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**Weighted estimates for multilinear maximal
functions and singular integral operators**

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**Weighted estimates for multilinear maximal
functions and singular integral operators**

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*A Gabriel,
por ser quien endulza la sal
y hace que salga el sol.*

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*Es la posibilidad de realizar un sueño
lo que hace la vida interesante*

El Alquimista, Paulo Coelho

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Abstract

The main purpose of this dissertation is the study of weighted norm inequalities for different operators in harmonic analysis in two different settings.

On one hand, the first part of this work is focused on the study of strong and weak weighted inequalities for Calderón–Zygmund operators in spaces of homogeneous type, which generalize the classic situation in the Euclidean space \mathbb{R}^n . Our goal is to obtain mixed bounds formed by at least two different A_p constants so that these mixed estimates are strictly smaller than the classic one-constant bounds. We also derive generalizations of the so-called John–Nirenberg inequality with precise constants and a sharp reverse Hölder inequality for A_∞ weights in cubes in order to get sharp bounds for Calderón–Zygmund operators and commutators of these operators with *BMO* symbols.

On the other hand, the last part of this monograph is devoted to the study of weighted inequalities and the determination of the sharp bounds for the multilinear maximal function and multilinear singular integral operators. First, we derive a sharp mixed $A_{\vec{p}} - A_\infty$ bound for the multilinear maximal function as well as other results that generalize some known one-weight and two-weight results to the multiple setting. Besides, we prove a control in norm of multilinear Calderón–Zygmund operators and multilinear singular integral operators with non-smooth kernels by multilinear sparse operators. As a consequence of these results we also give an analogue of the A_2 theorem in the multilinear context for both classes of operators.

Next, we study the improvement of the boundedness to the stronger condition of compactness of the commutators of different families of operators with symbols in a subspace of *BMO*. First, we will focus in the study of compactness of commutators of a class of bilinear operators that extends the case of bilinear Calderón–Zygmund operators. We also study the case of the commutators of a more singular family of bilinear fractional integrals that can be seen as fractional versions of the bilinear Hilbert transform. Finally, we also determine the classes of multiple weights for which compactness of commutators of bilinear Calderón–Zygmund operators and their iterates in weighted Lebesgue spaces still hold.

Resumen

El principal objetivo de esta tesis es el estudio de desigualdades con pesos para diferentes operadores del análisis armónico en dos ambientes diferentes: los espacios de tipo homogéneo y en el contexto euclídeo multilineal.

La primera parte del presente trabajo se centra en el estudio de acotaciones fuertes y débiles con pesos de operadores de Calderón–Zygmund que generalizan a espacios de tipo homogéneo la situación del espacio euclídeo \mathbb{R}^n . Nuestro objetivo es la obtención de cotas formadas por al menos dos constantes A_p diferentes de manera que estas cotas mixtas sean estrictamente más pequeñas que las clásicas formadas por una constante. Asimismo, se generalizan desigualdades como la de John–Nirenberg y la desigualdad de Hölder al revés, las cuales serán herramientas fundamentales de cara a determinar acotaciones óptimas para los operadores de Calderón–Zygmund y los conmutadores de estos operadores con funciones de BMO .

La segunda parte de esta monografía se centra en el estudio de las desigualdades con pesos para la función maximal multilineal y operadores integrales singulares multilineales así como la determinación de las constantes óptimas para la acotación de dichos operadores. Con respecto al problema de determinación de las constantes óptimas para la función maximal multilineal, se consigue una cota mixta que mezcla la constante múltiple $A_{\vec{p}}$ con un producto de constantes A_{∞} . También se extienden total o parcialmente otros resultados en el contexto múltiple de uno y dos pesos como los teoremas de S. Buckley y E. Sawyer, respectivamente. Asimismo, se establece el control en norma de los operadores multilineales de Calderón–Zygmund e integrales singulares multilineales con núcleos no suaves por operadores multilineales de tipo *sparse*. Como consecuencia de este resultado se deriva un análogo del teorema A_2 para ambos tipos de operadores.

Finalmente, se estudia la compacidad de los conmutadores de diferentes operadores con símbolos en un subespacio de BMO . Por un lado, el estudio se centra en los conmutadores de una clase de operadores bilineales que extiende el caso de los operadores de Calderón–Zygmund. También se estudia el caso de los conmutadores de una familia de operadores bilineales fraccionarios más singulares que pueden verse como la versión fraccionaria de la transformada de Hilbert bilineal. Por otro lado, se estudian las clases de pesos múltiples para los cuales se tiene que los conmutadores de operadores de Calderón–Zygmund bilineales son compactos en espacios de Lebesgue con pesos.

Contents

Introduction	xiii
1 Preliminaries	1
1.1 Lebesgue spaces	1
1.2 Dyadic grids and sparse families	3
1.3 Main operators	5
1.3.1 Maximal operators	6
1.3.2 Singular integral operators	7
1.3.3 Fractional integral operators	9
1.3.4 Commutators with <i>BMO</i> functions	10
1.4 Weights	11
1.5 <i>BMO</i> and John-Nirenberg inequality	15
1.6 Local mean oscillation decomposition	16
2 A_1 theory on spaces of homogeneous type	19
2.1 Basics on spaces of homogeneous type	19
2.2 Useful inequalities in spaces of homogeneous type	24
2.2.1 Sharp reverse Hölder inequality for A_∞ weights	24
2.2.2 John-Nirenberg inequality and related lemmas	27
2.3 Strong estimates for Calderón–Zygmund operators	30
2.4 Weak estimates for Calderón–Zygmund operators	38
2.5 Sharp estimates for commutators	43
3 Weighted bounds for the multisublinear maximal operator	47
3.1 Some basics on multilinear theory of weights	47
3.2 One-weight estimates	49
3.2.1 Sharp $A_{\vec{p}} - A_\infty$ multilinear estimate	49
3.2.2 Buckley’s multilinear theorem	54
3.3 Two-weight estimates	57
3.3.1 A multilinear Carleson lemma	57
3.3.2 A multilinear Sawyer’s theorem	59
3.3.3 A multilinear B_p theorem	63
3.3.4 A mixed $A_{\vec{p}} - W_{\vec{p}}^\infty$ theorem	64

