x^* \in S$ belongs to Ω_n and P_{Ω_n} is nonexpansive, we deduce that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|P_{\Omega_n}(I - \gamma_n V_n)x_n - x^*\|^2 \\ &\leq \|(I - \gamma_n V_n)x_n - x^*\|^2 \\ &= \|(x_n - x^*) - \gamma_n V_n x_n\|^2 \\ &= \|x_n - x^*\|^2 - 2\gamma_n \langle x_n - x^*, V_n x_n \rangle + \gamma_n^2 \|V_n x_n\|^2. \quad (1.4.44) \end{aligned}$$

Since V_n is $(1/L)$-ism by Lemma 1.1.20 and $V_n x^* = 0$, we get

$$\langle x_n - x^*, V_n x_n \rangle = \langle x_n - x^*, V_n x_n - V_n x^* \rangle \geq \frac{1}{L} \|V_n x_n\|^2.$$

It follows from (1.4.44) that

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - \gamma_n \left(\frac{2}{L} - \gamma_n \right) \|V_n x_n\|^2. \quad (1.4.45)$$

This implies that the sequence $\{x_n\}$ is Fejér monotone with respect to S, therefore, by Lemma 1.1.24, it is bounded. On the other hand, we deduce that

$$\|V_n x_n\| = \left\| \sum_{i=1}^{N} \alpha_i (I - P_{C_i^n}) x_n + \sum_{j=1}^{M} \beta_j A^* (I - P_{Q_j^n}) A x_n \right\| \to 0. \quad (1.4.46)$$

By Proposition 1.1.8, since $x^* \in C_i^n$ and $Ax^* \in Q_j^n$, we get

$$\begin{aligned} \langle (I - P_{C_i^n}) x_n, x_n - x^* \rangle &= \langle (I - P_{C_i^n}) x_n, x_n - P_{C_i^n} x_n \rangle \\ &\quad + \langle (I - P_{C_i^n}) x_n, P_{C_i^n} x_n - x^* \rangle \\ &\geq \|(I - P_{C_i^n}) x_n\|^2 \quad (1.4.47) \end{aligned}$$

and

$$\begin{aligned}\langle (I-P_{Q_j^n})Ax_n, Ax_n - Ax^*\rangle &= \langle (I-P_{Q_j^n})Ax_n, Ax_n - P_{Q_j^n}Ax_n\rangle \\ &\quad + \langle (I-P_{Q_j^n})Ax_n, P_{Q_j^n}Ax_n - Ax^*\rangle \\ &\geq \|(I-P_{Q_j^n})Ax_n\|^2. \end{aligned} \tag{1.4.48}$$

Combining (1.4.47) and (1.4.48) we obtain

$$\begin{aligned}\langle x_n - x^*, V_n x_n\rangle &= \sum_{i=1}^{N}\alpha_i\langle x_n - x^*, (I-P_{C_i^n})x_n\rangle + \sum_{j=1}^{M}\beta_j\langle x_n - x^*, A^*(I-P_{Q_j^n})Ax_n\rangle \\ &\geq \sum_{i=1}^{N}\alpha_i\|(I-P_{C_i^n})x_n\|^2 + \sum_{j=1}^{M}\beta_j\|(I-P_{Q_j^n})Ax_n\|^2.\end{aligned}$$

This together with (1.4.46) ensures that, for each $1 \leq i \leq N$ and $1 \leq j \leq M$,

$$\lim_{n\to\infty}\|(I-P_{C_i^n})x_n\| = 0, \quad \lim_{n\to\infty}\|(I-P_{Q_j^n})Ax_n\| = 0. \tag{1.4.49}$$

Now, since $x_{n+1} = P_{\Omega_n}(x_n - \gamma_n V_n x_n)$, by Proposition 1.1.8, we have

$$\langle (x_n - \gamma_n V_n x_n) - x_{n+1}, x^* - x_{n+1}\rangle \leq 0.$$

It turns out that, using (1.4.46),

$$\langle x_n - x_{n+1}, x^* - x_{n+1}\rangle \leq \gamma_n\langle V_n x_n, x^* - x_{n+1}\rangle \leq \gamma_n\|x^* - x_{n+1}\|\|V_n x_n\| \to 0.$$

Therefore, the identity

$$\|x_{n+1} - x_n\|^2 = \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\langle x_n - x_{n+1}, x^* - x_{n+1}\rangle$$

implies that

$$\|x_{n+1} - x_n\| \to 0. \tag{1.4.50}$$

Remember that $\omega_w(x_n)$ is the set of all weak accumulation points of the bounded sequence $\{x_n\}$. We now prove

- Claim $\omega_w(x_n) \subseteq S$.

As a matter of fact, take $\hat{x} \in \omega_w(x_n)$ and let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$. Then $x_{n_k+1} \rightharpoonup \hat{x}$.

To prove that $\hat{x} \in S$, we must check that $\hat{x} \in C_i$ and $A\hat{x} \in Q_j$ for any $1 \le i \le N$ and $1 \le j \le M$. To see this, we first show that $\hat{x} \in \Omega$. Indeed, the fact that $x_{n_k+1} \in \Omega_{n_k}$ yields that

$$\omega(x_{n_k}) \le \langle \zeta_{n_k}, x_{n_k+1} - x_{n_k} \rangle. \tag{1.4.51}$$

Since ω is bounded on bounded subsets, by Lemma 1.4.17, $\{\zeta_n\}$ is bounded. Hence, from (1.4.50) and (1.4.51) together with the weak lower semicontinuity of ω, it turns out

$$\omega(\hat{x}) \le \liminf_{k\to\infty} \omega(x_{n_k}) \le \liminf_{k\to\infty} \|\zeta_{n_k}\| \, \|x_{n_k+1} - x_{n_k}\| = 0.$$

Namely, $\hat{x} \in \Omega$.

By definition of $C_i^{n_k}$ and $Q_j^{n_k}$, we have that

$$c_i(x_{n_k}) + \langle \xi_i^{n_k}, P_{C_i^{n_k}} x_{n_k} - x_{n_k} \rangle \le 0 \tag{1.4.52}$$

and

$$q_j(Ax_{n_k}) + \langle \eta_j^{n_k}, P_{Q_j^{n_k}} Ax_{n_k} - Ax_{n_k} \rangle \le 0. \tag{1.4.53}$$

Since $\{\xi_n\}$ and $\{\eta_n\}$ are bounded by virtue of Lemma 1.4.17, using (1.4.49) and the weak lower semicontinuity of c_i and q_j, we derive from (1.4.52) and (1.4.53) that

$$c_i(\hat{x}) \le \liminf_{k\to\infty} c_i(x_{n_k+1}) \le \liminf_{k\to\infty} \|\xi_{n_k}\| \, \|P_{C_i^{n_k}} x_{n_k} - x_{n_k}\| = 0$$

and

$$q_j(A\hat{x}) \le \liminf_{k\to\infty} q_j(Ax_{n_k+1}) \le \liminf_{k\to\infty} \|\eta_{n_k}\| \, \|P_{Q_j^{n_k}} Ax_{n_k} - Ax_{n_k}\| = 0.$$

Hence $\hat{x} \in C_i$ and $A\hat{x} \in Q_j$, for any $1 \le i \le N$ and $1 \le j \le M$. Thus $\hat{x} \in S$.

Therefore, due to the Fejér monotonicity of $\{x_n\}$ with respect to S (see (1.4.45)), we can apply Lemma 1.1.24 to conclude that $\{x_n\}$ converges weakly to a point in S. $\square$

Now we present general perturbation techniques in the direction of the approaches studied in Section 1.3. These techniques consist of taking approximate sets which involve the ρ-distance between two closed convex sets A and B of a Hilbert space,

$$d_\rho(A,B) = \sup\{\|P_A x - P_B x\| : x \in H,\ \|x\| \le \rho\}.$$

Let Ω_n, $\{C_i^n\}$ and $\{Q_j^n\}$ be closed convex sets which are viewed as perturbations for the closed convex sets Ω, $\{C_i\}$ and $\{Q_j\}$, respectively. Define functions f and f_n by

$$f(x) = \frac{1}{2}\sum_{i=1}^{N} \alpha_i \|x - P_{C_i}x\|^2 + \frac{1}{2}\sum_{j=1}^{M} \beta_j \|Ax - P_{Q_j}(Ax)\|^2 \tag{1.4.54}$$

and, respectively,

$$f_n(x) = \frac{1}{2}\sum_{i=1}^{N} \alpha_i \|x - P_{C_i^n}x\|^2 + \frac{1}{2}\sum_{j=1}^{M} \beta_j \|Ax - P_{Q_j^n}(Ax)\|^2, \tag{1.4.55}$$

where $\{\alpha_i\}$ and $\{\beta_j\}$ are families of positive real numbers. Recall that these functions are convex and differentiable with gradients

$$\nabla f(x) = \sum_{i=1}^{N} \alpha_i (I - P_{C_i})x + \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})Ax \tag{1.4.56}$$

and, respectively,

$$\nabla f_n(x) = \sum_{i=1}^{N} \alpha_i (I - P_{C_i^n})x + \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j^n})Ax. \tag{1.4.57}$$

Furthermore, ∇f and ∇f_n are both Lipschitz mappings with the same Lipschitz constant L given by (1.4.17). Then, given $x_0 \in H_1$ we define the sequence $\{x_n\}$ generated by the perturbed Mann type iterative algorithm

$$\begin{aligned} x_{n+1} &= (1-\gamma_n)x_n + \gamma_n P_{\Omega_n}(I - \gamma\nabla f_n)x_n \\ &= (1-\gamma_n)x_n \\ &\quad + \gamma_n P_{\Omega_n}\left(x_n - \gamma\left(\sum_{i=1}^{N} \alpha_i (I - P_{C_i^n})x_n + \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j^n})Ax_n\right)\right). \end{aligned} \tag{1.4.58}$$

Theorem 1.4.19. *Assume that the following conditions are satisfied.*

(i) $0 < \gamma < 2/L$.

(ii) $\gamma_n \in [0, 4/(2+\gamma L)]$ *for all* $n \geq 0$ *(note that* γ_n *may be bigger than one since* $0 < \gamma < 2/L$*) and*

$$\sum_{n=0}^{\infty} \gamma_n \left(\frac{4}{2+\gamma L} - \gamma_n \right) = \infty. \tag{1.4.59}$$

(iii) *for each* $\rho > 0$, $1 \leq i \leq N$, *and* $1 \leq j \leq M$, *there hold* $\sum_{n=0}^{\infty} \gamma_n d_\rho(\Omega_n, \Omega) < \infty$, $\sum_{n=0}^{\infty} \gamma_n d_\rho(C_i^n, C_i) < \infty$, *and* $\sum_{n=0}^{\infty} \gamma_n d_\rho(Q_j^n, Q_j) < \infty$.

Then $\{x_n\}$ *generated by the algorithm* (1.4.58) *converges weakly to a solution to the CMSSFP.*

Proof. Define the mappings T and T_n by

$$Tx = P_\Omega(I - \gamma \nabla f)x \quad \text{and} \quad T_n x = P_{\Omega_n}(I - \gamma \nabla f_n)x. \tag{1.4.60}$$

Thus the algorithm (1.4.58) can be rewritten as

$$x_{n+1} = (1-\gamma_n)x_n + \gamma_n T_n x_n. \tag{1.4.61}$$

Since $0 < \gamma < 2/L$, both T and T_n are α-averaged with

$$\alpha = \frac{2+\gamma L}{4} < 1. \tag{1.4.62}$$

Therefore, we can write

$$T = (1-\alpha)I + \alpha U, \quad T_n = (1-\alpha)I + \alpha U_n, \tag{1.4.63}$$

where U and U_n are nonexpansive.

Note that the solution set S of the CMSSFP (1.4.19) is the fixed point set $\mathrm{Fix}\,(T)$ of T (and of $\mathrm{Fix}\,(U)$). Besides, the algorithm (1.4.61) can further be rewritten as

$$x_{n+1} = (1-\tau_n)x_n + \tau_n U_n x_n, \tag{1.4.64}$$

where $\tau_n = \alpha\gamma_n \in (0,1)$ for all $n \geq 0$ and satisfies the property (by virtue of (1.4.59)):

$$\sum_{n=1}^{\infty} \tau_n(1-\tau_n) = \infty.$$

Recall that the ρ-distance between two mappings T' and $\tilde{T}$ is defined as

$$D_\rho(T', \tilde{T}) = \sup\{\|T'x - \tilde{T}x\| : \|x\| \leq \rho\}.$$

We now compute the ρ-distance from U_n to U for $\rho > 0$,

$$\begin{aligned}
\alpha D_\rho(U_n, U) &= D_\rho(T_n, T) \\
&= \sup\{\|P_{\Omega_n}(I - \gamma\nabla f_n)x - P_\Omega(I - \gamma\nabla f)x\| : \|x\| \leq \rho\} \\
&\leq \sup\{\|P_{\Omega_n}(x - \gamma\nabla f_n(x)) - P_{\Omega_n}(x - \gamma\nabla f(x))\| : \|x\| \leq \rho\} \\
&\quad + \sup\{\|P_{\Omega_n}(x - \gamma\nabla f(x)) - P_\Omega(x - \gamma\nabla f(x))\| : \|x\| \leq \rho\} \\
&\leq \gamma \sup\{\|\nabla f_n(x) - \nabla f(x)\| : \|x\| \leq \rho\} + \sup\{\|P_{\Omega_n}y - P_\Omega y\| : \|y\| \leq \tilde{\rho}\} \\
&\leq \gamma \sum_{i=1}^{N} \alpha_i \sup\{\|P_{C_i^n}x - P_{C_i}x\| : \|x\| \leq \rho\} \\
&\quad + \gamma\|A^*\| \sum_{j=1}^{M} \beta_j \sup\{\|P_{Q_j^n}Ax - P_{Q_j}Ax\| : \|x\| \leq \rho\} + d_{\tilde{\rho}}(\Omega_n, \Omega) \\
&\leq \gamma \sum_{i=1}^{N} \alpha_i d_\rho(C_i^n, C_i) + \gamma\|A\| \sum_{j=1}^{M} \beta_j d_{\|A\|\rho}(Q_j^n, Q_j) + d_{\tilde{\rho}}(\Omega_n, \Omega) \quad (1.4.65)
\end{aligned}$$

where

$$\tilde{\rho} = \sup\{\|x - \gamma\nabla f(x)\| : \|x\| \leq \rho\} < \infty.$$

By assumption (iii), it follows that $\sum_{n=0}^{\infty} \tau_n D_\rho(U_n, U) < \infty$.

Therefore, we can apply Corollary 2.3 of [124] to the algorithm (1.4.64) to conclude that the sequence $\{x_n\}$ converges weakly to a fixed point of U (and of T) which is a solution to the CMSSFP (1.4.19). □

Finally, by using the results provided in Section 1.3, we present a perturbation iterative method which converges strongly to a solution to the CMSSFP.

Given an initial guess $x_0 \in H_1$ and a positive sequence $\{\gamma_n\}_{n=0}^{\infty}$, let $\{x_n\}$ be generated by the perturbed iterative algorithm

$$x_{n+1} = \gamma_n u + (1 - \gamma_n) T_n x_n, \tag{1.4.66}$$

where $u \in H_1$ and the mappings T_n and T are defined by (1.4.60).

Theorem 1.4.20. *Assume that the following conditions are satisfied.*

(a) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(b) *for each* $\rho > 0$, $1 \le i \le N$, *and* $1 \le j \le M$, *there hold* $\sum_{n=0}^{\infty} d_\rho(\Omega_n, \Omega) < \infty$, $\sum_{n=0}^{\infty} d_\rho(C_i^n, C_i) < \infty$, *and* $\sum_{n=0}^{\infty} d_\rho(Q_j^n, Q_j) < \infty$;

(c) $0 < \gamma < 2/L$, *where* $L = \sum_{i=1}^{N} \alpha_i + \|A\|^2 \sum_{j=1}^{M} \beta_j$ *is the Lipschitz constant of* ∇f *and* ∇f_n.

Then $\{x_n\}$ *generated by the algorithm* (1.4.66) *converges strongly to the solution to CMSSFP* (1.4.19) *which is the nearest to* u.

Proof. As we proved in Theorem 1.4.19 the mappings T_n and T can be rewritten as in (1.4.63). Then algorithm (1.4.66) turns into

$$x_{n+1} = \gamma_n u + (1 - \gamma_n)((1 - \alpha)x_n + \alpha U_n x_n). \tag{1.4.67}$$

where $\{U_n\}$ is a family of nonexpansive mapping satisfying that, for any $\rho > 0$,

$$\sum_{n=1}^{\infty} D_\rho(U_n, U) < \infty,$$

thanks to inequality (1.4.65) and hypothesis (b). On the other hand, recall that $\mathrm{Fix}(T) = \mathrm{Fix}(U)$. Thus Corollary 1.3.6 allows us to conclude that the sequence $\{x_n\}$ converges strongly to $x^* = P_{\mathrm{Fix}\,(T)} u$, the solution to CMSSFP (1.4.19) which is the closest to u. $\square$

Minimum-Norm Solution

Let us focus now on the problem of finding the minimum-norm solution to the MSSFP, that is the solution $x^\dagger$ to the MSSFP which has the least norm among all solutions. In other words,

$$\|x^\dagger\| = \min\{\|x\| : x \in S\}. \tag{1.4.68}$$

We claim that $x^\dagger$ exists and is unique because S is closed and convex (see Remark 1.4.14) and $x^\dagger = P_S(0)$. Therefore, the fact that $x^\dagger = P_S(0)$ means that we can approximate the minimum-norm solution to the MSSFP by means of the Halpern type iterative methods which converge strongly to the closest solution to the problem to the arbitrary point $u \in H_1$, just considering $u = 0$. An example of this is the following result.

Theorem 1.4.21. *(Xu [124]) Assume that* $0 < \gamma < 2/L'$ *with* L' *given by* (1.4.17). *Let* $\{\gamma_n\}$ *be a sequence in* $(0,1)$ *satisfying the conditions*

(i) $\lim_n \gamma_n = 0$*;*

(ii) $\sum_n \gamma_n = \infty$*;*

(iii) $\sum_n |\gamma_n - \gamma_{n+N}| < \infty$ *or* $\lim_n(\gamma_n/\gamma_{n+N}) = 1$.

Define a sequence $\{x_n\}$ *by the iterative algorithm*

$$x_{n+1} = (1-\gamma_{n+1})P_{C_{[n+1]}}\left(I - \gamma\sum_{j=1}^{M}\beta_j A^*(I - P_{Q_j})A\right)x_n. \tag{1.4.69}$$

Then $\{x_n\}$ *converges strongly to the minimum-norm solution* $x^\dagger$ *to the MSSFP.*

It is straightforward to see that the sequence defined by $\gamma_n = 1/n$, for all $n \geq 1$, satisfies conditions (i)-(iii).

We now propose a different way to approximate $x^\dagger$ by regularization. Consider the constrained problem CMSSFP (1.4.19), imposing that the solution belongs to a closed convex subset Ω. Then we want to find $x^\dagger$ satisfying (1.4.68) where now S denotes the solution set of the CMSSFP.

Let f be the function given by (1.4.15) with gradient ∇f given by (1.4.16). Then

$$S = \operatorname*{argmin}_{x\in\Omega}\{f(x)\} \tag{1.4.70}$$

Note that in general the minimization problem (1.4.18), equivalent to the CMSSFP, is ill-posed. So regularization is necessary. For each parameter $\alpha > 0$, consider the regularized objective function

$$f_\alpha(x) = f(x) + \frac{1}{2}\alpha\|x\|^2 = \frac{1}{2}\sum_{i=1}^{N}\alpha_i\|x - P_{C_i}x\|^2 + \frac{1}{2}\sum_{j=1}^{M}\beta_j\|Ax - P_{Q_j}Ax\|^2 + \frac{1}{2}\alpha\|x\|^2. \tag{1.4.71}$$

The gradient of f_α is

$$\nabla f_\alpha(x) = \sum_{i=1}^{N}\alpha_i(I - P_{C_i})x + \sum_{j=1}^{M}\beta_j A^*(I - P_{Q_j})Ax + \alpha x. \tag{1.4.72}$$

It is easily seen that ∇f_α is L_α-Lipschitz continuous with constant

$$L_\alpha := \alpha + \sum_{i=1}^{N}\alpha_i + \sum_{j=1}^{M}\beta_j\|A\|^2. \tag{1.4.73}$$

Moreover, we can prove that ∇f_α is also strongly monotone.

It turns out that the regularized minimization problem

$$\min_{x\in\Omega} f_\alpha(x) \tag{1.4.74}$$

has a unique solution which is denoted by x_α.

Theorem 1.4.22. *Let $\{x_\alpha\}$ be the net defined by* (1.4.74)*. Then, as $\alpha \to 0$, $\{x_\alpha\}$ converges strongly to $x^\dagger$, the minimum-norm solution to the CMSSFP.*

Proof. For any $\hat{x} \in S$, we have

$$f(\hat{x}) + \frac{\alpha}{2}\|x_\alpha\|^2 \le f(x_\alpha) + \frac{\alpha}{2}\|x_\alpha\|^2 = f_\alpha(x_\alpha) \le f_\alpha(\hat{x}) = f(\hat{x}) + \frac{\alpha}{2}\|\hat{x}\|^2.$$

It follows that, for all $\alpha > 0$ and $\hat{x} \in S$,

$$\|x_\alpha\| \leq \|\hat{x}\|. \tag{1.4.75}$$

Therefore $\{x_\alpha\}$ is bounded. Assume $\alpha_j \to 0$ is such that $x_{\alpha_j} \rightharpoonup \tilde{x}$. Then the weak lower semicontinuity of f implies that, for any $x \in \Omega$,

$$\begin{aligned} f(\tilde{x}) &\leq \liminf_{j\to\infty} f(x_{\alpha_j}) \leq \liminf_{j\to\infty} f_{\alpha_j}(x_{\alpha_j}) \\ &\leq \liminf_{j\to\infty} f_{\alpha_j}(x) = \liminf_{j\to\infty} \left[f(x) + \frac{\alpha_j}{2}\|x\|^2\right] \\ &= f(x). \end{aligned}$$

This means that $\tilde{x} \in S$. Since the norm is weak lower semicontinuous, we get from (1.4.75) that $\|\tilde{x}\| \leq \|\hat{x}\|$ for all $\hat{x} \in S$; hence $\tilde{x} = x^\dagger$. This is sufficient to ensure that $x_\alpha \rightharpoonup x^\dagger$. To see that the convergence is strong, noting that (1.4.75) holds for $x^\dagger$, we compute

$$\begin{aligned} \|x_\alpha - x^\dagger\|^2 &= \|x_\alpha\|^2 - 2\langle x_\alpha, x^\dagger\rangle + \|x^\dagger\|^2 \\ &\leq 2(\|x^\dagger\|^2 - \langle x_\alpha, x^\dagger\rangle). \end{aligned}$$

Since $x_\alpha \rightharpoonup x^\dagger$, we get $\|x_\alpha - x^\dagger\|^2 \to 0$. Therefore $x_\alpha \to x^\dagger$. □

The advantage of the regularized solution x_α lies in the fact that it can be obtained via the Banach contraction principle. As a matter of fact, since the gradient ∇f_α is α-strongly monotone and L_α-Lipschitz, x_α is the unique fixed point in Ω of the contraction

$$\begin{aligned} T_\alpha &:= P_\Omega(I - \gamma\nabla f_\alpha) \qquad (1.4.76) \\ &= P_\Omega\left((1-\alpha\gamma)I - \gamma\sum_{i=1}^{N}\alpha_i(I - P_{C_i}) - \gamma\sum_{j=1}^{M}\beta_j A^*(I - P_{Q_j})A\right), \end{aligned}$$

where $0 < \gamma < 2\alpha/L_\alpha^2$ (see Lemma 1.4.8). It follows that, for any $x_0 \in \Omega$, Picard iteration $\{T_\alpha^n x_0\}$ converges strongly to x_α (see Lemma 1.4.8(i)).

Hence $x^\dagger$ can be obtained via two steps: (i) getting x_α through Picard iteration $\{T_\alpha^n x_0\}$ and (ii) letting α go to 0 to get $x^\dagger$ via Theorem 1.4.22. Next we show that these two steps can be combined to create an iterative method that generates a sequence converging in norm to $x^\dagger$.

Theorem 1.4.23. *Given an initial point $x_0 \in \Omega$. Define a sequence $\{x_n\}$ by the iterative algorithm*

$$\begin{aligned} x_{n+1} &= P_\Omega(I - \gamma_n \nabla f_{\alpha_n})x_n \qquad (1.4.77) \\ &= P_\Omega\left((1-\alpha_n\gamma_n)x_n - \gamma_n \sum_{i=1}^{N} \alpha_i (I - P_{C_i})x_n - \gamma_n \sum_{j=1}^{M} \beta_j A^*(I - P_{Q_j})Ax_n\right), \end{aligned}$$

where the sequences $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfy the conditions:

(i) $0 < \gamma_n < \alpha_n / L_{\alpha_n}^2$ *for all (large enough)* $n \geq 0$*;*

(ii) $\alpha_n \to 0$ *(hence* $\gamma_n \to 0$ *as well);*

(iii) $\sum_{n=1}^{\infty} \alpha_n \gamma_n = \infty$*;*

(iv) $(|\gamma_n - \gamma_{n-1}| + |\alpha_n\gamma_n - \alpha_{n-1}\gamma_{n-1}|)/(\alpha_n\gamma_n)^2 \to 0.$

Then $x_n \to x^\dagger$*.*

Proof. Bearing in mind that by Lemma 1.4.8(i) the mapping $P_\Omega(I - \gamma\nabla f_\alpha)$ is a contraction with coefficient $1 - \frac{1}{2}\alpha\gamma$ whenever $0 < \gamma < \alpha/L_\alpha^2$, we deduce that, for $\hat{x} \in S$,

$$\begin{aligned} \|x_{n+1} - \hat{x}\| &= \|P_\Omega(I - \gamma_n\nabla f_{\alpha_n})x_n - P_\Omega(I - \gamma_n \nabla f)\hat{x}\| \\ &\leq \|P_\Omega(I - \gamma_n\nabla f_{\alpha_n})x_n - P_\Omega(I - \gamma_n \nabla f_{\alpha_n})\hat{x}\| \\ &\quad + \|P_\Omega(I - \gamma_n\nabla f_{\alpha_n})\hat{x} - P_\Omega(I - \gamma_n \nabla f)\hat{x}\| \\ &\leq (1 - \frac{1}{2}\alpha_n\gamma_n)\|x_n - \hat{x}\| + \alpha_n\gamma_n\|\hat{x}\| \\ &\leq \max\{\|x_n - \hat{x}\|, 2\|\hat{x}\|\}. \end{aligned}$$

This implies by induction that

$$\|x_n - \hat{x}\| \leq \max\{\|x_0 - \hat{x}\|, 2\|\hat{x}\|\}, \quad n \geq 0.$$

Hence, (x_n) is bounded.

Now let $z_n := z_{\alpha_n}$ be the unique fixed point of the contraction T_{α_n} defined by (1.4.76). By Theorem 1.4.22, we have that $z_n \to x^\dagger$. It remains to prove that

$\|x_{n+1} - z_n\| \to 0$. Using (1.4.77), the fact that $z_n = P_C(I - \gamma_n \nabla f_{\alpha_n}) z_n$ and the fact that $P_C(I - \gamma_n \nabla f_{\alpha_n})$ is a contraction with coefficient $1 - \frac{1}{2}\alpha_n\gamma_n$, we derive that

$$\begin{aligned} \|x_{n+1} - z_n\| &\leq (1 - \frac{1}{2}\alpha_n\gamma_n)\|x_n - z_n\| \\ &\leq (1 - \frac{1}{2}\alpha_n\gamma_n)\|x_n - z_{n-1}\| + \|z_n - z_{n-1}\|. \end{aligned} \tag{1.4.78}$$

On the other hand, we have

$$\begin{aligned} \|z_n - z_{n-1}\| &= \|P_C(I - \gamma_n \nabla f_{\alpha_n}) z_n - P_C(I - \gamma_{n-1} \nabla f_{\alpha_{n-1}}) z_{n-1}\| \\ &\leq \|P_C(I - \gamma_n \nabla f_{\alpha_n}) z_n - P_C(I - \gamma_n \nabla f_{\alpha_n}) z_{n-1}\| \\ &\quad + \|P_C(I - \gamma_n \nabla f_{\alpha_n}) z_{n-1} - P_C(I - \gamma_{n-1} \nabla f_{\alpha_{n-1}}) z_{n-1}\| \\ &\leq (1 - \frac{1}{2}\alpha_n\gamma_n)\|z_n - z_{n-1}\| \\ &\quad + \|(I - \gamma_n \nabla f_{\alpha_n}) z_{n-1} - (I - \gamma_{n-1} \nabla f_{\alpha_{n-1}}) z_{n-1}\| \\ &\leq (1 - \frac{1}{2}\alpha_n\gamma_n)\|z_n - z_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|\nabla f(z_{n-1})\| \\ &\quad + |\alpha_n\gamma_n - \alpha_{n-1}\gamma_{n-1}|\|z_{n-1}\| \\ &\leq (1 - \frac{1}{2}\alpha_n\gamma_n)\|z_n - z_{n-1}\| \\ &\quad + \frac{M}{2}(|\gamma_n - \gamma_{n-1}| + |\alpha_n\gamma_n - \alpha_{n-1}\gamma_{n-1}|), \end{aligned} \tag{1.4.79}$$

where M is big enough so that $M > 2\max\{\|z_n\|, \|\nabla f(z_n)\|\}$ for all $n \geq 1$. It follows from (1.4.79) that

$$\|z_n - z_{n-1}\| \leq M \frac{|\gamma_n - \gamma_{n-1}| + |\alpha_n\gamma_n - \alpha_{n-1}\gamma_{n-1}|}{\alpha_n\gamma_n}. \tag{1.4.80}$$

Substituting (1.4.80) into (1.4.78) we get

$$\|x_{n+1} - z_n\| \leq (1 - \frac{1}{2}\alpha_n\gamma_n)\|x_n - z_{n-1}\| + M \frac{|\gamma_n - \gamma_{n-1}| + |\alpha_n\gamma_n - \alpha_{n-1}\gamma_{n-1}|}{\alpha_n\gamma_n}. \tag{1.4.81}$$

By virtue of the conditions (iii) and (iv), we can apply Lemma 1.1.21 to the relation (1.4.81) to get $\|x_{n+1} - z_n\| \to 0$. $\square$

Remark 1.4.24. If we take

$$\alpha_n = \frac{1}{(n+1)^\alpha}, \quad \gamma_n = \frac{1}{(n+1)^\gamma},$$

where α and γ are such that $0 < \alpha < \gamma < 1$ and $2\alpha + \gamma < 1$, then we can prove that conditions (i)-(iv) of Theorem 1.4.23 are all satisfied.

