Drawing the double circle on a grid of minimum size

S. Bereg*[1], R. Fabila-Monroy†[2], D. Flores-Peñaloza‡[3], M.A. Lopez§[4], and P. Pérez-Lantero¶[5]

[1]Department of Computer Science, University of Texas at Dallas, USA.
[2]CINVESTAV, Instituto Politécnico Nacional, Mexico.
[3]Departamento de Matemáticas, Facultad de Ciencias, UNAM, Mexico.
[4]Department of Computer Science, University of Denver, USA.
[5]Escuela de Ingeniería Civil en Informática, Universidad de Valparaíso, Chile.

Abstract

In 1926, Jarník introduced the problem of drawing a convex n-gon with vertices having integer coordinates. He constructed such a drawing in the grid $[1, c\cdot n^{3/2}]^2$ for some constant $c > 0$, and showed that this grid size is optimal up to a constant factor. We consider the analogous problem of drawing the double circle, and prove that it can be done within the same grid size. Moreover, we give an $O(n \log n)$-time algorithm to construct such a point set.

1 Introduction

Given $n \geq 3$, a *double circle* is a set $P = \{p_0, p_1, \ldots, p_{n-1}, p'_0, p'_1, \ldots, p'_{n-1}\}$ of $2n$ planar points in general position such that: (1) $p_0, p_1, \ldots, p_{n-1}$ are precisely the vertices of the convex hull of P labelled in counterclockwise order around the boundary; (2) point p'_i is close to the segment joining p_i with p_{i+1}; (3) the line passing through p_i and p'_i separates p_{i+1} from P; and (4) the line passing through p'_i and p_{i+1} separates p_i from P (see Figure 1). Subindices are taken modulo n. The double circle has been considered in combinatorial geometry and it is conjectured to have the least number of triangulations [1, 2].

Drawing an n-vertex convex polygon with integer vertices can be easily done by considering the n points $(1, 1), (2, 4), (3, 9), \ldots, (n, n^2)$ as the vertices of the polygon. In this case the *size* of the integer point set is equal to $n^2 - 1 = \Theta(n^2)$, where size refers to the smallest N such that the point set can be translated to lie in the grid $[0, N]^2$. In 1926, Jarník [5] showed how to draw an n-vertex convex polygon with size $N = O(n^{3/2})$ and proved that this bound is optimal. In recent years the so-called Jarník polygons and extensions of them have been studied [3, 6].

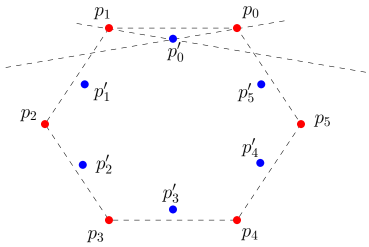

Figure 1: A double circle of twelve points.

Given any integer point (i, j), we say that (i, j) is *visible* (from the origin) if the interior of the line segment joining the origin and (i, j) contains no lattice points. Observe that (i, j) is visible if and only if $\gcd(i, j) = 1$, where $\gcd(i, j)$ denotes the greatest common divisor of i and j. We consider points as vectors as well, and vice versa. A Jarník polygon is formed by choosing a natural number Q, and taking the set V_Q of visible vectors (i, j) such that $\max\{|i|, |j|\} \leq Q$ [4, 5, 6]. The polygon is then the unique (up to translation) convex polygon whose edges, viewed as vectors, are precisely the elements of V_Q, that is, the vertices can be obtained by starting from an arbitrary point and adding the vectors of V_Q, one by one, in counterclockwise order, to the previously computed vertex (see Figure 2).

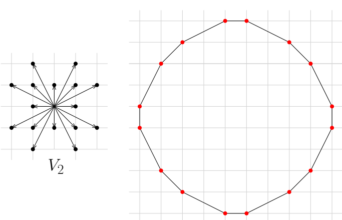

Figure 2: A Jarník polygon (right) and its generating vectors V_2 (left).