3.3.5	A sufficient condition for the two-weighted boundedness of \mathcal{M}	66
4	Weighted bounds for multilinear singular integral operators	69
4.1	Basics on multilinear Calderón–Zygmund operators	69
4.2	Local mean oscillation estimate for multilinear Calderón–Zygmund operators	72
4.3	Control by multilinear sparse operators	74
4.4	A_2 theorem for multilinear Calderón–Zygmund operators	77
4.5	Basics on multilinear generalized Calderón–Zygmund operators	79
4.6	A_2 theorem for multilinear generalized Calderón–Zygmund operators	84
4.7	Further remarks and open questions	88
5	Compactness of commutators of bilinear singular integrals	91
5.1	Basics on compactness properties of bilinear operators	91
5.2	Compactness for commutators of the class $\{T_\alpha\}$	94
5.3	Separate compactness for commutators of the class $\{BI_\alpha\}$	99
5.4	Compactness of bilinear CZO: weighted case	106
	References	121
	List of symbols and notation	131

Introduction

The origin of the modern theory of weighted inequalities can be traced back to the works of R. Hunt, B. Muckenhoupt, R. Wheeden, R. Coifman, and C. Fefferman in the decade of the 70's. The basic problem concerning weighted inequalities consists in determining under which conditions a given operator, initially bounded on $L^p(\mathbb{R}^n)$, is bounded on $L^p(\mathbb{R}^n, \mu)$, where μ is an absolutely continuous measure with respect to Lebesgue measure, i.e. $d\mu = w dx$. Here, w denotes a non-negative locally integrable function on \mathbb{R}^n that is positive almost everywhere, that is called a *weight*.

A sustained research period was started with the groundbreaking work of Muckenhoupt [91]. In this work he characterized the class of weights u, v for which the following weak inequality for the Hardy–Littlewood maximal operator and for $1 \leq p < \infty$ holds

$$(1) \quad \sup_{\lambda > 0} \lambda^p \int_{\{Mf > \lambda\}} u(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) dx, \quad f \in L^p(v).$$

This condition on the weights is known as A_p condition, namely

$$[u, v]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q u(x) dx \right) \left(\frac{1}{|Q|} \int_Q v(x)^{-\frac{1}{p-1}} \right)^{p-1} < \infty, \quad p > 1,$$

where the supremum is taken over all the cubes in \mathbb{R}^n . Note that when $p = 1$, the term $(\int_Q v(x)^{-\frac{1}{p-1}})^{p-1}$ must be understood as $(\inf_Q v)^{-1}$. Although weights in the A_p class are also known as Muckenhoupt weights, it is worth mentioning that variant of this condition was previously introduced by Rosenblum in [102]. In the particular case $u = v$ and $p > 1$, Muckenhoupt also proved that the following strong estimate

$$\int_{\mathbb{R}^n} (Mf(x))^p v(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) dx, \quad f \in L^p(v),$$

holds if and only if v satisfies the A_p condition.

From that point on, the interest of harmonic analysts focused on studying weighted inequalities for the classical operators such as the Hilbert and Riesz transforms and other singular integral operators leading to a wide literature on one-weight norm inequalities.

However, the problem of finding a condition on the weights u, v satisfying the strong estimate above was much more complicated. It was not until 1982 that E. Sawyer [103] characterized the two weight inequality, showing that $M : L^p(v) \rightarrow L^p(u)$ if and only if the pair of weights (u, v) satisfies the following testing condition known as

Sawyer's S_p condition

$$(2) \quad [u, v]_{S_p} = \sup_Q \left(\frac{\int_Q M(\chi_Q \sigma)^p u dx}{\sigma(Q)} \right)^{1/p} < \infty,$$

where $\sigma = v^{1-p'}$ and $1 < p < \infty$. Observe that condition (2) involves the operator under study itself and, for this reason, it is difficult either to check or use it to construct examples of weights for applications. This difficulty together with the fact that these conditions are just defined for particular operators motivated the development of different sufficient conditions, close in form to the A_p condition.

The classical results mentioned so far did not reflect the quantitative dependence of the $L^p(w)$ operator norm in terms of the relevant constant involving the weights since they were qualitative properties. Therefore, the relevant question then was to determine the precise sharp bounds of a given operator in $L^p(w)$, whenever $w \in A_p$.

The first author who studied this problem for the Hardy–Littlewood maximal operator was S. Buckley, a Ph.D. student of R. Fefferman, who proved in [16],

$$(3) \quad \|M\|_{L^p(w)} \leq C p' [w]_{A_p}^{\frac{1}{p-1}},$$

where C is a dimensional constant. We say that the above inequality is sharp in the sense that we cannot replace the exponent on the weight constant by an smaller one. Buckley also proved another quantitative result related to the weak estimate for the Hardy–Littlewood maximal operator as an application of the classical covering lemmas. More precisely,

$$(4) \quad \|M\|_{L^p(w) \rightarrow L^{p,\infty}(w)} \leq C [w]_{A_p}^{1/p},$$

where C is a dimensional constant. In fact, it can be easily proved that the operator norm and the weight constant in (4) are comparable, whereas in (3) this result is false (see [61] for further details).