Chapter 2

Iterative methods in Hadamard manifolds

In this chapter, we develop a theory of monotone operators and approximation of fixed points of nonexpansive mappings. We start by introducing the basic knowledge of Riemannian geometry. Section 2.2 is devoted to introducing different classes of monotone set-valued vector fields and proving the connection with upper semicontinuity and accretivity. In Section 2.3 we study several nonexpansive type mappings, in particular, firmly nonexpansive and pseudo-contractive mappings. In Section 2.4 we study the existence of singularities of monotone vector fields and establish the equivalence of this problem with a fixed point problem by means of the concept of a resolvent. We also analyze the asymptotic behavior of the resolvent by using the notion and properties of the Yosida regularization; as a consequence we obtain some existence results of singularities under boundary conditions. Regarding the approximation of singularities, in Section 2.5, we provide a proximal point algorithm for maximal monotone vector fields. In Section 2.6, we study the convergence of Picard iteration for firmly nonexpansive mappings, Mann and Halpern iterations for nonexpansive mappings and a viscosity approximation method. In order to illustrate the application of these methods, we provide some numerical examples for Mann and Halpern iterations. Finally, Section 2.7 focuses on some applications to different problems: minimization problems, minimax problems, variational inequalities and equilibrium problems.

2.1 Theoretical framework

The object of this section is to familiarize the reader with the classical language and some fundamental theorems in Hadamard manifolds, needed to understand the work presented in this chapter. To this end we introduce some concepts and results on differential manifolds; then the basic notions of Riemannian geometry, such as metric, geodesic and parallel transport; and finally the objects and facts that characterize the Hadamard manifolds, which is the setting we will focus on to develop our analysis. A complete description of these concepts can be found in any textbook on Riemannian geometry, for instance [37, 105].

2.1.1 Differentiable manifolds

Definition 2.1.1. A *differentiable manifold* of dimension n is a set M and a family of injective mappings $\mathbf{x}_\alpha : U_\alpha \to M$ of open sets $U_\alpha \subseteq \mathbb{R}^n$ such that:

(1) $\bigcup_\alpha \mathbf{x}_\alpha(U_\alpha) = M$.

(2) For any pair α, β, with $\mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta) = W$, the sets $\mathbf{x}_\alpha^{-1}(W)$ and $\mathbf{x}_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the mappings $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$ are differentiable.

(3) The family $\{(U_\alpha, \mathbf{x}_\alpha)\}$ is maximal relative to the conditions (1) and (2).

The pair $(U_\alpha, \mathbf{x}_\alpha)$ (or the mapping $\mathbf{x}_\alpha$) with $x \in \mathbf{x}_\alpha(U_\alpha)$ is called a *parametrization* of M at x. A family $\{(U_\alpha, \mathbf{x}_\alpha)\}$ satisfying (1) and (2) is called a *differentiable structure* on M. In general, with a certain abuse of language, since given a differentiable structure on M, we can easily complete it to a maximal one, we say that a differentiable manifold is a set provided with a differentiable structure. Note that a differentiable structure induces on M a natural topology defining $A \subseteq M$ to be an open set in M if and only if $\mathbf{x}_\alpha^{-1}(A \cap \mathbf{x}_\alpha(U_\alpha))$ is an open set in $\mathbb{R}^n$ for all α. The Euclidean space $\mathbb{R}^n$, with the differentiable structure given by the identity is a trivial example.

Definition 2.1.2. Given two differentiable manifolds M_1 and M_2 of dimension n and m respectively, a mapping $\phi : M_1 \to M_2$ is differentiable at $x \in M_1$ if given a parametrization $\mathbf{y} : V \subseteq \mathbb{R}^m \to M_2$ with $\phi(x) \in y(V)$ there exists a parametrization

$\mathbf{x} : U \subseteq \mathbb{R}^n \to M_1$ with $x \in \mathbf{x}(U)$ such that $\phi(\mathbf{x}(U)) \subseteq \mathbf{y}(V)$ and the mapping $\mathbf{y}^{-1} \circ \phi \circ \mathbf{x} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{x}^{-1}(x)$.

Once we have extended the idea of differentiability to mappings between manifolds, we can define the notion of tangent vector.

Definition 2.1.3. Let M be a differentiable manifold of dimension n. A *curve* in M is a differentiable function $\gamma : (-\epsilon, \epsilon) \to M$. A curve is said to be smooth if it is of class $\mathcal{C}^\infty$, that is infinitely differentiable. Suppose that $\gamma(0) = x \in M$, and let D be the set of functions on M that are differentiable at x. The *tangent vector* to the curve γ at $t = 0$ is a function $\gamma'(0) : D \to \mathbb{R}$ given by

$$\gamma'(0)f = \left.\frac{d(f \circ \gamma)}{dt}\right|_{t=0}, \quad f \in D.$$

And we say that the tangent vector at x is the tangent vector at $t = 0$ of some curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = x$. The set of all tangent vectors to M at x, denoted by T_xM, forms a vector space of dimension n called tangent space of M at x. The set $TM = \bigcup_{x \in M} T_xM$ provided with a differentiable structure is a differentiable manifold and will be called the *tangent bundle* of M. A vector field A on M is a mapping of M into the tangent bundle TM, that is, it associates to each point $x \in M$ a vector $A(x) \in T_xM$.

2.1.2 Riemannian manifolds

The Riemannian geometry can be seen as a natural development of the differential geometry of surfaces in $\mathbb{R}^3$. Then, departing from a differentiable manifold M, we can introduce a way of measuring the length of tangent vectors by means of an inner product, which leads to special curves behaving as if they were "the straight lines" of M.

Definition 2.1.4. A *Riemannian metric* on a differential manifold M is a correspondence which associates to each point x of M an inner product $\langle \, , \, \rangle$, (that is, a symmetric bilinear positive-definite form) on the tangent space T_xM, which varies differentiably in the following sense: for any vector fields A and B, which are differentiable in a neighborhood V of M, the function $\langle A, B \rangle$ is differentiable on

V. A differentiable manifold with a Riemannian metric will be called *Riemannian manifold* and its corresponding norm will be denoted by $\|\cdot\|$.

Definition 2.1.5. Given a smooth curve $\gamma : [a, b] \to M$ joining x to y (i.e. $\gamma(a) = x$ and $\gamma(b) = y$), we can define the *length* of γ by using the metric as

$$L(\gamma) = \int_a^b \|\gamma'(t)\| dt.$$

Then the *Riemannian distance* $d(x, y)$, which induces the original topology on M, is defined by minimizing this length over the set of all such curves joining x to y,

$$d(x, y) := \inf\{L(\gamma) : \gamma \text{ joining } x \text{ to } y\}.$$

Remark 2.1.6. From now on, M is assumed to be connected so that the set of curves joining x to y is always nonempty.

Definition 2.1.7. Let ∇ be the *Levi-Civita connection* associated to $(M, \langle\, ,\, \rangle)$ and $\nabla_A B$ the *covariant derivative* of the vector fields A by B (see [105] for more details). Given a smooth curve γ in M a vector field A is said to be *parallel* along γ if $\nabla_{\gamma'} A = 0$. If γ' itself is parallel along γ, we say that γ is a *geodesic*, and in this case $\|\gamma'\|$ is constant. When $\|\gamma'\| = 1$, γ is called *normalized.* A geodesic joining x to y in M is said to be *minimal* if its length equals $d(x, y)$.

Note that given a point $x \in M$ and $u \in T_x M$ there exists a neighborhood U of u in $T_x M$ such that for any $v \in U$ we have a unique geodesic γ defined on an interval satisfying $\gamma(0) = x$ and $\gamma'(0) = v$. We denote this geodesic, starting at x with velocity v, by $\gamma_v(., x)$.

Definition 2.1.8. The *parallel transport* on the tangent bundle TM along γ with respect to ∇ is defined by

$$P_{\gamma,\gamma(b),\gamma(a)}(v) := A(\gamma(b)), \quad \forall a, b \in \mathbb{R} \text{ and } v \in T_{\gamma(a)}M,$$

where A is the unique vector field satisfying $\nabla_{\gamma'(t)} A = 0$ for all t and $A(\gamma(a)) = v$.

Remark 2.1.9. It can be proved that for any $a, b \in \mathbb{R}$, $P_{\gamma,\gamma(b),\gamma(a)}$ is an isometry from $T_{\gamma(a)}M$ to $T_{\gamma(b)}M$. Note that, for any a, b, b_1, $b_2 \in \mathbb{R}$,

$$P_{\gamma,\gamma(b_2),\gamma(b_1)} \circ P_{\gamma,\gamma(b_1),\gamma(a)} = P_{\gamma,\gamma(b_2),\gamma(a)} \quad \text{and} \quad P^{-1}_{\gamma,\gamma(b),\gamma(a)} = P_{\gamma,\gamma(a),\gamma(b)}.$$

For the sake of simplicity, we will write $P_{y,x}$ instead of $P_{\gamma,y,x}$ in the case when γ is a minimal geodesic joining x to y and no confusion arises.

Definition 2.1.10. A Riemannian manifold M is said to be complete, if for any point $x \in M$, all geodesics emanating from x are defined for all $t \in \mathbb{R}$.

By the Hopf-Rinow Theorem we know that if M is a complete Riemannian manifold then any pair of points in M can be joined by a minimal geodesic. Moreover, a complete Riemannian manifold (M, d) is a complete metric space and bounded closed subsets are compact. The concept of completeness allows us to study the global behavior of a Riemannian manifold M by looking at how geodesics run on M.

Definition 2.1.11. Assuming that M is a complete Riemannian manifold, the *exponential map* at $x \in M$, $\exp_x : T_xM \to M$ is defined by

$$\exp_x v = \gamma_v(1, x), \;\; v \in T_xM,$$

where recall that $\gamma_v(., x)$ is the geodesic starting at x with velocity v. Then, for any value of t, $\exp_x tv = \gamma_v(t, x)$. Note that the map $\exp_x$ is differentiable on T_xM for any $x \in M$.

2.1.3 Hadamard manifolds

The notion of sectional curvature in a Riemannian manifold plays an important role in the development of geometry. This concept measures in some sense the amount that a Riemannian manifold deviates from being Euclidean. It was introduced by Riemann as a natural generalization of the Gaussian curvature of surfaces. A few years later, an explicit formula was given by Christoffel by using the Levi-Civita connection. We do not include the technical definition of sectional curvature; see references given at the beginning of this chapter for explicit definitions. In particular, we are interested in Riemannian manifolds of nonpositive sectional curvature, whose basic geometrical characterization is gathered in Proposition 2.1.14.

Definition 2.1.12. A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a *Hadamard manifold*.

Throughout the remainder of this chapter, we will always assume that M is an m-dimensional Hadamard manifold. The following well-known result will be essential for the development of this chapter. It can be found, for example, in [105, pag. 221, Theorem 4.1].

Proposition 2.1.13. *Let $x \in M$. Then, $\exp_x : T_xM \to M$ is a diffeomorphism, and for any two points $x, y \in M$ there exists a unique normalized geodesic joining x to y, which is a minimal geodesic.*

This proposition says that M is diffeomorphic to the Euclidean space $\mathbb{R}^m$. Thus, M has the same topology and differential structure as $\mathbb{R}^m$. Moreover, Hadamard manifolds and Euclidean spaces have some similar geometrical properties.

One of the most important properties of Hadamard manifolds is described in the following proposition, which can be taken from [105, pag. 223, Proposition 4.5]. Recall that a geodesic triangle $\Delta(x_1, x_2, x_3)$ of a Riemannian manifold is a set consisting of three points x_1, x_2, x_3, and three minimal geodesics joining these points.

Proposition 2.1.14. *Let $\Delta(x_1, x_2, x_3)$ be a geodesic triangle in M. Denote, for each $i = 1, 2, 3$ (mod 3), by $\gamma_i : [0, l_i] \to M$ the geodesic joining x_i to x_{i+1}, and set $l_i := L(\gamma_i)$, $\alpha_i := \angle(\gamma_i'(0), -\gamma_{i-1}'(l_{i-1}))$. Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \le \pi, \tag{2.1.1}$$

$$l_i^2 + l_{i+1}^2 - 2l_i l_{i+1} \cos\alpha_{i+1} \le l_{i-1}^2. \tag{2.1.2}$$

In terms of the distance and the exponential map, the inequality (2.1.2) can be rewritten as

$$\mathrm{d}^2(x_i, x_{i+1}) + \mathrm{d}^2(x_{i+1}, x_{i+2}) - 2\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2}\rangle \le \mathrm{d}^2(x_{i-1}, x_i), \tag{2.1.3}$$

since

$$\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2}\rangle = \mathrm{d}(x_i, x_{i+1})\mathrm{d}(x_{i+1}, x_{i+2}) \cos\alpha_{i+1}.$$

The following lemma collects some properties of the exponential map and the parallel transport, which will be very useful in the next sections. Its technical proof can be found in [62].

Lemma 2.1.15. *Let $x_0 \in M$ and $\{x_n\} \subset M$ be such that $x_n \to x_0$. Then the following assertions hold.*

(i) *The differential of the exponential map at the origin,*

$$\frac{d}{dt}\bigg|_{t=0} \exp_p tu = \gamma_u'(0) = u,$$

is the identity.

(ii) *For any $y \in M$,*

$$\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y \qquad \text{and} \qquad \exp_y^{-1} x_n \to \exp_y^{-1} x_0.$$

(iii) *If $\{v_n\}$ is a sequence such that $v_n \in T_{x_n}M$ and $v_n \to v_0$, then $v_0 \in T_{x_0}M$.*

(iv) *Given the sequences $\{u_n\}$ and $\{v_n\}$ satisfying $u_n, v_n \in T_{x_n}M$, if $u_n \to u_0$ and $v_n \to v_0$ with $u_0, v_0 \in T_{x_0}M$, then*

$$\langle u_n, v_n \rangle \to \langle u_0, v_0 \rangle.$$

(v) *For any $u \in T_{x_0}M$, the function $F : M \to TM$ defined by $Fix(x) = P_{x,x_0}u$ for each $x \in M$ is continuous on M.*

Let us introduce in following sections some fundamental notions and results of convex analysis in Hadamard manifolds, as well as some metric properties. References on this topic are [114, 113, 105, 91].

Projections onto convex sets

Definition 2.1.16. A subset $C \subseteq M$ is said to be *convex* if for any two points x and y in C, the geodesic joining x to y is contained in C, that is, if $\gamma : [a,b] \to M$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma((1-t)a + tb) \in C$ for all $t \in [0,1]$.

As in linear metric spaces, we can define a projection map onto closed convex sets.

Definition 2.1.17. The *projection* onto a set C is the set-valued mapping defined by

$$P_C(x) = \{x_0 \in C : d(x, x_0) \leq d(x, y) \text{ for all } y \in C\}, \quad \forall x \in M.$$

Proposition 2.1.18. *(Walter [114]) For any point $x \in M$, given a closed convex set $C \subseteq M$, $P_C(x)$ is a singleton and the following inequality holds for all $y \in C$:*

$$\langle \exp^{-1}_{P_C(x)} x,\ \exp^{-1}_{P_C(x)} y\rangle \leq 0.$$

Corollary 2.1.19. *(Ferreira et al. [41]) If M is a Hadamard manifold with constant curvature, given $x \in M$ and $v \in T_xM$, the set*

$$L_{x,v} := \{y \in M : \langle \exp^{-1}_x y, v\rangle \leq 0\}$$

is convex.

Remark 2.1.20. The previous result remains true if the curvature is nonconstant but the dimension of the manifold is 2.

From now on, C will denote a nonempty closed convex set in M, unless explicitly stated otherwise.

Convex functions

Definition 2.1.21. Let $f : M \to \overline{\mathbb{R}}$ be a proper extended real-valued function. The domain of the function f is denoted by $\mathcal{D}(f)$ and defined by $\mathcal{D}(f) := \{x \in M : f(x) \neq +\infty\}$. The function f is said to be convex if for any geodesic γ in M, the composition function $f \circ \gamma : \mathbb{R} \to \overline{\mathbb{R}}$ is convex, that is,

$$(f \circ \gamma)(ta + (1-t)b) \leq t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b)$$

for any $a, b \in \mathbb{R}$ and $0 \leq t \leq 1$.

Definition 2.1.22. The *subdifferential* of a function $f : M \to \overline{\mathbb{R}}$ at $x \in M$ is the set-valued mapping $\partial f : M \to 2^{TM}$ defined by

$$\partial f(x) = \{u \in T_xM : \langle u, \exp^{-1}_x y\rangle \leq f(y) - f(x),\ \forall y \in M\},$$

and its elements are called *subgradients*.

The subdifferential $\partial f(x)$ at a point $x \in M$ is a closed convex (possible empty) set. The existence of subgradients for convex functions is guaranteed by the following proposition taken from [42].

Proposition 2.1.23. *Let M be a Hadamard manifold and $f : M \to \mathbb{R}$ a convex function. Then, for any $x \in M$, the subdifferential $\partial f(x)$ of f at x is nonempty. That is, the domain of the subdifferential is $\mathcal{D}(\partial f) = M$.*

Fix a point $x \in M$ and define the mapping $\rho_x : M \to \mathbb{R}$ by

$$\rho_x(y) := \frac{1}{2} d^2(x, y).$$

Then this mapping is $\mathcal{C}^\infty$ and satisfies the following property; see [105].

Proposition 2.1.24. *In a Hadamard manifold M, the mapping ρ_x is strictly convex and its gradient at y is*

$$\partial \rho_x(y) = -\exp_y^{-1} x.$$

Metric properties

The following proposition describes the convexity property of the distance function (cf. [105, pag. 222, Proposition 4.3]).

Proposition 2.1.25. *Let $\mathrm{d} : M \times M \to \mathbb{R}$ be the distance function. Then $\mathrm{d}(\cdot, \cdot)$ is a convex function with respect to the product Riemannian metric; that is, given any pair of geodesics $\gamma_1 : [0, 1] \to M$ and $\gamma_2 : [0, 1] \to M$, the following inequality holds for all $t \in [0, 1]$:*

$$\mathrm{d}(\gamma_1(t), \gamma_2(t)) \leq (1 - t)\mathrm{d}(\gamma_1(0), \gamma_2(0)) + t\mathrm{d}(\gamma_1(1), \gamma_2(1)).$$

In particular, for each $x \in M$, the function $\mathrm{d}(\cdot, x) : M \to \mathbb{R}$ is a convex function on M.

The following relation between geodesic triangles and triangles in $\mathbb{R}^2$ can be found in [11, pag. 24].

Lemma 2.1.26. *Let $\Delta(x, y, z)$ be a geodesic triangle in M Hadamard space. Then, there exists $x', y', z' \in \mathbb{R}^2$ such that*

$$d(x, y) = \|x' - y'\|, \quad d(y, z) = \|y' - z'\|, \quad d(z, x) = \|z' - x'\|.$$

The triangle $\Delta(x', y', z')$ is called the comparison triangle of the geodesic triangle $\Delta(x, y, z)$, which is unique up to isometry of M. The following lemma can be proved from elementary geometry. This is also a direct application of the Alexandrov's Lemma in $\mathbb{R}^2$ (see [11, pag. 25]).

Lemma 2.1.27. *Consider four distinct points $x, y, z, q \in \mathbb{R}^2$. Suppose that x and y lie on opposite sides of the line through z and q. Consider the triangles $\Delta(z, x, q)$ and $\Delta(z, y, q)$. Let β be the angle of $\Delta(z, y, q)$ at the vertex y, and let θ_1 and θ_2 be the angles of $\Delta(z, x, q)$ and $\Delta(z, y, q)$ at the vertex q, respectively. Let y' be the point such that $d(z, y') = d(z, y)$ and $d(x, y') = d(x, q) + d(q, y)$. Let β' be the angles of $\Delta(z, x, y')$ at the vertex y' (see Figure 2.1). If $\theta_1 + \theta_2 \geq \pi$, then*

$$\beta \leq \beta'.$$

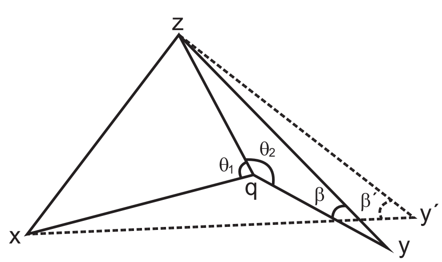

Figure 2.1

The next result shows a relationship between a geodesic triangle and its comparison triangle which expresses the geometric idea of a manifold having nonpositive sectional curvature.

Lemma 2.1.28. *Let $\Delta(x, y, z)$ be a geodesic triangle in a Hadamard space M and $\Delta(x', y', z')$ be its comparison triangle.*

(1) *Let α, β, γ (resp. α', β', γ') be the angles of $\Delta(x, y, z)$ (resp. $\Delta(x', y', z')$) at the vertices x, y, z (resp. x', y', z'). Then, the following inequalities hold:*

$$\alpha' \geq \alpha, \quad \beta' \geq \beta, \quad \gamma' \geq \gamma. \tag{2.1.4}$$

(2) *Let q be a point in the geodesic joining x to y and q' its comparison point in the interval $[x', y']$. Suppose that $d(q,x) = \|q' - x'\|$ and $d(q,y) = \|q' - y'\|$; see Figure* 2.2. *Then the following inequality holds:*

$$d(q,z) \leq \|q' - z'\|. \tag{2.1.5}$$

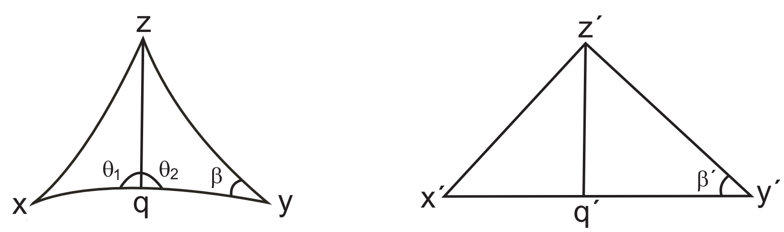

Figure 2.2

Proof. (1) We only prove the inequality $\beta' \geq \beta$. To do this, consider the triangle $\Delta(x', y', z')$ in $\mathbb{R}^2$. Then, by the law of cosines we have that

$$\|z' - y'\|^2 + \|y' - x'\|^2 - 2\|z' - y'\|\|y' - x'\| \cos \beta' = \|z' - x'\|^2.$$

By (2.1.2), one has that

$$d^2(z,y) + d^2(y,x) - 2d(z,y)d(y,x) \cos \beta \leq d^2(z,x).$$

It follows from Lemma 2.1.26 that

$$\cos \beta \leq \cos \beta',$$

and $\beta' > \beta$ (because $\beta,\ \beta' \in [0,\pi]$).

(2) We fix a geodesic joining q to z. Let θ_1 and θ_2 denote respectively the angles of $\Delta(z,x,q)$ and $\Delta(z,y,q)$ at the vertex q. Let β and β' be the angles of $\Delta(z,x,y)$ and $\Delta(z', x', y')$ at the vertex y and y' respectively. See Figure 2.2.

Consider comparison triangles $\Delta(\bar{z}, \bar{x}, \bar{q})$ and $\Delta(\bar{z}, \bar{y}, \bar{q})$ for the geodesic triangles $\Delta(z,x,q)$ and $\Delta(z,y,q)$ respectively, such that they share the same edge $[\bar{z}, \bar{q}]$, and

$\bar{x}$, $\bar{y}$ lie on opposite sides of the line which passes through $\bar{z}$ and $\bar{q}$. Let $\bar{\theta}_1$ and $\bar{\theta}_2$ be the angles of $\Delta(\bar{z}, \bar{x}, \bar{q})$ and $\Delta(\bar{z}, \bar{y}, \bar{q})$ at the vertex $\bar{q}$, respectively. Denote $\bar{\beta}$ the angle at the vertex $\bar{y}$. From the inequalities (2.1.4) we deduce that

$$\bar{\theta}_1 + \bar{\theta}_2 \geq \theta_1 + \theta_2 = \pi.$$

Thus Lemma 2.1.27 is applicable to getting that $\bar{\beta} \leq \beta'$. Therefore, using the law of cosines, we have $d(q, z) \leq \|q' - z'\|$. □

The following lemma is a consequence of the inequality (2.1.5) and the parallelogram identity in a Euclidean space $\mathbb{R}^n$:

$$\|x - y\|^2 + \|x + y\|^2 = 2(\|x\|^2 + \|y\|^2), \tag{2.1.6}$$

for all $x, y \in \mathbb{R}^n$.

Lemma 2.1.29. *For all $x, y, z \in M$ and $m \in M$ with $d(x, m) = d(y, m) = d(x, y)/2$, one has*

$$d^2(z, m) \leq \frac{1}{2}\, d^2(z, x) + \frac{1}{2}\, d^2(z, y) - \frac{1}{4}\, d^2(x, y). \tag{2.1.7}$$

From the well-known "law of cosines" in $\mathbb{R}^2$ and inequality (2.1.5) we deduce the following inequality, which is a general characteristic of the spaces with nonpositive curvature (see [11]).

Proposition 2.1.30. *For any $x, y, z \in M$ the following inequality holds,*

$$\langle \exp_x^{-1} y, \exp_x^{-1} z \rangle + \langle \exp_y^{-1} x, \exp_y^{-1} z \rangle \geq d^2(x, y).$$

2.2 Monotone and accretive vector fields

Let $\mathcal{X}(M)$ denote the set of all set-valued vector fields $A : M \to 2^{TM}$ such that $A(x) \subseteq T_xM$ for each $x \in \mathcal{D}(A)$, where $\mathcal{D}(A)$ denotes the domain of A defined by

$$\mathcal{D}(A) = \{x \in M : A(x) \neq \emptyset\}.$$

The concepts of monotonicity and strict monotonicity of single-valued vector fields defined on a Riemannian manifold were introduced by Németh in [81]. In [36], the strong monotonicity was defined. The authors of [41] provided an example of class of monotone vector fields, those which are gradients of convex functions. The complementary vector field of a mapping was introduced and proved to be monotone when T is nonexpansive in [80]. For more examples and relations between different kinds of generalized monotone vector fields in Riemannian manifolds see [81, 82, 83, 53].

The concept of monotone set-valued vector field was first introduced in [35] where it was shown that the subdifferential operator of a Riemannian convex function is a monotone set-valued vector field. We provide the notion of maximal monotonicity for set-valued vector fields and gather the previous concepts in the setting of Hadamard manifolds in the following definition. Note that they can be rewritten in general Riemannian manifold in terms of geodesics.

Definition 2.2.1. A vector field $A \in \mathcal{X}(M)$ is said to be

- *monotone* if for any $x, y \in \mathcal{D}(A)$,

$$\langle u, \exp_x^{-1} y\rangle \leq \langle v, -\exp_y^{-1} x\rangle, \quad \forall u \in A(x) \text{ and } \forall v \in A(y); \tag{2.2.1}$$

- *strictly monotone* if for any $x, y \in \mathcal{D}(A)$ with $x \neq y$, the strict inequality in (2.2.1) holds;

- *strongly monotone* if there exists $\rho > 0$ such that, for any $x, y \in \mathcal{D}(A)$,

$$\langle u, \exp_x^{-1} y\rangle - \langle v, -\exp_y^{-1} x\rangle \leq -\rho \mathrm{d}^2(x, y), \quad \forall u \in A(x) \text{ and } \forall v \in A(y); \tag{2.2.2}$$

- *maximal monotone* if it is monotone and for any $x \in M$ and $u \in T_xM$, the following implication holds:

$$\langle u, \exp_x^{-1} y\rangle \leq \langle v, -\exp_y^{-1} x\rangle,\ \forall y \in \mathcal{D}(A) \text{ and } v \in A(y) \Longrightarrow u \in A(x). \tag{2.2.3}$$

Remark 2.2.2. By definition, if A is a monotone vector field and $x \in \operatorname{int} \mathcal{D}(A)$ then, for each $v \in T_xM$, there exists a constant $\mu > 0$ such that $\langle u, v\rangle \leq \mu$ for all $u \in A(x)$. This means that $A(x)$ is bounded for any $x \in \operatorname{int} \mathcal{D}(A)$.

In order to characterize maximal monotone vector fields, the notions of upper semicontinuity and upper Kuratowski semicontinuity, as well as local boundedness, for operators in Banach spaces, (cf. [107, pag. 55]) are extended to the setting of Hadamard manifolds in the following definition.

Definition 2.2.3. Given $A \in \mathcal{X}(M)$ and $x_0 \in \mathcal{D}(A)$, the vector field A is said to be

- *upper semicontinuous* at x_0 if for any open set V satisfying $A(x_0) \subseteq V \subseteq T_{x_0}M$, there exists an open neighborhood $U(x_0)$ of x_0 such that $P_{x_0,x}A(x) \subseteq V$ for any $x \in U(x_0)$;

- *upper Kuratowski semicontinuous* at x_0 if for any sequences $\{x_k\} \subset \mathcal{D}(A)$ and $\{u_k\} \subset TM$ with each $u_k \in A(x_k)$, the relations $\lim_{k\to\infty} x_k = x_0$ and $\lim_{k\to\infty} u_k = u_0$ imply $u_0 \in A(x_0)$;

- *locally bounded* at x_0 if there exists an open neighborhood $U(x_0)$ of x_0 such that the set $\cup_{x\in U(x_0)} A(x)$ is bounded.

- upper semicontinuous (resp. upper Kuratowski semicontinuous, locally bounded) on M if it is upper semicontinuous (resp. upper Kuratowski semicontinuous, locally bounded) at each $x_0 \in \mathcal{D}(A)$.

Remark 2.2.4.

Remark 2.2.5. Clearly, the upper semicontinuity implies the upper Kuratowski semicontinuity. The converse is also true if A is locally bounded on M. Indeed, from the definition of upper semicontinuity we deduce that if A is locally bounded at x_0 but is not upper semicontinuous at x_0, then there exists $\{x_n\} \subset M$ and $v_n \in A(x_n)$ for any $n \geq 0$, such that $x_n \to x_0$, $v_n \to v_0 \in T_{x_0}M$ and $v_0 \notin A(x_0)$.

The following proposition shows that the maximality implies the upper Kuratowski semicontinuity. Let $x_0 \in \mathcal{D}(A)$ and let $T : T_{x_0}M \to 2^{T_{x_0}M}$ be the mapping defined by

$$T(u) = P_{x_0,\exp_{x_0} u} A(\exp_{x_0} u), \quad \forall u \in T_{x_0}M. \tag{2.2.4}$$

Proposition 2.2.6. *Let $A \in \mathcal{X}(M)$. Consider the following assertions.*

(i) *A is maximal monotone.*

(ii) *For each $x_0 \in \mathcal{D}(A)$, the mapping $T : T_{x_0}M \to 2^{T_{x_0}M}$ defined by* (2.2.4) *is upper Kuratowski semicontinuous on $T_{x_0}M$.*

(iii) *A is upper Kuratowski semicontinuous on M.*

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii)*.*

Proof. (i)$\Longrightarrow$(ii). Suppose that (i) holds. Let $x_0 \in M$ and $u_0 \in T_{x_0}M$. Let $\{u_n\} \subset T_{x_0}M$ and $\{v_n\} \subset T_{x_0}M$ with each $v_n \in T(u_n)$ be such that $u_n \to u_0$ and $v_n \to v_0$ for some $v_0 \in T_{x_0}M$. We have to verify that $v_0 \in T(u_0)$. To this end, set $x_n = \exp_{x_0} u_n$ and $\bar{v}_n = P_{x_n,x_0} v_n$ for each $n \geq 0$. Then by Lemma 2.1.15, $x_n \to \bar{x} := \exp_{x_0} u_0$ and $\bar{v}_n \in A(x_n)$ for each $n \geq 0$. Furthermore, we have that $\bar{v}_n \to P_{\bar{x},x_0} v_0$ because

$$P_{x_n,x_0}(v_n - v_0) \to 0, \quad P_{x_n,x_0} v_0 \to P_{\bar{x},x_0} v_0$$

and

$$\bar{v}_n = P_{x_n,x_0}(v_n - v_0) + P_{x_n,x_0} v_0.$$

On the other hand, by monotonicity,

$$\langle \bar{v}_n, \exp_{x_n}^{-1} y \rangle + \langle v, \exp_y^{-1} x_n \rangle \leq 0, \quad \forall y \in M \text{ and } v \in A(y). \tag{2.2.5}$$

Taking limit, as $n \to \infty$, (2.2.5) yields that

$$\langle P_{\bar{x},x_0} v_0, \exp_{\bar{x}}^{-1} y \rangle + \langle v, \exp_y^{-1} \bar{x} \rangle \leq 0, \quad \forall y \in M \text{ and } v \in A(y). \tag{2.2.6}$$

Since A is maximal monotone, $P_{\bar{x},x_0} v_0 \in A(\bar{x})$. Therefore, by the definition of T and the fact that $\bar{x} = \exp_{x_0} u_0$, one has that $v_0 \in T(u_0)$.