We study how to draw a $2n$-point double circle with integer points using the smallest size N. We present

*Email: besp@utdallas.edu.
†Email: ruyfabila@math.cinvestav.edu.mx. Partially supported by grant 153984 (CONACyT, Mexico).
‡Email: dflorespenaloza@gmail.com. Partially supported by grants 168277 (CONACyT, Mexico) and IA102513 (PAPIIT, UNAM, Mexico).
§Email: mlopez@du.edu.
¶Email: pablo.perez@uv.cl. Partially supported by grant CONICYT, FONDECYT/Iniciación 11110069 (Chile).

an $O(n \log n)$-time algorithm that correctly constructs the double circle with size within $O(n^{3/2})$, where that bound is also optimal. In Section 2 we show our algorithm, and in Section 3 its correctness is proved. Finally, in Section 4, we state future work.

2 Double circle construction

Observe that a simple construction with quadratic size is as follows: Consider the function $f(x) = x^2 + x$. For $i = 1, \ldots, 2n-1$, add the point $(i, f(i))$ if i is odd, and the point $(i, f(i)+2)$ otherwise. The final point is $(n, \frac{f(2n-1)+f(1)}{2} - 1) = (n, 2n^2 - n)$, i.e., the point just below the midpoint of the segment connecting $(1, f(1))$ and $(2n-1, f(2n-1))$. The size of the resulting point set is $N = f(2n-1) - f(1) = (2n-1)^2 + (2n-1) - 2 = 4n^2 - 2n - 2 = \Theta(n^2)$.

We say that a sequence V of vectors is *symmetric* if V contains an even number of vectors sorted counterclockwise around the origin, and for every vector a in V its opposite vector $-a$ is also in V. Observe that any sequence of vectors defining a Jarník polygon is symmetric. For any sequence $V = [v_1, v_2, \ldots, v_{2t}]$ of $2t$ vectors let the point set $\mathcal{P}(V) := \{p_1, p_2, \ldots, p_{2t}\}$, where $p_1 = v_1$ and $p_i = p_{i-1} + v_i$ for $i = 2, \ldots, 2t$. Note that if we sort the elements of V_Q around the origin then the elements of $\mathcal{P}(V_Q)$ are the vertices of the Jarník polygon. Furthermore, if V is symmetric then the elements of $\mathcal{P}(V)$ are in convex position. Let sequence $\text{alt}(V) := [v_2, v_1, v_4, v_3, \ldots, v_{2t}, v_{2t-1}]$ (see Figure 3 for an example with $t = 8$). For any scalar λ let the sequence $\lambda V := [\lambda v_1, \lambda v_2, \ldots, \lambda v_{2t}]$.

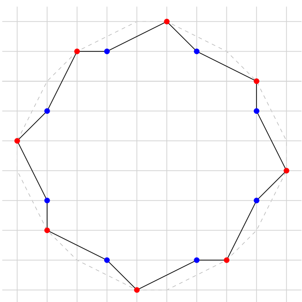

Figure 3: $\mathcal{P}(\text{alt}(V_4))$.

The idea is to generate a suitable symmetric sequence V of $2n$ vectors and then build the point set $\mathcal{P}(\text{alt}(V))$ as the double circle point set, up to some transformation of the elements of $\text{alt}(V)$. A (not optimal) example is $V = [(1,1), (1,2), \ldots, (1,n), (-1,-1), (-2,-2), \ldots, (-1,n)]$ for even $n \geq 4$. The point set $\mathcal{P}(\text{alt}(V))$ is in fact a double circle but its size is equal to $1 + 2 + \ldots + n = \Theta(n^2)$ (see Figure 4).

