Cocyclic Hadamard matrices over Latin rectangles

Raúl M. Falcón
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(Joint work with V. Álvarez, M. D. Frau, M. B. Güemes and F. Gudiel).

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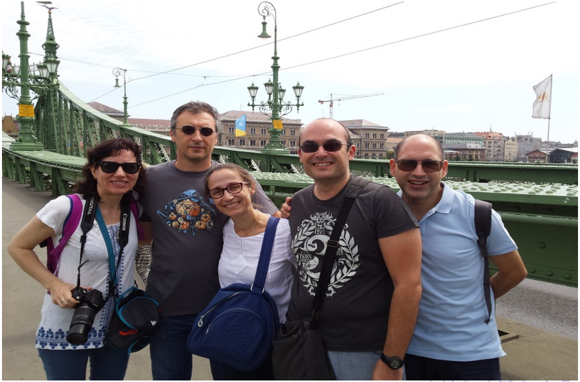

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I. Preliminaries.

Cocyclic Hadamard matrices.

Kathy J. Horadam.

Warwick De Launey.

Cocycle over a finite group G: $\psi : G \times G \rightarrow \langle 1, -1 \rangle$ such that $\psi(ij,k)\psi(i,j)\psi(j,k)\psi(i,jk) = 1$, for all $i,j,k \in G$.

- **Cocyclic matrix:** $M_\psi = (\psi(i,j))_{i,j \in G}$.
- M_ψ Hadamard $\rightarrow$ **Cocyclic Hadamard matrix over** G.

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Kathy J. Horadam.

Warwick De Launey.

Cocycle over a finite group G:
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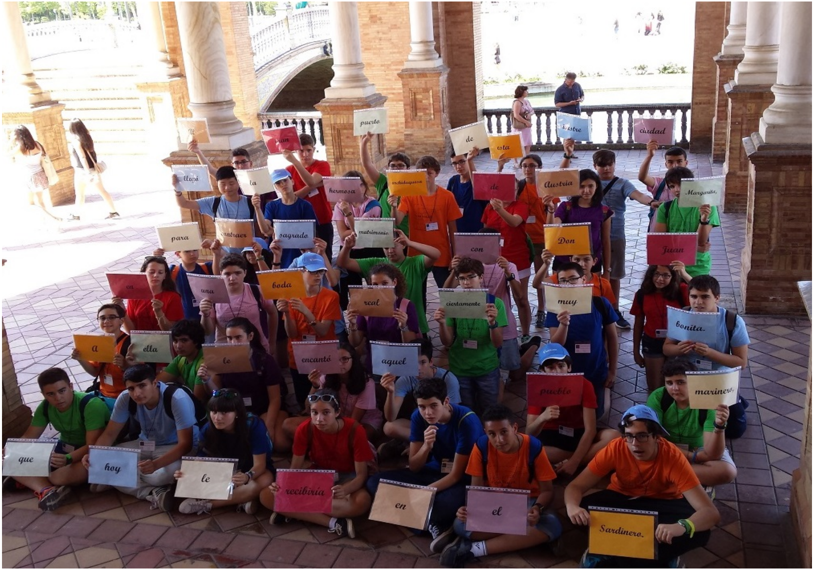
Dane L. Flannery.

Cocyclic Hadamard matrices over

$$D_{4t} := \langle\, a, b \mid a^{2t} = b^2 = (ab)^2 = 1 \,\rangle,$$

for all positive integer $t \leq 11$.



Andalusian Mathematical Olympiad. Seville, 2016.

Latin rectangles ($\mathcal{R}_{r,n}$).

Let r, n be two positive integers such that $r \leq n$.

- **Latin rectangle**: $r \times n$ array where
  1. each cell contains one symbol of $[n] := \{1, \ldots, n\}$.
  2. each symbol occurs once per row and at most once per column.

$L \equiv$

1	4	3	2
3	2	1	4
2	3	4	1

$\in \mathcal{R}_{3,4}$.