(ii)$\Longrightarrow$(iii). Let $x_0 \in M$. By (ii), the mapping $T : T_{x_0}M \to 2^{T_{x_0}M}$ defined by (2.2.4) is upper Kuratowski semicontinuous on $T_{x_0}M$. Since $\exp_{x_0}^{-1} : M \to T_{x_0}M$

is a diffeomorphism, it follows that the composition $T \circ \exp_{x_0}^{-1}$ is upper Kuratowski semicontinuous on M. Since

$$A(x) = P_{x,x_0}(T \circ \exp_{x_0}^{-1})(x), \quad \forall x \in M,$$

one sees that A is upper Kuratowski semicontinuous on M as P_{x,x_0} is an isometry. □

Recall the well-known result that maximal monotonicity and upper semicontinuity are equivalent for a set-valued operator with closed and convex values in a Hilbert space (cf. [87]). To extend this result to set-valued vector fields on Hadamard manifolds, we first need to prove the following lemma.

Lemma 2.2.7. *Suppose that $A \in \mathcal{X}(M)$ is maximal monotone and that $\mathcal{D}(A) = M$. Then A is locally bounded on M.*

Proof. Let $x_0 \in M$. Suppose on the contrary that A is not locally bounded at x_0. Then there exist sequences $\{x_n\} \subset \mathcal{D}(A)$ and $\{v_n\} \subset TM$ with each $v_n \in A(x_n)$ such that, $x_n \to x_0$ but $\|v_n\| \to \infty$. Note that $A(x_0)$ is bounded by Remark 2.2.2. Hence

$$\rho := \sup\{\|u\| : u \in A(x_0)\} < \infty.$$

Taking $v_0 \in A(x_0)$, we define

$$u_n = (1 - t_n)P_{x_n,x_0}v_0 + t_n v_n, \quad \forall n \geq 1$$

where $\{t_n\} \subset [0, 1]$ such that $\|u_n\| = \rho + 1$ for each $n \geq 1$. This means that $\{u_n\}$ is bounded and $t_n \to 0$. Without loss of generality, assume that $u_n \to u_0$ for some $u_0 \in T_{x_0}M$. Then $\|u_0\| = \rho + 1$ and $u_0 \notin A(x_0)$. On the other hand, for any $y \in \mathcal{D}(A)$ and $v \in A(y)$, one has that, for each $n \geq 1$,

$$\begin{aligned} \langle u_n, \exp_{x_n}^{-1} y\rangle + \langle v, \exp_y^{-1} x_n\rangle &= (1 - t_n)\big(\langle P_{x_n,x_0}v_0, \exp_{x_n}^{-1} y\rangle + \langle v, \exp_y^{-1} x_n\rangle\big) \\ &\quad + t_n\big(\langle v_n, \exp_{x_n}^{-1} y\rangle + \langle v, \exp_y^{-1} x_n\rangle\big) \\ &\leq (1 - t_n)\big(\langle P_{x_n,x_0}v_0, \exp_{x_n}^{-1} y\rangle + \langle v, \exp_y^{-1} x_n\rangle\big), \end{aligned}$$

where the last inequality holds because $t_n \geq 0$ and $\langle v_n, \exp_{x_n}^{-1} y\rangle + \langle v, \exp_y^{-1} x_n\rangle \leq 0$ thanks to the monotonicity of A. Now, letting $n \to \infty$, we get that

$$\langle u_0, \exp_{x_0}^{-1} y\rangle + \langle v, \exp_y^{-1} x_0\rangle \leq \langle v_0, \exp_{x_0}^{-1} y\rangle + \langle v, \exp_{x_0}^{-1} y\rangle \leq 0,$$

by Lemma 2.1.15 and the monotonicity of A. Since A is maximal monotone, it follows that $u_0 \in A(x_0)$, which is a contradiction. □

Theorem 2.2.8. *Suppose that $A \in \mathcal{X}(M)$ is monotone and that $\mathcal{D}(A) = M$. Then the following statements are equivalent.*

(i) *A is maximal monotone.*

(ii) *For any $x_0 \in M$, the mapping $T : T_{x_0}M \to 2^{T_{x_0}M}$ defined by (2.2.4) is upper semicontinuous on $T_{x_0}M$, and $T(u)$ is closed and convex for each $u \in T_{x_0}M$.*

(iii) *A is upper semicontinuous on M, and $A(x)$ is closed and convex for each $x \in M$.*

Proof. (i)⇒(ii). Assuming that (i) holds, by Lemma 2.2.7, we have that A is locally bounded. Given $x_0 \in M$, let T be defined by (2.2.4). Then T is locally bounded because the mapping $\exp_{x_0} : T_{x_0}M \to M$ is a diffeomorphism and the parallel transport $P_{x_0,\exp_{x_0}}$ is an isometry. Furthermore, T is upper Kuratowski semicontinuous on $T_{x_0}M$ by Proposition 2.2.6. Thus, bearing in mind Remark 2.2.5, we conclude that T is upper semicontinuous on $T_{x_0}M$.

It remains to prove that $T(u)$ is closed and convex for each $u \in T_{x_0}M$. To this end, let $u \in T_{x_0}M$ and $x = \exp_{x_0} u$. For the sake of simplicity, we use $G(A)$ to denote the graph of A defined by

$$G(A) := \{(y, v) \in M \times TM : \ v \in A(y)\}.$$

By the maximality of A, we see that

$$T(u) = P_{x_0,x}A(x) = P_{x_0,x} \bigcap_{(y,v)\in G(A)} \big\{w \in T_xM : \langle w, \exp_x^{-1} y\rangle + \langle v, \exp_y^{-1} x\rangle \le 0\big\}.$$

Therefore $T(u)$ is closed and convex.

(ii)⇒(iii). Let $x_0 \in M$. It suffices to prove that A is upper semicontinuous at x_0 because $A(x_0) = T(0)$ is closed and convex by (ii), where $T : T_{x_0}M \to 2^{T_{x_0}M}$ is defined by (2.2.4). For this purpose, consider the set-valued mapping $S : M \to 2^{T_{x_0}M}$ defined by

$$S(x) = P_{x_0,x}A(x), \quad \forall x \in M.$$

It is clear that A is upper semicontinuous at x_0 if and only if so is S. Since the mapping T is upper semicontinuous on $T_{x_0}M$ by (ii) and since $\exp_{x_0}^{-1} : M \to T_{x_0}M$ is a diffeomorphism, it follows that the composition $T \circ \exp_{x_0}^{-1}$ is upper semicontinuous on M. Noting that

$$S(x) = P_{x_0,x}A(x) = (T \circ \exp_{x_0}^{-1})(x), \quad \forall x \in M,$$

one sees that S is upper semicontinuous on M and so at x_0.

(iii)⇒(i). Suppose that (iii) holds but A is not maximal. Then there exist $x_0 \in M$ and $u_0 \in T_{x_0}M \setminus A(x_0)$ such that

$$\langle u_0, \exp_{x_0}^{-1} y\rangle \le \langle v, -\exp_y^{-1} x_0\rangle, \quad \forall y \in M \text{ and } \forall v \in A(y). \tag{2.2.7}$$

Note that $A(x_0)$ is a convex closed set by (iii), so the well-known separation theorem is applicable and there exists $h \in T_{x_0}M$ such that

$$\langle u_0, h\rangle > \alpha = \sup_{u \in A(x_0)} \langle u, h\rangle.$$

Define $V := \{u \in T_{x_0}M : \langle u, h\rangle < \langle u_0, h\rangle\}$. Then V is an open set containing $A(x_0)$. By (iii), A is upper semicontinuous at x_0; thus there exists a neighborhood $U(x_0)$ of x_0 such that $P_{x_0,x}A(x) \subseteq V$ for each $x \in U(x_0)$. Now, set $x_t := \exp_{x_0} th$ for each $t > 0$. Then $x_t \to x_0$, as $t \to 0$. Hence $x_t \in U(x_0)$ and $P_{x_0,x_t}A(x_t) \subseteq V$ for all $t > 0$ small enough. This means that we can take some $t > 0$ such that

$$\langle P_{x_0,x_t}v, h\rangle < \langle u_0, h\rangle, \quad \forall v \in A(x_t).$$

Since $th = \exp_{x_0}^{-1} x_t$, the previous inequality turns into

$$\langle P_{x_0,x_t}v, \exp_{x_0}^{-1} x_t\rangle < \langle u_0, \exp_{x_0}^{-1} x_t\rangle, \quad \forall v \in A(x_t).$$

Therefore, for each $v \in A(x_t)$,

$$\langle v, -\exp_{x_t}^{-1} x_0\rangle = \langle v, P_{x_t,x_0} \exp_{x_0}^{-1} x_t\rangle = \langle P_{x_0,x_t}v, \exp_{x_0}^{-1} x_t\rangle < \langle u_0, \exp_{x_0}^{-1} x_t\rangle,$$

which contradicts (2.2.7). □

We now extend the classical definition of accretive operators on Banach spaces to vector fields defined on Hadamard manifolds.

Definition 2.2.9. Let $A \in \mathcal{X}(M)$ and $\alpha > 0$. The vector field A is said to be

- *accretive* if for any $x, y \in \mathcal{D}(A)$ and each $r \geq 0$ we have that
$$d(x,y) \leq d(\exp_x(ru), \exp_y(rv)), \text{ for each } u \in A(x) \text{ and } v \in A(y); \tag{2.2.8}$$
- *strictly accretive* if for any $x, y \in \mathcal{D}(A)$ with $x \neq y$ and each $r \geq 0$, the strict inequality in (2.2.8) holds;
- *α-strongly accretive* if for any $x, y \in \mathcal{D}(A)$ and each $r \geq 0$ we have that
$$(1+\alpha r)d(x,y) \leq d(\exp_x(ru), \exp_y(rv)), \text{ for each } u \in A(x) \text{ and } v \in A(y); \tag{2.2.9}$$
- *m-accretive* if it is accretive and
$$\bigcup_{x \in \mathcal{D}(A)} \left(\bigcup_{u \in A(x)} \exp_x u \right) = M. \tag{2.2.10}$$

Note that these definitions make also sense in the setting of Riemannian manifolds. However, it is in the particular case of a Hadamard manifold where the notions of accretivity and monotonicity can be proved to be equivalent. The following lemma is an essential tool to study this equivalence.

Lemma 2.2.10. *Let $x, y \in M$ with $x \neq y$, $u \in T_xM$ and $v \in T_yM$. Then*
$$\left(\frac{\mathrm{d}}{\mathrm{d}s} d(\exp_x su, \exp_y sv) \right)_{s=0} = \frac{1}{d(x,y)} \left(-\langle u, \exp_x^{-1} y \rangle + \langle v, -\exp_y^{-1} x \rangle \right). \tag{2.2.11}$$

Proof. Let $\varepsilon > 0$ and $f : (-\varepsilon, \varepsilon) \times [0,1] \to M$ be the function defined by
$$f(s,t) = \exp_{\exp_x su} t(\exp^{-1}_{\exp_x su} \exp_y sv), \text{ for each } (s,t) \in (-\varepsilon, \varepsilon) \times [0,1].$$

Let γ be the geodesic joining x to y. It follows that
$$\gamma'(0) = \exp_x^{-1} y \quad \text{and} \quad \gamma'(1) = -\exp_y^{-1} x. \tag{2.2.12}$$

Since the exponential map is differentiable and $f(0,t)=\gamma(t)$, f is a variation of γ and $V(t)=\frac{\partial f}{\partial s}(0,t)$ is the variational field of f. In particular,

$$V(0)=\frac{\partial f}{\partial s}(0,0)=u \quad \text{and} \quad V(1)=\frac{\partial f}{\partial s}(0,1)=v. \tag{2.2.13}$$

Note that, for each $s\in(-\varepsilon,\varepsilon)$, the parameterized curve $f_s:[0,1]\to M$ given by $f_s(t)=f(s,t)$ is a geodesic. Then $\left\|\frac{\partial f}{\partial t}(s,t)\right\|$ is a constant. Moreover

$$\left\|\frac{\partial f}{\partial t}(s,t)\right\|=\|\exp^{-1}_{\exp_x su}\exp_y sv\|=d(\exp_x su,\exp_y sv). \tag{2.2.14}$$

Define $L:(-\varepsilon,\varepsilon)\to\mathbb{R}$ by

$$L(s)=\int_0^1\left\|\frac{\partial f}{\partial t}(s,t)\right\|\mathrm{d}t, \quad \text{for each } s\in(-\varepsilon,\varepsilon). \tag{2.2.15}$$

Therefore, by [105, p. 38, Proposition 2.5],

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}s}L(s)\right)_{s=0} &= \frac{1}{l(\gamma)}\left(-\int_0^1\left\langle V(t),\frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\rangle\mathrm{d}t\right)\\ &\quad+\frac{1}{l(\gamma)}\left(\left\langle V(1),\frac{\mathrm{d}\gamma}{\mathrm{d}t}(1)\right\rangle-\left\langle V(0),\frac{\mathrm{d}\gamma}{\mathrm{d}t}(0)\right\rangle\right)\\ &= \frac{1}{d(x,y)}\left(\left\langle\frac{\partial f}{\partial s}(0,1),\gamma'(1)\right\rangle-\left\langle\frac{\partial f}{\partial s}(0,0),\gamma'(0)\right\rangle\right)\end{aligned} \tag{2.2.16}$$

where the second equality holds because γ is a geodesic and $\frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{d}\gamma}{\mathrm{d}t}=0$. Then, bearing in mind that

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}d(\exp_x su,\exp_y sv)\right)_{s=0}=\left(\frac{\mathrm{d}}{\mathrm{d}s}L(s)\right)_{s=0},$$

equality (2.2.11) follows from (2.2.12), (2.2.13) and (2.2.16). □

In view of the previous definitions and Lemma 2.2.10, we deduce the following characterization of monotonicity.

Corollary 2.2.11. *Let $A\in\mathcal{X}(M)$ and $\alpha>0$. Then the following assertions hold.*

(i) *A is monotone if and only if for any $x, y \in \mathcal{D}(A)$,*

$$\left(\frac{d}{ds} d(\exp_x(su), \exp_y(sv))\right)_{s=0} \geq 0, \text{ for each } u \in A(x) \text{ and } v \in A(y). \tag{2.2.17}$$

(ii) *A is strictly monotone if and only if for any $x, y \in \mathcal{D}(A)$ with $x \neq y$,*

$$\left(\frac{d}{ds} d(\exp_x(su), \exp_y(sv))\right)_{s=0} > 0, \text{ for each } u \in A(x) \text{ and } v \in A(y). \tag{2.2.18}$$

(iii) *A is α-strongly monotone if and only if for any $x, y \in \mathcal{D}(A)$,*

$$\left(\frac{d}{ds} d(\exp_x(su), \exp_y(sv))\right)_{s=0} \geq \alpha d(x, y), \text{ for each } u \in A(x) \text{ and } v \in A(y). \tag{2.2.19}$$

Remark 2.2.12. It is worth mentioning that, in [53], Iwamiya and Okochi defined a monotone vector field $A \in \mathcal{X}(M)$ on a more general Riemannian manifold by requiring that, for any $x, y \in \mathcal{D}(A)$ and for each $u \in A(x),\ v \in A(y)$, the inequality

$$\left(\frac{\mathrm{d}}{\mathrm{d}s} d(\exp_x(su), \exp_y(sv))\right)_{s=0} := \lim_{s \to 0} \frac{d(\exp_x(su), \exp_y(sv)) - d(x, y)}{s} \geq 0$$

holds. When M is a Hadamard manifold, we deduce from Corollary 2.2.11 that this definition coincides with the one we presented in this section.

Theorem 2.2.13. *Let $A \in \mathcal{X}(M)$ and $\alpha > 0$. Then the following assertions hold.*

(i) *A is accretive if and only if A is monotone.*

(ii) *A is α-strongly accretive if and only if A is α-strongly monotone.*

(iii) *If A is m-accretive, then A is maximal monotone.*

Proof.

(i) If A is accretive, for any $x, y \in \mathcal{D}(A)$ and each $u \in A(x)$, $v \in A(y)$, we have that

$$d(x,y) \le d(\exp_x(ru), \exp_y(rv)), \tag{2.2.20}$$

for all $r \ge 0$, which implies that

$$\left(\frac{\mathrm{d}}{\mathrm{d}s} d(\exp_x(su), \exp_y(sv))\right)_{s=0} \ge 0. \tag{2.2.21}$$

Then, by Corollary 2.2.11, A is monotone. Assume now that A is monotone. Let $x, y \in \mathcal{D}(A)$ and $u \in A(x)$, $v \in A(y)$. Define a mapping $g : [0, +\infty) \to [0, +\infty)$ by

$$g(r) = d(\exp_x ru, \exp_y rv) \quad \text{for each } r \in [0, +\infty).$$

Since M is a Hadamard manifold, we get that $g(\cdot)$ is a convex function by Proposition 2.1.25. Hence,

$$\left(\frac{d}{dr} d(\exp_x(ru), \exp_y(rv))\right)_{r=0} = \inf_{r \ge 0} \frac{g(r) - g(0)}{r}. \tag{2.2.22}$$

By Corollary 2.2.11(i), we have that A is monotone if and only if

$$\left(\frac{d}{dr} d(\exp_x(ru), \exp_y(rv))\right)_{r=0} \ge 0. \tag{2.2.23}$$

Then, by (2.2.22), (2.2.23) is equivalent to

$$g(r) - g(0) \ge 0, \quad \text{for each } r \ge 0,$$

which means that A is accretive.

(ii) If A is α-strongly accretive, for any $x, y \in \mathcal{D}(A)$ and each $u \in A(x)$, $v \in A(y)$, we have that

$$(1 + \alpha r) d(x,y) \le d(\exp_x(ru), \exp_y(rv)), \tag{2.2.24}$$

for all $r \ge 0$, which means that

$$\left(\frac{\mathrm{d}}{\mathrm{d}s} d(\exp_x su, \exp_y sv)\right)_{s=0} \ge \alpha d(x,y). \tag{2.2.25}$$

Then, by Corollary 2.2.11 A is strongly monotone. Assume that A is α-strongly monotone. Let $x, y \in \mathcal{D}(A)$ and $u \in A(x)$, $v \in A(y)$. Define a mapping $g : [0, +\infty) \to [0, +\infty)$ by

$$g(r) = d(\exp_x ru, \exp_y rv) - \alpha\, r\, d(x, y), \qquad \text{for each } r \in [0, +\infty).$$

Then $g(\cdot)$ is a convex function. Hence,

$$\left(\frac{d}{dr} d(\exp_x(ru), \exp_y(rv))\right)_{r=0} - \alpha d(x, y) = \inf_{r \geq 0} \frac{g(r) - g(0)}{r}. \tag{2.2.26}$$

Recall that by Corollary 2.2.11(ii), we have that A is α-strongly monotone if and only if

$$\left(\frac{d}{dr} d(\exp_x(ru), \exp_y(rv))\right)_{r=0} - \alpha d(x, y) \geq 0. \tag{2.2.27}$$

Then, by (2.2.26), (2.2.27) is equivalent to

$$g(r) - g(0) \geq 0, \qquad \text{for each } r \geq 0,$$

which means that A is α-strongly accretive.

(iii) Assume that A is m-accretive. In particular, A is accretive so by (i) A is monotone. In order to prove the maximality we take $x \in M$ and $u \in T_x M$ such that, for any $y \in \mathcal{D}(A)$ and $v \in A(y)$,

$$\langle u, \exp_x^{-1} y \rangle \leq -\langle v, \exp_y^{-1} x \rangle. \tag{2.2.28}$$

Defining the vector field $B : M \to 2^{TM}$ by $B(y) = A(y)$ if $x \neq y$ and $B(x) = u$, since A is monotone, the inequality (2.2.28) implies that B is monotone. Then B is accretive, so, for $r = 1$ in the definition, we have that

$$d(x, y) \leq d(\exp_x u, \exp_y v), \tag{2.2.29}$$

for any $y \in \mathcal{D}(B)$ and $v \in B(y)$, since $u \in B(x)$. On the other hand, since A is m-accretive, by (2.2.10), there exists $y \in \mathcal{D}(A)$ and $v \in A(y)$ such that

$$\exp_x u = \exp_y v. \tag{2.2.30}$$

Then from inequality (2.2.29) we deduce that $x = y$ and therefore $u = v \in A(x)$, establishing that A is maximal monotone.

□

Remark 2.2.14. The converse of assertion (iii) in the previous theorem is true when $\mathcal{D}(A) = M$. It will be proved in Corollary 2.4.5.

2.3 Nonexpansive type mappings

In this section we study some properties of the nonexpansive type mappings that will be essential to characterize the monotone vector fields. In particular, we introduce the definition of firmly nonexpansive mappings on Hadamard manifolds and provide an analysis of these mappings. By using the concept of complementary vector field, we also establish a connection with the class of pseudo-contractive mappings introduced by Reich and Shafrir, [101], in the more general setting of hyperbolic spaces.

Recall that, given a subset $C \subseteq M$, $T : C \to M$ is a nonexpansive mapping if for any $x, y \in C$,

$$d(T(x), T(y)) \leq d(x, y). \tag{2.3.1}$$

Let us denote the fixed point set of T by

$$\mathrm{Fix}\,(T) := \{x \in C : x = T(x)\}. \tag{2.3.2}$$

From either Brouwer's theorem or the fixed point property for CAT(0) spaces (cf. [59]), the existence of fixed points is ensured provided that C is bounded. Kirk, in [60], proved the following result in complete CAT(0) spaces. However, we include the proof in the setting of Hadamard manifolds for the sake of completeness.

Proposition 2.3.1. *Let $T : C \to M$ be a nonexpansive mapping defined on a closed convex set $C \subseteq M$. Then the fixed point set Fix(T) is closed and convex.*

Proof. Let us start by proving that $\mathrm{Fix}\,(T)$ is closed. Let $\{x_n\}$ be a sequence in $\mathrm{Fix}\,(T)$ that converges to $x \in M$. Then $x \in C$ by closeness of C, and $T(x_n) \to T(x)$ by continuity (nonexpansivity) of T. On the other hand, since $\{x_n\} \subset \mathrm{Fix}\,(T)$, $T(x_n) \to x$. Altogether, we have that $T(x) = x$. Therefore, $\mathrm{Fix}\,(T)$ is closed.

In order to prove the convexity, let $x, y \in \mathrm{Fix}\,(T)$ and let γ be the geodesic joining x to y. Given a point q in the geodesic γ, that is $q = \gamma(t)$ for some $t \in [0, 1]$, we just need to prove that $q \in \mathrm{Fix}\,(T)$. Since C is convex, $T(q)$ is well-defined. Consider the geodesic triangle $\Delta(x, y, Tq)$ with comparison triangle $\Delta(x', y', (Tq)')$ in $\mathbb{R}^2$, and the corresponding point to the point q, $q' = (1 - t)x' + ty'$; that is $d(x', q') = d(x, q)$ and $d(q', y') = d(q, y)$, see Figure 2.2. On the other hand, recall the fact that for any $\alpha \in \mathbb{R}^+$ and any $a, b \in \mathbb{R}^2$, we have the equality

$$\|\alpha a + (1 - \alpha)b\|^2 = \alpha\|a\|^2 + (1 - \alpha)\|b\|^2 - \alpha(1 - \alpha)\|a - b\|^2. \tag{2.3.3}$$

Then, from Lemma 2.1.28, equality (2.3.3) and the nonexpansivity of T, we obtain

$$\begin{aligned}
d^2(q,T(q)) &\leq d^2(q',(T(q))') \\
&= \|(1-t)(x'-(T(q))')+t(y'-(T(q))')\|^2 \\
&= (1-t)\|x'-(T(q))'\|^2+t\|y'-(T(q))'\|^2-t(1-t)\|x'-y'\|^2 \\
&= (1-t)d^2(x,T(q))+td(y,T(q))-t(1-t)d^2(x,y) \\
&\leq (1-t)d^2(x,q)+td(y,q)-t(1-t)d^2(x,y) \\
&= (1-t)\|x'-q'\|^2+t\|y'-q'\|^2-t(1-t)\|x'-y'\|^2 \\
&= \|(1-t)x'+ty'-q'\|^2 \\
&= 0.
\end{aligned}$$

So $q=T(q)$ for any q in the geodesic joining $x \in \text{Fix}\,(T)$ to $y \in \text{Fix}\,(T)$, thus $\text{Fix}\,(T)$ is convex. □

2.3.1 Firmly nonexpansive mappings

The notion of firm nonexpansivity was previously defined on a Banach space [17, 18] and the Hilbert ball with the hyperbolic metric [48], so-called firmly nonexpansive mapping of the first kind in the latter case. In fact, the following analysis shows that in the framework of Hadamard manifolds this class of mappings verifies similar properties to those ones well-known in Hilbert spaces.

Definition 2.3.2. Given a mapping $T : C \subseteq M \to M$, we say that T is *firmly nonexpansive* if for any $x, y \in C$, the function $\theta : [0,1] \to [0,\infty]$ defined by

$$\theta(t) = d(\gamma_1(t), \gamma_2(t)), \tag{2.3.4}$$

is nonincreasing, where γ_1 and γ_2 denote the geodesics joining x to $T(x)$ and y to $T(y)$, respectively.

Remark 2.3.3. From the definition we readily deduce that any firmly nonexpansive mapping T is nonexpansive.

Proposition 2.3.4. *Let $T : C \subseteq M \to M$. Then the following assertions are equivalent.*

(i) *T is firmly nonexpansive.*

(ii) *For any $x, y \in C$ and $t \in [0,1]$*

$$d(T(x), T(y)) \leq d(\exp_x t \exp_x^{-1} Tx, \exp_y t \exp_y^{-1} Ty). \tag{2.3.5}$$

(iii) *For any $x, y \in C$*

$$\langle \exp_{T(x)}^{-1} T(y), \exp_{T(x)}^{-1} x\rangle + \langle \exp_{T(y)}^{-1} T(x), \exp_{T(y)}^{-1} y\rangle \leq 0. \tag{2.3.6}$$

Proof. Given $x, y \in C$, let $\theta : [0,1] \to [0,\infty]$ be the function defined by (2.3.4). Note that from Proposition 2.1.25 we deduce that θ is a convex function.

Let us start by proving that the derivative at 1^- of the function θ can be formulated as

$$(\theta)'(1) = \langle \exp_{T(x)}^{-1} T(y), \exp_{T(x)}^{-1} x\rangle + \langle \exp_{T(y)}^{-1} T(x), \exp_{T(y)}^{-1} y\rangle. \tag{2.3.7}$$

Let $u = \exp_{T(x)}^{-1} x \in T_{T(x)}M$ and $v = \exp_{T(y)}^{-1} y \in T_{T(y)}M$. Then the function θ can be written as

$$\theta(t) = d(\exp_{T(x)}(1-t)u, \exp_{T(y)}(1-t)v). \tag{2.3.8}$$

Let γ be the geodesic joining $T(x)$ to $T(y)$, that is, for any $r \in [0,1]$

$$\gamma(r) = \exp_{T(x)} r \exp_{T(x)}^{-1} T(y). \tag{2.3.9}$$

Given $\varepsilon > 0$ we define the function $f : (-\varepsilon, \varepsilon) \times [0,1] \to M$ by

$$f(s,r) = \exp_{\exp_{Tx} su} r(\exp_{\exp_{Tx} su}^{-1} \exp_{Ty} sv) \quad \text{for each } (s,r) \in (-\varepsilon,\varepsilon) \times [0,1]. \tag{2.3.10}$$

Note that $f(0,r) = \gamma(r)$. Then, since the exponential map is differentiable and f is a variation of γ, $V(r) = \frac{\partial f}{\partial s}(0,r)$ is the variational field of f. In particular,

$$V(0) = \frac{\partial f}{\partial s}(0,0) = u \quad \text{and} \quad V(1) = \frac{\partial f}{\partial s}(0,1) = v. \tag{2.3.11}$$

Note that for each $s \in (-\varepsilon, \varepsilon)$, the parameterized curve $f_s : [0,1] \to M$ given by $f_s(r) = f(s,r)$ is a geodesic and therefore $\left\| \frac{\partial f}{\partial r}(s,r) \right\|$ is a constant. In particular, from (2.3.8),

$$\left\| \frac{\partial f}{\partial r}(s,r) \right\| = \| \exp^{-1}_{\exp_{T(x)} su} \exp_{T(y)} sv \| = d(\exp_{T(x)} su, \exp_{T(y)} sv) = \theta(1-s). \tag{2.3.12}$$

Define $L : (-\varepsilon, \varepsilon) \to \mathbb{R}$ by

$$L(s) = \int_0^1 \left\| \frac{\partial f}{\partial r}(s,r) \right\| \mathrm{d}r, \qquad \text{for each } s \in (-\varepsilon, \varepsilon). \tag{2.3.13}$$

Recall that $\left\| \frac{\partial f}{\partial t}(s,r) \right\|$ is a constant, thus

$$L^2(s) = \int_0^1 \left\| \frac{\partial f}{\partial t}(s,t) \right\|^2 \mathrm{d}t = \theta^2(1-s).$$

Therefore, by the first variation formula stated in [105, p. 89, Proposition 2.2] (where E denotes the energy integral $E(s) = \frac{1}{2}L^2(s)$)

$$\begin{aligned} \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}s}L^2(s)\right)_{s=0} &= -\int_0^1 \left\langle V(r), \frac{\mathrm{D}}{\mathrm{d}r}\frac{\mathrm{d}\gamma}{\mathrm{d}r} \right\rangle \mathrm{d}r - \left\langle V(0), \frac{\mathrm{d}\gamma}{\mathrm{d}r}(0) \right\rangle + \left\langle V(1), \frac{\mathrm{d}\gamma}{\mathrm{d}r}(1) \right\rangle \\ &= -\left\langle \frac{\partial f}{\partial s}(0,0), \gamma'(0) \right\rangle + \left\langle \frac{\partial f}{\partial s}(0,1), \gamma'(1) \right\rangle \qquad (2.3.14) \\ &= -\left\langle \exp^{-1}_{T(x)} x, \exp^{-1}_{T(x)} T(y) \right\rangle + \left\langle \exp^{-1}_{T(y)} y, -\exp^{-1}_{T(y)} T(x) \right\rangle, \end{aligned}$$

where the second equality holds because $\frac{\mathrm{D}}{\mathrm{d}r}\frac{\mathrm{d}\gamma}{\mathrm{d}r} = 0$, since γ is a geodesic. Then the fact that

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}L^2(s)\right)_{s=0} = -\left(\frac{\mathrm{d}}{\mathrm{d}s}\theta^2(t)\right)_{t=1} = -(\theta^2)'(1) = -2\theta'(1), \tag{2.3.15}$$

together with inequality (2.3.14), implies that equality (2.3.7) holds.