The construction in which the resulting point set is a double circle of size $O(n^{3/2})$ is based on the next

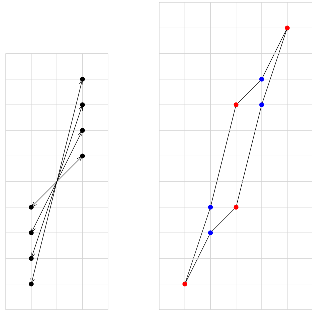

Figure 4: A naive construction for $n = 4$ showing both vectors (left) and the resulting point set (right).

two algorithms:

VisibleVectors(n): With input $n \geq 3$, the symmetric sequence V of $2n$ visible vectors, sorted counterclockwise around the origin, is generated so as to satisfy the next two invariants. Let $\mathrm{B}_t := \{p \in \mathbb{Z}^2 : \|p\|_1 \leq t\}$, $k := \max_{v \in V} \|v\|_1$, and (even) s be the number of visible vectors of B_{k-1}: (i) all visible vectors of B_{k-1} are in V, and (ii) the other elements of V are generated as follows, until $2n - s$ elements are obtained: for $i = 1, \ldots, k-1$ generate vectors $(i, k-i)$, $(-i, -(k-i))$, $(-i, k-i)$, $(i, -(k-i))$ in this order, if and only if $\gcd(i, k-i) = 1$. Refer to Algorithm 2.1 for a pseudo-code.

BuildDoubleCircle(n): With input $n \geq 3$, build a $2n$-point double circle. First, set sequences V :=VisibleVectors(n) and $[v'_1, v'_2, \ldots, v'_{2n}] := \text{alt}(V)$. Then, the sequence $W = [w_1, w_2, \ldots, w_{2n}]$ of $2n$ vectors is created as follows: for $i = 1, 3, \ldots, 2n-1$ set $w_i = (1-\lambda)v'_i + \lambda v'_{i+1}$ and $w_{i+1} = \lambda v'_i + (1-\lambda)v'_{i+1}$, where $\lambda = 1/3$. Finally, build the $2n$-point set $\mathcal{P}((1/\lambda)W)$ as the double circle.

Algorithm 2.1: VisibleVectors(n)

$k \leftarrow 1, V \leftarrow [(1,0), (-1,0), (0,1), (0,-1)]$
repeat
  $k \leftarrow k + 1$
  for $i \leftarrow 1$ **to** $k - 1$
  do
    $j \leftarrow k - i$
    if $\text{GCD}(i, j) = 1$
    then
      if $\text{LENGTH}(V) < 2n$
        then $V \leftarrow V + [(i,j), (-i,-j)]$
      if $\text{LENGTH}(V) < 2n$
        then $V \leftarrow V + [(-i,j), (i,-j)]$
until $\text{LENGTH}(V) = 2n$
Sort V counterclockwise around origin
return (V)

3 Construction correctness

Let $V = [v_1, v_2, \ldots, v_{2n}]$ be the (circular) sequence of vectors obtained by executing VisibleVectors(n), for $n \geq 3$. For every $i = 1, 3, 5, \dots, 2n-1$ we say that the pair of vectors v_i, v_{i+1} is a pair of alt(V).

Lemma 1 (Chapter 2 of [4]) *Given a natural number Q, the number $|V_Q|$ of vertices of the Jarník polygon is equal to*

$$4 + 4 \sum_{\substack{i=1 \\ gcd(i,j)=1}}^{Q} \sum_{j=1}^{Q} 1 = \frac{24Q^2}{\pi^2} + O(Q \log Q).$$

The size $S(Q)$ of the Jarník polygon is equal to

$$\begin{aligned} 1 + 2 \sum_{\substack{i=1 \\ gcd(i,j)=1}}^{Q} \sum_{j=1}^{Q} i &= 1 + 2 \sum_{\substack{i=1 \\ gcd(i,j)=1}}^{Q} \sum_{j=1}^{Q} j \\ &= \frac{6Q^3}{\pi^2} + O(Q^2 \log Q). \end{aligned}$$