- **Latin square**: $r = n \rightarrow$ **Quasigroup** $([n], \cdot)$.
- **Reduced**: Symbols appearing in its first row and its first column are displayed in natural order.

$L \equiv$

1	2	3	4
2	4	1	3
3	1	4	2

$\in \mathcal{R}_{3,4}$.

  - $r = n \rightarrow$ **Loop** (quasigroup with unit element).

II. Cocycles over Latin rectangles.

Cocycles over Latin rectangles.

Let $L = (l_{ij}) \in \mathcal{R}_{r,n}$.

- $S(L) := [r] \cup \{l_{ij} \mid i,j \leq r\} \subseteq [n]$.

$L \equiv$

2	3	4	1	6	7	8	5
3	5	6	8	7	2	1	4

$\Rightarrow S(L) = \{1, 2, 3, 5\}$.

Cocycles over Latin rectangles.

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- **(Binary) cocycle over** L:

$$\psi \colon S(L) \times [n] \to \langle -1 \rangle \text{ such that}$$

$$\psi(l_{ij}, k)\psi(i, j)\psi(j, k)\psi(i, l_{jk}) = 1, \text{ for all } i, j \leq r \text{ and } k \leq n.$$

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$L \equiv$

2	3	4	1	6	7	8	5
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- $\mathcal{C}_L := \{\text{Cocycles over } L\}$.

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- $\mathcal{C}_L :=$ {Cocycles over L}.
- **Cocyclic matrix**: $M_\psi := (\psi(e_i, j))$.

$$\begin{pmatrix} + & - & - & + & + & + & - & - \\ - & + & - & + & + & - & + & - \\ - & - & + & + & + & - & - & + \\ + & + & + & - & + & - & - & - \end{pmatrix}.$$

Cocyclic Hadamard matrices over Latin rectangles.

$$L \equiv \begin{array}{|c|c|c|c|c|c|c|c|}\hline 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 \\ \hline 3 & 5 & 6 & 8 & 7 & 2 & 1 & 4 \\ \hline \end{array} \Rightarrow S(L) = \{e_1, e_2, e_3, e_4\}.$$

$$\Downarrow$$

$$M_\psi \equiv \begin{pmatrix} + & - & - & + & + & + & - & - \\ - & + & - & + & + & - & + & - \\ - & - & + & + & + & - & - & + \\ + & + & + & - & + & - & - & - \end{pmatrix} \Rightarrow M_\psi M_\psi^t = 8\,\mathrm{Id}_4 = n\,\mathrm{Id}_{|S(L)|}.$$

Cocyclic Hadamard matrices over Latin rectangles.

$L \equiv$

2	3	4	1	6	7	8	5
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$\Rightarrow S(L) = \{e_1, e_2, e_3, e_4\}.$

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- $\overline{\mathcal{R}}_{r,n} := \{L \in \mathcal{R}_{r,n} : \ |S(L)| = n\}$.

$$L \equiv \begin{array}{|c|c|c|c|c|c|c|c|}\hline 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 \\ \hline 3 & 5 & 6 & 8 & 7 & 2 & 1 & 4 \\ \hline \end{array} \Rightarrow S(L) = \{e_1, e_2, e_3, e_4\}.$$

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$$M_\psi \equiv \begin{pmatrix} + & - & - & + & + & + & - & - \\ - & + & - & + & + & - & + & - \\ - & - & + & + & + & - & - & + \\ + & + & + & - & + & - & - & - \end{pmatrix} \Rightarrow M_\psi M_\psi^t = 8\,\mathrm{Id}_4 = n\,\mathrm{Id}_{|S(L)|}.$$

- $\overline{\mathcal{R}}_{r,n} := \{L \in \mathcal{R}_{r,n} \colon \; |S(L)| = n\}$.

Let $L \in \overline{\mathcal{R}}_{r,n}$.