By definition the mapping T is firmly nonexpansive if and only if the convex function θ is nonincreasing. On the one hand, this is equivalent to $\theta(1) \leq \theta(t)$, for any $t \in [0,1]$, which is exactly the inequality (2.3.5), proving the equivalence between (i) and (ii). On the other hand, since θ is differentiable at 1 and convex, the definition of firmly nonexpansive mapping is equivalent to $\theta'(1) \leq 0$, and then to inequality (2.3.6) because of equality (2.3.7). Then (i) is proved to be equivalent to (iii). □

The following result provides us with an important example of class of firmly nonexpansive mappings, the metric projection.

Corollary 2.3.5. *The metric projection onto a closed convex subset $C \subseteq M$ is a firmly nonexpansive mapping.*

Proof. Let $T = P_C$ be the metric projection onto C. For any $x, y \in M$, by Proposition 2.1.18, we have that

$$\langle \exp_{T(x)}^{-1} T(y), \exp_{T(x)}^{-1} x \rangle \leq 0 \tag{2.3.16}$$

because $T(y) \in C$, and analogously, since $T(x) \in C$,

$$\langle \exp_{T(y)}^{-1} T(x), \exp_{T(y)}^{-1} y \rangle \leq 0. \tag{2.3.17}$$

By summing inequalities (2.3.16) and (2.3.17) we obtain inequality (2.3.6) which implies the firmly nonexpansivity of P_C by Proposition 2.3.4. □

Given a nonexpansive mapping T, as it happens in Hilbert spaces, there exists another example of class of firmly nonexpansive mappings constituted by an associated family of mappings $\{G_t : 0 \leq t < 1\}$, whose fixed point set coincides with the fixed point set of T.

Indeed, given $T : C \to C$ nonexpansive and $x \in C$, for any $t \in [0, 1)$, let T_t be the mapping defined by

$$T_t y = \exp_x t \exp_x^{-1} Ty, \tag{2.3.18}$$

for any $y \in C$. It is straightforward to see that T_t is a contraction for any $t \in [0, 1)$. Then the Banach contraction principle implies that there exists a unique fixed point of T_t, which is being denoted by x_t. By means of the approximating curve $\{x_t\}$, for any $t \in [0, 1)$, we define the mapping $G_t : C \to C$ by

$$G_t(x) =: x_t = \exp_x t \exp_x^{-1} T(G_t(x)),$$

for all $x \in C$. Then the following result holds.

Proposition 2.3.6. *Let $t \in [0, 1)$.*

(i) *The mapping G_t is firmly nonexpansive.*

(ii) $Fix(G_t) = Fix(T)$.

Proof.

(i) Let $x, y \in K$. From the convexity of the distance, we have that

$$\mathrm{d}(G_t(x), G_t(y)) \leq (1-t)\mathrm{d}(x,y) + t\mathrm{d}(G_t(x), G_t(y)),$$

which implies the nonexpansivity of G_t. Now consider the geodesics γ_1 and γ_2 joining x to $G_t(x)$ and y to $G_t(y)$, respectively. Let $s \in [0,1]$ and set $r := \frac{(1-s)t}{1-st} \in [0,1)$. Then $r \in [0,1)$ and satisfies that

$$G_t(x) = \exp_{\gamma_1(s)} r \, \exp^{-1}_{\gamma_1(s)} T(G_t(x)) \tag{2.3.19}$$

and

$$G_t(y) = \exp_{\gamma_2(s)} r \, \exp^{-1}_{\gamma_2(s)} T(G_t(y)). \tag{2.3.20}$$

To show this fact, note that by the definition of the mapping G_t the geodesics γ_1 and γ_2 are contained in the geodesics α_1 and α_2 joining x to $T(G_t(x))$ and y to $T(G_t(y))$, respectively. In particular, the point $\gamma_1(s)$ belongs to the geodesic α_1.

Since

$$\mathrm{d}(x, \gamma_1(s)) = s\,\mathrm{d}(x, G_t(x)) = s\,t\,\mathrm{d}(x, T(G_t(x))), \tag{2.3.21}$$

it follows that

$$\mathrm{d}(\gamma_1(s), T(G_t(x)))) = (1-st)\,\mathrm{d}(x, T(G_t(x))). \tag{2.3.22}$$

Therefore, using the definitions of $\gamma_1(s)$, inequalities (2.3.21) and (2.3.22), we obtain that

$$\begin{aligned}
\mathrm{d}(\gamma_1(s), G_t(x)) &= (1-s)\,\mathrm{d}(x, G_t(x)) \\
&= (1-s)\,t\,\mathrm{d}(x, T(G_t(x))) \\
&= (1-s)\,t\,\frac{1}{1-st}\,\mathrm{d}(\gamma_1(s), T(G_t(x))) \\
&= r\,\mathrm{d}(\gamma_1(s), T(G_t(x))).
\end{aligned}$$

This means that (2.3.19) is satisfied. Similarly, (2.3.20) is satisfied. Thus, by the convexity of the distance and the nonexpansivity of G_t, we obtain that

$$\mathrm{d}(G_t(x), G_t(y)) \leq (1-r)\,\mathrm{d}(\gamma_1(s), \gamma_2(s)) + r\,\mathrm{d}(G_t(x), G_t(y)).$$

This implies that, for any $s \in [0,1]$

$$\mathrm{d}(G_t(x), G_t(y)) \leq \mathrm{d}(\gamma_1(s), \gamma_2(s)). \tag{2.3.23}$$

Hence, inequality (2.3.5) holds and G_t is firmly nonexpansive by Proposition 2.3.4.

(ii) The following equivalences prove the equality.

$$\begin{aligned} x = G_t(x) &\Leftrightarrow x = \exp_x (1-t) \exp_x^{-1} Tx \\ &\Leftrightarrow 0 = \exp_x^{-1} Tx \\ &\Leftrightarrow x = Tx. \end{aligned}$$

□

2.3.2 Pseudo-contractive mappings

In the setting of the Hilbert spaces there is a class of nonexpansive type mappings, the so called pseudo-contractive mappings, which are closely related to the class of monotone operators. Given a subset C of a Hilbert space H, a set-valued mapping $T : C \to 2^H$ is said to be pseudo-contractive if for all $x, y \in C$, for any $u \in T(x), v \in T(y)$ and $r \geq 0$,

$$\|x - y\| \leq \|(1+r)(x-y) - r(u-v)\|.$$

This concept was introduced by Browder and Petryshyn, in [**?**], and they proved that a mapping T is pseudo-contractive if and only if the operator $I - T$ is monotone. This means that the problem of solving an equation involving monotone operators may be formulated as a fixed point problem for a pseudo-contractive mapping on a Hilbert space.

We define the concept of pseudo-contractive mappings in the setting of Hadamard manifolds, by using the notion of complementary vector field which was introduced by Németh, in [80], in order to provide a relation between nonexpansive mappings and monotone vector fields.

Definition 2.3.7. [80] Let $T : C \subseteq M \to M$. The vector field $A \in \mathcal{X}(M)$ defined by

$$A(x) = -\exp_x^{-1} T(x), \tag{2.3.24}$$

for any $x \in C$, is said to be the complementary vector field of T.

Definition 2.3.8. Let $T : C \subseteq M \to M$ and $\alpha > 0$. Then T is said to be

- *pseudo-contractive* if its complementary vector field is accretive;
- *α-strongly pseudo-contractive* if its complementary vector field is α-strongly accretive.

Remark 2.3.9. The definition of pseudo-contractive mappings coincides with the one introduced by Reich and Shafrir in the more general setting of hyperbolic spaces [101].

In view of Definition 2.3.8 and Theorem 2.2.13, we deduce the following result.

Corollary 2.3.10. *Let $T : C \subseteq M \to M$ and $\alpha > 0$. Then the following assertions hold.*

(i) *T is pseudo-contractive if and only if its complementary vector field is monotone.*

(ii) *If T is α-strongly pseudo-contractive, then its complementary vector field is α-strongly monotone.*

(iii) *Conversely, if the complementary vector field of T is α-strongly monotone, then T is α'-strongly pseudo-contractive, where $0 < \alpha' < \alpha$.*

Remark 2.3.11. If T is a nonexpansive mapping, it was proved in [80] that the complementary vector field of T is monotone. Hence, by Corollary 2.3.10, we deduce that any nonexpansive mapping is pseudo-contractive.

2.4 Singularities, resolvent and Yosida approximation of vector fields

Let $A \in \mathcal{X}(M)$ be a set-valued vector field with domain $\mathcal{D}(A)$. The existence of singularities of vector fields is a relevant problem with numerous applications in other areas. In particular, it will be a crucial fact for the resolvent operator to be well-defined in the setting of Hadamard manifolds.

Definition 2.4.1. Given a vector field $A \in \mathcal{X}(M)$, we say that $x \in \mathcal{D}(A)$ is a *singularity* of A if $0 \in A(x)$. The set of all singularities of A is denoted by $A^{-1}(0)$.

2.4.1 Singularities of strongly maximal vector fields

Concerning the existence of singularities for monotone vector fields, as a direct consequence of Definition 2.2.1, we first deduce the following proposition.

Proposition 2.4.2. *Let $A \in \mathcal{X}(M)$ be strictly monotone. Then A has at most one singularity.*

In [36, 41] it was proved that differentiable strongly monotone single-valued vector fields on Hadamard manifolds with $\mathcal{D}(A) = M$ have at least one singularity; that is, since the strong monotonicity implies the strictly monotonicity, existence and uniqueness are ensured. This result can be improved and extended to the set-valued case for maximal strongly monotone vector fields, by using the equivalence established in Theorem 2.2.8 and the following coercivity result for finite-dimensional Banach spaces; see, for example, [87, pag. 115].

Proposition 2.4.3. *Let X be a finite-dimensional Banach space and $T : X \to 2^{X^*}$ be an upper semicontinuous set-valued operator. Suppose that $\mathcal{D}(T) = X$ and that T satisfies the following coercivity condition:*

$$\lim_{\|z\| \to \infty} \frac{\langle w, z \rangle}{\|z\|} = +\infty, \quad \forall w \in T(z). \tag{2.4.1}$$

Then, there exists $x \in X$ such that $0 \in T(x)$.

Theorem 2.4.4. *Let $A \in \mathcal{X}(M)$ be a maximal strongly monotone vector field with $\mathcal{D}(A) = M$. Then there exists a unique singularity of A.*

Proof. Since the strong monotonicity implies the strict monotonicity, the uniqueness of the singularity follows from Proposition 2.4.2. Now we will prove the existence of singularity. To this end, let $x_0 \in M$ and let $T : T_{x_0}M \to 2^{T_{x_0}M}$ be defined by (2.2.4), that is,

$$T(u) = P_{x_0,\exp_{x_0} u} A(\exp_{x_0} u), \quad \forall u \in T_{x_0}M.$$

Then T is upper semicontinuous by Theorem 2.2.8. Moreover, by the strong monotonicity, there exists $\rho > 0$ such that for any $x \in \mathcal{D}(A)$

$$\langle u, \exp_{x_0}^{-1} x\rangle - \langle v, -\exp_x^{-1} x_0\rangle \le -\rho \mathrm{d}^2(x_0, x), \quad \forall u \in A(x_0) \text{ and } \forall v \in A(x),$$

which is equivalent to

$$\langle P_{x_0,x}v - u, \exp_{x_0}^{-1} x\rangle \ge \rho \mathrm{d}^2(x_0, x), \quad \forall u \in A(x_0) \text{ and } \forall v \in A(x). \tag{2.4.2}$$

Letting $u \in T_{x_0}M$ and $w \in T(u)$, we set $x = \exp_{x_0} u$ and assume that $w = P_{x_0,x}v$ for some $v \in A(x)$. Fix $v_0 \in A(x_0)$. Then by (2.4.2), we get that

$$\begin{aligned} \langle w, u\rangle &= \langle P_{x_0,x}v - v_0, \exp_{x_0}^{-1} x\rangle + \langle v_0, \exp_{x_0}^{-1} x\rangle \\ &\ge \rho \mathrm{d}^2(x_0, x) + \langle v_0, \exp_{x_0}^{-1} x\rangle \\ &\ge \rho\|u\|^2 - \|v_0\|\|u\|. \end{aligned}$$

Therefore,

$$\lim_{\|u\|\to\infty} \frac{\langle w, u\rangle}{\|u\|} = +\infty, \quad \forall w \in T(u).$$

This shows that T satisfies the coercivity condition (2.4.1). Consequently, Proposition 2.4.3 is applicable to concluding that there exists a point $u \in T_{x_0}M$ such that $0 \in T(u)$. Then, $x := \exp_{x_0} u \in M$ is a singularity of A. □

Corollary 2.4.5. *Let $A \in \mathcal{X}(M)$ be maximal monotone with $\mathcal{D}(A) = M$. Then A is m-accretive.*

Proof. Since A is monotone, by Theorem 2.2.13 (i), it is accretive as well. In order to proved that A is m-accretive we need to show that (2.2.10) is true. To this end, given $y \in M$ let us prove that

$$y \in \bigcup_{x\in\mathcal{D}(A)} \left(\bigcup_{u\in A(x)} \exp_x u \right). \tag{2.4.3}$$

Define the set-valued vector field $B : M \to 2^{TM}$ by

$$B(x) = A(x) - \exp_x^{-1} y \quad \text{for each } x \in M. \tag{2.4.4}$$

It can be proved by Proposition 2.1.30 that B is maximal strongly monotone and $\mathcal{D}(B) = M$. Therefore, by Theorem 2.4.4, there exists a unique singularity x_0 of B; that is, $0 \in B(x_0) = A(x_0) - \exp_{x_0}^{-1} y$. Hence there exists $u_0 \in A(x_0)$ such that $u_0 = \exp_{x_0}^{-1} y$, which means that $y = \exp_{x_0} u_0$. Then (2.4.3) is true. □

2.4.2 Resolvent and Yosida approximation of a vector field

We are now in a position to define the notions of the resolvent and the Yosida approximation of a vector field $A \in \mathcal{X}(M)$.

Definition 2.4.6. Given $\lambda > 0$, the *resolvent* of $A \in \mathcal{X}(M)$ of order λ is the set-valued mapping $J_\lambda : M \to 2^M$ defined as

$$J_\lambda(x) = \{z \in M |\ x \in \exp_z \lambda A z\}. \tag{2.4.5}$$

Definition 2.4.7. For any $\lambda > 0$, the *Yosida approximation* $A \in \mathcal{X}(M)$ of order λ is the set-valued vector field $A_\lambda : M \to 2^{TM}$ defined as

$$A_\lambda(x) = -\frac{1}{\lambda} \exp_x^{-1} J_\lambda(x), \tag{2.4.6}$$

in other words, it is the complementary vector field of the resolvent (cf. [81]) multiplied by the constant $\frac{1}{\lambda}$.

Remark 2.4.8. Note that, for every $\lambda > 0$, the domains of the resolvent J_λ and the Yosida approximation A_λ are the range of the vector field defined by $x \mapsto \exp_x \lambda A x$. We will denote this range as

$$\mathcal{R}(\exp_\cdot \lambda A(\cdot)) = \bigcup_{x \in \mathcal{D}(A)} \left(\bigcup_{u \in A(x)} \exp_x \lambda u \right).$$

The resolvent and Yosida approximation were implicitly defined in the setting of differential manifolds, in particular, in Finsler manifold by J. Hoyos in [51] and in Hilbert manifolds by Iwamiya and Okochi in [53]. As a matter of fact, these two definitions can be proved to coincide with the corresponding concepts defined on Hadamard manifolds. However, it turns out that in these settings where the resolvent and the Yosida approximation are defined is still unknown, whereas we are proving next that under certain monotonicity conditions these operators have full domain in a Hadamard manifold. Moreover, in the following theorem the relation between the firm nonexpansivity of the resolvent and the monotonicity of the vector field is stated as well.

Theorem 2.4.9. *Let $A \in \mathcal{X}(M)$. Then, for any $\lambda > 0$,*

(i) *the vector field A is monotone if and only if J_λ is single-valued and firmly nonexpansive;*

(ii) *if $\mathcal{D}(A) = M$, the vector field A is maximal monotone if and only if J_λ is single-valued, firmly nonexpansive and the domain $\mathcal{D}(J_\lambda) = M$;*

(iii) *if A is monotone*
$$Fix\,(J_\lambda) = A^{-1}(0).$$

Proof.

(i) Given $x \in \mathcal{D}(J_\lambda)$, note that
$$z \in J_\lambda(x) \Leftrightarrow x \in \exp_z \lambda Az \Leftrightarrow 0 \in \lambda Az - \exp_z^{-1} x. \tag{2.4.7}$$

This means that $z \in M$ belongs to the resolvent of A at x if and only if it is a singularity of the vector field $B : M \to TM$ defined as
$$B(y) = \lambda Ay - \exp_y^{-1} x \tag{2.4.8}$$
for each $y \in M$.

If we assume that A is monotone, from the definition of B and Proposition 2.1.30 we deduce that B is strictly monotone, which implies the uniqueness of singularity as it can be readily seen. Then the resolvent is single-valued. To prove the firm nonexpansivity, by Proposition 2.3.4, it suffices to verify that for any $x, y \in M$
$$\langle \exp_{J_\lambda(x)}^{-1} J_\lambda(y), \exp_{J_\lambda(x)}^{-1} x\rangle + \langle \exp_{J_\lambda(y)}^{-1} J_\lambda(x), \exp_{J_\lambda(y)}^{-1} y\rangle \leq 0. \tag{2.4.9}$$

Considering $J_\lambda(x), J_\lambda(y) \in M$, by definition of resolvent we know that

$$\exp^{-1}_{J_\lambda(x)} x \in \lambda A(J_\lambda(x)),\ \exp^{-1}_{J_\lambda(y)} y \in \lambda A(J_\lambda(y)).$$

Thus the monotonicity of A implies that

$$\langle \exp^{-1}_{J_\lambda(x)} J_\lambda(y), \frac{1}{\lambda} \exp^{-1}_{J_\lambda(x)} x\rangle + \langle \exp^{-1}_{J_\lambda(y)} J_\lambda(x), \frac{1}{\lambda} \exp^{-1}_{J_\lambda(y)} y\rangle \leq 0,$$

and (2.4.9) is proved.

Conversely, suppose now that J_λ is firmly nonexpansive. To show that A is monotone, for any $x, y \in \mathcal{D}(A)$ and $u \in A(x)$, $v \in A(y)$, by definition of the resolvent J^A_λ, we can write $x = J_\lambda(\exp_x \lambda u)$ and $y = J_\lambda(\exp_y \lambda v)$. By using the characterization (2.3.6) of firmly nonexpansive mappings considering the points $\exp_x \lambda u$ and $\exp_y \lambda v$ we obtain that

$$\langle \exp^{-1}_x y, \lambda u\rangle + \langle \exp^{-1}_y x, \lambda v\rangle \leq 0,$$

which implies that

$$\langle u, \exp^{-1}_x y\rangle \leq -\langle v, \exp^{-1}_y x\rangle \leq 0,$$

and the monotonicity of A is proved.

(ii) Assuming that $\mathcal{D}(A) = M$, if A is maximal monotone, it can be proved that the vector field B defined as (2.4.8) is strongly monotone and maximal monotone (see [36]). Then, for any $x \in M$, Theorem 2.4.1 assures the existence and uniqueness of a singularity of B, that is, an element of $J_\lambda(x)$. Thus J_λ is single-valued and firmly nonexpansive, as we proved in (i), and moreover $\mathcal{D}(J_\lambda) = M$.

Now, suppose T is firmly nonexpansive and $\mathcal{D}(T) = M$. By (i) we know that $T = J_\lambda$ where A, the vector field defined in (2.4.10), is monotone. By Theorem 2.2.13 we deduce that A is accretive. So the full domain of the resolvent of order $\lambda = 1$ implies that A is m-accretive. Again by Theorem 2.2.13 we have that A is maximal monotone.

(iii) If A is monotone J_λ is single-valued and by the equivalence (2.4.7) we deduce that $z \in \mathrm{Fix}\,(J_\lambda)$ if and only if $0 \in A(z)$.

□

Remark 2.4.10. Note that in the previous theorem we have proved indeed that, for each $\lambda > 0$, any firmly nonexpansive T with full domain $\mathcal{D}(T) = M$ is the resolvent J_λ^A of the maximal monotone vector field A defined by

$$A(x) := \frac{1}{\lambda} \exp_x^{-1} T^{-1}x, \quad \forall x \in M. \tag{2.4.10}$$

Indeed, the resolvent of the vector field A is the mapping T:

$$J_\lambda(x) = \{z \in M |\, x \in \exp_z \lambda Az\} = \{z \in M |\, x \in T^{-1}z\} = Tx.$$

From the proof of (ii) in Theorem 2.4.9 and remark 2.4.8, we can deduce the following result which constitutes a counterpart of Minty's theorem [76] in the setting of Hadamard manifolds.

Corollary 2.4.11. *Let $A \in \mathcal{X}(M)$ be monotone such that $\mathcal{D}(A) = M$, and let $\lambda > 0$. Then A is maximal monotone if and only if $\mathcal{R}(\exp_{\cdot} \lambda A(\cdot)) = M$.*

As a byproduct of Theorem 2.4.9 (iii) we obtain the following result about the structure of the set of singularities of a maximal monotone vector field. As far as we know the only result of this type was proved in [41] under the assumption that A is smooth.

Corollary 2.4.12. *Let $A \in \mathcal{X}(M)$ be monotone with closed convex domain $\mathcal{D}(A)$ such that $\mathcal{D}(A) \subseteq \mathcal{D}(J_\lambda)$. Then $A^{-1}(0)$ is closed and convex.*

Proof. Thanks to Theorem 2.4.9, the set $A^{-1}(0)$ coincides with the fixed point set of the resolvent J_λ, for any $\lambda > 0$, which is firmly nonexpansive. Then the result can be deduced from Proposition 2.3.1, since we have that if we set $T = J_\lambda$,

$$\mathrm{Fix}\,(T) = A^{-1}(0) \subseteq \mathcal{D}(A) \subseteq \mathcal{D}(T).$$

□

We present now some properties of the Yosida approximation of a vector field. In particular, the relation with the resolvent will be fundamental for the study of the asymptotic behavior of the resolvent as the order $\lambda \to +\infty$ or $\lambda \to 0$.

Proposition 2.4.13. *Let $A \in \mathcal{X}(M)$ be monotone and let $\lambda > 0$. Then*

(1) *for any $x \in \mathcal{D}(A_\lambda)$*
$$A_\lambda(x) \in P_{x,J_\lambda(x)} A J_\lambda(x); \tag{2.4.11}$$

(2) *for any $x \in \mathcal{D}(A_\lambda) \bigcap \mathcal{D}(A)$*
$$\|A_\lambda(x)\| \leq |Ax|, \tag{2.4.12}$$
where $|Ax| = \inf\{\|v\| : v \in Ax\}$;

(3) *the Yosida approximation A_λ is a monotone vector field. Moreover, if A is maximal then so is A_λ.*

Proof. Note that $\mathcal{D}(A_\lambda) = \mathcal{D}(J_\lambda)$ and, since A is monotone, its resolvent is single-valued and so is its Yosida approximation.

(1) Let $x \in \mathcal{D}(A_\lambda)$ and $v = A_\lambda(x)$. Then $v = -\frac{1}{\lambda} \exp_x^{-1} J_\lambda(x)$, which is equivalent to write
$$-\lambda v = \exp_x^{-1} J_\lambda(x) = -P_{x,J_\lambda(x)} \exp_{J_\lambda(x)}^{-1} x.$$
But, by definition of resolvent, we have that $x \in \exp_{J_\lambda(x)} \lambda A J_\lambda(x)$, or equivalently, $\exp_{J_\lambda(x)}^{-1} x \in \lambda A J_\lambda(x)$. Therefore,
$$v \in P_{x,J_\lambda(x)} A J_\lambda(x).$$

(2) Let $x \in \mathcal{D}(A_\lambda) \bigcap \mathcal{D}(A)$. We know that
$$\|A_\lambda(x)\| = \frac{1}{\lambda} \| \exp_x^{-1} J_\lambda(x)\| = \frac{1}{\lambda} d(x, J_\lambda(x)).$$
For any $u \in A(x)$, which exists since $x \in \mathcal{D}(A)$, we can write $x = J_\lambda(\exp_x \lambda u)$ because it is equivalent to $\exp_x \lambda u \in \exp_x \lambda A(x)$. Then, by the nonexpansivity of the resolvent, we obtain
$$\|A_\lambda(x)\| = \frac{1}{\lambda}\, d(J_\lambda(\exp_x \lambda u), J_\lambda(x)) \leq \frac{1}{\lambda}\, d(\exp_x \lambda u, x) = \frac{1}{\lambda}\, \| \exp_x^{-1} \exp_x \lambda u\|.$$
Then $\|A_\lambda(x)\| \leq \|u\|$ holds for any $u \in A(x)$; hence, we obtain the inequality required.

(3) The nonexpansivity of the resolvent implies the monotonicity of the Yosida approximation of any order $\lambda > 0$ because it is defined as its complementary vector field multiplied by a constant, see [81]. Since the resolvent and the exponential map are continuous, so is the Yosida approximation A_λ. Then, assuming that A is maximal monotone with full domain, A_λ has full domain and maximality is consequence of Theorem 2.2.8 thanks to the upper semi-continuity and the fact that it is single-valued.

□

2.4.3 Asymptotic behavior of the resolvent of a vector field

Asymptotic behavior of J_λ as $\lambda \to \infty$

Let $A \in \mathcal{X}(M)$ be monotone. Recall that, by Theorem 2.4.9, for any $\lambda > 0$ the resolvent J_λ is a single-valued and nonexpansive mapping. Given a point $x \in \mathcal{D}(J_\lambda)$, we want to study the behavior of the sequence $\{J_\lambda\}$ when we let λ go to $+\infty$. This question was solved in the setting of Banach spaces in relation with the problem of approximating zeros of monotone and accretive operators, see [95, 111, 56]. Here, in the framework of Hadamard manifolds, we provide a similar answer with the help of the following lemma.

Lemma 2.4.14. *Let $A \in \mathcal{X}(M)$ be monotone such that $\mathcal{D}(A) \subseteq \mathcal{D}(J_r)$ for some $r > 0$. Let $x \in \mathcal{D}(J_\lambda)$ for any $\lambda > 0$.*

(i) *If there exists a sequence $\{t_n\}$ with $t_n \to \infty$, such that $\lim_{n\to\infty} J_{t_n}(x) = y$, then the limit $y \in A^{-1}(0)$.*

(ii) *If there exist two sequences $\{t_n\}$ and $\{s_n\}$ with $t_n \to \infty$ and $s_n \to \infty$, such that $\lim_{n\to\infty} J_{t_n}(x) = y$ and $\lim_{n\to\infty} J_{s_n}(x) = z$, then $y = z$.*

Proof.

(i) If the limit $\lim_{n\to\infty} J_{t_n}(x) = y$ exist, this means that the sequence $\{J_{t_n}(x)\}$ is bounded and $\|\exp_x^{-1} J_{t_n}(x)\| \leq M$ for some $M > 0$. Let $r > 0$ such that

$\mathcal{D}(A) \subseteq \mathcal{D}(J_r)$. By using the definition of the Yosida approximation A_r and the inequality (2.4.12), we obtain that

$$\begin{aligned} d(J_r(J_{t_n}(x)), J_{t_n}(x)) &= \| \exp^{-1}_{J_{t_n}(x)} J_r(J_{t_n}(x)) \| \\ &= r\|A_r(J_{t_n}(x))\| \\ &\leq r|A(J_{t_n}(x))|. \end{aligned} \tag{2.4.13}$$

From the property (2.4.11) in Proposition 2.4.13, $A_{t_n}(x) \in P_{J_{t_n}(x),x} AJ_{t_n}(x)$, or equivalently, $P_{x,J_{t_n}(x)} A_{t_n}(x) \in AJ_{t_n}(x)$. Then the norm $|A(J_{t_n}(x))| \leq \|P_{x,J_{t_n}(x)} A_{t_n}(x)\| = \|A_{t_n}(x)\|$ and from inequality (2.4.13) it follows that

$$d(J_r(J_{t_n}(x)), J_{t_n}(x)) \leq r\|A_{t_n}(x)\| = \frac{r}{t_n} \| \exp^{-1}_x J_{t_n}(x)\| \leq \frac{r}{t_n} M.$$

Letting $n \to \infty$, we obtain that $y = J_r(y)$, that is, $y \in \operatorname{Fix}(J_r) = A^{-1}(0)$.

(ii) By the property (2.4.11) in Proposition 2.4.13, for any $n \geq 0$, $P_{x,J_{t_n}(x)} A_{t_n}(x) \in AJ_{t_n}(x)$, and by (ii) we know that $0 \in A(z)$. Since A is monotone,

$$\langle P_{J_{t_n}(x),x} A_{t_n}(x), \exp^{-1}_{J_{t_n}(x)} z \rangle \leq 0;$$

therefore, since $A_{t_n}(x) = -\exp^{-1}_x J_{t_n}(x)$,

$$\langle \exp^{-1}_{J_{t_n}(x)} x, \exp^{-1}_{J_{t_n}(x)} z \rangle \leq 0.$$

Taking limit when $n \to \infty$ we obtain

$$\langle \exp^{-1}_y x, \exp^{-1}_y z \rangle \leq 0. \tag{2.4.14}$$

Changing the roles of t_n with s_n and z with y, we get the inequality

$$\langle \exp^{-1}_z x, \exp^{-1}_z y \rangle \leq 0. \tag{2.4.15}$$

Thus, by using Proposition 2.1.30, the law of cosines for Hadamard manifolds, the previous inequalities (2.4.14) and (2.4.15) implies that

$$d^2(y,z) \leq \langle \exp^{-1}_y x, \exp^{-1}_y z \rangle + \langle \exp^{-1}_z x, \exp^{-1}_z y \rangle \leq 0.$$

□

Theorem 2.4.15. *Let $A \in \mathcal{X}(M)$ be monotone with closed convex domain $\mathcal{D}(A)$ such that $\mathcal{D}(A) \subseteq \mathcal{D}(J_r)$ for some $r > 0$ and let $x \in \mathcal{D}(J_\lambda)$ for all $\lambda > 0$. Then,*

(i) *if $A^{-1}(0) \neq \emptyset$,*
$$\lim_{\lambda\to\infty} J_\lambda(x) = P_{A^{-1}(0)}(x),$$
where $P_{A^{-1}(0)}(x)$ is the projection over the set of singularities $A^{-1}(0)$ of the point x;

(ii) *if $A^{-1}(0) = \emptyset$,*
$$\lim_{\lambda\to\infty} \|J_\lambda(x)\| = +\infty.$$

Proof.