Following the spirit of Buckley's results, a similar problem was studied by J. Wittwer, another Ph.D. student of R. Fefferman, for the martingale operator and the square function in [113] and [114], respectively. Later on, regarding the two-weight problem for the Hardy–Littlewood maximal function, K. Moen found in [87] a quantitative form of E. Sawyer's result in terms of Sawyer's S_p condition (2). Namely

$$(5) \quad \|M\|_{L^p(v) \rightarrow L^p(u)} \approx [u, v]_{S_p}.$$

Although maximal functions are relevant operators in harmonic analysis, singular integrals are probably the central operators in this field. The term singular integral refers to a wide class of operators that are (formally) defined, as integral operators in the following way

$$Tf(x) = \int K(x, y)f(y)dy,$$

where K is a singular kernel in the sense that it is not locally integrable. The prototype or most representative example of this class of operators is the Hilbert transform in the real line, namely

$$Hf(x) = \frac{1}{\pi}p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$

In the light of the previous results, the relevant problem then was trying to determine the sharp constant in the corresponding weighted inequality for Calderón–Zygmund singular integral operators. Concerning this problem, the next relevant step in this direction was given by K. Astala, T. Iwaniec and E. Saksman in [7]. They studied the Beurling transform (also known as the Ahlfors-Beurling transform) defined as follows

$$Bf(z) = p.v. \int_{\mathbb{C}} \frac{f(w)}{(w-z)^2} dw.$$

This Calderón–Zygmund operator is one of the most important singular integral operators related to complex variables, quasi-conformal mappings and the regularity theory of the Beltrami equation. In fact, in [7] the authors were interested in finding the smallest $q < 2$ such that the solutions of the Beltrami equation

$$\bar{\partial}f = \mu\partial f$$

that belong to the Sobolev space $W_{loc}^{1,q}$ also belong to the better space $W_{loc}^{1,2}$ (i.e. the solutions are quasi-regular). Here μ is a bounded function such that $\|\mu\|_{\infty} = k < 1$. Lately, K. Astala [6] proved that $q > k + 1$ is sufficient. On the other hand, T. Iwaniec and G.J. Martin [65] found examples showing that, in general, the result does not hold for $q < k + 1$.

In [7] the authors also pointed out that in the case $q = k + 1$, the quasi-regularity would be a consequence of a linear bound of $\|B\|_{L^p(w)}$ for $p \geq 2$ in terms of the weight constant. In fact, they conjectured the following bound for the Beurling operator

$$(6) \quad \|B\|_{L^p(w)} \leq c_p[w]_{A_p}, \quad p \geq 2,$$

which was proved by S. Petermichl and A. Volberg in [101]. This conjecture revealed the importance of finding a bound on the norm of a given operator in terms of the

weight constant. Another feature of the theory is the relevance of the case $p = 2$. It is due to the fact that, as a consequence of Rubio de Francia's extrapolation theorem obtained in [37], it suffices to obtain a linear bound in the case $p = 2$ since it is the starting point to derive sharp bounds for all p . We refer the interested reader to [34] for a simpler proof of the precise extrapolation theorem, which was inspired by the work of Duoandikoetxea [39].

The next important advance in this area was due to S. Petermichl [99] who proved the optimal bounds for the Hilbert transform. Shortly after, she extended this result to the Riesz transforms in [100]. Lately, O. Beznosova proved the analogous linear bound for discrete paraproduct operators in [14].

It was then that the so-called A_2 conjecture gathered more importance. This conjecture claimed that the dependence for a Calderón–Zygmund operator will be linear on the A_2 constant, namely

$$(7) \quad \|T\|_{L^2(w)} \leq C[w]_{A_2}.$$

As mentioned before, from (7) it is possible to extrapolate to get the A_p dependence. More precisely,

$$(8) \quad \|T\|_{L^p(w)} \leq C[w]_{A_p}^{\max(1, \frac{1}{p-1})},$$

where the dimensional constant C depends also on p and T .

In 2010, the sharp A_2 bound for a large family of Haar shift operators that included dyadic operators was obtained by M. Lacey, S. Petermichl and M.C. Reguera in [71]. After that, D. Cruz-Uribe, J.M. Martell and C. Pérez proved a more flexible result in [34] that could be applied to many different operators and whose proof avoids Bellman functions as well as two-weight norm inequalities.

After many intermediate results by others, the A_2 conjecture was solved in full generality by T. Hytönen in [58] using a very different and interesting probabilistic approach. Shortly after, A.K. Lerner gave a simpler and beautiful proof in [77] based on the use of dyadic sparse operators and the so-called local mean oscillation formula. Lately, K. Moen [89] derived sharp weighted bounds for sparse operators for all p , $1 < p < \infty$, avoiding the use of extrapolation.

After the solution of the A_2 conjecture, several improvements of this and other results were obtained in [61] by T. Hytönen and C. Pérez. The underlying idea of this work was to replace a portion of the A_2 constant by another smaller constant defined

in terms of the A_∞ constant given by

$$(9) \quad [w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q).$$

This functional was implicitly considered by N. Fujii in [45] to provide a characterization of the A_∞ class of weights and later it was rediscovered by M. Wilson in [112]. It is smaller than the more classical A_∞ condition due to Hrusčev

$$[w]_{A_\infty}^H = \sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \exp \left(\frac{1}{|Q|} \int_Q \log w^{-1} \right),$$

as it was shown in [61] for the particular case of weights of the form $w = t\chi_E + \chi_{\mathbb{R} \setminus E}$ with $t \geq 3$. On the one hand, in [61] an improvement of Buckley's estimate for the Hardy–Littlewood maximal function is proved. Namely, for $p > 1$,

$$(10) \quad \|M\|_{L^p(w)} \leq Cp'([w]_{A_p}[\sigma]_{A_\infty})^{1/p},$$

where C is a dimensional constant and $\sigma = w^{1-p'}$. This result improves significantly Buckley's bound since

$$([w]_{A_p}[w]_{A_\infty})^{1/p} \lesssim ([w]_{A_p}[w]_{A_p}^{\frac{1}{p-1}})^{1/p} \lesssim [w]_{A_p}^{\frac{1}{p-1}}.$$

On the other hand, in [61] the A_2 theorem (as well as its L^p counterpart) was improved obtaining the following mixed sharp $A_2 - A_\infty$ estimate for singular integral operators

$$(11) \quad \|T\|_{L^2(w)} \leq C[w]_{A_2}^{1/2}([w^{-1}]_{A_\infty} + [w]_{A_\infty})^{1/2},$$

which is the starting point for proving analogous sharp bounds for other operators such as commutators and their iterates as well.