- $\mathcal{H}_L := \{M_\psi \text{ Hadamard} \colon \; \psi \in \mathcal{C}_L\}$.

$$L \equiv \begin{array}{|c|c|c|c|c|c|c|c|}\hline 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 \\ \hline 3 & 5 & 6 & 8 & 7 & 2 & 1 & 4 \\ \hline \end{array} \Rightarrow S(L) = \{e_1, e_2, e_3, e_4\}.$$

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$$M_\psi \equiv \begin{pmatrix} + & - & - & + & + & + & - & - \\ - & + & - & + & + & - & + & - \\ - & - & + & + & + & - & - & + \\ + & + & + & - & + & - & - & - \end{pmatrix} \Rightarrow M_\psi M_\psi^t = 8\,\mathrm{Id}_4 = n\,\mathrm{Id}_{|S(L)|}.$$

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$$L \equiv \begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & 1 \\ \hline 3 & 4 & 1 & 2 \\ \hline \end{array} \in \overline{\mathcal{R}}_{2,4} \rightarrow \begin{pmatrix} + & + & - & - \\ + & - & + & - \\ - & + & + & - \\ - & - & - & - \end{pmatrix} \in \mathcal{H}_L.$$

Cocyclic Hadamard matrices over Latin rectangles.

$L \equiv$

2	3	4	1	6	7	8	5
3	5	6	8	7	2	1	4

$\Rightarrow S(L) = \{e_1, e_2, e_3, e_4\}.$

$\Downarrow$

$$M_\psi \equiv \begin{pmatrix} + & - & - & + & + & + & - & - \\ - & + & - & + & + & - & + & - \\ - & - & + & + & + & - & - & + \\ + & + & + & - & + & - & - & - \end{pmatrix} \Rightarrow M_\psi M_\psi^t = 8\,\mathrm{Id}_4 = n\,\mathrm{Id}_{|S(L)|}.$$

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Let $L \in \overline{\mathcal{R}}_{r,n}$.

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$L \equiv$

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	5	6	7	8	1	2
4	3	7	8	1	2	6	5

$$\in \overline{\mathcal{R}}_{4,8} \rightarrow \begin{pmatrix} + & + & + & + & + & + & + & + \\ + & - & - & + & + & - & - & + \\ + & - & - & + & - & + & + & - \\ + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & - & + & - & - & + & - & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & - & - & + & + \end{pmatrix} \in \mathcal{H}_L.$$

Cocyclic Hadamard matrices over Latin rectangles (groups).

$$L \equiv \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 1 & 4 & 3 \\ \hline 3 & 4 & 1 & 2 \\ \hline 4 & 3 & 2 & 1 \\ \hline \end{array} \in \overline{\mathcal{R}}_{4,4} \Rightarrow S(L) = \{1,2,3,4\}.$$

$$\begin{pmatrix} + & + & + & + \\ + & + & - & - \\ + & - & - & + \\ + & - & + & - \end{pmatrix} \in \mathcal{H}_L.$$

Cocyclic Hadamard matrices over Latin rectangles.

Problem (1)

$L \in \overline{\mathcal{R}}_{r,n} \Rightarrow \mathcal{H}_L \neq \emptyset$?

Problem (2)

Let M be a Hadamard matrix of order n. Does there exist $L \in \overline{\mathcal{R}}_{r,n}$ such that $M \in \mathcal{H}_L$?

III. Cocyclic Hadamard matrices over Latin rectangles.

Lemma

Let $L = (l_{ij}) \in \mathcal{R}_{r,n}$ *and* $\psi \in \mathcal{C}_L$.

- $l_{ij} = i \Rightarrow \psi(l_{ki}, j) = \psi(i, j)$, *for all* $k \leq r$.
- $l_{ij} = j$ *for some* $i, j \leq r \Rightarrow \psi(i, k) = \psi(i, j)$, *for all* $k \leq n$.

1	2	4	3
3	4	2	1

$$\rightarrow \begin{pmatrix} + & + & + & + \\ - & + & - & + \\ + & - & - & + \\ + & + & - & - \end{pmatrix}$$

1	2	3	4
2	1	4	3
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$$\rightarrow \begin{pmatrix} + & + & + & + \\ + & + & - & - \\ + & - & - & + \\ + & - & + & - \end{pmatrix}$$

Cocycles over Latin rectangles.