Lemma 2 *V is symmetric and point set $\mathcal{P}(V)$ has size $O(n^{3/2})$.*

Proof. Observe that for every vector a in V, $-a$ is also in V since in algorithm VisibleVectors the vectors are added to sequence V in pairs, and each pair consists of two opposite vectors. Then V becomes symmetric once the elements of V are sorted counterclockwise around the origin. On the other hand $V_{\lfloor \frac{k-1}{2} \rfloor} \subset V \subset V_k$, where $k = \max_{v \in V} \|v\|_1$. Then we have $|V_{\lfloor \frac{k-1}{2} \rfloor}| \leq 2n \leq |V_k|$, which implies $k = \Theta(\sqrt{n})$ by Lemma 1. By the same lemma we obtain:

$$\begin{aligned} \sum_{i=1}^{n} x(v_i), \sum_{i=1}^{n} y(v_i) &< 1 + 2 \sum_{\substack{i=1 \\ gcd(i,j)=1}}^{k} \sum_{j=1}^{k} i \\ &= S(k) \\ &= \Theta(k^3) = \Theta(n^{3/2}) \end{aligned}$$

Hence, the size of $\mathcal{P}(V)$ is $O(n^{3/2})$. □

Let o denote the origin of coordinates. Given two points p, q let $\ell(p, q)$ denote the line passing through p and q and *directed* from p to q, and pq denote the segment joining p and q. Given three points $p = (x_p, y_p)$, $q = (x_q, y_q)$, and $r = (x_r, y_r)$, let $\Delta(p, q, r)$ denote the triangle with vertices at p, q, and r; $A(p, q, r)$ denote de area of $\Delta(p, q, r)$; and turn(p, q, r) denote the so-called *geometric turn* (going from p to r passing through q) where

$$\text{turn}(p, q, r) = \begin{vmatrix} x_p & y_p & 1 \\ x_q & y_q & 1 \\ x_r & y_r & 1 \end{vmatrix}$$

and $A(p, q, r) = \frac{1}{2} |\text{turn}(p, q, r)|$. Extending this notation, let $\Delta(p, q) := \Delta(o, p, q)$, $A(p, q) := A(o, p, q)$, and turn$(p, q) :=$ turn(o, p, q). We use the so-called Pick's theorem:

Theorem 3 (Pick's theorem [7]) *The area of any simple polygon H with lattice vertices is equal to $i + b/2 - 1$, where i and b are the numbers of lattice points in the interior and the boundary of H, respectively.*

Lemma 4 *For every two consecutive vectors a_1, a_2 of V we have $A(a_1, a_2) = 1/2$.*

Proof. Suppose $\Delta(a_1, a_2)$ contains a lattice point p different from o, a_1, and a_2. Then p cannot belong to segments oa_1 and oa_2, and segment op contains a visible point q (possibly equal to p). If $\|q\|_1 < \max\{\|a_1\|_1, \|a_2\|_1\}$ then q must belong to V by invariant (i) of algorithm VisibleVectors. Otherwise, we have $\|q\|_1 = \max\{\|a_1\|_1, \|a_2\|_1\}$. Suppose w.l.o.g. that $\|a_1\|_1 < \|a_2\|_1$, and let the point q' denote the intersection of $\ell(o, q)$ with the segment s connecting a_1 to a_2. Observe that $q' = \delta a_1 + (1-\delta) a_2$ for some $\delta \in (0, 1)$, and further that $\|q\|_1 \leq \|q'\|_1 = \|\delta a_1 + (1-\delta) a_2\|_1 \leq \delta \|a_1\|_1 + (1-\delta)\|a_2\|_2 < \|a_2\|_1$, which is a contradiction. Then we must have that $\|q\|_1 = \|a_1\|_1 = \|a_2\|_1$, which implies that q, a_1, a_2 belong to a same quadrant since in this case q is at the interior of the segment s. Therefore, q must belong to V by invariant (ii) of algorithm VisibleVectors. In both cases, the fact that q belongs to V contradicts the fact that a_1 and a_2 are consecutive vectors of V. Hence $A(a_1, a_2) = 1/2$ by Pick's theorem. □

Let $\lambda \in (0, 1/2)$. Given a pair a, b of vectors let $h(\lambda, a, b) := (1-\lambda)a + \lambda b$ (see Figure 5).

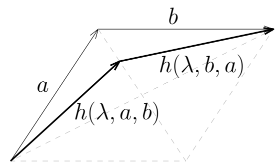

Figure 5: Two vectors a and b, and the vectors $h(\lambda, a, b)$ and $h(\lambda, b, a)$.

