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BANACH ALGEBRAS WITH TRIVIAL COHOMOLOGY

RACHID EL HARTI

ABSTRACT. A plausible conjecture is that Banach algebras with trivial cohomology have to be semisimple and finite dimensional. In this paper, we show that a Hermitian Banach $*$-algebra has trivial cohomology if and only if it is $*$-isomorphic to a finite direct sum of full matrix algebras.

1. INTRODUCTION

Let $(\mathcal{A}, \|\cdot\|, *)$ be a Banach algebra $\mathcal{A}$ with an identity and some involution. $\mathcal{A}$ is called a Hermitian Banach $*$-algebra if for every element a in $\mathcal{A}$, a^*a has real spectrum. It is said to be a C^*-algebra if its norm $\|\cdot\|$ is a C^*-norm, i.e, $\|\cdot\|$ satisfies: $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$. Obviously, a C^*-algebra is a Hermitian Banach $*$-algebra. A state τ is a linear form on $\mathcal{A}$ such that $\tau(1) = \|\tau\| = 1$. A $*$-representation of $\mathcal{A}$ on a pre-Hilbert space X is a $*$-homomorphism of $\mathcal{A}$ into the algebra $B(X)$ of bounded operator on X.

Given a Banach algebra, $\mathcal{A}$, we define a Banach left $\mathcal{A}$-module $\mathcal{X}$ to be a Banach space which is a structure left module $\mathcal{A}$ such that the linear map

$$(a, x) \in \mathcal{A} \times \mathcal{X} \quad \longmapsto \quad ax \in \mathcal{X}$$

is continuous. Right modules are defined analogously. A Banach $\mathcal{A}$-bimodule is a Banach space with a structure bimodule over $\mathcal{A}$ such that the linear map

$$(a, x, b) \in \mathcal{A} \times \mathcal{X} \times \mathcal{A} \quad \longmapsto \quad axb \in \mathcal{X}$$

is continuous. A Banach $\mathcal{A}$-submodule of a given Banach $\mathcal{A}$-module $\mathcal{X}$ is at the same time a closed subspace of $\mathcal{X}$ and an algebraic $\mathcal{A}$-submodule of $\mathcal{X}$. A Banach left $\mathcal{A}$-module morphism $\theta : \mathcal{X} \to \mathcal{Y}$ is a continuous linear map

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between two left Banach $\mathcal{A}$-modules such that $\theta(ax) = a\theta(x)$ for all $a \in \mathcal{A}$ and all $x \in \mathcal{X}$. Banach right $\mathcal{A}$-module morphisms and Banach $\mathcal{A}$-bimodule morphisms are defined analogously. For each Banach $\mathcal{A}$-bimodule $\mathcal{X}$, the dual $\mathcal{X}^*$ is naturally a Banach bimodule over $\mathcal{A}$ with the module actions defined by $aT(x) = T(xa)$ and $Ta(x) = T(ax)$, for all $a \in \mathcal{A}$, $T \in \mathcal{X}^*$, and $x \in \mathcal{X}$; where $T(x)$ denotes the evaluation of T at x.

A derivation from $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is a linear operator $D : \mathcal{A} \to \mathcal{X}$ which satisfies $D(ab) = D(a)b + aD(b)$, $\forall a, b \in \mathcal{A}$. Notice that for any $x \in \mathcal{X}$, the mapping $\delta_x : \mathcal{A} \to \mathcal{X}$ defined by $\delta_x(a) = ax - xa$, $a \in \mathcal{A}$, is a continuous derivation called an inner derivation. An n-linear bounded operator $T : \mathcal{A} \times \overset{n)}{\cdots} \times A \to \mathcal{X}$ is called a n-cochain. The set of n-cochains forms a Banach space denoted by $C^n(\mathcal{A}, \mathcal{X})$. The standard cohomological complex is