Another type of quantitative results can be derived assuming different conditions on the weight. For instance, if $w \in A_1$, in [79] it was proved that

$$(12) \quad \|T\|_{L^p(w)} \leq C_T p p' [w]_{A_1}.$$

Observe that there is a linear growth for all p in the above result since we are assuming stronger conditions on the weight. In particular, there is no blow-up in the exponent as it happens in (8), where the weight is assumed to be in the A_p class. In [79] the sharpness of (12) is proved, as well as a weak $(1, 1)$ estimate where the dependence on the constant is of the form $[w]_{A_1}(\log(e + [w]_{A_1}))$. As mentioned before, the linear

growth has been improved in [61] by using a mixed $A_1 - A_\infty$ bound. This result is better since $[w]_{A_\infty} \leq [w]_{A_1}$. More precisely, it was proved that

$$(13) \quad \|Tf\|_{L^p(w)} \leq C_T p p' [w]_{A_1}^{1/p} [w]_{A_\infty}^{1/p'} \|f\|_{L^p(w)},$$

and

$$(14) \quad \|Tf\|_{L^{1,\infty}(w)} \leq c_T [w]_{A_1} \log(e + [w]_{A_\infty}) \|f\|_{L^1(w)}.$$

Another relevant class of singular integrals was introduced and studied by R. Coifman, R. Rochberg and G. Weiss in [32], motivated by the works of A.P. Calderón on the Cauchy transform along Lipschitz curves. If T is a singular integral operator associated with a kernel K and b is a locally integrable function in \mathbb{R}^n (often called *symbol*), we define the commutator of T with b as follows

$$[T, b]f(x) = \int_{\mathbb{R}^n} (b(y) - b(x))K(x, y)f(y)dy.$$

Although these operators were initially related to the generalizations of the factorization theorem for Hardy spaces in several variables on the unit disk [32], they are interesting for many other reasons such as their applications to partial differential equations.

In [32] it was shown that in the general case, when T is a Calderón–Zygmund operator, $b \in BMO$ is a sufficient condition for $[T, b]f$ to be bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Conversely, if for every $j = 1, \dots, n$, $[R_j, b]$ is bounded on $L^p(\mathbb{R}^n)$ for some p , $1 < p < \infty$, where R_j is the j -th Riesz transform given by

$$R_j f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} p.v. \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} f(x-y)dy, \quad 1 \leq j \leq n,$$

then $b \in BMO$.

Even though commutators behave in some sense as Calderón–Zygmund operators when considering $L^p(\mathbb{R}^n)$ estimates, they are different since $[T, b]$ in general is not of weak type $(1, 1)$ when $b \in BMO$ as observed by C. Pérez in [94]. In fact, C. Pérez proved a weak estimate of type $L(\log L)$ for commutators.

Another interesting phenomenon concerning commutators is the so-called smoothing effect. Indeed, there is an improvement of boundedness to the stronger condition of compactness of these operators when commuting with a special class of symbols. In [108], A. Uchiyama showed that linear commutators of Calderón–Zygmund operators and pointwise multiplication with a symbol belonging to an appropriate subspace of

the John-Nirenberg space BMO are compact. This result improved the boundedness properties obtained by Coifman, Rochberg and Weiss in [32]. Compactness has different applications: deriving a Fredholm alternative for equations with coefficients in appropriate L^p spaces with $1 < p < \infty$ [66]; the theory of compensated compactness of Coifman, Lions, Meyer and Semmes [29]; or the integrability theory of Jacobians [64].

Outline

The aim of this dissertation is to extend the A_1 theory of weights to spaces of homogeneous type as well as to give multilinear analogues of some of the above mentioned results following the spirit of the theory of multiple weights developed in [80].

This work is organized as follows:

In Chapter 1 we present some notions and results in the linear setting and in the context of the Euclidean space \mathbb{R}^n that will lead the reader to a complete understanding of the more general concepts that we are dealing with along this dissertation. More precisely, we start defining our natural framework, the Lebesgue spaces. We also recall the definition and some boundedness properties of the main operators in our study, as well as the definition of the A_p classes of weights and the more relevant constants in this theory. Finally, some basic facts on BMO functions, non-increasing rearrangements and the local mean oscillation formula of Lerner are also listed.

In Chapter 2 we will be working in the more general setting of the spaces of homogeneous type that generalizes the Euclidean situation in \mathbb{R}^n with the Lebesgue measure. The purpose of this chapter is to extend some well-known weak and strong sharp mixed inequalities mentioned before concerning Calderón–Zygmund operators, assuming stronger conditions on the weights, i.e. $w \in A_1$. That is, we want to find bounds for these operators formed by at least two different A_p constants since these mixed estimates are strictly smaller than the original one-constant bound. We also extend (11) in order to prove sharp bounds for commutators of Calderón–Zygmund operators and their iterates with BMO functions. It is worth mentioning the use of the new techniques of Lerner to derive in a simpler manner a Coifman–Fefferman type inequality, which will be very useful to get precise constants. Furthermore, we extend two well-known inequalities to the homogeneous setting: the sharp reverse Hölder

inequality for A_∞ weights in dyadic cubes and a precise version of the John-Nirenberg inequality.

In Chapter 3 we present a variety of results related to the multilinear maximal function. Namely, we generalize the A_p - A_∞ bound presented in [61] to the multilinear setting proving its sharpness. From this result we give some partial results related to the multilinear version of Buckley's bound. We also prove a multilinear Carleson embedding lemma which is the key tool for proving the rest of the results in this chapter. More precisely, a multilinear version of Sawyer's theorem assuming that the multiple weights satisfy a sort of Reverse Hölder property and some multilinear two-weight estimates for the multilinear maximal operator that generalize those proved in [61].

In Chapter 4 we establish the control in norm of multilinear Calderón–Zygmund operators and multilinear singular integral operators with non-smooth kernels by multilinear sparse operators. As an application, we derive a multilinear analogue of the so-called A_2 theorem for both classes of operators. Some related remarks concerning the multilinear version of the A_p theorem as well as some open questions are also listed at the end of this chapter.