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1	2	4	3
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$$\rightarrow \begin{pmatrix} + & + & + & + \\ - & + & - & + \\ + & - & - & + \\ + & + & - & - \end{pmatrix}$$

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$$\rightarrow \begin{pmatrix} + & + & + & + \\ + & + & - & - \\ + & - & - & + \\ + & - & + & - \end{pmatrix}$$

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Cocyclic Hadamard matrices over a Latin rectangle.

Proposition

Let $L = (l_{ij}) \in \overline{\mathcal{R}}_{r,n}$ *be such that* $\mathcal{H}_L \neq \emptyset$.

- $l_{ij} = j$, *for some* $i, j \leq r \Rightarrow l_{ik} = k$, *for all* $k \leq r$.

  $\Downarrow$
- $\#\{k \leq n \mid l_{ik} = k\} > n/2 \Rightarrow l_{ki} = k$, *for all* $i \leq r$.

1	2	4	3
3	4	2	1

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1	3	5	4	7	8	2	6
3	5	8	2	1	6	7	4
4	7	6	1	8	3	2	5

$\in \overline{\mathcal{R}}_{3,8} \Rightarrow \mathcal{H}_L = \emptyset.$

Cocyclic Hadamard matrices over a Latin rectangle.

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Cocyclic Hadamard matrices over a Latin rectangle.

Theorem

Let $L \in \overline{\mathcal{R}}_{n,n}$ be such that $\mathcal{H}_L \neq \emptyset$. Then,

- *L is the multiplication table of a loop.*
- *If $\psi \in \mathcal{C}_L$ and e is the unit element of the loop, then*

$$\psi(e,i) = \psi(j,e), \forall i,j \leq n.$$

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

$$\rightarrow \begin{pmatrix} + & + & + & + \\ + & + & - & - \\ + & - & - & + \\ + & - & + & - \end{pmatrix}$$

Cocyclic Hadamard matrices over a Latin rectangle.

Theorem

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$$\begin{pmatrix} + & + & + & + \\ - & + & - & + \\ + & - & - & + \\ + & + & - & - \end{pmatrix} \in \mathcal{H}_L, \text{ for none } L \in \mathcal{R}_{4,4}.$$

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Does there exist a **non-associative loop** $L \in \overline{\mathcal{R}}_{r,n}$ *such that* $\mathcal{H}_L \neq \emptyset$*?*

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1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
4	3	2	1	8	7	6	5
5	6	8	7	3	4	2	1
6	5	7	8	4	3	1	2
7	8	6	5	1	2	4	3
8	7	5	6	2	1	3	4

$$\in \overline{\mathcal{R}}_{8,8} \rightarrow \begin{pmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & - & - & + & + \\ + & + & - & - & + & + & - & - \\ + & - & - & + & - & + & + & - \\ + & - & - & + & + & - & - & + \\ + & - & + & - & - & + & - & + \\ + & - & + & - & + & - & + & - \end{pmatrix}$$

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Cocyclic Hadamard matrices over a Latin rectangle.

Theorem

Every non-associative loop of order 8 *that gives rise to a cocyclic Hadamard matrix is isomorphic to*

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
4	3	2	1	8	7	6	5
5	6	8	7	3	4	2	1
6	5	7	8	4	3	1	2
7	8	6	5	1	2	4	3
8	7	5	6	2	1	3	4

$\in \overline{\mathcal{R}}_{8,8}.$

$$M_{\psi_1} \equiv \begin{pmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & - & - & + & + \\ + & + & - & - & + & + & - & - \\ + & - & - & + & - & + & + & - \\ + & - & - & + & + & - & - & + \\ + & - & + & - & - & + & - & + \\ + & - & + & - & + & - & + & - \end{pmatrix} \quad M_{\psi_2} \equiv \begin{pmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & - & - & + & + \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & - & - & + & - & + & + & - \\ + & - & + & - & + & - & + & - \\ + & - & + & - & - & + & - & + \end{pmatrix}$$