$$0 \to \mathcal{X} \xrightarrow{\delta^0} C^1(\mathcal{A}, \mathcal{X}) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} C^n(\mathcal{A}, \mathcal{X}) \xrightarrow{\delta^n} C^{n+1}(\mathcal{A}, \mathcal{X}) \xrightarrow{\delta^{n+1}} \cdots ,$$

where δ^n is defined as follows for $n \geq 1$:

$$\begin{aligned}(\delta^n T)(a_1, \ldots, a_{n+1}) =& a_1 T(a_2, \ldots, a_{n+1}) \\ &+ \sum_{k=1}^{n} (-1)^k T(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{n+1}) \\ &+ (-1)^{n+1} T(a_1, \ldots, a_n) a_{n+1}\end{aligned}$$

and

$$\delta^0(x)(a) = \delta_x(a) = xa - ax.$$

The spaces $\ker \delta^n$ and $\operatorname{im} \delta^{n-1}$ are denoted by $Z^n(\mathcal{A}, \mathcal{X})$ and $B^n(\mathcal{A}, \mathcal{X})$ respectively, and their elements are called n-dimensional cocycles and n-dimensional coboundaries respectively.

The quotient $Z^n(\mathcal{A}, \mathcal{X})/B^n(\mathcal{A}, \mathcal{X})$ is called the n-dimensional cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}$ and denoted by $H^n(\mathcal{A}, \mathcal{X})$ (see [3]). Note that $H^n(\mathcal{A}, \mathcal{X}) = 0$ means that every bounded n-linear map T with $\delta^n T = 0$ is of the form $\delta^{n-1} S$ for some bounded $(n-1)$-linear map S. Note also that $Z^1(\mathcal{A}, \mathcal{X})$ is the space of all derivations from $\mathcal{A}$ to $\mathcal{X}$ and that $B^1(\mathcal{A}, \mathcal{X})$ is the space of all inner derivations. Thus, $H^1(\mathcal{A}, \mathcal{X}) = 0$ means that $Z^1(\mathcal{A}, \mathcal{X}) = B^1(\mathcal{A}, \mathcal{X})$ and consequently all derivations from $\mathcal{A}$ to $\mathcal{X}$ are inner.

Fixed a Banach algebra $\mathcal{A}$, the statement $H^1(\mathcal{A}, \mathcal{X}) = 0$ for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$ imply the same statement for all $n \geq 1$.

Definition 1.1. *We say that a Banach algebra $\mathcal{A}$ has trivial cohomology if $H^1(\mathcal{A}, \mathcal{X})$ vanishes for each Banach $\mathcal{A}$-bimodule $\mathcal{X}$.*

Obviously, a finite-dimensional algebra $\mathcal{A}$ has trivial cohomology if and only if it is semisimple (i.e, its Jacobson radical is trivial) and so, it is a finite direct sum of full matrix algebras. In view of the difficulties to find an infinite dimensional Banach algebra with trivial cohomology, the following question has been frequently asked (see [6, p. 180]):

Question. *Is every Banach algebra with trivial cohomology a finite direct sum of full matrix algebras?*

In the paper mentioned above, J.L. Taylor gives a partial answer to this question by proving that Banach algebras with trivial cohomology and the bounded approximation property are finite direct sum of full matrix algebras. Note that not every Hermitian Banach $*$-algebra or C^*-algebra does satisfy the bounded approximation property, so it is important to know what happens whenever this class of algebras have trivial cohomology. The following theorem, which is the main result of this paper, solves the above question for the class of Hermitian Banach $*$-algebras.

Theorem 1.2. *Let $\mathcal{A}$ be a Hermitian Banach $*$-algebra. Then, $\mathcal{A}$ has trivial cohomology if and only if there are $n_1, n_2, \dots, n_k \in \mathbb{N}$ such that*

$$\mathcal{A} \cong \mathbb{M}_{n_1}(\mathbb{C}) \oplus \mathbb{M}_{n_2}(\mathbb{C}) \oplus \cdots \oplus \mathbb{M}_{n_k}(\mathbb{C})$$

where $\mathbb{M}_{n_i}(\mathbb{C})$ is the algebra of complex $n_i \times n_i$-matrices.