(i) Assume that $A^{-1}(0) \neq \emptyset$. Note that since A is monotone with closed convex domain $\mathcal{D}(A)$ such that $\mathcal{D}(A) \subseteq \mathcal{D}(J_\lambda)$, by Corollary 2.4.12, $A^{-1}(0)$ is closed and convex. Therefore the metric projection over $A^{-1}(0)$ is well-defined on M. The fact that $\mathrm{Fix}\,(J_\lambda) = A^{-1}(0) \neq \emptyset$, for all $\lambda > 0$, implies that there exists $p \in \mathrm{Fix}\,(J_\lambda)$ such that
$$d(J_\lambda(x), p) \leq d(x, p),$$
for any $\lambda > 0$. Therefore, the sequence $\{J_\lambda(x)\}$ is bounded and we can ensure the existence of $\{t_n\}$ with $t_n \to \infty$ and $\lim_{n\to\infty} J_{t_n}(x) = y$, where y is a cluster point belonging to $A^{-1}(0)$ by Lemma 2.4.14(i). Now, Lemma 2.4.14(ii) ensures that $y \in A^{-1}(0)$ is the unique cluster point of the sequence $\{J_\lambda(x)\}$ and thus the convergence point. It remains to be proved that $y = P_{A^{-1}(0)}(x)$. By replacing z with $P_{A^{-1}(0)}(x)$ in inequality (2.4.14) we obtain
$$\langle \exp_y^{-1} x, \exp_y^{-1} P_{A^{-1}(0)}(x)\rangle \leq 0. \tag{2.4.16}$$
Since $y \in A^{-1}(0)$, from the property of the projection in Proposition 2.1.18, the inequality
$$\langle \exp^{-1}_{P_{A^{-1}(0)}(x)} x, \exp^{-1}_{P_{A^{-1}(0)}(x)} y\rangle \leq 0 \tag{2.4.17}$$

holds. Then, Proposition 2.1.30 together with inequalities (2.4.16) and (2.4.17) imply that

$$\begin{aligned} d^2(y, P_{A^{-1}(0)}(x)) &\leq \langle \exp_y^{-1} x, \exp_y^{-1} P_{A^{-1}(0)}(x)\rangle \\ &\quad + \langle \exp_{P_{A^{-1}(0)}(x)}^{-1} x, \exp_{P_{A^{-1}(0)}(x)}^{-1} y\rangle \\ &\leq 0. \end{aligned}$$

(ii) If $A^{-1}(0) = \emptyset$, let us assume that $\lim_{\lambda\to\infty} \|J_\lambda(x)\| \neq +\infty$. Then we can consider a positive sequence $\{\lambda_n\}$ such that $\lambda_n \to \infty$ and $\{J_{\lambda_n}(x)\}$ is bounded. This implies that the sequence $\{J_{\lambda_n}(x)\}$ possesses a cluster point y which, by Lemma 2.4.14, is forced to be in $A^{-1}(0) = \emptyset$.

□

Asymptotic behavior of J_λ as $\lambda \to 0$

In this subsection, instead of letting $\lambda \to \infty$ we make $\lambda \to 0$, and then it turns out that the behavior of the resolvent J_λ is completely different, as we prove in the following theorem.

Theorem 2.4.16. *Let $A \in \mathcal{X}(M)$ be monotone. Then, if $x \in \mathcal{D}(A) \bigcap \mathcal{D}(J_\lambda)$, for all $\lambda > 0$,*

$$\lim_{\lambda\to 0} J_\lambda(x) = x.$$

Proof. By using the Yosida approximation A_λ and inequality (2.4.12), we have that

$$d(J_\lambda x, x) = \| \exp_x^{-1} J_\lambda x\| = t\|A_\lambda x\| \leq \lambda |Ax|.$$

Since $|Ax| = \inf\{\|v\| : v \in Ax\}$, there exists $M > 0$ such that $|A(x)| \leq M$. Thus, letting $\lambda \to 0$ the statement of the theorem is proved. □

2.4.4 Singularities of monotone vector fields under boundary conditions

The existence of singularities of a maximal monotone vector field was analyzed in Section 2.4.1 under the strong monotonicity condition as it was established in Theorem 2.4.4. Dealing with this problem, as a consequence of Lemma 2.4.14 and Theorem 2.4.15 we obtain the following result.

Theorem 2.4.17. *Let $A \in \mathcal{X}(M)$ be monotone. Then the following assertions are equivalent.*

(i) $A^{-1}(0) \neq \emptyset$;

(ii) $\{J_\lambda(x)\}$ *is bounded for each* $x \in \mathcal{D}(J_\lambda)$ *for all* $\lambda > 0$;

(iii) $\liminf_{\lambda\to\infty} \|J_\lambda(x)\| < \infty$ *for some* $x \in \mathcal{D}(J_\lambda)$ *for all* $\lambda > 0$.

Proof. This proof is enclosed in the one of Theorem 2.4.15. But note that in this case we do not need to require any condition on the domain. For the sake of completeness we are including the reasoning here.

Let $x \in \mathcal{D}(J_\lambda)$. Assume that $A^{-1}(0) \neq \emptyset$. Since A is monotone, by Theorem 2.4.9, $\text{Fix}\,(J_\lambda) = A^{-1}(0) \neq \emptyset$. Then there exists $p \in \text{Fix}\,(J_\lambda)$ such that

$$d(J_\lambda(x), p) \leq d(x, p),$$

for any $\lambda > 0$. Then the sequence $\{J_\lambda(x)\}$ is bounded, implying statement (ii). In this situation, we can ensure the existence of $\{t_n\}$ with $t_n \to \infty$ and $\lim_{n\to\infty} J_{t_n}(x) = y \in M$. However, Lemma 2.4.14 ensures that there exist a unique cluster point of the sequence $\{J_\lambda(x)\}$ and thus $\lim_{\lambda\to\infty} \|J_\lambda(x)\| = y < \infty$. We assume now that (iii) holds. This means that there exists $\{t_n\}$ with $t_n \to \infty$ and $\lim_{n\to\infty} J_{t_n}(x) = y \in M$. From Lemma 2.4.14 we obtain that $y \in A^{-1}(0)$ and (i) holds. □

Next we provide a result of existence of singularities under some boundary conditions. For that we need the following lemma and, given $x_0 \in M$ and $\epsilon > 0$, we denote

$$B(x_0, \epsilon) := \{y \in M : d(x, y) \leq \epsilon\}$$

$$\partial B(x_0, \epsilon) := \{y \in M : d(x, y) = \epsilon\}.$$

Lemma 2.4.18. *Let $A \in \mathcal{X}(M)$ be maximal monotone with full domain $\mathcal{D}(A) = M$. Let $x_0 \in M$ and $\epsilon > 0$ such that*

$$|A(x_0)| < |A(x)| \tag{2.4.18}$$

for any $x \in \partial B(x_0, \epsilon)$. Then $J_\lambda(x_0) \in B(x_0, \epsilon)$ for all $\lambda > 0$.

Proof. Let $\lambda > 0$. From Proposition 2.4.13 (i) we know that $A_\lambda(x_0) \in P_{x_0,J_\lambda(x_0)} A J_\lambda(x_0)$. Then, by Proposition 2.4.13 (ii), we deduce that

$$|A(J_\lambda(x_0))| = |P_{x_0,J_\lambda(x_0)} A(J_\lambda(x_0))| \leq \|A_\lambda(x_0)\| \leq |A(x_0)|. \tag{2.4.19}$$

So our boundary condition implies the fact that $J_\lambda(x_0) \notin \partial B(x_0, \epsilon)$. By Theorem 2.4.16, the sequence $\{J_\lambda(x_0)\}$ converges to $x_0 \in B(x_0, \epsilon)$. Therefore if the function $\lambda \mapsto J_\lambda(x_0)$ is continuous it follows that $J_\lambda(x_0) \in B(x_0, \epsilon)$ for any $\lambda > 0$. Let us prove the continuity of this function. For that, given a sequence $\lambda_n \to \lambda_0$ as $n \to \infty$, we need to show that $z_n = J_{\lambda_n}(x_0) \to z_0 = J_{\lambda_0}(x_0)$. First of all, this sequence $\{z_n\}$ is bounded. Indeed, by definition of the Yosida approximation and Proposition 2.4.13 (ii),

$$d(z_n, x_0) = \|exp_{x_0}^{-1} J_{\lambda_n}(x_0)\| = \lambda_n |A_\lambda(x_0)| \leq \lambda_n \|A(x_0)\|. \tag{2.4.20}$$

As it was proved in Theorem 2.4.16 the set $A(x_0)$ is bounded. Then, since $\lambda_n \to \lambda_0$, the sequence $\{z_n\}$ is bounded. This implies, by Lemma 2.4.14, that $z_n \to z^* \in A^{-1}(0)$. It remains to prove that $z^* = z_0$; equivalently, that

$$x_0 \in \exp_{z^*}^{-1} \lambda_0 A(z^*). \tag{2.4.21}$$

As $z_n = J_{\lambda_n}(x_0)$, $x_0 \in \exp_{z_n}^{-1} \lambda_n A(z_n)$; that is,

$$x_0 = \exp_{z_n}^{-1} \lambda_n y_n \tag{2.4.22}$$

with $y_n \in A(z_n)$. Since A is maximal monotone with full domain, Lemma 2.2.7 implies that A is locally bounded; that is, for $x_0 \in M$, there exists an open neighborhood $U(x_0)$ of x_0 such that the set $\cup_{x \in U(x_0)} A(x)$ is bounded. Then, by taking a subsequence of $\{y_n\}$ if need be but using the same notation, we can assume that $\{y_n\}$ is convergent to $y \in M$. On the other hand, the maximality of A implies the upper semicontinuity by Theorem 2.2.8, which means that $y \in A(z^*)$ because $z_n \to z^*$, $y_n \to y$ and $y_n \in A(z_n)$. Therefore, taking limit in the equality (2.4.22) we obtain that (2.4.21) holds. □

Theorem 2.4.19. *Let $A \in \mathcal{X}(M)$ be maximal monotone with full domain $\mathcal{D}(A) = M$. Let $x_0 \in M$ and $\epsilon > 0$ such that*

$$|A(x_0)| < |A(x)| \tag{2.4.23}$$

for any $x \in \partial B(x_0, \epsilon)$. Then there exists a singularity $y \in B(x_0, \epsilon)$.

Proof. From Lemma 2.4.18 we know that, for all $\lambda > 0$, $J_\lambda(x_0) \in B(x_0, \epsilon)$. This means that the sequence $\{J_\lambda(x_0)\}$ is bounded. So by Lemma 2.4.14 we deduce that $\{J_\lambda(x_0)\}$ converges to a singularity $y \in \overline{B(x_0, \epsilon)}$. However, since $y \in A^{-1}(0)$, $A(y) = 0$ and, therefore, $|A(y)| = 0$. Then the boundary condition (2.4.23) implies that $y \notin \partial B(x_0, \epsilon)$ because otherwise $|A(x_0)| < 0$. Thus there exists a singularity $y \in B(x_0, \epsilon)$. □

The following result of existence of fixed point for single-value continuous pseudo-contractive mappings is the counterpart of Theorem 1 in [57], proved by Kirk and Schöneberg in the setting of Hilbert spaces.

Corollary 2.4.20. *Let $T : M \to M$ be a continuous pseudo-contractive mapping. Let $x_0 \in M$ and $\epsilon > 0$ such that*

$$d(x_0, T(x_0)) < d(x, T(x)) \tag{2.4.24}$$

for any $x \in \partial B(x_0, \epsilon)$. Then there exists a fixed point of T in $B(x_0, \epsilon)$.

Proof. Let A be the complementary vector field of T; that is, $A(x) = -\exp_x^{-1} T(x)$, for any $x \in M$. Since T is single-valued, so is A, then this means that $|A(x)| = \|\exp_x^{-1} T(x)\| = \mathrm{d}(x, T(x))$, for any $x \in M$. Therefore, by hypothesis (2.4.24), condition (2.4.23) is satisfied. On the other hand, the fact that T is a continuous pseudo-contractive mapping implies that A is maximal monotone. Thus Theorem 2.4.19 assures the existence of a singularity of A in $\mathbf{B}(x_0, \epsilon)$ which is a fixed point of T. □

2.5 Proximal point algorithm for monotone vector fields

Let $A \in \mathcal{X}(M)$ be a set-valued vector field with domain $\mathcal{D}(A)$. In this section we present an iterative method to approximate a singularity of A, which is motivated by the proximal point algorithm introduced and studied in the setting of Hilbert spaces by Martinet [75], Moreau [78] and Rockafellar [103]. Let $x_0 \in \mathcal{D}(A)$ and $\{\lambda_n\} \subset \mathbb{R}^+$. We define a sequence $\{x_n\}$ by means of the recursive formula

$$0 \in A(x_{n+1}) - \lambda_n \exp^{-1}_{x_{n+1}} x_n. \tag{2.5.1}$$

Remark 2.5.1. Note that the algorithm (2.5.1) is an implicit method. So a basic problem is study whether this algorithm is well-defined. For each $n \geq$, define the vector field $B_n \in \mathcal{X}(M)$ by

$$B_n(x) := A(x) - \lambda_n \exp^{-1}_x x_n, \quad \forall x \in \mathcal{D}(A).$$

In the case when $A \in \mathcal{X}(M)$ is monotone, each B_n can be proved to be strongly monotone by Proposition 2.1.30. Moreover, if A is maximal monotone, it is readily seen that so is each B_n. Thus, in view of Proposition 2.4.2 and Theorem 2.4.4, the following assertions hold when A is monotone.

(i) The algorithm (2.5.1) is well-defined if and only if $B_n^{-1}(0) \neq \emptyset$ for each $n \geq 0$.

(ii) If $\mathcal{D}(A) = M$ and A is maximal monotone, then the algorithm (2.5.1) is well-defined.

In the following theorem we prove the convergence of the algorithm 2.5.1 for upper Kuratowski semicontinuous monotone vector fields, provided that the algorithm is well-defined.

Theorem 2.5.2. *Let $A \in \mathcal{X}(M)$ be such that $A^{-1}(0) \neq \emptyset$. Suppose that A is monotone and upper Kuratowski semicontinuous. Let $\{\lambda_n\} \subset \mathbb{R}^+$ satisfy*

$$\sup\{\lambda_n : n \geq 0\} < \infty. \tag{2.5.2}$$

Let $x_0 \in \mathcal{D}(A)$ and suppose that the sequence $\{x_n\}$ generated by the algorithm (2.5.1) *is well-defined. Then $\{x_n\}$ converges to a singularity of A.*

Proof. We first prove that $\{x_n\}$ is Fejér monotone with respect to $A^{-1}(0)$. For this purpose, let $x \in A^{-1}(0)$ and $n \geq 0$. Then $0 \in A(x)$ and $\lambda_n \exp^{-1}_{x_{n+1}} x_n \in A(x_{n+1})$ by (2.5.1). This together with the monotonicity of A implies that

$$\langle \lambda_n \exp^{-1}_{x_{n+1}} x_n, \exp^{-1}_{x_{n+1}} x \rangle \leq \langle 0, -\exp^{-1}_{x} x_{n+1} \rangle = 0. \tag{2.5.3}$$

Consider the geodesic triangle $\Delta(x_n, x_{n+1}, x)$. By inequality (2.1.3) of the Comparison Theorem for triangles, we have that

$$\mathrm{d}^2(x_{n+1}, x) + \mathrm{d}^2(x_{n+1}, x_n) - 2\langle \exp^{-1}_{x_{n+1}} x_n, \exp^{-1}_{x_{n+1}} x \rangle \leq \mathrm{d}^2(x_n, x).$$

It follows from (2.5.3) that

$$\mathrm{d}^2(x_{n+1}, x) + \mathrm{d}^2(x_{n+1}, x_n) \leq \mathrm{d}^2(x_n, x). \tag{2.5.4}$$

This clearly implies that $\mathrm{d}^2(x_{n+1}, x) \leq \mathrm{d}^2(x_n, x)$; therefore $\{x_n\}$ is Fejér monotone with respect to $A^{-1}(0)$. Furthermore, by (2.5.4), we get

$$\mathrm{d}^2(x_{n+1}, x_n) \leq \mathrm{d}^2(x_n, x) - \mathrm{d}^2(x_{n+1}, x). \tag{2.5.5}$$

Since the sequence $\{\mathrm{d}(x_n, x)\}$ is bounded and monotone, it is also convergent. Hence $\lim_{n\to\infty} \mathrm{d}(x_{n+1}, x_n) = 0$ by (2.5.5).

Thus, by Lemma 1.1.24, to complete the proof, we only need to prove that any cluster point of $\{x_n\}$ belongs to $A^{-1}(0)$. Let $\widehat{x}$ be a cluster point of $\{x_n\}$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \to \widehat{x}$. Hence $\mathrm{d}(x_{n_k}, x_{n_k+1}) \to 0$ by the assertion just proved and so $x_{n_k+1} \to \widehat{x}$. It follows that

$$u_{n_k+1} := \lambda_{n_k} \exp^{-1}_{x_{n_k+1}} x_{n_k} \to 0 \tag{2.5.6}$$

since $\{\lambda_n\}$ is bounded by assumption (2.5.2). By the algorithm (2.5.1), we obtain that $u_{n_k+1} \in A(x_{n_k+1})$ for each k. Combining this with (2.5.6) implies that $0 \in A(\widehat{x})$ because A is upper Kuratowski semicontinuous at $\widehat{x}$, that is, $\widehat{x} \in A^{-1}(0)$. □

In the case when $\mathcal{D}(A) = M$, the maximal monotonicity is equivalent to the upper semicontinuity by Theorem 2.2.8. Moreover the maximality implies that $\{x_n\}$ generated by (2.5.1) is well-defined by Remark 2.5.1. Therefore the following corollary is direct.

Corollary 2.5.3. *Let $A \in \mathcal{X}(M)$ be such that $A^{-1}(0) \neq \emptyset$ and $\mathcal{D}(A) = M$. Suppose that A is maximal monotone. Let $\{\lambda_n\} \subset \mathbb{R}^+$ satisfy* (2.5.2). *Then, for any $x_0 \in M$, the sequence $\{x_n\}$ generated by the algorithm* (2.5.1) *is well-defined and converges to a singularity of A.*

This corollary is an extension to Hadamard manifolds of the corresponding convergence theorem for the proximal point algorithm in Hilbert spaces (see [103]).

2.6 Iterative algorithms for nonexpansive type mappings

The study of the asymptotic behavior of nonexpansive type mappings is one of the most active research areas in nonlinear analysis. As we mentioned in Chapter 1, most of the investigations in this direction have focused on the case when T is a self-mapping defined on a closed convex subset C of a normed linear space. Besides Picard iteration $\{T^n(x)\}$ which converges when T is either a contraction or firmly nonexpansive, basically two types of algorithms has been considered: Halpern and Mann algorithms (see Chapter 1 Section 1.1.2). Because of the convex structure of both algorithms, few results have been obtained out of the setting of linear spaces. Our objective in this section is to study the convergence of different iterative methods for nonexpansive type mappings defined on Hadamard manifolds. First of all, we prove the convergence of Picard iteration for firmly nonexpansive mappings. Then we define and study Mann and Halpern iterations for nonexpansive mappings defined on Hadamard manifolds. Finally a viscosity approximation method will be presented in this framework. In order to illustrate the application of these methods, in particular, Mann and Halpern iterations, we provide some numerical examples.

2.6.1 Picard iteration for firmly nonexpansive mappings

As it happens in Banach spaces and the Hilbert ball with the hyperbolic metric [48, 100], the class of firmly nonexpansive mappings is characterized by the good asymptotic behavior of the sequence of Picard iterates $\{T^n x\}$.

Theorem 2.6.1. *Let $T : C \to C$ be a firmly nonexpansive mapping such that its fixed point set $Fix(T) \neq \emptyset$. Then for each $x \in C$, the sequence of iterates $\{T^n(x)\}$ converges to a fixed point of T.*

Proof. Let $x_n = T^n(x)$ for any $n \geq 0$. Note that C itself is a complete metric space. Thus, by Lemma 1.1.24, it suffices to prove that $\{x_n\}$ is Fejér monotone with respect to $\mathrm{Fix}\,(T)$ and that all cluster points of $\{x_n\}$ belong to $\mathrm{Fix}\,(T)$.To this end, let $n \geq 0$ and $y \in \mathrm{Fix}\,(T)$ be fixed. Since T is nonexpansive,

$$d(x_{n+1}, y) = d(T(x_n), T(y)) \leq d(x_n, y).$$

Hence $\{x_n\}$ is Fejér monotone with respect to $\mathrm{Fix}\,(T)$. Now let x be a cluster point of $\{x_n\}$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \to x$. On the other hand, one has that

$$\begin{aligned} d(x,T(x)) &\le d(x,x_{n_k}) + d(x_{n_k},T(x_{n_k})) + d(T(x_{n_k}),T(x)) \\ &\le 2d(x_{n_k},x) + d(x_{n_k},T(x_{n_k})), \end{aligned}$$

Then we just need to prove that

$$\lim_{n\to\infty} d(x_n,T(x_n)) = 0, \tag{2.6.1}$$

because if so, taking limit, we obtain that $d(x,Tx) = 0$, which means that $x \in \operatorname{Fix}(T)$.

Let $y \in \operatorname{Fix}(T)$. Since $\{x_n\}$ is Fejér monotone with respect to $\operatorname{Fix}(T)$, there exists the limit $\lim_{n\to\infty} d(x_n,y) = \lim_{n\to\infty} d(T(x_n),y) = d$. Given $n \ge 0$ fixed, let $\gamma_n : [0,1] \to M$ be the geodesic joining x_n to $T(x_n)$. Then $\gamma_n(1/2) = m_n$ verifies $d(m_n,x_n) = d(m_n,T(x_n)) = d(x_n,T(x_n))/2$. Since T is firmly nonexpansive

$$d(T(x_n),y) \le d(m_n,y) \le d(x_n,y).$$

Then $\lim_{n\to\infty} d(m_n,y) = d$. By inequality (2.1.7) of Lemma 2.1.29, we obtain

$$\frac{1}{4}\, d^2(x_n,T(x_n)) \le \frac{1}{2}\, d^2(x_n,y) + \frac{1}{2}\, d^2(T(x_n),y) - d^2(m_n,y).$$

Taking limit as $n \to \infty$ we have that (2.6.1) holds. □

In the case when the mapping T is just nonexpansive, we know that in general Picard iteration $\{T^n(x)\}$ does not converge, as we can observe by considering the Euclidean space $\mathbb{R}$ and the mapping $T(x) = -x$, since the sequence $\{T^n(x)\}$ does not convergence unless $x = 0$. However, we proved in Proposition 2.3.6 that there exists a family of firmly nonexpansive mappings $\{G_t : 0 \le t < 1\}$ whose fixed point sets coincide with the fixed point set of the nonexpansive mapping T. Then we can use the family $\{G_t\}$ for approximating a fixed point of T, considering the sequence defined by Picard iteration $x_{n+1} = G_t(x_n)$ for any $t \in [0,1)$.

Moreover, if we fix a point $x \in C$, the approximating curve $\{x_t\}$ defined by the unique fixed point of the contraction T_t in (2.3.18) converges to a fixed point of T as $t \to 1$. In fact, this was proved by Kirk [59] in the more general setting of CAT(0) spaces and constitutes an extension of the convergence of Browder algorithm (1.1.18) as we mention in the following section.

2.6.2 Halpern algorithm for nonexpansive mappings

Let C be a closed convex subset of M and $T : C \to C$ a nonexpansive mapping. In order to solve the problem of finding a fixed point of T out of the setting of linear spaces, Kirk, in [59], provided an implicit algorithm for approximating fixed points of nonexpansive mappings. More precisely, he studied such an algorithm in a complete CAT(0) space though the convergence result is formulated in the following theorem for the special case of a Hadamard manifold.

Theorem 2.6.2. *[59] Suppose that $C \subseteq M$ is bounded besides closed and convex. Let $T : C \to C$ be nonexpansive, $x \in C$, and for each $t \in [0,1)$, let x_t be the unique point such that*

$$x_t = \exp_x (1-t) \exp_x^{-1} T(x_t)$$

(which exists by the Banach contraction principle). Then $\lim_{t\to 0} x_t = \overline{x}$, the unique nearest point to x in $Fix(T)$.

In an Euclidean space $\mathbb{R}^n$, this iteration scheme turns into $x_t = (1-t)x + tT(x_t)$, which coincides with the implicit Browder iteration (1.1.18) seen in Chapter 1. We also pointed out the existence of a host of works about the convergence of the explicit Halpern iteration (1.1.17) in a Banach space X. We now present an analogue of this algorithm to approximate fixed points for nonexpansive mappings on Hadamard manifolds. Let $x_0,\ z \in M$ and let $\{\alpha_n\} \subset (0,1)$. Consider the iteration scheme

$$x_{n+1} = \exp_z (1-\alpha_n) \exp_z^{-1} T(x_n), \quad \forall n \geq 0; \tag{2.6.2}$$

or equivalently,

$$x_{n+1} = \gamma_n (1-\alpha_n), \quad \forall n \geq 0,$$

where $\gamma_n : [0,1] \to M$ is the geodesic joining z to $T(x_n)$ (i.e. $\gamma(0) = z$ and $\gamma(1) = T(x_n)$). Indeed, this algorithm coincides with Halpern algorithm in the particular case of an Euclidean space, and we can prove its convergence under the same conditions. Then we consider the following hypothesis.

(H1) $\lim_{n\to\infty} \alpha_n = 0$;
(H2) $\sum_{n\geq 0} \alpha_n = \infty$;
(H3) $\sum_{n\geq 0} |\alpha_{n+1} - \alpha_n| < \infty$;
(H4) $\lim_{n\to\infty} (\alpha_n - \alpha_{n-1})/\alpha_n = 0$.

Theorem 2.6.3. *Let C be a closed convex subset of M and $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let $z, x_0 \in M$. Suppose that $\{\alpha_n\} \in (0,1)$ satisfies $(H1)$, $(H2)$ and, $(H3)$ or $(H4)$. Then the sequence $\{x_n\}$ generated by the algorithm* (2.6.2) *converges strongly to $P_{Fix(T)}z$.*

Proof. Let $n \geq 0$ and let $\gamma_n : [0,1] \to M$ denote the geodesic joining z to $T(x_n)$. We divide the proof into four steps.

Step 1. $\{x_n\}$ and $\{T(x_n)\}$ are bounded.

We only prove the boundedness of $\{x_n\}$ since the boundedness of $\{\psi(x_n)\}$ is a direct consequence. To this end, take $x \in \operatorname{Fix}(T)$ and fix $n \geq 0$. Then, by the convexity of the distance function and the nonexpansivity of T, we have that

$$\begin{aligned} d(x_{n+1}, x) &= d(\gamma_n(1-\alpha_n), x) \\ &\leq \alpha_n d(\gamma_n(0), x) + (1-\alpha_n) d(\gamma_n(1), x) \\ &= \alpha_n d(z, x) + (1-\alpha_n) d(T(x_n), x) \\ &\leq \alpha_n d(z, x) + (1-\alpha_n) d(x_n, x). \end{aligned}$$

Then by mathematical induction we deduce that

$$d(x_{n+1}, x) \leq \max\{d(z, x), d(x_0, x)\}, \; n \geq 0,$$

which implies the boundness of $\{x_n\}$ and, therefore, of $\{T(x_n)\}$.

Step 2. $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

By Step 1, we can find a constant ρ such that

$$d(x_n, x_{n-1}) \leq \rho \quad \text{and} \quad d(z, T(x_n)) \leq \rho, \quad \forall n \geq 0. \tag{2.6.3}$$

By using the convexity of the distance function, we have that, for each $n \geq 0$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\gamma_n(1-\alpha_n), \gamma_{n-1}(1-\alpha_{n-1})) \\ &\leq d(\gamma_n(1-\alpha_n), \gamma_{n-1}(1-\alpha_n)) + d(\gamma_{n-1}(1-\alpha_n), \gamma_{n-1}(1-\alpha_{n-1})) \\ &\leq (1-\alpha_n) d(T(x_n), T(x_{n-1})) + |\alpha_n - \alpha_{n-1}| d(z, T(x_{n-1})). \end{aligned}$$

This together with (2.6.3) and the nonexpansivity of T implies that

$$d(x_{n+1}, x_n) \leq (1-\alpha_n) d(x_n, x_{n-1}) + \rho|\alpha_n - \alpha_{n-1}|, \quad \forall n \geq 0. \tag{2.6.4}$$

Thus, if (H4) holds, we apply Lemma 1.1.21 (with $\beta_n = \alpha_n$ and $b_n = \rho|\alpha_n - \alpha_{n-1}|/\alpha_n$ for each $n \geq 0$) to conclude that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. As to the case when (H3) holds, let $k \leq n$. By (2.6.4), one gets that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \prod_{i=k}^{n}(1-\alpha_i)d(x_k, x_{k-1}) + \rho\sum_{i=k}^{n}|\alpha_i - \alpha_{i-1}| \\ &\leq \rho\prod_{i=k}^{n}(1-\alpha_i) + \rho\sum_{i=k}^{n}|\alpha_i - \alpha_{i-1}|. \end{aligned}$$

Letting $n \to \infty$ implies that,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) \leq \rho\prod_{i=k}^{\infty}(1-\alpha_i) + \rho\sum_{i=k}^{\infty}|\alpha_i - \alpha_{i-1}|. \tag{2.6.5}$$

Condition (H2) implies that $\lim_{k\to\infty}\prod_{i=k}^{\infty}(1-\alpha_i) = 0$; while Condition (H3) implies that $\lim_{k\to\infty}\sum_{i=k}^{\infty}|\alpha_i - \alpha_{i-1}| = 0$. Hence, letting $k \to \infty$ in (2.6.5), we get $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$ in the case when (H3) holds.

Step 3. $\limsup_{n\to\infty}\langle \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} z,\ \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} T(x_n)\rangle \leq 0$.

By Step 1, $\{\langle \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} z,\ \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} T(x_n)\rangle\}$ is bounded; hence its upper limit exists. Thus we can find a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\limsup_{n\to\infty}\langle \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} z, \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} T(x_n)\rangle = \lim_{k\to\infty}\langle \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} z, \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} T(x_{n_k})\rangle. \tag{2.6.6}$$

Without loss of generality, we may assume that $x_{n_k} \to \bar{x}$ for some $\bar{x} \in C$ because $\{x_n\}$ is bounded by Step 1 and C is closed. By the convexity of the distance function and the definition of the algorithm,

$$d(x_{n_k+1}, T(x_{n_k})) \leq \alpha_n d(z, T(x_{n_k})).$$

Since $\{d(z, T(x_{n_k}))\}$ is bounded by Step 1, it follows that $\lim_{k\to\infty} d(x_{n_k+1}, T(x_{n_k})) = 0$ as $\alpha_{n_k} \to 0$ by (H1). Noting that

$$d(x_{n_k}, T(x_{n_k})) \leq d(x_{n_k+1}, x_{n_k}) + d(x_{n_k+1}, T(x_{n_k})),$$

one sees that $\lim_{n\to\infty} d(x_{n_k}, T(x_{n_k})) = 0$. Therefore

$$d(\bar{x}, T(\bar{x})) \leq d(\bar{x}, x_{n_k}) + d(x_{n_k}, T(x_{n_k})) + d(T(x_{n_k}), T(\bar{x})) \to 0,$$

which means that $\bar{x} \in \mathrm{Fix}\,(T)$. Then, Proposition 2.1.18 implies that

$$\langle \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} z, \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} \bar{x} \rangle \leq 0.$$

Therefore

$$\lim_{k\to\infty} \langle \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} z, \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} T(x_{n_k}) \rangle \;=\; \langle \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} z, \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} \bar{x} \rangle \leq 0.$$

Combining this with (2.6.6), we complete the proof of Step 3.