In Chapter 5 we study the compactness of commutators of different classes of bilinear singular integrals with symbols in CMO , a subspace of the space of functions of bounded mean oscillation. First, we will concentrate on the study of the compactness of the commutators in a class of bilinear operators that extends the case of bilinear Calderón–Zygmund operators studied in [11]. Second, we will study the commutators of a more singular family of bilinear fractional integrals that can be seen as fractional versions of the bilinear Hilbert transform obtaining that, although the smoothing phenomenon is still present, in this case it is weaker. Finally, as all these compactness results rely on the Fréchet-Kolmogorov-Riesz characterization of precompact sets in unweighted Lebesgue spaces, we also are interested in the case when the Lebesgue measure dx is replaced by an absolutely continuous measure with respect to it. More precisely, we study of the compactness in weighted Lebesgue spaces of commutators of bilinear Calderón–Zygmund operators and their iterates with CMO symbols determining the suitable classes of multiple weights for which it still works.

This dissertation contains results from the articles [3], [21], [35], [36], [10] and [9]. Note that some proofs in Chapter 5 are presented in a different manner from that in the corresponding article in order to provide a different and more comprehensive proof.

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Preliminaries

In this chapter we introduce some of notions and results in the linear setting and in the context of the Euclidean space \mathbb{R}^n that will lead the reader to a complete understanding of the concepts that we are dealing with along this dissertation. It is assumed that they are mostly well known to the reader. For further details, see for instance, [8, 38, 48, 68].

Firstly, we start defining the Lebesgue spaces where our work is mainly developed. The notion of sparseness of a family of cubes is also recalled. Next, we remind the definition and some boundedness properties of the main operators in our study. We also describe the A_p classes of weights as well as the most important weight constants involved in the problem of finding sharp norm bounds. Finally, some basics on *BMO* functions, non-increasing rearrangements and the local mean oscillation formula of Lerner are also listed.

1.1 Lebesgue spaces

In this section we review the definitions and some properties of the main spaces where this thesis is mainly developed. First, we introduce the notions of Lebesgue and weak Lebesgue spaces, as well as some properties of these spaces. We also recall some basic properties of the associate space of a Banach function space that generalize the ones listed for Lebesgue spaces which will be used in the sequel.

Definition 1.1.1. *Let (X, μ) be a measure space.*

1. *The space $L^p(X, \mu)$, $1 \leq p < \infty$, is defined as the set of μ -measurable functions from X to \mathbb{C} whose p -th powers are integrable. The norm of a function f in $L^p(X, \mu)$ is defined by*

$$\|f\|_{L^p(X, \mu)} = \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}.$$

2. *In the particular case when $p = \infty$, $L^\infty(X, \mu)$ denotes the space of essentially bounded functions from X to \mathbb{C} , that is, the collection of functions f for which*

$$\|f\|_{L^\infty(X, \mu)} := \operatorname{ess\,sup}_{x \in X} |f(x)| < \infty,$$

where the essential supremum is defined as follows

$$\operatorname{ess\,sup}_{x \in X} |f(x)| := \inf \{ \alpha > 0 : \mu(\{x \in X : |f(x)| > \alpha\}) = 0 \}.$$

$L^p(X, \mu)$, for $1 \leq p \leq \infty$, are Banach spaces equipped with their corresponding norms. They satisfy the Hölder's inequality. Namely,

$$(1.1) \quad \int_X |f(x)g(x)|d\mu(x) \leq \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p} \left(\int_X |g(x)|^{p'} d\mu(x) \right)^{1/p'},$$

where p' denotes the conjugate exponent of p and satisfy the following relationship

$$(1.2) \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Note that (1.1) is sharp in the sense that

$$(1.3) \quad \|g\|_{L^{p'}(X, \mu)} = \sup \left\{ \int_X |fg|d\mu : f \in L^p, \|f\|_{L^p(X, \mu)} \leq 1 \right\},$$

for all $g \in L^{p'}(X, \mu)$ and for all p and p' satisfying (1.2). Therefore, the space $L^{p'}(X, \mu)$ is described explicitly in terms of $L^p(X, \mu)$. This is not surprising since $L^p(X, \mu)$ and $L^{p'}(X, \mu)$ are associate Banach function spaces. In general, given a Banach function space X , we define its associate space X' that consists of all measurable functions f for which

$$\|f\|_{X'} = \sup_{\|g\|_X \leq 1} \int_{\mathbb{R}^n} |f(x)g(x)|dx < \infty.$$

It is easy to prove that this definition implies the following Hölder inequality

$$(1.4) \quad \int_{\mathbb{R}^n} |f(x)g(x)|dx \leq \|f\|_X \|g\|_{X'}.$$

Furthermore, [8, p. 13],

$$(1.5) \quad \|f\|_X = \sup_{\|g\|_{X'}=1} \int_{\mathbb{R}^n} |f(x)g(x)|dx.$$

By Fatou's lemma [8, p. 5], if $f_n \rightarrow f$ a.e., and if $\liminf_{n \rightarrow \infty} \|f_n\|_X < \infty$, then $f \in X$, and

$$(1.6) \quad \|f\|_X \leq \liminf_{n \rightarrow \infty} \|f_n\|_X.$$

Properties (1.4), (1.5) and (1.6) will be essential in the sequel. A general account of Banach function spaces is addressed in [8, Ch. 1].

Let us remark that Hölder's inequality in Lebesgue spaces could be generalized to act over m functions: consider $m + 1$ exponents, p_1, \dots, p_m, p such that verify the so-called homogeneity or Hölder's condition:

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}.$$

Then

$$(1.7) \quad \|f_1 \cdot \dots \cdot f_m\|_{L^p(X, \mu)} \leq \prod_{i=1}^m \|f_i\|_{L^{p_i}(X, \mu)}.$$

Next, we remind the Minkowski's integral inequality. Let f be an integrable function on the product space $(X, \mu) \times (Y, \nu)$ where μ, ν are σ -finite and $p \geq 1$. Then,

$$(1.8) \quad \left[\int_Y \left(\int_X |f(x, y)| d\mu(x) \right)^p d\nu(y) \right]^{1/p} \leq \int_X \left[\int_Y |f(x, y)|^p d\nu(y) \right]^{1/p} d\mu(x).$$

Other useful spaces in which we will also be working are weak Lebesgue spaces defined as follows.