2. Preliminaries

Given $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ Banach $\langle$left, bi-$\rangle$ $\mathcal{A}$-modules and $\theta : \mathcal{X} \to \mathcal{Y}$, $\beta : \mathcal{Y} \to \mathcal{Z}$ $\langle$left, bi-$\rangle$ module morphisms, the sequence

$$\Sigma : 0 \to \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \to 0$$

is a short exact sequence if θ is one-to-one, $\operatorname{im} \beta = \mathcal{Z}$, and $\operatorname{im} \theta = \ker \beta$. A short exact sequence Σ is called admissible if β has a continuous right inverse or, equivalently, if $\ker \beta$ has a Banach space complement in $\mathcal{Y}$. An admissible short exact sequence splits if the right inverse of β is a Banach $\langle$left, bi-$\rangle$ module morphism or, equivalently, if $\ker \beta$ has a Banach space complement in $\mathcal{Y}$ which is an $\mathcal{A}$-submodule.

Proposition 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ be a continuous homomorphism with dense range. If $\mathcal{A}$ has trivial cohomology, then so has $\mathcal{B}$. In particular, if $\mathcal{I}$ is a closed two-sided ideal of a Banach algebra $\mathcal{A}$ with trivial cohomology, then so is $\mathcal{A}/\mathcal{I}$.*

Proof. Assume that $\mathcal{A}$ has trivial cohomology. Let $\mathcal{X}$ be a Banach $\mathcal{B}$-bimodule. Consider on $\mathcal{X}$, the structure of $\mathcal{A}$-bimodule defined by $ax = \theta(a)x$ and $xa = x\theta(a)$. Since θ is continuous, $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule. Now, let $D : \mathcal{B} \to \mathcal{X}$ be a continuous derivation. It is easy to see that $D \circ \theta$ is a continuous derivation from $\mathcal{A}$ to the Banach $\mathcal{A}$-bimodule $\mathcal{X}$ and, thus, it is inner. Therefore, there exists $x \in \mathcal{X}$ such that $D(\theta(a)) = ax - xa = \theta(a)x - x\theta(a)$ for all $a \in \mathcal{A}$. Since $\theta(\mathcal{A})$ is dense in $\mathcal{B}$, we have $D(b) = bx - xb$ for all $b \in \mathcal{B}$. It follows that D is inner and thus, $\mathcal{B}$ has trivial cohomology. □

Notice that a Banach algebra $\mathcal{A}$ with trivial cohomology is unital and every admissible short exact sequence of $\langle$left, bi-$\rangle$ Banach modules on $\mathcal{A}$ splits ([1, Theorem 6.1]). This is the reason why the following proposition holds:

Proposition 2.2. *Let $\mathcal{A}$ be a Banach algebra with trivial cohomology and $\mathcal{I}$ be a closed $\langle$left, two-sided$\rangle$ ideal of $\mathcal{A}$ which has a Banach space complement. Then there exists a closed $\langle$left, two-sided$\rangle$ ideal $\mathcal{J}$ of $\mathcal{A}$ such that*

$$\mathcal{A} = \mathcal{I} \oplus \mathcal{J}.$$

Proof. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{I}$ be a closed $\langle$left, two-sided$\rangle$ ideal of $\mathcal{A}$ which has a Banach space complement. Then the short exact sequence

$$\Sigma : 0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{A}/\mathcal{I} \to 0$$

is admissible. Moreover, if $\mathcal{A}$ has trivial cohomology, then Σ splits and $\mathcal{I}$ has a Banach space complement which is a $\langle$left, two-sided$\rangle$ ideal. □

Theorem 2.3. *Let $\mathcal{A}$ be a Banach algebra with trivial cohomology. Assume that each maximal left ideal of $\mathcal{A}$ is complemented as a Banach space in $\mathcal{A}$. Then, there are $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that*

$$\mathcal{A} \cong \mathbb{M}_{n_1}(\mathbb{C}) \oplus \mathbb{M}_{n_2}(\mathbb{C}) \oplus \cdots \oplus \mathbb{M}_{n_k}(\mathbb{C}).$$