Step 4. $\lim_{n\to\infty} d(x_n, P_{\mathrm{Fix}\,(T)}z) = 0$.
Let us set

$$b_n = \alpha_n d^2(z, P_{\mathrm{Fix}\,(T)}z) + 2\langle \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} z, \exp^{-1}_{P_{\mathrm{Fix}\,(T)}z} T(x_n) \rangle, \quad \forall n \geq 0. \tag{2.6.7}$$

Then $\lim_{n\to\infty} b_n \leq 0$ by Step 3. Thus, by Lemma 1.1.21, it suffices to verify that

$$d^2(x_{n+1}, P_{\mathrm{Fix}\,(T)}z) \leq (1-\alpha_n) d^2(x_n, P_{\mathrm{Fix}\,(T)}z) + \alpha_n b_n, \quad \forall n \geq 0. \tag{2.6.8}$$

To this end, we fix $n \geq 0$ and set $p = T(x_n),\ q = P_{\mathrm{Fix}\,(T)}z$. Consider the geodesic triangle $\Delta(z, p, q)$ and its comparison triangle $\Delta(z', p', q')$. Then

$$d(z, P_{\mathrm{Fix}\,(T)}z) = d(z, q) = \|z' - q'\| \quad \text{and} \quad d(T(x_n), P_{\mathrm{Fix}\,(T)}z) = d(p, q) = \|p' - q'\|.$$

Recall from (2.6.2) that $x_{n+1} = \exp_z (1-\alpha_n) \exp_z^{-1} T(x_n) = \exp_z (1-\alpha_n) \exp_z^{-1} p$. The comparison point of x_{n+1} is $x'_{n+1} = \alpha_n z' + (1-\alpha_n) p'$. Let β and β' denote the angles at q and q', respectively. Then $\beta \leq \beta'$ by Lemma 2.1.28(1) and so $\cos\beta' \leq \cos\beta$. Then, by Lemma 2.1.28(2) we have

$$\begin{aligned}
d^2(x_{n+1}, P_{\mathrm{Fix}\,(T)}z) &\leq \|x'_{n+1} - q'\|^2 \\
&= \|\alpha_n (z' - q') + (1-\alpha_n)(p' - q')\|^2 \\
&= \alpha_n^2 \|z' - q'\|^2 + (1-\alpha_n)^2 \|p' - q'\|^2 \\
&\quad + 2\alpha_n (1-\alpha_n) \|z' - q'\| \|p' - q'\| \cos\beta' \\
&\leq \alpha_n^2 d^2(z, P_{\mathrm{Fix}\,(T)}z) + (1-\alpha_n)^2 d^2(T(x_n), P_{\mathrm{Fix}\,(T)}z) \\
&\quad + 2\alpha_n (1-\alpha_n) d(z, P_{\mathrm{Fix}\,(T)}z) d(T(x_n), P_{\mathrm{Fix}\,(T)}z) \cos\beta \\
&\leq (1-\alpha_n) d^2(x_n, P_{\mathrm{Fix}\,(T)}z) + \alpha_n b_n,
\end{aligned}$$

where b_n is defined by (2.6.7). Hence (2.6.8) is proved and the proof is complete. □

2.6.3 Mann algorithm for nonexpansive mappings

Mann iteration (1.1.16) and some of the convergence results in Banach spaces presented in Chapter 1 have been extended to the framework of more general metric spaces by Goebel-Kirk [58, 46] and Reich-Shafrir [101]. They provided an iterative method for finding fixed points of nonexpansive mappings on spaces of *hyperbolic type* which includes Hadamard manifolds as a particular case. The algorithm is defined by

$$x_{n+1} \in [x_n, T(x_n)] \quad \text{such that} \quad d(x_n, T(x_n)) = (1-\alpha_n) d(x_n, x_{n+1}), \tag{2.6.9}$$

where $[x_n, T(x_n)]$ denotes the metric segment joining x_n to $T(x_n)$. More precisely, under the assumption that $\{\alpha_n\}$ is bounded away from 0 and 1, Reich and Shafrir proved the convergence of this iteration to a fixed point of T defined on the Hilbert ball with the hyperbolic metric.

Motivated by these results, we introduce Mann iteration (2.6.9) in Hadamard manifolds M by means of the recursive formula

$$x_{n+1} = \exp_{x_n}(1-\alpha_n)\exp_{x_n}^{-1} T(x_n), \quad \forall n \geq 0, \tag{2.6.10}$$

We next prove that the sequence $\{x_n\}$ generated by Mann algorithm (2.6.10) converges to a fixed point of T when $\{\alpha_n\}$ satisfies the condition:

$$\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty. \tag{2.6.11}$$

Theorem 2.6.4. *Let $C \subseteq M$ be a closed convex set and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Suppose that $\{\alpha_n\} \subset (0,1)$ satisfy condition* (2.6.11)*. Then, for any $x_0 \in C$, the sequence $\{x_n\}$ generated by algorithm* (2.6.10) *converges to a fixed point of T.*

Proof. Note that C itself is a complete metric space. Let $n \geq 0$ and $p \in \mathrm{Fix}(T)$ be fixed. Let $\gamma : [0,1] \to M$ denote the geodesic joining x_n to $T(x_n)$. Then

$x_{n+1} = \gamma(1-\alpha_n)$. By the convexity of the distance function and the nonexpansivity of T, we have

$$d(x_{n+1}, p) = d(\gamma(1-\alpha_n), p) \leq \alpha_n d(x_n, p) + (1-\alpha_n) d(T(x_n), p) \leq d(x_n, p).$$

Hence $\{x_n\}$ is Fejér monotone with respect to $\operatorname{Fix}(T)$. Thus, by Lemma 1.1.24, it suffices to prove that all cluster points of $\{x_n\}$ belong to $\operatorname{Fix}(T)$. To this end, let x be a cluster point of $\{x_n\}$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \to x$. We next prove that

$$\lim_{n\to\infty} d(x_n, T(x_n)) = 0. \tag{2.6.12}$$

Let $p \in \operatorname{Fix}(T)$ and $n \geq 0$. Let $\Delta(x_n, q, p)$ be the geodesic triangle with vertices x_n, $q := T(x_n)$ and p. From Lemma 2.1.26 there exists a comparison triangle $\Delta(x_n', q', p')$ which preserves the length of the edge. Recall that $x_{n+1} = \gamma(1-\alpha_n)$ and $x_{n+1}' := (1-\alpha_n)x_n' + \alpha_n q'$ is its comparison point. By Lemma 2.1.28(2),

$$\begin{aligned} d^2(x_{n+1}, p) &\leq \|x_{n+1}' - p'\|^2 \\ &= \|\alpha_n(x_n' - p') + (1-\alpha_n)(q' - p')\|^2 \\ &= \alpha_n\|x_n' - p'\|^2 + (1-\alpha_n)\|q' - p'\|^2 - \alpha_n(1-\alpha_n)\|x_n' - q'\|^2 \\ &= \alpha_n d^2(x_n, p) + (1-\alpha_n) d^2(T(x_n), p) - \alpha_n(1-\alpha_n) d^2(x_n, T(x_n)) \\ &\leq d^2(x_n, p) - \alpha_n(1-\alpha_n) d^2(x_n, T(x_n)). \end{aligned}$$

It follows that

$$\alpha_n(1-\alpha_n) d^2(x_n, T(x_n)) \leq d^2(x_n, p) - d^2(x_{n+1}, p)$$

and

$$\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) d^2(x_n, T(x_n)) < \infty, \tag{2.6.13}$$

which implies that

$$\liminf_{n\to\infty} d(x_n, T(x_n)) = 0 \tag{2.6.14}$$

because otherwise, $d(x_n, T(x_n)) \geq a \ \forall n \geq 0$ for some $a > 0$, and then,

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) d(x_n, T(x_n)) \geq a \sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$$

which is a contradiction with (2.6.13).

On the other hand, using the nonexpansivity of T and the convexity of the distance function,

$$\begin{aligned} d(x_{n+1}, T(x_{n+1})) &\leq d(x_{n+1}, T(x_n)) + d(T(x_n), T(x_{n+1})) \\ &\leq d(x_{n+1}, T(x_n)) + d(x_n, x_{n+1}) \\ &= \alpha_n d(x_n, T(x_n)) + (1-\alpha_n) d(x_n, T(x_n)) \\ &= d(x_n, T(x_n)). \end{aligned}$$

This means that $\{d(x_n, T(x_n))\}$ is a monotone sequence. By combining this and (2.6.14) we get that (2.6.12) holds. Then, since

$$\begin{aligned} d(x, T(x)) &\leq d(x, x_{n_k}) + d(x_{n_k}, T(x_{n_k})) + d(T(x_{n_k}), T(x)) \\ &\leq 2d(x_{n_k}, x) + d(x_{n_k}, T(x_{n_k})), \end{aligned}$$

by taking limit, we deduce that $d(x, Tx) = 0$, which means that $x \in \operatorname{Fix}(T)$. $\square$

2.6.4 Viscosity approximation method for nonexpansive mappings

Recall that the viscosity approximation method of selecting a particular fixed point of a nonexpansive mapping T was presented in Section 1.1.2 in the setting of Banach spaces. It consists of two algorithms (implicit and explicit) that, under suitable conditions, converge strongly to the unique fixed point of T, $p \in \operatorname{Fix}(T)$, which solves the variational inequality

$$\langle (I-\psi)q, x-q \rangle \geq 0, \quad \forall x \in \operatorname{Fix}(T), \tag{2.6.15}$$

where ψ is a contraction.

The purpose of this section is to establish the convergence of a viscosity method for nonexpansive mappings in the setting of a Hadamard manifold. Let C be a closed convex subset of a Hadamard manifold M, $T : C \to C$ a nonexpansive self-mapping and $\psi : C \to C$ a contraction. Assume that the fixed point set $\operatorname{Fix}(T)$ is nonempty. We next prove the convergence of an explicit algorithm to a fixed point of T which solves the variational inequality

$$\langle \exp_{\bar{x}}^{-1} \psi(\bar{x}), \exp_{\bar{x}}^{-1} x \rangle \leq 0, \quad \forall x \in \operatorname{Fix}(T). \tag{2.6.16}$$

Let $x_0 \in M$, $\{\alpha_n\} \subset (0,1)$. Consider the iteration scheme

$$x_{n+1} = \exp_{\psi(x_n)}\big((1-\alpha_n)\exp^{-1}_{\psi(x_n)} T(x_n)\big), \quad \forall n \geq 0; \tag{2.6.17}$$

or equivalently,

$$x_{n+1} = \gamma_n(1-\alpha_n), \quad \forall n \geq 0,$$

where $\gamma_n : [0,1] \to M$ is the geodesic joining $\psi(x_n)$ to $T(x_n)$ (i.e. $\gamma(0) = \psi(x_n)$ and $\gamma(1) = T(x_n)$).

For the convergence theorem we consider the conditions (H1)-(H4) on the sequence $\{\alpha_n\}$ enumerated in Section 2.6.2.

Theorem 2.6.5. *Let $C \subseteq M$ be a closed convex ser and $T : C \subseteq\to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let $x_0 \in M$ and $\psi : C \to C$ a ρ-contraction. Suppose that $\{\alpha_n\} \in (0,1)$ satisfies* (H1)*,* (H2) *and,* (H3) *or* (H4)*. Then the sequence $\{x_n\}$ generated by the algorithm* (2.6.17) *converges to $\bar{x} \in C$, the unique fixed point of the contraction $P_{Fix(T)}\psi$.*

Moreover, the convergence point $\bar{x}$ is a solution of the variational inequality (2.6.16)*.*

Proof. As a matter of fact, if $\bar{x} \in C$ is the unique fixed point of $P_{\mathrm{Fix}\,(T)}\psi$, then

$$\langle \exp^{-1}_{\bar{x}} \psi(\bar{x}), \exp^{-1}_{\bar{x}} x\rangle = \langle \exp^{-1}_{P_{\mathrm{Fix}\,(T)}\psi(\bar{x})} \psi(\bar{x}), \exp^{-1}_{P_{\mathrm{Fix}\,(T)}\psi(\bar{x})} x\rangle \leq 0,$$

by Proposition 2.1.18.

For $n \geq 0$, let $\gamma_n : [0,1] \to M$ denote the geodesic joining $\psi(x_n)$ to $T(x_n)$. Then

$$x_{n+1} = \gamma_n(1-\alpha_n).$$

We divide the proof into four steps.

Step 1. $\{x_n\}$ and $\{\psi(x_n)\}$ are bounded.

We only prove the boundedness of $\{x_n\}$ since the boundedness of $\{\psi(x_n)\}$ is a direct consequence. To this end, take $x \in \mathrm{Fix}\,(T)$ and fix $n \geq 0$. Then, by the convexity of the distance function and the nonexpansivity of T, we have that

$$\begin{array}{rcl} d(x_{n+1},x) & = & d(\gamma_n(1-\alpha_n),x) \\ & \leq & \alpha_n d(\psi(x_n),x)+(1-\alpha_n)d(x_n,x) \\ & \leq & \alpha_n(\rho d(x_n,x)+d(\psi(x),x))+(1-\alpha_n)d(x_n,x) \\ & = & (1-\alpha_n(1-\rho))d(x_n,x)+\alpha_n d(\psi(x),x) \\ & \leq & \max\{d(x_n,x),\dfrac{1}{1-\rho}d(\psi(x),x)\}. \end{array}$$

Then, by mathematical induction, for any $n\geq 0$ we have that

$$d(x_{n+1},x)\leq \max\{d(x_0,x),\frac{1}{1-\rho}d(\psi(x),x)\}.$$

Step 2. $\lim_{n\to\infty} d(x_{n+1},x_n)=0$.

Since $\{x_n\}$ is bounded by Step 1, we can find a constant L such that for all $n\geq 0$,

$$d(x_n,x_{n-1})\leq L \text{ and } d(\psi(x_n),x_n)\leq L. \qquad (2.6.18)$$

By using the convexity of the distance function, we have that

$$\begin{array}{rcl} d(x_{n+1},x_n) & \leq & d(\gamma_n(1-\alpha_n),\gamma_{n-1}(1-\alpha_{n-1})) \\ & \leq & d(\gamma_n(1-\alpha_n),\gamma_{n-1}(1-\alpha_n))+d(\gamma_{n-1}(1-\alpha_n),\gamma_{n-1}(1-\alpha_{n-1})) \\ & \leq & \alpha_n d(\psi(x_n),\psi(x_{n-1}))+(1-\alpha_n)d(x_n,x_{n-1}) \\ & & +|\alpha_n-\alpha_{n-1}|d(\psi(x_{n-1}),x_{n-1}) \\ & = & (1-\alpha_n(1-\rho))d(x_n,x_{n-1})+|\alpha_n-\alpha_{n-1}|d(\psi(x_{n-1}),x_{n-1}). \end{array}$$

This together with (2.6.18) and denoting $\bar{\alpha}_n=\alpha_n(1-\rho)$ implies that

$$d(x_{n+1},x_n)\leq (1-\bar{\alpha}_n)d(x_n,x_{n-1})+L|\alpha_n-\alpha_{n-1}|. \qquad (2.6.19)$$

Thus, if (H4) holds, by Lemma 1.1.21 (with $\beta_n=\bar{\alpha}_n$ and $b_n=L|\alpha_n-\alpha_{n-1}|/\alpha_n$ for each $n\geq 0$) we conclude that $\lim_{n\to\infty} d(x_{n+1},x_n)=0$. As to the case when (H3) holds, let $k \leq n$. By inequalities (2.6.19) and (2.6.18), one gets that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \prod_{i=k}^{n}(1-\bar{\alpha}_i)d(x_k, x_{k-1}) + L\sum_{i=k}^{n}|\alpha_i - \alpha_{i-1}| \\ &\leq L\prod_{i=k}^{n}(1-\bar{\alpha}_i) + L\sum_{i=k}^{n}|\alpha_i - \alpha_{i-1}|. \end{aligned}$$

Letting $n \to \infty$,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) \leq L\prod_{i=k}^{\infty}(1-\bar{\alpha}_i) + L\sum_{i=k}^{\infty}|\alpha_i - \alpha_{i-1}|. \tag{2.6.20}$$

Condition (H2) implies that $\lim_{k\to\infty}\prod_{i=k}^{\infty}(1-\bar{\alpha}_i) = 0$; while by condition (H3) we deduce that $\lim_{k\to\infty}\sum_{i=k}^{\infty}|\alpha_i - \alpha_{i-1}| = 0$. Hence, letting $k \to \infty$ in (2.6.20), we get $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$ in the case when (H3) holds.

Step 3. $\limsup_{n\to\infty}\langle \exp_{\bar{x}}^{-1}\psi(\bar{x}),\ \exp_{\bar{x}}^{-1} x_n\rangle \leq 0$, where $\bar{x}$ is the unique fixed point of the contraction $P_{\mathrm{Fix}\,(T)}\psi$.

By Step 1, $\{\langle \exp_{\bar{x}}^{-1}\psi(\bar{x}),\ \exp_{\bar{x}}^{-1} x_n\rangle\}$ is bounded; hence its upper limit exists. Thus we can find a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\limsup_{n\to\infty}\langle \exp_{\bar{x}}^{-1}\psi(\bar{x}),\ \exp_{\bar{x}}^{-1} x_n\rangle = \lim_{k\to\infty}\langle \exp_{\bar{x}}^{-1}\psi(\bar{x}),\ \exp_{\bar{x}}^{-1} x_{n_k}\rangle. \tag{2.6.21}$$

Without loss of generality, we may assume that $x_{n_k} \to x^*$ for some $x^* \in M$ because $\{x_n\}$ is bounded by Step 1. By the convexity of the distance function and the definition of the algorithm,

$$d(x_{n_k+1}, T(x_{n_k})) \leq \alpha_n d(\psi(x_{n_k}), T(x_{n_k})).$$

Since $\{d(\psi(x_{n_k}), T(x_{n_k}))\}$ is bounded by Step 1, it follows that the limit $\lim_{k\to\infty} d(x_{n_k+1}, T(x_{n_k})) = 0$, as $\alpha_{n_k} \to 0$ by (H1). Noting that

$$d(x_{n_k}, T(x_{n_k})) \leq d(x_{n_k+1}, x_{n_k}) + d(x_{n_k+1}, T(x_{n_k})),$$

one gets that $\lim_{n\to\infty} d(x_{n_k}, T(x_{n_k})) = 0$. Therefore

$$d(\bar{x}, T(\bar{x})) \leq d(\bar{x}, x_{n_k}) + d(x_{n_k}, T(x_{n_k})) + d(T(x_{n_k}), T(\bar{x})) \to 0,$$

which means that $x^* \in \mathrm{Fix}\,(T)$. Then, since $\langle \exp_{\bar{x}}^{-1}\psi(\bar{x}), \exp_{\bar{x}}^{-1} x\rangle \leq 0$ for any $x \in \mathrm{Fix}\,(T)$, we obtain that

$$\lim_{k\to\infty} \langle \exp_{\bar{x}}^{-1}\psi(\bar{x}),\ \exp_{\bar{x}}^{-1} x_{n_k}\rangle \;=\; \langle \exp_{\bar{x}}^{-1}\psi(\bar{x}),\ \exp_{\bar{x}}^{-1} x^*\rangle \leq 0.$$

Combining this with (2.6.21), we complete the proof of Step 3.

Step 4. $\lim_{n\to\infty} d(x_n,\bar{x}) = 0$.

We fix $n \geq 0$ and set $r = \psi(x_n)$, $p = T(x_n)$, $q = \bar{x}$. Consider the geodesic triangle $\Delta(r,p,q)$ and its comparison triangle $\Delta(r',p',q') \subset \mathbb{R}^2$. Then

$$d(\psi(x_n),\bar{x}) = d(r,q) = \|r'-q'\| \quad \text{and} \quad d(T(x_n),\bar{x}) = d(p,q) = \|p'-q'\|.$$

Recall that $x_{n+1} = \exp_r(1-\alpha_n)\exp_r^{-1} p$. The comparison point of x_{n+1} in $\mathbb{R}^2$ is $x'_{n+1} = \alpha_n r' + (1-\alpha_n)p'$. Let β and β' denote the angles at q and q', respectively. Therefore $\beta \leq \beta'$ by Lemma 2.1.28(1) and then $\cos\beta' \leq \cos\beta$. Thus, by nonexpansivity of T and Lemma 2.1.28(2) we have

$$\begin{aligned}
d^2(x_{n+1},\bar{x}) &\leq \|x'_{n+1}-q'\|^2 \\
&= \|\alpha_n(r'-q') + (1-\alpha_n)(p'-q')\|^2 \\
&= \alpha_n^2\|r'-q'\|^2 + (1-\alpha_n)^2\|p'-q'\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)\|r'-q'\|\|p'-q'\|\cos\beta' \\
&\leq \alpha_n^2 d^2(\psi(x_n),\bar{x}) + (1-\alpha_n)^2 d^2(T(x_n),\bar{x}) \\
&\quad +2\alpha_n(1-\alpha_n) d(\psi(x_n),\bar{x}) d(T(x_n),\bar{x})\cos\beta \\
&\leq \alpha_n^2 d^2(\psi(x_n),\bar{x}) + (1-\alpha_n)^2 d^2(x_n,\bar{x}) \\
&\quad +2\alpha_n(1-\alpha_n)\big(d(\psi(\bar{x}),\bar{x}) + d(\psi(x_n),\psi(\bar{x}))\big) d(x_n,\bar{x})\cos\beta \\
&\leq \alpha_n^2 d^2(\psi(x_n),\bar{x}) + (1-\alpha_n)^2 d^2(x_n,\bar{x}) \\
&\quad +2\alpha_n(1-\alpha_n)\big(\langle \exp_{\bar{x}}^{-1}\psi(\bar{x}),\ \exp_{\bar{x}}^{-1} x_n\rangle + \rho d^2(x_n,\bar{x})\big) \\
&= (1-2\alpha_n+\alpha_n^2+2\alpha_n(1-\alpha_n)\rho) d^2(x_n,\bar{x}) + \alpha_n^2 d^2(\psi(x_n),\bar{x}) \\
&\quad +2\alpha_n(1-\alpha_n)\langle \exp_{\bar{x}}^{-1}\psi(\bar{x}),\ \exp_{\bar{x}}^{-1} x_n\rangle \\
&= (1-\gamma_n) d^2(x_n,\bar{x}) + \gamma_n b_n,
\end{aligned}$$

where $b_n = 1/\gamma_n(\alpha_n^2 d^2(\psi(x_n),\bar{x}) + 2\alpha_n(1-\alpha_n)\langle \exp_{\bar{x}}^{-1}\psi(\bar{x}),\ \exp_{\bar{x}}^{-1} x_n\rangle$ and $\gamma_n = 2\alpha_n - \alpha_n^2 - 2\alpha_n(1-\alpha_n)\rho$. Then $\lim_{n\to\infty} b_n \leq 0$ by Step 3, $\lim\gamma_n = 0$ by hypothesis (H1) and $\sum_{n=0}^{\infty}\gamma_n = \infty$ by hypothesis (H2). Thus, by Lemma 1.1.21 $\lim_{n\to\infty} d(x_n,\bar{x}) = 0$ and the proof is complete. □

2.6.5 Numerical example

Let $\mathbb{E}^{m,1}$ denote the vector space $\mathbb{R}^{m+1}$ endowed with the symmetric bilinear form defined by

$$\langle x,y\rangle = \sum_{i=1}^{m} x_i y_i - x_{m+1}y_{m+1}, \quad \forall x=(x_i),\ y=(y_i)\in\mathbb{R}^{m+1}.$$

This bilinear form is called the Lorentz metric. The *hyperbolic m-space* $\mathbb{H}^m$ is defined by

$$\{x=(x_1,...,x_{m+1})\in\mathbb{E}^{m,1} : \langle x,x\rangle = -1, x_{m+1}>0\};$$

that is the upper sheet of the hyperboloid $\{x\in\mathbb{E}^{m,1} : \langle x,x\rangle = -1\}$. Note that $x_{m+1}\geq 1$ for any $x\in\mathbb{H}^m$, with equality if and only if $x_i = 0$ for all $i=1,...,m$. The metric of $\mathbb{H}^m$ is induced by the Lorentz metric $\langle\cdot,\cdot\rangle$ and it will be denoted by the same symbol. Then $\mathbb{H}^m$ is a Hadamard manifold with sectional curvature -1 (cf. [11] and [41]). Furthermore, the normalized geodesic $\gamma:\mathbb{R}\to\mathbb{H}^m$ starting from $x\in\mathbb{H}^m$ is given by

$$\gamma(t) = (\cosh t)x + (\sinh t)v, \quad \forall t\in\mathbb{R}, \tag{2.6.22}$$

where $v\in T_x\mathbb{H}^m$ is a unit vector; while the distance d on $\mathbb{H}^m$ is

$$d(x,y) = \operatorname{arccosh}(-\langle x,y\rangle), \quad \forall x,y\in\mathbb{H}^m. \tag{2.6.23}$$

Then, the exponential map can be expressed as

$$\exp_x(rv) = (\cosh r)x + (\sinh r)v,$$

for any $r\in\mathbb{R}^+$, $x\in\mathbb{H}^m$ and any unit vector $v\in T_x\mathbb{H}^m$. To get the expression of the inverse exponential map, we write for any $x,y\in\mathbb{H}^m$,

$$y = \exp_x\big(d(x,y)\frac{\exp_x^{-1} y}{d(x,y)}\big) = (\cosh d(x,y))x + (\sinh d(x,y))\frac{\exp_x^{-1} y}{d(x,y)}.$$

Therefore, by definition of the distance (2.6.23), we obtain

$$\exp_x^{-1} y = \operatorname{arccosh}(-\langle x, y\rangle)\frac{y + \langle x, y\rangle x}{\sqrt{\langle x, y\rangle^2 - 1}}, \quad \forall x, y \in \mathbb{H}^m.$$

Thus, using this expressions, both algorithms in previous sections can be formulated in a simple way in the hyperbolic space $\mathbb{H}^m$. Write

$$r(x, y) = \operatorname{arccosh}(-\langle x, T(y)\rangle)$$

and

$$V(x, y) = \frac{T(y) + \langle x, T(y)\rangle x}{\sqrt{\langle x, T(y)\rangle^2 - 1}}.$$

Then Halpern algorithm (2.6.2) has the form

$$x_{n+1} = \cosh((1-\alpha_n)r(u, x_n))u + \sinh((1-\alpha_n)r(u, x_n))V(u, x_n), \quad \forall n \geq 0;$$

whereas Mann algorithm (2.6.10) has the form

$$x_{n+1} = \cosh((1-\alpha_n)r(x_n, x_n))x_n + \sinh((1-\alpha_n)r(x_n, x_n))V(x_n, x_n), \quad \forall n \geq 0.$$

We present an example in $\mathbb{H}^3$, where these methods are implemented for two concrete mappings.

Example 2.6.6. *Let $M = \mathbb{H}^3$ and let $T_1, T_2 : M \to M$ be the nonexpansive mappings respectively defined by*

$$T_1(x) = (-x_1, -x_2, -x_3, x_4)$$

and

$$T_2(x) = (-x_1, x_2, x_3, x_4),$$

for any $x = (x_1, x_2, x_3, x_4) \in \mathbb{H}^3$. Then $Fix(T_1) = \{(0, 0, 0, 1)\}$ and

$$Fix(T_2) = \{(x_1, x_2, x_3, x_4) \in \mathbb{H}^3 : x_1 = 0, x_2^2 + x_3^2 = x_4^2 - 1\}.$$

For both algorithms we are going to consider the sequence of parameters $\alpha_n = \frac{1}{n+3}$, for each $n \geq 0$, and the point

$$u = (0.60379247919382, 0.27218792496996, 0.19881426776106, 1.21580374135624)$$

for Halpern iteration. We consider three random initial points x_0:

$$x_0^1 = (0.69445440978475, 1.01382609280137, 0.99360871330745, 1.87012527625153);$$

$$x_0^2 = (0.82054041398189, 1.78729906182707, 0.11578260956854, 2.20932797928782);$$

$$x_0^3 = (0.93181457846166, 0.46599434167542, 0.41864946772751, 1.50356127641534).$$

The numerical results are illustrated in the following graphics. We measure the error of the nth step in both algorithms by means of the distance $d_n = d(x_n, x^*)$, where $x^* = (0, 0, 0, 1)$ is the unique fixed point of the mapping T_1, in Graphic 1; whereas in Graphic 2, $d_n = d(x_{n+1}, x_n)$ denotes the distance between two consecutive iterates x_{n+1} and x_n for the mapping T_2.

From the numerical results, one can observe that Mann iteration seems to converge much quicker than Halpern iteration. Moreover, as it is predicted from the theoretical results, the measure of the errors in Graphic 1 shows that the sequence $\{x_n\}$ is Fejér monotone with respect to $\mathrm{Fix}\,(T_1)$ just in the case of Mann algorithm.

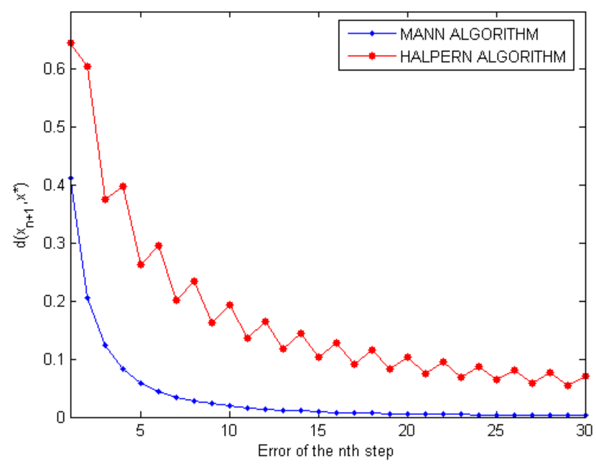

Figure 2.1: Graphic 1

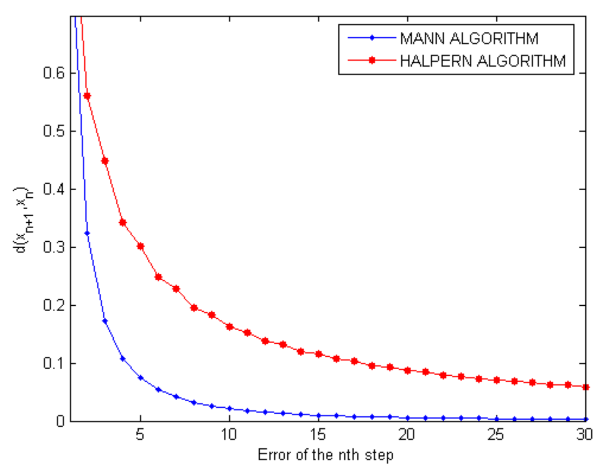

Figure 2.2: Graphic 2

2.7 Applications

This section is devoted to some applications of the convergence result obtained for the proximal point algorithm (2.5.1) and Picard iteration. The first one is on constrained minimization problems, followed by the application on a minimax problem consisting of finding a saddle-point. Then variational inequality problems are solved. Finally we define the resolvent of a bifunction and get convergence results for equilibrium problems.

2.7.1 Constrained optimization problems

Some nonconvex constrained optimization problems can be solved after being written as convex problems in Riemann manifolds, see for example [36, 41, 104, 113]. More specifically in [41] an algorithm that can be used for solving any constrained problem in $\mathbb{R}^n$ with a convex objective function and constraint set being a constant curvature Hadamard manifold is presented. Regarding the problem of how to determine the sectional curvature of a manifold given in an implicit form, very nice results have been recently obtained by Rapcsk in [91]. In this section we apply our results to a constrained minimization problem.