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6	5	7	8	4	3	1	2
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8	7	5	6	2	1	3	4

$\in \overline{\mathcal{R}}_{8,8}$.

$$M_{\psi_3} \equiv \begin{pmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & + & - & - & - & - & + & + \\ + & - & + & - & - & + & - & + \\ + & - & + & - & + & - & + & - \\ + & - & - & + & + & - & - & + \\ + & - & - & + & - & + & + & - \end{pmatrix} \quad M_{\psi_4} \equiv \begin{pmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & + & - & - & - & - & + & + \\ + & - & + & - & + & - & + & - \\ + & - & + & - & - & + & - & + \\ + & - & - & + & - & + & + & - \\ + & - & - & + & + & - & - & + \end{pmatrix}$$

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Lemma

Let L_1 and L_2 be two Latin rectangles in $\mathcal{R}_{r,n}$ that are isotopic by means of an isotopism (f, g, g) such that $f(i) = g(i)$, for all $i \leq r$. Then, there exists a 1-1 correspondence between the sets $\mathcal{C}_{L_1}$ and $\mathcal{C}_{L_2}$.

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$$\alpha : \mathcal{C}_{L_1} \to \mathcal{C}_{L_2}$$
$$\alpha(\psi)(i,j) = \psi(g^{-1}(i), g^{-1}(j)).$$

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Let L_1 and L_2 be two Latin rectangles in $\mathcal{R}_{r,n}$ that are **isomorphic**. *Then, there exists a 1-1 correspondence between the sets $\mathcal{C}_{L_1}$ and $\mathcal{C}_{L_2}$.*

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Proposition

Let L_1 and L_2 be two isomorphic Latin rectangles in $\overline{\mathcal{R}}_{r,n}$. Then, there exists a 1-1 correspondence between $\mathcal{H}_{L_1}$ and $\mathcal{H}_{L_2}$.

Cocyclic Hadamard matrices over a Latin rectangle.

r	n	L	$\mathcal{H}_L$			
1	1	1	+			
	2	21	+ −	− −		
2	2	12 21	+ +	+ −		
	4	1243 3421	+ + + +	+ − − +	+ − + −	+ + − −
			+ + + +	− + − +	+ − − +	+ + − −
			+ + + +	− + + −	+ − + −	− − + +
			+ + + +	+ − + −	+ − − +	− − + +
		2341 3412	+ + − −	+ − + −	− + + −	− − − −
			+ − − +	− + − +	− − + +	+ + + +
		2134 3421	+ − + −	− − + +	+ − − +	+ + + +
		2314 3421	+ − − +	− + − +	− − − −	+ + − −
		2314 4123	+ − + −	− − + +	− + + −	+ + + +
3	4	1234 2143 3412	+ + + +	+ + − −	+ − − +	+ − + −
			+ + + +	+ + − −	+ − + −	+ − − +
			+ + + +	+ − + −	+ − − +	+ + − −
			+ + + +	+ − + −	+ + − −	+ − − +
			+ + + +	+ − − +	+ − + −	+ + − −
			+ + + +	+ − − +	+ + − −	+ − + −
		1234 2143 3421	+ + + +	+ + − −	+ − − +	+ − + −
			+ + + +	+ + − −	+ − + −	+ − − +
		2341 3412 4123	+ + − −	+ − + −	− + + −	− − − −
			+ − − +	− + − +	− − + +	+ + + +
4	4	1234 2143 3412 4321	+ + + +	+ + − −	+ − − +	+ − + −
			+ + + +	+ + − −	+ − + −	+ − − +
			+ + + +	+ − + −	+ − − +	+ + − −
			+ + + +	+ − + −	+ + − −	+ − − +
			+ + + +	+ − − +	+ − + −	+ + − −
			+ + + +	+ − − +	+ + − −	+ − + −
		1234 2143 3421 4312	+ + + +	+ + − −	+ − − +	+ − + −
			+ + + +	+ + − −	+ − + −	+ − − +

IV. Further work.

Further work.