Definition 1.1.2. For $0 < p < \infty$, the space weak $L^{p, \infty}(X, \mu)$ is defined as the set of all μ -measurable functions from X to \mathbb{C} such that

$$\|f\|_{L^{p, \infty}(X, \mu)} = \sup \{t > 0 : t\mu(\{x \in X : |f(x)| > t\})^{1/p} < \infty\}.$$

Observe that, by definition, the weak $L^\infty(X, \mu)$ space is the space $L^\infty(X, \mu)$. Weak Lebesgue spaces are not Banach spaces but quasi-Banach spaces and they satisfy that $L^p(X, \mu) \subset L^{p, \infty}(X, \mu)$.

In the particular case when X is \mathbb{R}^n or a subset of \mathbb{R}^n and $d\mu = dx$ is the Lebesgue measure, we can omit the measure writing $L^p(\mathbb{R}^n)$ or simply L^p ($L^{p, \infty}$, resp.) instead of $L^p(\mathbb{R}^n, dx)$ ($L^{p, \infty}(\mathbb{R}^n, \mu)$, resp.).

When μ is an absolutely continuous measure with respect to Lebesgue measure, that is, $d\mu = w dx$, we write $L^p(w)$ or $L^{p, \infty}(w)$ to denote the corresponding *weighted Lebesgue space*. The measurable non-negative function w appearing above is called *weight* (see Section 1.4 for more details).

1.2 Dyadic grids and sparse families

In this section we recall the concept of sparseness of a family of cubes within a dyadic grid that will play an important role along this dissertation. We define the *standard*

dyadic grid in \mathbb{R}^n , which we will denote by \mathcal{D} , as the collection of cubes

$$2^{-k}([0, 1]^n + j), \quad k \in \mathbb{Z}, j \in \mathbb{Z}^n.$$

By a *general dyadic grid* \mathcal{D} we mean a collection of cubes with the following properties:

1. For any $Q \in \mathcal{D}$ its sidelength ℓ_Q is 2^k , for some $k \in \mathbb{Z}$.
2. $Q \cap R \in \{Q, R, \emptyset\}$ for any $Q, R \in \mathcal{D}$.
3. The cubes of a fixed sidelength 2^k form a partition of \mathbb{R}^n .

We say that a collection of cubes $\{Q_j^k\}$ is a *sparse family* of cubes if:

1. The cubes Q_j^k are disjoint in j , with k fixed.
2. If $\Omega_k = \cup_j Q_j^k$, then $\Omega_{k+1} \subset \Omega_k$.
3. $|\Omega_{k+1} \cap Q_j^k| \leq \frac{1}{2}|Q_j^k|$.

Observe that we can associate to a sparse family of cubes $\{Q_j^k\}$ a pairwise disjoint family of sets defined as follows:

$$E_j^k = Q_j^k \setminus \Omega_{k+1},$$

satisfying

$$|Q_j^k| \leq 2|E_j^k|.$$

The following property, which can be found in [61, 36], will be useful in the sequel.

Proposition 1.2.1. *There are 2^n dyadic grids \mathcal{D}_α such that for any cube $Q \subset \mathbb{R}^n$ there exists a cube $Q_\alpha \in \mathcal{D}_\alpha$ such that $Q \subset Q_\alpha$ and $\ell_{Q_\alpha} \leq 6\ell_Q$.*

Associated to a sparse family $S \subset \mathcal{D}$, we define a sparse operator as follows

$$(1.9) \quad \mathcal{A}_{S, \mathcal{D}}(f) = \sum_{Q \in S} \left(\int_Q f \right) \chi_Q.$$

As mentioned in the introduction these operators play an important role in showing the sharp bound for Calderón–Zygmund operators (described in Section 1.3 below) within the proof of the A_2 conjecture following Lerner’s approach.

1.3 Main operators

In this section, we recall the definition as well as some boundedness properties of the main operators in the linear setting that we will use in the following as well as to understand the multilinear operators appearing along this dissertation.

Since we will mainly work with linear or sublinear operators that satisfy strong or weak inequalities we consider necessary to remind first that notions.

Definition 1.3.1. *We say that an operator T is:*

1. *Linear if $T(f + g) = T(f) + T(g)$ and $T(\lambda f) = \lambda T(f)$.*
2. *Sublinear if $|T(f + g)| \leq |T(f)| + |T(g)|$ and $|T(\lambda f)| = |\lambda| |T(f)|$,*

for all functions f, g and $\lambda \in \mathbb{C}$.

Definition 1.3.2. *Let (X, μ) and (Y, ν) be measure spaces and let T be an operator defined from $L^p(X, \mu)$ into the space of measurable functions from Y to \mathbb{C} . We say that:*

1. *T is strong (p, q) if $\|Tf\|_{L^q(Y, \nu)} \lesssim \|f\|_{L^p(X, \mu)}$.*
2. *T is weak (p, q) if $\|Tf\|_{L^{q, \infty}(Y, \nu)} \lesssim \|f\|_{L^p(X, \mu)}$.*

It is clear from the definition that a strong (p, q) operator is also a weak (p, q) operator. When $(X, \mu) = (Y, \nu)$ in the above definition of weak (p, p) operator, we get the so called Chebyshev inequality, that is,

$$\mu(\{x \in X : |f(x)| > \lambda\}) \lesssim \left(\frac{\|f\|_{L^p(X, \mu)}}{\lambda} \right)^p.$$

In general, given two Lebesgue spaces $L^p(X, \mu)$ and $L^q(Y, \nu)$, we say that T is a bounded operator from $L^p(X, \mu)$ to $L^q(Y, \nu)$ (and we denote it by $T : L^p(X, \mu) \rightarrow L^q(Y, \nu)$) if for all functions $f \in L^p(X, \mu)$ we have

$$\|T(f)\|_{L^q(Y, \nu)} \leq C \|f\|_{L^p(X, \mu)},$$

where C is a constant. We will denote the operator norm $\|T\|_{L^p(X, \mu) \rightarrow L^q(Y, \nu)}$ by

$$\|T\|_{L^p(X, \mu) \rightarrow L^q(Y, \nu)} := \sup_{\|f\|_{L^p(X, \mu)}=1} \|Tf\|_{L^q(Y, \nu)}.$$

In the particular case when $(X, \mu) = (Y, \nu)$ we will denote the operator norm as $\|T\|_{L^p(X, \mu)}$.