Recall that M is a Hadamard manifold. Let $f : M \to \overline{\mathbb{R}}$ be a proper lower semicontinuous convex function. For simplicity, we write $C = \mathcal{D}(f)$. Then C is closed convex subset of M. Let $x \in M$ and let

$$T_xC = \{u \in T_xM : \exp_x tu \in C \text{ for some } t > 0\}.$$

Then T_xC is a convex cone. Define the directional derivative of f at x in direction $u \in T_xM$ by

$$f'(x,u) = \lim_{t\to 0+} \frac{f(\exp_x tu) - f(x)}{t}.$$

It can be proved that $f'(x,\cdot)$ is subadditive and positively homogeneous on T_xC. Moreover, we have by definition that $\mathcal{D}(f'(x,\cdot)) = T_xC$.

The *subdifferential* of f at x is defined by

$$\partial f(x) = \{u \in T_xM : \langle u, \exp_x^{-1} y\rangle \leq f(y) - f(x),\ \forall y \in M\}.$$

Then $\partial f(x)$ is a closed convex (possible empty) set, and

$$u \in \partial f(x) \Longleftrightarrow \langle u, h\rangle \leq f'(x,h) \quad \text{for each } h \in T_xM. \tag{2.7.1}$$

The proofs of the above assertions can be found in [113]. Furthermore, it is clear from the definition that

$$\partial f(x) = \partial(f'(x, \cdot))(0). \tag{2.7.2}$$

Throughout the whole subsection, we always assume that $\mathcal{D}(\partial f) \neq \emptyset$.

Theorem 2.7.1. *Let f be a proper lower semicontinuous convex function on M. The subdifferential $\partial f(\cdot)$ is a monotone and upper Kuratowski semicontinuous set-valued vector field. Furthermore, if in addition $\mathcal{D}(f) = M$, then the subdifferential ∂f of f is a maximal monotone vector field.*

Proof. The monotonicity of ∂f is a consequence of the definition of the subdifferential, which was previously showed in [35]. Indeed, for any $x, y \in \mathcal{D}(f)$, $u \in \partial f(x)$ and $v \in \partial f(y)$, we have that

$$\langle u, \exp_x^{-1} y\rangle \leq f(y) - f(x) \leq \langle v, -\exp_y^{-1} x\rangle.$$

To prove the upper Kuratowski semicontinuity, let $x_0 \in \mathcal{D}(f)$, $\{x_n\} \subset \mathcal{D}(f)$ and $\{v_n\} \subset TM$ with each $v_n \in \partial f(x_n)$ be such that $x_n \to x_0$ and $v_n \to v_0$ for some $v_0 \in TM$. Then, by definition,

$$\langle v_n, \exp_{x_n}^{-1} y\rangle \leq f(y) - f(x_n), \quad \forall y \in M.$$

Taking lower limits in the previous inequality, we get that

$$\langle v_0, \exp_{x_0}^{-1} y\rangle \leq f(y) - f(x_0), \quad \forall y \in M \tag{2.7.3}$$

because $\lim_{n\to\infty}\langle v_n, \exp_{x_n}^{-1} y\rangle = \langle v_0, \exp_{x_0}^{-1} y\rangle$ by Lemma 2.1.15 and, by the lower semicontinuity of f, $f(x_0) \leq \liminf_{n\to\infty} f(x_n)$. Thus (2.7.3) means that $v_0 \in \partial f(x_0)$ and proves the upper Kuratowski semicontinuity of ∂f.

Finally, we assume additionally that $\mathcal{D}(f) = M$. By Theorem 2.2.8 and Remark 2.2.5, it suffices to prove that ∂f is locally bounded. To do this, let $x_0 \in M$. By [49], we know that f is locally Lipschitz. Therefore, there exist $\epsilon > 0$ and $L > 0$ such that

$$|f(y) - f(x)| \leq L\, \mathrm{d}(x, y), \quad \forall x, y \in \mathbf{U}(x_0, \epsilon), \tag{2.7.4}$$

where $\mathbf{U}(x_0, \epsilon)$ denotes the open metric ball with center x_0 and radius ϵ. For each $x \in \mathbf{U}(x_0, \epsilon)$, there exists $r > 0$ such that $\mathbf{U}(x, r) \subseteq \mathbf{U}(x_0, \epsilon)$. Hence, thanks to

(2.7.4) and the definition of the subdifferential, we get, for each $x \in \mathbf{U}(x_0, \epsilon)$ and each $v \in \partial f(x)$,

$$\langle v, \exp_x^{-1} y\rangle \leq |f(y) - f(x)| \leq L\mathrm{d}(x, y) \leq Lr, \quad \forall y \in \mathbf{U}(x, r).$$

This implies that $\|v\| \leq L$ and so ∂f is locally bounded because $x \in \mathbf{U}(x_0, \epsilon)$ and $v \in \partial f(x)$ are arbitrary. The proof is complete.

□

Recall that $f : M \to (-\infty, +\infty]$ is a proper lower semicontinuous convex function. Consider the non constrained minimization problem

$$\min_{x\in M} f(x). \tag{2.7.5}$$

We use S_f to denote the solution set of (2.7.5), that is,

$$S_f := \{x \in M : \ f(x) \leq f(y), \ \forall y \in M\}.$$

It is easy to check that

$$x \in S_f \Longleftrightarrow 0 \in \partial f(x). \tag{2.7.6}$$

Applying the algorithm (2.5.1) to the set-valued vector field ∂f, we get the following proximal point algorithm for optimization problem (2.7.5):

$$0 \in \partial f(x_{n+1}) - \lambda_n \exp_{x_{n+1}}^{-1} x_n, \quad \forall n \geq 0. \tag{2.7.7}$$

Remark 2.7.2. Let $y \in M$ and $\lambda > 0$. We define a real-valued convex function $\phi_{\lambda,y}$ by

$$\phi_{\lambda,y}(x) = \frac{\lambda}{2}\mathrm{d}(y, x)^2, \quad \forall x \in M.$$

Consider the following algorithm for finding a solution of the optimization problem (2.7.5):

$$x_{n+1} \in S_{f+\phi_{\lambda_n,x_n}}, \quad \forall n \geq 0, \tag{2.7.8}$$

which was presented and studied by Ferreira and Oliveira in [42] in the special case when f is a real-valued convex function on M. By [114], the derivative of $\phi_{\lambda,y}$ is given by

$$\phi'_{\lambda,y}(x) = -\lambda \exp_x^{-1} y, \quad \forall x \in M.$$

Using (2.7.1), it is a routine to verify that

$$\partial(f+\phi_{\lambda,y})(x)=\partial f(x)-\frac{\lambda}{2}\exp_x^{-1}y,\quad \forall x\in\mathcal{D}(f). \tag{2.7.9}$$

Hence, by (2.7.6), the proximal point algorithms (2.7.7) and (2.7.8) are equivalent.

The following theorem on the convergence of the proximal point algorithm (2.7.7) is a consequence of Theorem 2.5.2 (cf. [42, Theorem 6.1]).

Theorem 2.7.3. *Let $f: M\to(-\infty,+\infty]$ be a proper lower semicontinuous convex function with the solution set $S_f\neq\emptyset$. Let $x_0\in M$ and $\{\lambda_n\}\subset\mathbb{R}^+$ satisfy* (2.5.2)*. Then, the sequence $\{x_n\}$ generated by the algorithm* (2.7.7) *is well-defined and converges to one point $x\in S_f$, a minimizer of f in M.*

Proof. By Theorem 2.7.1, ∂f is monotone and upper Kuratowski semicontinuous. Thus, it suffices to show that the algorithm (2.7.7) is well-defined. Since $S_f\neq\emptyset$, it follows that $\inf_{x\in M}f(x)>-\infty$. We claim that $S_{f+\phi_{\lambda,y}}\neq\emptyset$ for each $\lambda>0$ and each $y\in M$. In fact, let $\{x_n\}\subset M$ be such that

$$\lim_{n\to\infty}(f(x_n)+\phi_{\lambda,y}(x_n))=\inf_{x\in M}(f(x)+\phi_{\lambda,y}(x)).$$

Then $\{x_n\}$ is bounded because, otherwise, $\limsup_{n\to\infty}(f(x_n)+\phi_{\lambda,y}(x_n))=+\infty$. Hence, without loss of generality, we may assume that $\{x_n\}$ converges to a point x. Then $x\in S_{f+\phi_{\lambda,y}}$ and $S_{f+\phi_{\lambda,y}}\neq\emptyset$. This together with Remark 2.7.2 implies that the algorithm (2.7.7) is well-defined and completes the proof. □

Now let $f: M\to\mathbb{R}$ be a convex function and C a closed and convex subset of M. Consider the following optimization problem with constrains.

$$\min_{x\in C}f(x). \tag{2.7.10}$$

Define $f_C:=f+\delta_C$, where δ_C is the indicate function defined by $\delta_C(x)=0$ if $x\in C$ and $\delta_C(x)=+\infty$ otherwise. Then, a point $x\in C$ is a solution of the problem (2.7.10) if and only if it is a solution of the problem (2.7.5) with f replaced by f_C. Let $N_C(x)$ denote the normal cone of the set C at $x\in C$:

$$N_C(x):=\{u\in T_xM:\langle u,\exp_x^{-1}y\rangle\leq 0,\quad \forall y\in C\}.$$

Then

$$N_C(x) = \partial\delta_{T_xC}(0) = \partial\delta_C(x), \quad \forall x \in C. \tag{2.7.11}$$

To apply Theorem 2.7.3, we need to establish the following clear fact on the subdifferential of f_C.

Proposition 2.7.4. *Let $f : M \to \mathbb{R}$ be a convex function and C a closed and convex subset of M. Then*

$$\partial f_C(x) = \partial f(x) + N_C(x), \quad \forall x \in C. \tag{2.7.12}$$

Proof. By definition it is obvious that

$$f'_C(x,u) = f'(x,u) + \delta_{T_xC}(u), \quad \forall u \in T_xM.$$

Applying (2.7.2), we obtain that

$$\partial f_C(x) = \partial f'_C(x,\cdot)|_{u=0} \quad \text{and} \quad \partial f(x) = \partial f'(x,\cdot)|_{u=0}. \tag{2.7.13}$$

Since $f'(x,\cdot)$ is a continuous convex function on T_xM, it follows from the well-known subdifferential sum rule (see for example [9]), that

$$\partial f'_C(x,\cdot)|_{u=0} = \partial f'(x,\cdot)|_{u=0} + \partial\delta_{T_xC}(0). \tag{2.7.14}$$

According to (2.7.11), (2.7.13) and (2.7.14), we obtain (2.7.12) and the proof is complete. □

Consider the following algorithm with initial point $x_0 \in C$:

$$0 \in \partial f(x_{n+1}) + N_C(x_{n+1}) - \lambda_n \exp^{-1}_{x_{n+1}} x_n, \quad \forall n \geq 0. \tag{2.7.15}$$

Then the theorem below is a direct consequence of Corollary 2.5.3.

Theorem 2.7.5. *Let $f : M \to \mathbb{R}$ be a convex function and C be a closed convex set of M such that the solution set of the optimization problem* (2.7.10) *is nonempty. Let $x_0 \in M$ and $\{\lambda_n\}$ satisfy* (2.5.2). *Then, the sequence $\{x_n\}$ generated by the algorithm* (2.7.15) *is well-defined and converges to a solution of the optimization problem* (2.7.10).

2.7.2 Saddle-points in a Minimax Problem

In the spirit of the works by Rockafellar ([102, 103]) on the convergence of the proximal point algorithm in terms of an associated maximal monotone operator for saddle-functions on the product Hilbert space $H_1 \times H_2$, the present subsection focused on the study of the convergence of the proximal point algorithm for saddle-functions on Hadamard manifolds.

Let M_1 and M_2 be Hadamard manifolds. A function $L : M_1 \times M_2 \to \mathbb{R}$ is called a *saddle-function* if $L(x, .)$ is convex on M_2 for each $x \in M_1$ and $L(., y)$ is concave, i.e $-L(., y)$ is convex, on M_1 for each $y \in M_2$. A point $\bar{z} = (\bar{x}, \bar{y}) \in M_1 \times M_2$ is called a *saddle-point* of L if

$$L(x, \bar{y}) \leq L(\bar{x}, \bar{y}) \leq L(\bar{x}, y), \quad \forall z = (x, y) \in M_1 \times M_2. \tag{2.7.16}$$

Associated with the saddle-function L, define the set-valued vector field $A_L : M_1 \times M_2 \to 2^{TM_1} \times 2^{TM_2}$ by

$$A_L(x, y) = \partial\big(-L(\cdot, y)\big)(x) \times \partial\big(L(x, \cdot)\big)(y), \quad \forall (x, y) \in M_1 \times M_2. \tag{2.7.17}$$

By [105, pag. 239, Problem 10], the product space $M = M_1 \times M_2$ is a Hadamard manifold and the tangent space of M at $z = (x, y)$ is $T_zM = T_xM_1 \times T_yM_2$. The corresponding metric is given by

$$\langle w, w'\rangle = \langle u, u'\rangle + \langle v, v'\rangle, \quad \forall w = (u, v), w' = (u', v') \in T_zM.$$

Note also that a geodesic in the product manifold M is the product of two geodesics in M_1 and M_2, respectively. Then, for any two points $z = (x, y)$ and $z' = (x', y')$ in M,

$$\exp_z^{-1} z' = \exp_{(x,y)}^{-1}(x', y') = (\exp_x^{-1} x', \exp_y^{-1} y').$$

Therefore, in view of the definition of monotonicity, the set-valued vector field $A : M_1 \times M_2 \to 2^{TM_1} \times 2^{TM_2}$ is monotone if and only if for any $z = (x, y), z' = (x', y')$, $w = (u, v) \in A(z)$ and $w' = (u', v') \in A(z')$,

$$\langle u, \exp_x^{-1} x'\rangle + \langle v, \exp_y^{-1} y'\rangle \leq \langle u', -\exp_{x'}^{-1} x\rangle + \langle v', -\exp_{y'}^{-1} y\rangle. \tag{2.7.18}$$

Theorem 2.7.6. *Let L be a saddle-function on $M = M_1 \times M_2$ and A_L the set-valued vector field defined by* (2.7.17)*. Then, A_L is maximal monotone.*

Proof. Consider two points $z = (x, y)$ and $z' = (x', y')$ in M, and $w = (u, v) \in A_L(z)$, $w' = (u', v') \in A_L(z')$. Since $\partial\big(-L(\cdot, y)\big)$ and $\partial\big(L(x, \cdot)\big)$ are monotone by Theorem 2.7.1, it follows from the definition of A_L that

$$\langle u, \exp_x^{-1} x' \rangle \leq \langle u', -\exp_{x'}^{-1} x \rangle \quad \text{and} \quad \langle v, \exp_y^{-1} y' \rangle \leq \langle v', -\exp_{y'}^{-1} y \rangle.$$

Hence (2.7.18) holds and A_L is monotone because $z, z' \in M$ and $w \in A_L(z)$, $w' \in A_L(z')$ are arbitrary.

To verify the maximality, let $z = (x, y) \in M_1 \times M_2$ and $w = (u, v) \in T_x M_1 \times T_y M_2$ be such that

$$\langle u, \exp_x^{-1} x' \rangle + \langle v, \exp_y^{-1} y' \rangle \leq \langle u', -\exp_{x'}^{-1} x \rangle + \langle v', -\exp_{y'}^{-1} y \rangle \tag{2.7.19}$$

for any $z' = (x', y') \in M_1 \times M_2$, $w' = (u', v') \in A_L(z')$. We have to prove that $w \in A_L(z)$, that is, $u \in \partial\big(-L(\cdot, y)\big)(x)$ and $v \in \partial\big(L(x, \cdot)\big)(y)$. Taking $y' = y$ in (2.7.19), we get

$$\langle u, \exp_x^{-1} x' \rangle \leq \langle u', -\exp_{x'}^{-1} x \rangle, \quad \forall x' \in M_1 \text{ and } u' \in \partial\big(-L(\cdot, y)\big)(x'). \tag{2.7.20}$$

Note that $\partial\big(-L(\cdot, y)\big)$ is maximal by Theorem 2.7.1, hence (2.7.20) implies that $u \in \partial\big(-L(\cdot, y)\big)(x)$. Similarly, taking $x' = x$ in (2.7.19), we get that $v \in \partial\big(-L(x, \cdot)\big)(y)$, as desired. □

It is straightforward to check that a point $\bar{z} = (\bar{x}, \bar{y}) \in M$ is a saddle point of L if and only if it is a singularity of A_L. Consider the following algorithm for A_L,

$$0 \in A_L(z_{n+1}) - \lambda_n \exp_{z_{n+1}}^{-1} z_n, \tag{2.7.21}$$

where $z_0 \in M_1 \times M_2$ and $\{\lambda_n\} \subset \mathbb{R}^+$. Thus applying Corollary 2.5.3 to the vector field A_L associated with the saddle-function L, we immediately obtain the following theorem.

Theorem 2.7.7. *Let $L : M = M_1 \times M_2 \to \mathbb{R}$ be a saddle-function and $A_L : M_1 \times M_2 \to 2^{TM_1} \times 2^{TM_2}$ be the associated maximal monotone vector field. Suppose that L has a saddle point. Let $z_0 \in M$ and let $\{\lambda_n\} \subset \mathbb{R}^+$ satisfy* (2.5.2)*. Then, the sequence $\{z_n\}$ generated by the algorithm* (2.7.21) *is well-defined and converges to a saddle point of L.*

2.7.3 Variational Inequalities

Let C be a convex subset of M and $V : C \to TM$ a single-valued vector field, that is, $V(x) \in T_xM$ for each $x \in C$. Following [84], the problem of finding $x \in C$ such that

$$\langle V(x),\ \exp_x^{-1} y\rangle \geq 0, \quad \forall y \in C, \tag{2.7.22}$$

is called a *variational inequality* on C. Clearly, a point $x \in C$ is a solution of the variational inequality (2.7.22) if and only if x satisfies that

$$0 \in V(x) + N_C(x),$$

that is, x is a singularity of the set-valued vector field $A := V + N_C$. Applying the algorithm (2.5.1) to A, we get the following proximal point algorithm with initial point x_0 for finding solutions of the variational inequality (2.7.22):

$$0 \in V(x_{n+1}) + N_C(x_{n+1}) - \lambda_n \exp_{x_{n+1}}^{-1} x_n, \quad \forall n \geq 0. \tag{2.7.23}$$

The remainder of this subsection is directed towards the study of the convergence of algorithm (2.7.23). To apply Theorem 2.5.2, one need to prove that the algorithm is well-defined. To this end and for the sake of completeness, we first include some lemmas. One of them (Lemma 2.7.11) is an extension of the well-known Brouwer fixed point Theorem to Hadamard manifolds. In [84] the author gives a similar but incomplete proof of what we present below. The following proposition is a direct consequence of [105, pag. 170, Theorem 5.5 and Lemma 5.4] (noting that M is a Hadamard manifold).

Proposition 2.7.8. *Let C be a convex compact subset of M. Then there exists a totally geodesic submanifold $N \subseteq C$ such that $C = \overline{N}$, the closure of N, and the following condition holds: for any $q \in C \setminus N$ and $p \in N$, $\exp_p t(\exp_p^{-1} q) \in N$ for all $t \in (0,1)$ and $\exp_p t(\exp_p^{-1} q) \notin C$ for any $t \in (1, +\infty)$.*

Remark 2.7.9. Following [105], $\operatorname{int} C := N$ is called the interior of C and $\operatorname{bd} C := C \setminus N$ the boundary of C. Moreover, if C is a compact convex set, then $\operatorname{bd} C \neq \emptyset$.

Lemma 2.7.10. *Let C be a convex compact subset of M and let $p_0 \in \operatorname{int} C$. Then*

$$\exp_{p_0}^{-1}(\operatorname{bd} C) = \operatorname{bd}(\exp_{p_0}^{-1} C) \quad \text{and} \quad \exp_{p_0}^{-1}(\operatorname{int} C) = \operatorname{int}(\exp_{p_0}^{-1} C). \tag{2.7.24}$$

Proof. Let $p_0 \in \operatorname{int} C$. Since $\exp_{p_0}^{-1}$ is bijection from C to $\exp_{p_0}^{-1} C$ and $(\operatorname{bd} C) \cap (\operatorname{int} C) = \emptyset$, it follows that

$$\exp_{p_0}^{-1}(C) = \exp_{p_0}^{-1}(\operatorname{bd} C) \cup \exp_{p_0}^{-1}(\operatorname{int} C).$$

Thus, to complete the proof, it suffices to prove that

$$\exp_{p_0}^{-1}(\operatorname{bd} C) \subseteq \operatorname{bd}(\exp_{p_0}^{-1} C) \quad \text{and} \quad \exp_{p_0}^{-1}(\operatorname{int} C) \subseteq \operatorname{int}(\exp_{p_0}^{-1} C). \tag{2.7.25}$$

To show the first inclusion, let $q \in \operatorname{bd} C$. Then $\exp_{p_0}^{-1} q \in \exp_{p_0}^{-1} C$. By Proposition 2.7.8, we see that $\exp_{p_0} t(\exp_{p_0}^{-1} q) \notin C$ for all $t > 1$. Hence $t(\exp_{p_0}^{-1} q) \notin \exp_{p_0}^{-1} C$ for any $t > 1$. Therefore, $\exp_{p_0}^{-1} q \in \operatorname{bd}(\exp_{p_0}^{-1} C)$ and the inclusion $\exp_{p_0}^{-1}(\operatorname{bd} C) \subseteq \operatorname{bd}(\exp_{p_0}^{-1} C)$ is proved.

Below we show the inclusion $\exp_{p_0}^{-1}(\operatorname{int} C) \subset \operatorname{int}(\exp_{p_0}^{-1} C)$. For simplicity, we use $\mathbf{U}_E(x, \epsilon)$ to denote the open ball at $x \in E$ with center ϵ in a metric space E. note that $\operatorname{int} C = N$ is the totally geodesic submanifold given by Proposition 2.7.8. Then, by [105, pag. 171] (noting that M is a Hadamard manifold), for any $q \in \operatorname{int} C$,

$$T_q N = \{v \in T_q M \setminus \{0\} : \exp_q tv/\|v\| \in N \text{ for some } t > 0\} \cup \{0\}. \tag{2.7.26}$$

and $\exp_q$ is a local diffeomorphism at 0 from $T_q N$ to N (cf. [105]), that is there exists an open ball $\mathbf{U}_{T_q N}(0, \epsilon) \subset T_q N$ at 0 such that $\exp_q(\mathbf{U}_{T_q N}(0, \epsilon)) \subseteq N$. This means that

$$q \in \operatorname{int} C \Longleftrightarrow \exp_q(\mathbf{U}_{T_q N}(0, \delta)) \subseteq \operatorname{int} C \text{ for some } \delta > 0. \tag{2.7.27}$$

Consequently, $\operatorname{bd} C$ is closed in M because C is closed. Let $q \in \operatorname{int} C$ and set $M_0 = \exp_{p_0}(T_{p_0} N)$. Then

$$\epsilon := \mathrm{d}(q, M_0 \setminus C) > 0. \tag{2.7.28}$$

In fact, otherwise, there exists a sequence $\{q_k\} \subset M_0 \setminus C$ such that $\lim_k \mathrm{d}(q_k, q) = 0$. By Proposition 2.7.8 and (2.7.26) (Noting $p_0 \in N$), we may assume that, for each k, $q_k = \exp_{p_0} t_k u_k$, where $t_k \geq 1$ and $u_k \in T_{p_0} N$ is such that $\exp_{p_0} u_n \in \operatorname{bd} C$. Since $\{t_k u_k\}$ is bounded and each $t_k \geq 1$, it follows that $\{u_k\}$ is bounded too. Since $\operatorname{bd} C$ is closed, it follows that $\liminf_k \|u_k\| > 0$. This together with the boundedness of $\{t_k u_k\}$ implies that $\{t_k\}$ is bounded. Without loss of generality, we may assume that $t_k \to t_0$ and $u_k \to u_0$. Then $t_0 \geq 1$ and $\exp_{p_0} u_0 \in \operatorname{bd} C$. Hence $\exp_{p_0} t_0 u_0 \notin N$ by

Proposition 2.7.8, which is a contradiction because $\exp_{p_0}(t_0 u_0) = x \in N$. Therefore, (2.7.28) is proved.

Since $\exp_{p_0}$ is continuous on $T_{p_0}N$, there exists $\delta > 0$ such that

$$\mathrm{d}(\exp_{p_0} v, q) = \mathrm{d}(\exp_{p_0} v, \exp_{p_0}^{-1} q) < \frac{\epsilon}{2} \quad \text{for each } v \in \mathbf{U}_{T_{p_0}N}(\exp_{p_0}^{-1} q, \delta).$$

It follows from (2.7.28) that $\exp_{p_0}(\mathbf{U}_{T_{p_0}N}(\exp_{p_0}^{-1} q, \delta)) \subseteq C$. This implies that $\exp_{p_0} q \in \operatorname{int}(\exp_{p_0}^{-1} C)$ and the proof is complete. □

Lemma 2.7.11. *Let C be a compact convex subset of M. Let $T : C \to C$ be a continuous mapping. Then T has a fixed point in C.*

Proof. Note that the fixed point property is a topological property, i.e., if X and Y are homeomorphic topological spaces and any continuous mapping on X has fixed points, then does any continuous mapping on Y. Let $p_0 \in \operatorname{int} C$ and, for simplicity, write $\tilde{C} = \exp_{p_0}^{-1} C$. Then, C is homeomorphic to $\tilde{C}$. By the Brouwer fixed point Theorem, it suffices to prove that $\tilde{C} \subset T_{p_0}N$ is homeomorphic to the closed unit ball of $T_{p_0}N$, denoted by $\mathbf{B}$. We define the function $\phi : \tilde{C} \to \mathbb{R}^+$ by

$$\phi(x) = \begin{cases} \|\widehat{0x} \bigcap \operatorname{bd} \tilde{C}\| & if\ x \neq 0, \\ 1 & if\ x = 0, \end{cases}$$

where $\widehat{0x} := \{tx : t \geq 0\}$ is the straight half-line joining 0 to x. By Proposition 2.7.8, the geodesic joining p_0 and $\exp_{p_0} x$ intersects $\operatorname{bd} C$ at just one point and so, by Lemma 2.7.10, $\widehat{0x}$ intersects $\operatorname{bd} \tilde{C}$ at just one point. This implies that ϕ is well-defined and continuous at each point $x \neq 0$. Indeed, let $x_0 \in \tilde{C} \setminus \{0\}$ and denote $\overline{x} = \widehat{0x} \bigcap \operatorname{bd} \tilde{C}$ for any $x \neq 0$. Then $\overline{x} = \frac{x}{\|x\|}\phi(x)$. Let $\{x_n\} \subset \tilde{C}$ be such that $x_n \to x_0$. Since ϕ is bounded, we can assume without lost of generality that $\phi(x_n) \to r \in \mathbb{R}^+$. Then,

$$\overline{x_n} \to \overline{y_0} = \frac{x_0}{\|x_0\|} r \in \operatorname{bd} \tilde{C}.$$

We must prove that $r = \phi(x_0)$ to get the continuity of ϕ. To this end, write

$$\overline{y_0} = \frac{x_0}{\|x_0\|}\phi(x_0)\frac{r}{\phi(x_0)} = \frac{r}{\phi(x_0)}\overline{x}. \tag{2.7.29}$$

If $r < \phi(x_0)$, then $\frac{r}{\phi(x_0)} < 1$. Thus, by Proposition 2.7.8 and Lemma 2.7.10, (2.7.29) implies that $\overline{y_0} \in \operatorname{int} \tilde{C}$ because $\overline{x} \in \operatorname{bd} C$, which contradicts that $\overline{y_0} \in \operatorname{bd} \tilde{C}$. Similarly, if $r > \phi(x_0)$, we get that $\overline{y_0} \notin \tilde{C}$, which again contradicts that $\overline{y_0} \in \operatorname{bd} \tilde{C}$. Therefore ϕ is continuous at each $x \neq 0$.

Now define the function $h : \tilde{C} \to \mathbf{B}$ by

$$h(x) = \frac{1}{\phi(x)} x, \quad \forall x \in \tilde{C}.$$

Note that $\|h(x)\| = \frac{1}{\|\phi(x)\|}\|x\| \leq 1$. Thus h is well-defined and continuous, whose inverse function $h^{-1}(y) = \phi(y)y$ is continuous too. Hence, h is a homeomorphism from $\tilde{C}$ to $\mathbf{B}$ and the proof is complete. □

Theorem 2.7.12. *Let C be a closed convex subset of M and $V : C \to TM$ a single-valued continuous monotone vector field. Let $x_0 \in C$ and $\{\lambda_n\} \subset \mathbb{R}^+$ satisfy* (2.5.2). *Suppose that the variational inequality* (2.7.22) *has a solution. Then the sequence $\{x_n\}$ generated by the algorithm* (2.7.23) *is well-defined and converges to a solution of the variational inequality* (2.7.22).

Proof. By Theorem 2.5.2 and Remark 2.5.1, we only need to prove that the sequence $\{x_n\}$ generated by the algorithm (2.7.23) is well-defined. Let $\lambda > 0$ and $y_0 \in C$. Consider the following variational inequality:

$$\langle V(x) - \lambda \exp_x^{-1} y_0,\ \exp_x^{-1} y\rangle \geq 0, \quad \forall y \in C. \tag{2.7.30}$$

For fixed $n \geq 0$, note that x_{n+1} satisfies (2.7.23) if and only if x_{n+1} is a solution of the variational inequality (2.7.30) with $\lambda = \lambda_n$ and $y_0 = x_n$. Thus, it suffices to prove that the variational inequality (2.7.30) has a solution. The proof is standard, see [84]. However, we keep the proof here for completeness. Let $R > 0$ be such that $\|V(y_0)\| - 2R\lambda < 0$ and set

$$C_R = \{x \in C :\ \mathrm{d}(x, y_0) \leq R\}.$$

Then C_R is a compact convex subset of M. Let $P_{C_R} : M \to C_R$ be the projection to C_R. Then, by [114], P_{C_R} is Lipschitz continuous and characterized by

$$\langle \exp_{P_{C_R}x}^{-1} x,\ \exp_{P_{C_R}x}^{-1} y\rangle \leq 0, \quad \forall x \in M \text{ and } y \in C_R. \tag{2.7.31}$$

Consider the continuous map $T : C_R \to C_R$ defined by

$$T(x) := P_{C_R}(\exp_x(-V(x) - \lambda \exp_x^{-1} y_0), \quad \forall x \in C_R.$$

By Lemma 2.7.11, T has a fixed point x_R. This together with (2.7.31) implies that

$$\langle V(x_R) - \lambda \exp_{x_R}^{-1} y_0,\ \exp_{x_R}^{-1} y \rangle \geq 0 \tag{2.7.32}$$

holds for any $y \in C_R$. Since $\langle V(x),\ \exp_x^{-1} y_0 \rangle \leq \langle -V(y_0),\ \exp_{y_0}^{-1} x \rangle$ by the monotonicity and $\langle \exp_x^{-1} y_0,\ \exp_x^{-1} y_0 \rangle = \langle \exp_{y_0}^{-1} x,\ \exp_{y_0}^{-1} x \rangle = \mathrm{d}(x, y_0)^2$, it follows that if $\mathrm{d}(x, y_0) = R$, then

$$\begin{aligned} \langle V(x) - \lambda \exp_x^{-1} y_0,\ \exp_x^{-1} y_0 \rangle &\leq \langle V(y_0), \exp_{y_0}^{-1} x \rangle - 2\lambda \mathrm{d}(x, y_0)^2 \\ &\leq (\|V(y_0)\| - 2R\lambda)R \\ &< 0. \end{aligned}$$

This means that $\mathrm{d}(x_R, y_0) < R$. Below we shall show that (2.7.32) holds for any $y \in C$. Granting this the proof is complete. Indeed, given $y \in C$, $y_t = \exp_{x_R} t(\exp_{x_R}^{-1} y) \in C_R$ for $t > 0$ sufficiently small. Consequently,

$$t\langle V(x_R) - \lambda \exp_{x_R}^{-1} y_0,\ \exp_{x_R}^{-1} y \rangle = \langle V(x_R) - \lambda \exp_{x_R}^{-1} y_0,\ \exp_{x_R}^{-1} y_t \rangle \geq 0.$$

Thus, (2.7.32) holds for $y \in C$. □

2.7.4 Equilibrium problem

Many practical problems can be formulated as the equilibrium problem of finding a point x in a space X satisfying

$$F(x, y) \geq 0, \quad \forall y \in X, \tag{2.7.33}$$

where $F : X \times X \to \mathbb{R}$ is a bifunction. A point x^* which solves the problem (2.7.33) is said to be an equilibrium point for the bifunction F. We denote the equilibrium point set of F by $\mathrm{EP}(F)$.