Lemma 5 *Let a_1, a_2, a_3, a_4 be four consecutive vectors of V such that a_1, a_2 and a_3, a_4 are pairs of* alt(V). *Let $\lambda \in (0, 1/2)$, $q_1 = h(\lambda, a_2, a_1)$, $q_2 = q_1 + h(\lambda, a_1, a_2)$, $q_3 = q_2 + h(\lambda, a_4, a_3)$, and $q_4 = q_3 + h(\lambda, a_3, a_4)$. Then q_2 is to the right of $\ell(o, q_1)$ and both q_3 and q_4 are to the left of $\ell(o, q_1)$.*

Proof. (Refer to Figure 6.) We have turn$(q_1, q_2) =$ turn$(q_1, q_1 + h(\lambda, a_1, a_2)) =$ turn$((1-\lambda)a_2 + \lambda a_1, a_1 + a_2) = (1-\lambda)$ turn$(a_2, a_1) + \lambda$ turn$(a_1, a_2) = 2(2\lambda - 1)A(a_1, a_2) < 0$, which implies that q_2 is to the right of the line $\ell(o, q_1)$. On the other hand:

$$\begin{aligned} & \text{turn}(q_1, q_3) \\ = \; & \text{turn}(h(\lambda, a_2, a_1), h(\lambda, a_2, a_1) + h(\lambda, a_1, a_2) + \\ & h(\lambda, a_4, a_3)) \\ = \; & \text{turn}(h(\lambda, a_2, a_1), h(\lambda, a_1, a_2)) + \\ & \text{turn}(h(\lambda, a_2, a_1), h(\lambda, a_4, a_3)) \end{aligned}$$

$$
\begin{aligned}
&= \operatorname{turn}((1-\lambda)a_2+\lambda a_1,(1-\lambda)a_1+\lambda a_2))+\\
&\quad \operatorname{turn}((1-\lambda)a_2+\lambda a_1,(1-\lambda)a_4+\lambda a_3))\\
&= (1-\lambda)^2\operatorname{turn}(a_2,a_1)+\lambda^2\operatorname{turn}(a_1,a_2)+\\
&\quad (1-\lambda)^2\operatorname{turn}(a_2,a_4)+\lambda(1-\lambda)\operatorname{turn}(a_2,a_3)+\\
&\quad \lambda(1-\lambda)\operatorname{turn}(a_1,a_4)+\lambda^2\operatorname{turn}(a_1,a_3)\\
&= 2\Big((2\lambda-1)A(a_1,a_2)+(1-\lambda)^2A(a_2,a_4)+\\
&\quad \lambda(1-\lambda)A(a_2,a_3)+\lambda(1-\lambda)A(a_1,a_4)+\\
&\quad \lambda^2A(a_1,a_3)\Big)\\
&= 2\Big(\frac{1}{2}(2\lambda-1)+(1-\lambda)^2A(a_2,a_4)+ \qquad (1)\\
&\quad \lambda(1-\lambda)A(a_2,a_3)+\lambda(1-\lambda)A(a_1,a_4)+\\
&\quad \lambda^2A(a_1,a_3)\Big)\\
&\geq (2\lambda-1)+(1-\lambda)^2+\lambda(1-\lambda)+ \qquad (2)\\
&\quad \lambda(1-\lambda)+\lambda^2\\
&= 2\lambda>0
\end{aligned}
$$

where equation (1) follows from Lemma 4 and equation (2) follows from the fact that by Pick's theorem the area of any non-empty triangle with lattice vertices is at least 1/2. Therefore, q_3 is to the left of $\ell(o,q_1)$. Similarly, since we have that $\operatorname{turn}(a_i,a_j)>0$ ($i=1,2$; $j=3,4$) then $\operatorname{turn}(h(\lambda,a_2,a_1),h(\lambda,a_3,a_4))>0$, which implies that q_4 is to the left of $\ell(o,q_1)$ given that q_3 is to the left of $\ell(o,q_1)$. By symmetry, it can be proved that $\operatorname{turn}(q_4,q_3,q_1)<0$ and $\operatorname{turn}(q_4,q_3,o)<0$, implying that both q_1 and o are to the right of $\ell(q_4,q_3)$. ☐