1.3.1 Maximal operators

One of the main objects in harmonic analysis is the well known Hardy–Littlewood maximal function defined as follows

$$(1.10) \quad Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken all over the cubes Q containing $x \in \mathbb{R}^n$. Note that M can be equivalently defined over cubes or balls. Unless otherwise indicated, we will use the definition over cubes.

With respect to its boundedness properties, we have the following result due to Hardy and Littlewood [55] and Wiener [110].

Theorem 1.3.1. *Let $1 < p < \infty$, then M is a strong (p, p) and a weak $(1, 1)$ operator.*

As mentioned before, the Hardy–Littlewood maximal function plays an important role in Calderón–Zygmund theory since in some sense it controls singular integral operators. Besides, this operator is involved in the Lebesgue differentiation theorem stated as follows.

Theorem 1.3.2. *Let f be a locally integrable function in \mathbb{R}^n . Then for a.e. $x \in \mathbb{R}^n$, we have that:*

1. $f(x) = \lim_{r \rightarrow 0} \frac{1}{|Q(x, r)|} \int_{Q(x, r)} f(y) dy.$
2. $|f(x)| \leq Mf(x).$

Although there are several variants of the Hardy–Littlewood maximal function (i.e. centered or uncentered versions, over dyadic cubes, etc.), here we only introduce two of the most important ones. Other maximal functions will be defined when necessary along this memory.

The first maximal function is known as the Fefferman-Stein *sharp maximal function* and it measures the average oscillation of a function over cubes at a certain point.

Definition 1.3.3. *Given a locally integrable function f in \mathbb{R}^n and $x \in \mathbb{R}^n$, we define the sharp maximal function by*

$$(1.11) \quad M^\# f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where f_Q denotes the average of f over the cube Q and the supremum in the above definition is taken all over the cubes Q that contain the point x .

One variation of the above maximal function is the *sharp-delta maximal function*, define as follows

$$(1.12) \quad M_{\delta}^{\sharp} f(x) := (M^{\sharp}(|f|^{\delta})(x))^{1/\delta}.$$

Finally, we define a family of maximal functions that generalizes the Hardy–Littlewood maximal function known as fractional maximal functions.

Definition 1.3.4. *Given $0 \leq \alpha < n$, we define the fractional maximal operator M_{α} by*

$$(1.13) \quad M_{\alpha} f(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| dy,$$

where the supremum is taken all over the cubes Q containing the point $x \in \mathbb{R}^n$.

It is clear that the case $\alpha = 0$ corresponds to the Hardy–Littlewood maximal function (1.10). Besides, this family of maximal functions has the corresponding boundedness properties. Namely,

Theorem 1.3.3. *Let $1 < p \leq \frac{n}{\alpha}$ and q satisfying the following equation*

$$(1.14) \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}.$$

Then, $M_{\alpha} : L^p(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n)$.

1.3.2 Singular integral operators

The term singular integral operator refers to a wide class of operators that are (formally) defined, as integral operators of the following form

$$Tf(x) = \int K(x, y) f(y) dy,$$

where the kernel K is singular in the sense that it is not locally integrable. The importance of this kind of operators lies in their intimate connection to the study of the convergence of the Fourier series among other reasons. It is well known that the study of the L^p boundedness of the conjugate function in the torus is equivalent to the study of the convergence of the Fourier series of functions in L^p ; since the Hilbert transform in the real line is a version of the conjugate function, it turns out that it plays an important role in the study of the convergence of the Fourier series in the real line as the conjugate function does in the torus.

The prototype or most representative example of this class of operators is the Hilbert transform in the real line. It is defined by

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy.$$

This operator, initially defined over the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, can be extended from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$ and from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. These two results are due to M. Riesz and Kolmogorov, respectively and can be checked out for instance in [38].

Originally, the study of the properties of the Hilbert transform was carried out by using complex analysis techniques. Nevertheless, with the development of Calderón and Zygmund's school, real analysis techniques replaced the complex ones and singular integral operators were introduced in different areas of mathematics as partial differentiation equations.

Initially, the singular integral operators studied by Calderón and Zygmund were of convolution type, such as the Hilbert or Riesz transforms. Notwithstanding, it was also interesting to study non-convolution operators as the Cauchy transform. From now on, we will concentrate in a particular class of singular integral operators named after these two authors: the Calderón–Zygmund operators.

Calderón–Zygmund operators

The original Calderón–Zygmund operators were convolution operators defined via singular kernels. Afterwards, Coifman and Meyer in [31] introduced the wider notion of Calderón–Zygmund operators that included non-convolution operators. This extension did not involve much more complications in the development of the theory, except for the lack of L^2 boundedness property of this operators via the Fourier transform.

Next, let us recall the definition of a Calderón–Zygmund operator and its boundedness properties.

Definition 1.3.5. *A linear operator T is a Calderón–Zygmund operator if:*

1. $T : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$.
2. *There exists a function K defined off the diagonal of $x = y$, such that*

$$T(f)(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy, \quad x \notin \text{supp}(f), \quad f \in C_c^\infty.$$

3. K satisfies the following size condition

$$(1.15) \quad |K(x, y)| \lesssim \frac{1}{|x - y|^n}.$$

4. K satisfies the following regularity conditions, for a certain $\delta > 0$:

$$(1.16) \quad |K(x, y) - K(x', y)| \lesssim \frac{|x - x'|^\delta}{(|x - y| + |x' - y|)^{n+\delta}},$$

if $|x - x'| \leq \frac{1}{2} \max\{|x - y|, |x' - y|\}$, and

$$(1.17) \quad |K(x, y) - K(x, y')| \lesssim \frac{|y - y'|^\delta}{(|x - y| + |x - y'|)^{n+\delta}},$$

if $|y - y'| \leq \frac{1}{2} \max\{|x - y|, |x - y'|\}$.