We present an approach to approximate an equilibrium point for a bifunction $F : M \times M \to \mathbb{R}$ in the framework of Hadamard manifolds. It involves the resolvent of the bifunction F, which is a nonexpansive mapping whose fixed point set coincides with the equilibrium point set of F. This allows us to use the convergence results for fixed points of nonexpansive mappings on Hadamard manifolds which exist in the literature. In particular, such a mapping presents a stronger property, the firm nonexpansivity, which was proved to imply the convergence of Picard iteration $\{T^n(x)\}$ in Section 2.6.1.

Resolvent of a bifunction

The definition of the resolvent of a bifunction in the setting of a Hilbert space H appears implicitly in [8] and was first given in [33]. In order to distinguish the resolvents of vector fields and the resolvents of bifunctions we denote the latter with a supper index, J^F.

Definition 2.7.13. Given a single-valued bifunction $F : C \times C \to \mathbb{R}$, where $C \subseteq M$ is a nonempty closed convex subset, the resolvent of F of order $\lambda > 0$ is the set-valued vector field $J^F : M \to 2^C$ defined by

$$J^F(x) = \{z \in C \mid \lambda F(z, y) - \langle \exp_z^{-1} x, \exp_z^{-1} y\rangle \geq 0,\ \forall y \in C\}. \tag{2.7.34}$$

Remark 2.7.14. The resolvent could be defined for a set-valued bifunction $F : C \times C \to 2^{\mathbb{R}}$ as follows,

$$J_\lambda^F : M \to 2^C : x \mapsto \{z \in C \mid \lambda u - \langle \exp_z^{-1} x, \exp_z^{-1} y\rangle \geq 0,\ \forall y \in C, \forall u \in F(z, y)\},$$

for any $\lambda > 0$, and the following theorem would remain true except for (iii) which needs F to be single-valued.

Theorem 2.7.15. *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:*

(1) *the bifunction F is monotone, that is, for any $(x, y) \in C \times C$,*

$$F(x, y) + F(y, x) \leq 0;$$

(2) for each $\lambda > 0$, J_λ^F is properly defined, i. e., the domain $\mathcal{D}(J_\lambda^F) \neq \emptyset$.

Then, for any $\lambda > 0$,

(i) *the resolvent* J_λ^F *is single-valued;*

(ii) *the resolvent* J_λ^F *is firmly nonexpansive;*

(iii) *the fixed point set of* J_λ^F *is the equilibrium point set of* F*,*

$$Fix(J_\lambda^F) = EP(F).$$

Proof.

(i) We fix $x \in \mathcal{D}(J_\lambda^F)$ and assume that there exist $z_1, z_2 \in J_\lambda^F(x)$. By definition this means that

$$\lambda F(z_1, z_2) - \langle \exp_{z_1}^{-1} x, \exp_{z_1}^{-1} z_2 \rangle \geq 0, \tag{2.7.35}$$

$$\lambda F(z_2, z_1) - \langle \exp_{z_2}^{-1} x, \exp_{z_2}^{-1} z_1 \rangle \geq 0. \tag{2.7.36}$$

Summing inequalities (2.7.35) and (2.7.36), by condition (1) and applying Proposition 2.1.30, the law of cosines, we get

$$d^2(z_1, z_2) \leq \langle \exp_{z_1}^{-1} x, \exp_{z_1}^{-1} z_2 \rangle + \langle \exp_{z_2}^{-1} x, \exp_{z_2}^{-1} z_1 \rangle \leq 0.$$

Therefore $z_1 = z_2$.

(ii) To prove that J_λ^F is firmly nonexpansive we consider $x_1, x_2 \in \mathcal{D}(J_\lambda^F)$. By definition of resolvent we get

$$\lambda F(J_\lambda^F x_1, J_\lambda^F x_2) - \langle \exp_{J_\lambda^F x_1}^{-1} x_1, \exp_{J_\lambda^F x_1}^{-1} J_\lambda^F x_2 \rangle \geq 0 \tag{2.7.37}$$

$$\lambda F(J_\lambda^F x_2, J_\lambda^F x_1) - \langle \exp_{J_\lambda^F x_2}^{-1} x_2, \exp_{J_\lambda^F x_2}^{-1} J_\lambda^F x_1 \rangle \geq 0. \tag{2.7.38}$$

If we sum inequalities (2.7.37) and (2.7.38), it results

$$\langle \exp_{J_\lambda^F x_1}^{-1} x_1, \exp_{J_\lambda^F x_1}^{-1} J_\lambda^F x_2 \rangle + \langle \exp_{J_\lambda^F x_2}^{-1} x_2, \exp_{J_\lambda^F x_2}^{-1} J_\lambda^F x_1 \rangle \leq 0,$$

for any $x_1, x_2 \in \mathcal{D}(J_\lambda^F)$, which is equivalent to say that J_λ^F is firmly nonexpansive as we proved in Proposition 2.3.4.

(iii) Given $x \in \mathcal{D}(J_\lambda^F)$,

$$x = J_\lambda^F x \Leftrightarrow F(x,y) - \langle \exp_x^{-1} x, \exp_x^{-1} y\rangle \geq 0 \ (\forall y \in C) \Leftrightarrow F(x,y) \geq 0 \ (\forall y \in C).$$

So $\mathrm{Fix}\,(J_\lambda^F) = \mathrm{EP}(F)$.

□

Remark 2.7.16. If $\mathcal{D}(J_\lambda^F)$ is closed and convex, the equilibrium point set $\mathrm{EP}(F)$ is closed and convex by virtue of conditions (iii), (ii) and the fact that the fixed point set of a nonexpansive mapping defined on a closed convex set is closed and convex, proved in Proposition 2.3.1.

The previous theorem allows us to approximate a solution to the equilibrium problem associated to a bifunction F, whenever it exists, by means of the resolvent and the sequence of iterates $\{(J_\lambda^F)^n x\}$ whose convergence is assured by Theorem 2.6.1 for firmly nonexpansive mappings.

Theorem 2.7.17. *Let $F : C \times C \to \mathbb{R}$ be a monotone bifunction such that $EP(F) \neq \emptyset$. Let $\lambda > 0$ and assume that the resolvent of F, J_λ^F, is properly defined with $C \subseteq \mathcal{D}(J_\lambda^F)$. Then, for each $x \in \mathcal{D}(J_\lambda^F)$, the sequence defined by*

$$x_{n+1} = (J_\lambda^F)^n x, \quad n \geq 0, \tag{2.7.39}$$

converges to an equilibrium point of F.

Moreover, as in Hilbert spaces, the resolvent of a bifunction constitutes a generalization of the resolvent of a monotone vector field or the Moreau-Yosida regularization of a convex function. In these cases, as we will show in the following examples, we know that the resolvent is properly defined and moreover, $J_\lambda^F(x) \neq \emptyset$ for any $x \in C$, that is $\mathcal{D}(J_\lambda^F) = M$.

Resolvent of a vector field

In the single-valued case the resolvent of a monotone vector field can be seen as the resolvent of a bifunction. Indeed, given $A \in \mathcal{X}(M)$ define the bifunction

$$F : M \times M \to 2^{\mathbb{R}} : (x,y) \mapsto F(x,y) = \langle Ax, \exp_x^{-1} y\rangle. \tag{2.7.40}$$

The monotonicity of A implies the monotonicity of F. On the other hand, for any $x \in M$ and $\lambda > 0$, the resolvent of F can be written as

$$J_\lambda^F(x) = \{z \in M \mid \langle \lambda A(z) - \exp_z^{-1} x, \exp_z^{-1} y \rangle \geq 0,\ \forall y \in M\ \}.$$

Thus, if $z \in J_\lambda^A(x)$ we have that $\lambda Az = \exp_z^{-1} x$ and then $z \in J_\lambda^F(x)$, i.e.,

$$J_\lambda^A(x) \subseteq J_\lambda^F(x),\ \forall x \in M.$$

Therefore the conditions in Theorem 2.7.15 hold for F defined as in (2.7.40), so the fact that J_λ^F is single-valued implies that

$$J_\lambda^A(x) = J_\lambda^F(x),\ \forall x \in M.$$

Moreau-Yosida regularization of a convex function

Let $f : M \to \mathbb{R}$ be a convex function. The Moreau-Yosida regularization $f_\lambda : M \to M$ of f is defined by

$$f_\lambda(x) = \underset{y \in M}{\operatorname{argmin}} \{\lambda f(y) + \frac{1}{2} \operatorname{d}^2(x, y)\}. \tag{2.7.41}$$

In [42] it was proved that there exists a unique point $y_\lambda = f_\lambda(x)$ for any $x \in M$ and $\lambda \geq 0$, which is characterized by

$$\exp_{y_\lambda}^{-1} x \in \partial f(y_\lambda). \tag{2.7.42}$$

Then the mapping f_λ is well-defined and single-valued. If we consider the set-valued vector field $A = \partial f$, we know from Theorem 2.7.1 that A is maximal monotone with full domain. By the characterization (2.7.42) we can readily check that the resolvent of A coincides with the Moreau-Yosida regularization, $f_\lambda = J_\lambda^A$, and therefore Theorem 2.4.9 recovers the fact that f_λ has full domain, and implies its firm nonexpansivity.

On the other hand, the Moreau-Yosida regularization of a convex function is the resolvent of the bifunction $F : M \times M \to \mathbb{R}$ defined by $F(x, y) = f(y) - f(x)$. Indeed, given $x \in M$, let $z = f_\lambda(x)$. This means that $\frac{1}{\lambda} \exp_z^{-1} x \in \delta f(z)$, and by definition of the subdifferential of f, for any $y \in M$,

$$\frac{1}{\lambda} \langle \exp_z^{-1} x, \exp_z^{-1} y \rangle \leq f(y) - f(x).$$

Equivalently,

$$\lambda F(x, y) - \langle \exp_z^{-1} x, \exp_z^{-1} y\rangle \geq 0,$$

so $z \in J_\lambda^F(x)$. Then F is properly defined. Since F is monotone as well, Theorem 2.7.15 ensures that J_λ^F is single-valued, therefore we get that $f_\lambda = J^F\lambda$.

Bibliography

[1] A. Aleyner and S. Reich, A note on explicit iterative constructions of sunny nonexpansive retractions in Banach spaces, *J. Nonlinear Convex Anal.* **6** (2005), 525-533.

[2] J. B. Baillon and G. Haddad, Quelques propriétés des opérateurs angle-bornés et n-cycliquement monotones, *Israel J. Math* **26** (1997), no. 2, 137-150.

[3] J. B. Baillon, R. E. Bruck and S. Reich, On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces, *Houston J. Math.* **4** (1978), 1-9.

[4] H. Bauschke, The approximation of fixed points compositions of nonexpansive mappings in Hilbert space, *J. Math Anal. Appl.* **202** (1996), 150-159.

[5] H. Bauschke and J. M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Rev.* **38** (1996), 367-426.

[6] H. H. Bauschke, P. L. Combettes and S. Reich, The asymptotic behavior of the composition of two resolvents, *Nonlinear Anal.* **60** (2005), 283-301.

[7] H. H. Bauschke, E. Matoušková and S. Reich, Projection and proximal point methods: convergence results and counterexamples, *Nonlinear Anal.* **56** (2004), 715-738.

[8] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student* **63** (1994), 123–145.

[9] J. M. Borwein and A. S. Lewis, Convex Analysis and Nonlinear Optimization, Theory and Examples, Springer, 2006.

[10] H. Brezis, G. Crandall and P. Pazy, Perturbations of nonlinear maximal monotone sets in Banach spaces, *Comm. Pure Appl Math.* **23** (1970), 123-144.

[11] M. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Springer-Verlag, Berlin, Heidelberg, New York, 1999.

[12] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Nat. Acad. Sci. USA* **54** (1965), 1041-1044.

[13] F. E. Browder, Existence and approximation of solutions of nonlinear variational inequalities, *Proc. Nat. Acad. Sci. U.S.A.* **56** (1966), 1080-1086.

[14] F. E. Browder, Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces, *Arch. Rational Mech. Anal.* **24** (1967), 82-90.

[15] F. E. Browder, Convergence theorems for sequences of nonlinear operators in Banach spaces, *Math. Zeitschr.* **100** (1967), 201-225.

[16] F. E. Browder and W. V. Petryshyn, The solution by iteration of nonlinear functional equations in Banach spaces, *Bull. Amer. Math. Soc.* **72** (1966), 571-575.

[17] R. E. Bruck, Convergence theorems for sequence of nonlinear operators in Banach spaces, *Math. Z.* **100** (1997), 201-225.

[18] R. E. Bruck, Nonexpansive projections on subsets of Banach spaces, *Pac. J. Math* **47** (1973), 341-355.

[19] R. E. Bruck, Asymptotic behavior of nonexpansive mapping, *Contemporary Mathematics* **18** (1983), 1-47.

[20] R. E. Bruck and S. Reich, Nonexpansive projections and resolvents of accretive operators in Banach spaces, *Houston J. Math.* **3** (1977), 459-470.

[21] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Problems* **20** (2004), 103-120.

[22] Y. Censor, T. Bortfeld, B. Martin, and A. Trofimov, A unified approach for inversion problems in intensity-modulated radiation therapy, *Phys. Med. Biol.* **51** (2005), 2353-2365.

[23] Y. Censor and T. Elfving, A multiprojection algorithm using Bregman projections in a product space, *Numerical Algorithms* **8** (1994), 221-239.

[24] Y. Censor, T. Elfving, N. Kopf and T. Bortfeld, The multiple-sets split feasibility problem and its applications for inverse problems, *Inverse Problems* **21** (2005), 2071-2084.

[25] Y. Censor, A. Motova and A. Segal, Perturbed projections and subgradient projections for the multiple-sets split feasibility problem, *J. Math Anal. Appl.* **327** (2007), no. 2, 1244-1256.

[26] Y. J. Cho, S. M. Kang and H. Zhou, Some control conditions on iterative methods, *Comm. Appl. Nonlinear Anal.* **12** (2005), no. 2, 27-34.

[27] I. Cioranescu, Geometry of Banach spaces, duality mappings and nonlinear problems, Kluwer Academic Publishers, 1990.

[28] C. E. Chidume and C. O. Chidume, Iterative approximation of fixed points of nonexpansive mappings, *J. Math. Anal. Appl.* **318** (2006), 288-295.

[29] C. Chidume, Geometric properties of Banach spaces and nonlinear iterations. Lecture Notes in Mathematics, 1965. Springer-Verlag London, London, 2009.

[30] V. Colao, L. Leustean, G. López and V. Martín-Márquez, Alternative iterative methods for nonexpansive mappings, rates of convergence and application, *Oberwolfach Preprints* (OWP) **19** (2009).

[31] P. L. Combettes, The convex feasibility problem in image recovery, Adv. Imaging Electron Phys. **95** (1996), 155-270.

[32] P. L. Combettes, Fejér-monotonicity in convex optimization, *Encyclopedia of Optimization* (C. A. Floudas and P. M. Pardalos, Eds.) **2** (2001), 106-114.

[33] P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert Spaces, *Journal of Nonlinear and Convex Analysis* **1** (2005), no. 6, 117-136.

[34] P. L. Combettes and S. A. Hirstoaga, Approximating curves for nonexpansive and monotone operators, *J. Convex Anal.* **13** (2006), no. 3-4, 633-646.

[35] J.X. Da Cruz Neto, O.P. Ferreira and L.R. Lucambio Pérez, Monotone point-to-set vector fields, *Balkan journal of Geometry and its Applications* **5** (2000), 69-79.

[36] J.X. Da Cruz Neto, O.P. Ferreira and L.R. Lucambio Pérez, Contributions to the study of monotone vector fields, *Acta Math. Hungarica* **94** (2002), no. 4, 307-320.

[37] M. P. DoCarmo, Riemannian Geometry, Boston, Birkhauser, 1992.

[38] T. Dominguez Benavides, G. Lopez Acedo and H. K. Xu, Iterative solutions for zeros of accretive operators, *Math. Nachr.* **248-249** (2003), 62-71.

[39] J. Dye and S. Reich, Unrestricted iterations of nonexpansive mappings in Hilbert space, *Nonlinear Anal.* **18** (1992), 199-207.

[40] I. Ekeland and R. Temam, Convex Analysis and Variational Problems, Amsterdam, North-Holland, 1976.

[41] O.P. Ferreira, L.R. Lucambio Pérez and S.Z. Németh, Singularities of monotone vector fields and an extragradient-type algorithm, *J. Global Optim.* **31** (2005), 133-151.

[42] O.P. Ferreira and P.R. Oliveira, Proximal point algorithm on Riemannian manifolds, *Optimization* **51** (2002), no. 2, 257-270.

[43] M. Fukushima, A relaxed projection method for variational inequalities, *Math. Programming* **35** (1986), no. 1, 58-70.

[44] J. Garcia-Falset, W. Kaczor, T. Kuczumov and S. Reich, Weak convergence theorems for asymptotically nonexpansive mappings and semigroups, *Nonlinear Analysis* **43** (2001), 377-401.

[45] A. Genel and J. Lindenstrauss, An example concerning fixed points, *Israel Journal of Math.* **22** (1975), no. 1, 81-86.

[46] K. Goebel and W. A. Kirk, Iteration processes for nonexpansive mappings, *Contemporary Mathematics* **21** (1983), 115-123.

[47] K. Goebel and W. A. Kirk, Topics in metric fixed point theory, Cambridge Studies in Advanced Mathematics, 28, Cambridge University Press, Cambridge, 1990.

[48] K. Goebel and S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New York, 1984.

[49] R.E. Greene and K. Shiohama, Convex function on complete noncompact manifolds: topological structure, *Invent. Math.* **63** (1981), 129-157.

[50] B. Halpern, Fixed points of nonexpanding maps, *Bull. Amer. Math. Soc.* **73** (1967), 591-597.

[51] J. J. Hoyos Guerrero, Differential Equations of Evolution and Accretive Operators on Finsler Manifolds, Thesis, University of Chicago, 1978.

[52] S. Ishikawa, Fixed point and iteration of a nonexpansive mapping in a Banach space, *Proc. Amer. Math. Soc.* **44** (1976), 147-150.

[53] T. Iwamiya and H. Okochi, Monotonicity, resolvents and Yosida approximation on Hilbert manifolds, *Nonlinear Anal.* **54** (2003), 205-214.

[54] J. Jost, Nonpositive curvature: geometric and analytic aspects, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1997.

[55] S. Kamimura and W. Takahashi, Approximating solutions of maximal monotone operators in Hilbert spaces, *Journal of Approximation Theory* **106** (2000), 226-240.

[56] K. Kido, Strong convergence of resolvent of monotone operators in Banach spaces, *Journal of Approximation Theory* **106** (2000), 226-240.

[57] W. A. Kirk and R. Schöneberg, Some results on pseudo-contractive mappings, *Pacific J. Math.* **71** (1977), 89-99.

[58] W. A. Kirk, Krasnoselskii's Iteration process in hyperbolic space, Numerical Functional Analysis and Optimization **4** (1981/82), no. 4, 371-381.

[59] W. A. Kirk, Geodesic Geometry and Fixed Point Theory, Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), 195–225, Univ. Sevilla Secr. Publ., Seville, 2003.

[60] W.A. Kirk, Geodesic geometry and fixed point theory. II. International Conference on Fixed Point Theory and Applications, 113-142, Yokohama Publ., Yokohama, 2004.

[61] M. A. Krasnosel'skij, Two remarks on the method of successive approximations (Russian), *Uspehi Mat. Nauk* **10** (1955), no. 1(63), 123-127.

[62] C. Li, G. López and V. Martín-Márquez, Monotone vector fields and the proximal point algorithm on Hadamard manifolds, *Journal of London Mathematical Society* **79** (2009), no. 2, 663-683.

[63] C. Li, G. López and V. Martín-Márquez, Iterative algorithms for nonexpansive mappings in Hadamard manifolds, *Taiwanese Journal of Mathematics*, **14**(2), 541–559, 2010.

[64] C. Li, G. López, V. Martín-Márquez and J. H. Wang, Resolvents of set-valued monotone vector fields in Hadamard manifolds, *Set-valued and variational analysis*, submitted for publication.

[65] P. L. Lions, Approximation des points fixes de contractions, *C. R. Acad. Sci. Ser. A-B Paris* **284** (1977), 1357-1359.

[66] T. C. Lim and H. K. Xu, Fixed point theorems for asymptotically nonexpansive mappings, *Nonlinear Analysis* **22** (1994), 1345-1355.

[67] G. López, V. Martín-Márquez and H. K. Xu, Perturbation techniques for nonexpansive mappings, *Nonlinear Analysis: Real world applications* **10** (2009), no. 4, 2369-2383.

[68] G. López, V. Martín-Márquez and H. K. Xu, Iterative algorithms for the multiple-sets split feasibility problem, Biomedical Mathematics: Promising Directions in Imaging, Therapy Planning and Inverse Problems, Medical Physics Publishing, Madison, WI, USA, 2010.

[69] G. López, V. Martín-Márquez and H. K. Xu, Halpern's iteration for nonexpansive mappings, *Nonlinear Analysis and Optimization I: Nonlinear Analysis*, Contemporary Mathematics, AMS, **513**, 187–207, 2010.

[70] P. E. Maingé, Approximation method for common fixed points of nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.* **325** (2007), 469-479.

[71] P. E. Maingé, A hybrid extragradient-viscosity method for monotone operators and fixed point problems, *SIAM J. Control Optim.* **47** (2008), no. 3, 1499-1515.

[72] W. R. Mann, Mean value methods in iteration, *Proceedings of American Mathematical Society* **4** (1953), no. 3, 506-510.

[73] G. Marino and H. K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.* **318** (2006), no. 1, 43-52.

[74] V. Martín-Márquez, Nonexpansive mappings and monotone vector fields in Hadamard manifolds, Communications in Applied Analysis **13** (2009), 633-646.

[75] B. Martinet, Régularisation d'inéquations variationelles par approximations successives, *Revue Française d'Informatique et de Recherche Operationelle* (1970), 154-159.

[76] G. J. Minty, On the monotonicity of the gradient of a convex function, *Pacific J. Math.* **14** (1964), 243-247.

[77] I. Miyadera, Nonlinear semigroups, Translations of Mathematical Monographs, 109 *American Mathematical Society*, Providence, RI, 1992.

[78] J. J. Moreau, Proximité et dualité dans un espace Hilbertien, *Bull. Soc. Math. France* **193** (1965), 273-299.

[79] A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.* **241** (2000), 46-55.

[80] S. Z. Németh, Monotonicity of the complementary vector field of a nonexpansive map, *Acta Math. Hungarica* **84** (1999), no. 3, 189-197.

[81] S. Z. Németh, Monotone vector fields, *Publ. Math. Debrecen* **54** (1999), 437-449.

[82] S. Z. Németh, Five kinds of monotone vector fields, *Pure Math. Appl.* **9** (1999), no. 3–4, 417-428.

[83] S. Z. Németh, Geodesic monotone vector fields, *Lobachevskii J. Math.* **5** (1999), 13-28.

[84] S. Z. Németh, Variational inqualities on Hadamard manifolds, *Nonlinear Anal.* **52** (2003), 1491-1498.

[85] O. Nevanlinna and S. Reich, Strong convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces, *Israel J. Math.* **32** (1979), 44-58.

[86] J. G. O'Hara, P. Pillay and H. K. Xu, Iterative approaches to convex feasibility problems in Banach Space, *Nonlinear Analysis* **64** (2006), no. 9, 2002-2042.

[87] D. Pascali and S. Sburlan, Nonlinear Mappings of Monotone Type, Editura Academ., 1978.

[88] R. R. Phelps, Convex sets and nearest points, *Proc. Amer. Math. Soc.* **8** (1957), 790-797.

[89] B. Polyak, Introduction to Optimization, Optimization Software Inc., New York, 1987.

[90] T. Rapcsk, Smooth nonlinear optimization in $\mathbb{R}^n$, Nonconvex Optimization and Its Applications, 19, Kluwer Academic Publisher, Dordrecht, 1997.

[91] T. Rapcsk, Sectional curvature in nonlinear optimization, *J. Global Optim.* **40** (2008), no. 1-3, 375-388.

[92] S. Reich, Asymptotic behavior of contractions in Banach spaces, *Journal of Mathematical Analysis and Applications* **44** (1973), 57-70.

[93] S. Reich, Extension problems for accretive sets in Banach spaces, *J. Funct. Anal.* **26** (1977), 378-395.

[94] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, *J. Math Anal. Appl.* **67** (1979), 274-276.

[95] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, *Journal of Mathematical Analysis and Applications* **75** (1980), no. 1, 287-292.

[96] S. Reich, Product formulas, nonlinear semigroups, and accretive operators, *J. Funct. Anal.* **36** (1980), no. 2, 147-168.

[97] S. Reich, Convergence, resolvent consistency, and the fixed point property for nonexpansive mappings, *Contemporary Math.* **18** (1983), 167-174.

[98] S. Reich, Book Review: Geometry of Banach spaces, duality mappings and nonlinear problems, *Bull. Amer. Math. Soc.* **26** (1992), 367-370.

[99] S. Reich, Approximating fixed points of nonexpansive mappings, *Panamerican. Math. J.* **4** (1994), no. 2, 23-28.

[100] S. Reich and I. Shafrir, The asymptotic behavior of firmly nonexpansive mappings, *Proc. Amer. Math. Soc.* **101** (1987), no. 2, 246-250.

[101] S. Reich and I. Shafrir, Nonexpansive iterations in hyperbolic spaces, *Nonlinear Anal.* **15** (1990), no. 6, 537-558.

[102] R. T. Rockafellar, Monotone operators associated with saddle-functions and minimax problems, *Nonlinear Functional Analysis, Part 1*, F.E. Browder ed., Symp. in Pure Math., Amer. Math. Soc. Prov., R.I. **18** (1970), 397-407.

[103] R. T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM J. Control Optim.* **14** (1976), no. 5, 877-898.

[104] R. T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, NJ, 1997.

[105] T. Sakai, Riemannian Geometry, Translations of Mathematical Monographs 149, American Mathematical Society, Providence, RI, 1996.

[106] T. Shimizu and W. Takahashi, Strong convergence to common fixed points of nonexpansive mappings, *J. Math. Anal. Appl.* **211** (1997), no. 1, 71-83.

[107] I. Singer, The Theory of Best Approximation and Functional Analysis, CBMS-NSF Regional Conf. Ser. in Appl. Math., 13, SIAM, Philadelphia, PA, 1974.

[108] T. Suzuki, A sufficient and necessary condition for Halpern-type strong convergence to fixed points of nonexpansive mappings, *Proc. Amer. Math. Soc.* **135** (2007), no. 1, 99-106.

[109] T. Suzuki, Browder's type convergence theorems for one-parameter semigroups of nonexpansive mappings in Banach spaces, *Israel J. Math.* **157** (2007), 239-257.

[110] T. Suzuki, Reich's problem concerning Halpern's convergence, *Archiv der Mathematik* **92** (2009), no. 6, 602-613.

[111] W. Takahashi and Y. Ueda, On Reich‘s strong convergence theorem for resolvents of accretive operators, *J. Math. Anal. Appl.* **104** (1984), 546-553.

[112] P. Tseng, On the convergence of the products of firmly nonexpansive mappings, *SIAM J. Optim.* **2** (1992), 425-434.

[113] C. Udriste, Convex Functions and Optimization Methods on Riemannian Manifolds, Mathematics and Its Applications, 297, Kluwer Academic Publisher, 1994.

[114] R. Walter, On the metric projection onto convex sets in Riemannian spaces, *Archiv der Mathematik* **25** (1974), 91-98.

[115] J. H. Wang, G. López, V. Martín-Márquez and C. Li, Monotone and accretive vector fields on Riemannian manifolds, *Journal of Optimization Theory and Applications*, accepted for publication: **146**(3), Sept. 2010.

[116] R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.* **58** (1992), 486-491.

[117] I. Yamada, The hybrid steepest descent method for the variational inequality over the intersection of fixed point sets of nonexpansive mappings, in *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, D. Butnariu, Y. Censor, and S. Reich, eds., Elsevier, Amsterdam, 473–504, 2001.

[118] I. Yamada and N. Ogura, Hybrid steepest descent method for the variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings, *Numer. Funct. Anal. Optim.* **25** (2004), 619–655.

[119] Q. Yang, On variable-step relaxed projection algorithm for variational inequalities, *J. Math. Anal. Appl.* **302** (2005), no. 1, 166-179.

[120] H. K. Xu, Iterative algorithms for nonlinear operators, *Journal of London Mathematical Society* **66** (2002), 240-256.

[121] H. K. Xu, An iterative approach to quadratic optimization, *J. Optim. Theory Appl.* **116** (2003), no. 3, 659-678.

[122] H. K. Xu, Remarks on an iterative method for nonexpansive mappings, *Communications on Applied Nonlinear Analysis* **10** (2003), no. 1, 67-75.

[123] H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.* **298** (2004), 279-291.

[124] H. K. Xu, A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem, *Inverse problems* **22** (2006), 2021-2034.

[125] H. K. Xu, Strong convergence of an iterative method for nonexpansive and accretive operators, *J. Math. Anal. Appl.* **314** (2006), 631-643.

[126] H. K. Xu, An averaged mapping approach to the gradient-projection algorithm, preprint.

[127] H. K. Xu and T. H. Kim, Convergence of hybrid steepest descent methods for variational inequalities, *J. Optim. Theory Appl.* **119** (2003), 185–201.

[128] Q. Yang and J. Zhao, Generalized KM theorems and their applications, *Inverse Problems* **22** (2006), 833-844.

[129] E. Zeidler, Nonlinear Functional Analysis and Applications, II/B. Nonlinear Monotone Operators, Springer Verlag, New York, 1990.

[130] J. Zhao and Q. Yang, Several solution methods for the split feasibility problem, *Inverse Problems* **21** (2005), 1791-1799.