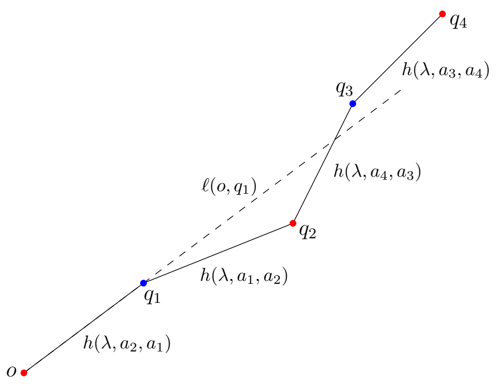

Figure 6: Proof of Lemma 5.

Theorem 6 *There is an $O(n\log n)$-time algorithm that for all $n\geq 3$ builds a double circle of $2n$ points in the grid $[0,N]^2$ where $N=O(n^{3/2})$.*

Proof. Execute the algorithm BuildDoubleCircle with input n, being V the result of calling VisiblePoints(n), building the point set P of $2n$ points. Observe that $\lambda=1/3$ implies that point $w_i/\lambda=3w_i$ is integer for $i=1\dots 2n$, and then all elements of P are integer points. By Lemma 5 point set P is a double circle. The size of $\mathcal{P}(V)$ is $O(n^{3/2})$ by Lemma 2, and since all elements of P belong to the polygon with vertices $\mathcal{P}(3V)$ the size N of P is also $O(n^{3/2})$. Finally, translate P to lie in the grid $[0,N]^2$. In algorithm VisiblePoints the time complexity is dominated by: (1) computing $\gcd(i,j)$ for $O((\sqrt{n})^2)=O(n)$ pairs i,j; and (2) sorting vectors V counterclockwise around the origin. In Case (1) the time complexity is $O(n\log n)$ since $\gcd(i,j)$ consumes $O(\log(\min\{i,j\}))=O(\log\sqrt{n})=O(\log n)$ time. Case (2) consumes $O(n\log n)$ time as well. Since the time complexity of VisiblePoints dominates the time complexity of the main algorithm BuildDoubleCircle, the result follows. ☐

4 Future work

We are working on extending these results to build other known point sets in integer points of small size, such as the double convex chain, the Horton set, and others. We plan to eventually release a software library supporting many of these constructions.

Acknowledgements

The problem studied here were introduced and partially solved during a visit to University of Valparaiso funded by project CONICYT Fondecyt/Iniciación 11110069 (Chile). The authors would like to thank anonymous referees for their valuable comments.

References

[1] O. Aichholzer, F. Hurtado, and M. Noy. A lower bound on the number of triangulations of planar point sets. *Computational Geometry*, 29(2):135–145, 2004.

[2] O. Aichholzer, D. Orden, F. Santos, and B. Speckmann. On the number of pseudo-triangulations of certain point sets. *Journal of Combinatorial Theory, Series A*, 115(2):254–278, 2008.

[3] I. Bárány and N. Enriquez. Jarník's convex lattice n-gon for non-symmetric norms. *Mathematische Zeitschrift*, 270:627–643, 2012.

[4] M. N. Huxley. *Area, lattice points, and exponential sums*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1996.

[5] V. Jarník. Über die Gitterpunkte auf konvexen Kurven. *Mathematische Zeitschrift*, 24:500–518, 1926.

[6] G. Martin. The Limiting Curve of Jarník's Polygons. *Transactions of the American Mathematical Society*, 355(12):4865–4880, 2003.

[7] G. Pick. Geometrisches zur Zahlenlehre. *Sitzungsberichte des Deutschen Naturwissenschaftlich-Medicinischen Vereines für Böhmen "Lotos" in Prag.*, 19:311–319, 1899.