Problem (2)

Let M be a Hadamard matrix of order n. Does there exist $L \in \overline{\mathcal{R}}_{r,n}$ such that $M \in \mathcal{H}_L$?

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Problem (5)

Let M be a Hadamard matrix of order n for which there exists $L \in \overline{\mathcal{R}}_{r,n}$ such that $M \in \mathcal{H}_L$. **Which is the minimum $r \in \mathbb{N}$ for which one such a Latin rectangle exists?**

Further work.

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Let M be a Hadamard matrix of order n. Does there exist $L \in \overline{\mathcal{R}}_{r,n}$ such that $M \in \mathcal{H}_L$?

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Let M be a Hadamard matrix of order n for which there exists $L \in \overline{\mathcal{R}}_{r,n}$ such that $M \in \mathcal{H}_L$. **Which is the minimum $r \in \mathbb{N}$ for which one such a Latin rectangle exists?**

Lemma

Let $L \in \overline{\mathcal{R}}_{r,n}$ be such that $\mathcal{H}_L \neq \emptyset$. Then, $r \leq n \leq r + r^2$.

Further work.

$$r \leq n \leq r^2 + r.$$

$r = 1; n = 2.$

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \rightarrow \begin{pmatrix} + & - \\ - & - \end{pmatrix}$$

$r = 2; n = 4.$

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 3 \\ \hline 3 & 4 & 2 & 1 \\ \hline\end{array} \rightarrow \begin{pmatrix} + & + & + & + \\ + & - & - & + \\ + & - & + & - \\ + & + & - & - \end{pmatrix}$$

$r = 3; n = 8.$

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 7 & 8 & 5 & 6 \\ \hline 2 & 5 & 8 & 3 & 1 & 6 & 7 & 4 \\ \hline 4 & 7 & 6 & 2 & 8 & 3 & 1 & 5 \\ \hline\end{array} \rightarrow \begin{pmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & - & - & + & + \\ + & + & - & - & + & + & - & - \\ + & - & - & + & - & + & + & - \\ + & - & - & + & + & - & - & + \\ + & - & + & - & - & + & - & + \\ + & - & + & - & + & - & + & - \end{pmatrix}$$

Further work.

$$r \leq n \leq r^2 + r.$$

$r = 3; n = 12\,?$

4	5	6					
7	8	9					
10	11	12					

$$\rightarrow \begin{pmatrix} + & + & + & + & + & + & + & + & + & + & + & + \\ + & - & + & - & + & + & + & - & - & - & + & - \\ + & - & - & + & - & + & + & + & - & - & - & + \\ + & + & - & - & + & - & + & + & + & - & - & - \\ + & - & + & - & - & + & - & + & + & + & - & - \\ + & - & - & + & - & - & + & - & + & + & + & - \\ + & - & - & - & + & - & - & + & - & + & + & + \\ + & + & - & - & - & + & - & - & + & - & + & + \\ + & + & + & - & - & - & + & - & - & + & - & + \\ + & + & + & + & - & - & - & + & - & - & + & - \\ + & - & + & + & + & - & - & - & + & - & - & + \\ + & + & - & + & + & + & - & - & - & + & - & - \end{pmatrix}$$

Further work.

n	M	r	L
1	1	1	1
2	2 0	1	21
		2	21 12
4	F 9 A C	2	1243 3421
		3	1234 2143 3421
		4	1234 2143 3421 4312
8	FF $F0$ $C3$ CC 96 99 $A5$ AA	3	12347856 25831674 47628315
		4	12345687 23851476 37526148 45187263
		5	12345678 21436587 34128765 43217856 56782143
		6	12345678 21436587 34128765 43217856 56782143 65871234
		7	12345678 21436587 34128765 43217856 56782143 65871234 78563412
		8	12345678 21436587 34128765 43217856 56782143 65871234 78563412 87654321
12	FFF $AE2$ 971 $CB8$ $A5C$ $92E$ 897 $C4B$ $E25$ $F12$ $B89$ $DC4$	3	

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Many thanks!! Köszönöm!!

Cocyclic Hadamard matrices over Latin rectangles