Proof. As stated above, the algebra $\mathcal{A}$ has an identity $1_{\mathcal{A}}$. Let $(\mathcal{M}_i)_{i\in I}$ be a family of all maximal left ideals of $\mathcal{A}$. By hypotesis each $\mathcal{M}_i$ is complemented as a Banach space. By Proposition 2.2, there exists a left ideal $\mathcal{J}_i$ such that $\mathcal{A} = \mathcal{M}_i \oplus \mathcal{J}_i$. Notice that by definition

$$\mathrm{Rad}(\mathcal{A}) = \bigcap_i \mathcal{M}_i$$

is the Jacobson radical of $\mathcal{A}$ and

$$\bigoplus_{i\in I} \mathcal{J}_i \subseteq \mathrm{Soc}(\mathcal{A}), \tag{1}$$

where $\mathrm{Soc}(\mathcal{A})$ is the socle of the algebra $\mathcal{A}$, i.e, it is the sum of all minimal left ideals of $\mathcal{A}$ and it coincides with the sum of all minimal right ideals of $\mathcal{A}$. Recall that every minimal left ideal of $\mathcal{A}$ is of the form $\mathcal{A}e$ where e is a minimal idempotent, i.e, $e^2 = e \neq 0$ and $e\mathcal{A}e = \mathbb{C}e$. On the other hand, for each finite family of minimal idempotents $(e_k)_{k\in K}$, we have

$$\mathcal{A} = \bigoplus_{k\in K} \mathcal{A}e_k \bigoplus \bigcap_{k\in K} \mathcal{A}(1_{\mathcal{A}} - e_k). \tag{2}$$

It follows from (1) and (2) that $\mathrm{Soc}(\mathcal{A})$ is dense in $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$. This shows that $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ is finite-dimensional. Therefore

$$\mathcal{A} = \mathrm{Rad}(\mathcal{A}) \oplus \mathrm{Soc}(\mathcal{A}).$$

If $\mathrm{Rad}(\mathcal{A}) \neq \{0\}$, this would mean that $\mathrm{Rad}(\mathcal{A})$ has an identity and this is impossible. Therefore, $\mathcal{A} = \mathrm{Soc}(\mathcal{A})$ and then it is a finite direct sum of certain full matrix algebras. □

3. Proof of Theorem 1.2

It suffices to show the only if part. First, suppose now that $\mathcal{A}$ is a C^*-algebra. Let $\mathcal{M}$ be a maximal left ideal. By [4, Theorem 5.3.5 and Theorem 5.2.4], the space $\mathcal{A}/\mathcal{M}$ is a Hilbert space. It follows that the short exact sequence

$$\Sigma : 0 \to \mathcal{M} \to \mathcal{A} \to \mathcal{A}/\mathcal{M} \to 0$$

is admissible and thus $\mathcal{M}$ has a Banach space complement. By Theorem 2.3, $\mathcal{A}$ is $*$-isomorphic to a finite direct sum of full matrix algebras.

Consider now the general case where $\mathcal{A}$ is a Hermitian Banach $*$-algebra with trivial cohomology. Let $\mathrm{T}(\mathcal{A})$ be the set of all states of $\mathcal{A}$ and let $R^*(\mathcal{A})$ be the $*$-radical of $\mathcal{A}$, i.e., the intersection of the kernels of all $*$-representations of $\mathcal{A}$ on Hilbert spaces. Since $\mathcal{A}$ is Hermitian and has an identity, $\mathrm{T}(\mathcal{A}) \neq \emptyset$ and so $R^*(\mathcal{A}) \neq \mathcal{A}$. Let $\tau \in \mathrm{T}(\mathcal{A})$ and $\mathcal{I}_\tau = \{a \in \mathcal{A} : \tau(b^*a) = 0 \text{ for all } b \in \mathcal{A}\}$. As shown, τ is positive, i.e $\tau(a^*a) \geq 0$ for all $a \in \mathcal{A}$. It is easy to check that $X_\tau = \mathcal{A}/\mathcal{I}_\tau$ is a pre-Hilbert space with respect to the induced inner product:

$$\langle a + \mathcal{I}_\tau, b + \mathcal{I}_\tau \rangle = \tau(b^*a).$$

For each $a \in \mathcal{A}$ define a linear operator $\theta_\tau(a)$ on X_τ by

$$\theta_\tau(a)(b + \mathcal{I}_\tau) = ab + \mathcal{I}_\tau.$$

Since $\mathcal{I}_\tau$ is a left ideal, $\theta_\tau(a)$ is well defined. It is easy to prove that $a \to \theta_\tau(a)$ is $*$-representation of $\mathcal{A}$ on X_τ.

Let H_τ be the Hilbert space completion of $\mathcal{A}/\mathcal{I}_\tau$ and let $\pi_\tau(a)$ denote the unique extension of $\theta_\tau(a)$ to a bounded linear operator on H_τ. Hence $a \to \pi_\tau(a)$ is a $*$-representation of $\mathcal{A}$ on H_τ. Let $\pi = \bigoplus_{\tau \in \mathrm{T}(\mathcal{A})} \pi_\tau$ and $H = \bigoplus_{\tau \in \mathrm{T}(\mathcal{A})} H_\tau$. Then π is a $*$-representation of $\mathcal{A}$ on H. Consider

$$\|\pi(a)\| = \sup_{\tau \in \mathrm{T}(\mathcal{A})} \|\pi_\tau(a)\|.$$

Then $\|\cdot\|$ is a C^*-norm on $\pi(\mathcal{A})$. Moreover, let $\mathcal{B}$ denote the closure of $(\pi(\mathcal{A}), \|\cdot\|)$, then $\pi : \mathcal{A} \to \mathcal{B}$ is a continuous mapping into the C^*-algebra $\mathcal{B}$ such that $\ker(\pi) = R^*(\mathcal{A})$. The Banach algebra $\mathcal{A}$ has trivial cohomology. By Proposition 2.1, the C^*-algebra $\mathcal{B}$ has also trivial cohomology and thus, it has to be finite dimensional. Notice that $\mathcal{A}/R^*(\mathcal{A})$ is $*$-isometric to the $*$-subalgebra $\pi(\mathcal{A})$ of $\mathcal{B}$. Thus, it follows that $\mathcal{A}/R^*(\mathcal{A})$ is finite-dimensional. Since $R^*(\mathcal{A})$ is a finite-codimensional closed two-sided $*$-ideal, there exists a closed two-sided ideal $\mathcal{K}$ such that

$$\mathcal{A} = R^*(\mathcal{A}) \oplus \mathcal{K}.$$

Next, note that $\|\pi(a)\|^2 = \sup\{\tau(a^*a), \quad \tau \in \mathrm{T}(\mathcal{A})\} \geq |a^*a|_\sigma$ where $|a|_\sigma$ is the spectral radius of $a \in \mathcal{A}$. By Pták [5], we obtain $\|\pi(a)\|^2 \geq |a|_\sigma^2$. So, if $a \in R^*(\mathcal{A})$, then $|a|_\sigma = 0$. Therefore every element of $R^*(\mathcal{A})$ is quasinilpotent. Notice that in general $\mathrm{Rad}(\mathcal{A}) \subseteq R^*(\mathcal{A})$. Since $R^*(\mathcal{A})$ is a closed two-sided $*$-ideal, we have $R^*(\mathcal{A}) = \mathrm{Rad}(\mathcal{A})$ and so, $\mathcal{A}$ is finite-dimensional and semisimple.

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Rachid El Harti, University Hassan I, FST de Settat, BP 577, 2600 Settat, Morocco

E-mail address: `elharti@ibnsina.uh1.ac.ma`