As we mentioned before, Hardy–Littlewood maximal function controls Calderón–Zygmund operators as it is shown in the following result due to J. Álvarez and C. Pérez [1].

Theorem 1.3.4. *Let T be a Calderón–Zygmund operator and $0 < \delta < 1$. Then, for any $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.*

$$(1.18) \quad M_\delta^\sharp(T(f))(x) \lesssim M(f)(x),$$

for $x \in \mathbb{R}^n$.

Observe that (1.18) is equivalent to the fact that Calderón–Zygmund operators are bounded operators from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. In fact, the endpoint estimate of T is necessary in the proof of (1.18) and, conversely, the endpoint estimate follows as a consequence of this result and the following Fefferman–Stein type inequality:

$$\|M_\delta(f)\|_{L^{1,\infty}} \leq C_n \|M_\delta^\sharp(f)\|_{L^{1,\infty}}, \quad \delta < \infty.$$

1.3.3 Fractional integral operators

We now introduce the family of fractional integral operators that are related to the study of the smoothness of functions and Sobolev embedding theorems (for further details, see [48] or [104]).

Definition 1.3.6. *Let $0 < \alpha < n$, we define the fractional integral operator or Riesz potential, by*

$$(1.19) \quad I_\alpha(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy.$$

Note that the function $|\cdot|^{-\alpha-n}$ is locally integrable for $0 < \alpha < n$, thus (1.19) is well defined by an absolutely convergent integral if $f \in S(\mathbb{R}^n)$.

The operator I_α is closely related to the fractional maximal function M_α . On one hand, I_α is pointwise bigger than M_α , that is

$$M_\alpha(f) \lesssim I_\alpha(f),$$

for all non-negative functions f . On the other hand, I_α has the same boundedness properties as M_α showed in [48]. More precisely,

Theorem 1.3.5. *Let $0 < \alpha < n$ and $1 \leq p < q < \infty$ satisfying (1.14). Then*

$$I_\alpha : L^1(\mathbb{R}^n) \longrightarrow L^{q,\infty}(\mathbb{R}^n)$$

and,

$$I_\alpha : L^p(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n).$$

when $p > 1$.

1.3.4 Commutators with BMO functions

The origin of commutators goes back to the generalization of the factorization theorem for Hardy spaces developed by Coifman, Rochberg and Weiss in [32] who were motivated by Calderón's study of the Cauchy transform along a Lipschitz curve. This kind of singular integral operators are interesting for several reasons; they are involved, for instance, in the study of partial differential equations. We can define the commutator of an operator T with a locally integrable function b by

$$[T, b]f(x) = T(bf) - bT(f).$$

If T is a Calderón-Zygmund operator with kernel K , then we can write

$$[T, b]f(x) = \int_{\mathbb{R}^n} (b(y) - b(x))K(x, y)f(y)dy.$$

As it was proved [32], these operators behave similarly to Calderón-Zygmund operators if we look at L^p estimates since for $1 < p < \infty$, $[T, b]$ is bounded from $L^p(\mathbb{R}^n)$ into itself if and only if $b \in BMO$ (see Section 1.5 for the definition of BMO). However, they are substantially different from singular integral operators since the proof of their L^p boundedness does not follow from the weak $(1, 1)$ inequality. Furthermore, it was

proved by C. Pérez in [94] that when $b \in BMO$, $[T, b]$ is not of weak type $(1, 1)$. In fact, commutators satisfy a weaker $L(\log L)$ estimate.

In general, we define the k -th order commutator with b , for an integer $k \geq 0$, as follows

$$[T, b]_k f(x) = \int_{\mathbb{R}^n} (b(y) - b(x))^k K(x, y) f(y) dy.$$

If $k = 1$, $[T, b]_1$ is the classical commutator $[T, b]$ already defined.

1.4 Weights

In this section we present some notions and results of the linear theory of weights that will help to understand the multiple theory of weights that we will study later on. It is well known that the origin of the modern theory of weights goes back to the works of Coifman, C. Fefferman, Hunt, Muckenhoupt and Wheeden in the 1970's.

It was natural to ask whether the boundedness properties of the operators, initially defined in $L^p(\mathbb{R}^n)$, could be extended to more general spaces. Even though this is a general and complicated question, we can deem a particular case where the answer is affirmative.

Let μ be an absolutely continuous measure with respect to Lebesgue measure, that is, $d\mu = w dx$ where w is a weight. Namely, w is a measurable locally integrable function defined in \mathbb{R}^n taking values in $(0, \infty)$ for almost each point. In this particular case, the study of two well-known problems motivated the introduction of the so-called A_p classes of weights. Namely,

1. **One-weight problem:** Given p , $1 < p < \infty$, determine the class of weights w on \mathbb{R}^n for which the Hardy–Littlewood maximal function is bounded from $L^p(w)$ into itself, that is, for which weights the following inequality holds:

$$(1.20) \quad \|Mf\|_{L^p(w)} \leq C \|f\|_{L^p(w)}.$$

2. **Two-weight problem:** Given p , $1 < p < \infty$, determine the classes of weights u and w on \mathbb{R}^n for which the Hardy–Littlewood maximal function is bounded from $L^p(w)$ into $L^p(u)$, that is, for which pairs of weights the following inequality holds:

$$(1.21) \quad \|Mf\|_{L^p(u)} \leq C \|f\|_{L^p(w)}$$

We can also consider the problems substituting the strong inequalities by weak inequalities. The answer to the strong one-weight problem for $1 < p < \infty$ goes back to the work of Muckenhoupt [91], where he characterized the class of weights for which (1.20) holds. This class of weights is known as the A_p class of weights which is defined as follows.

Definition 1.4.1. *A weight w is in the A_p class, $1 < p < \infty$, if*

$$(1.22) \quad [w]_{A_p} := \sup_Q \frac{1}{|Q|} \int_Q w(x) dx \left(\frac{1}{|Q|} \int_Q w^{1-p'} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all the cubes Q in \mathbb{R}^n and $[w]_{A_p}$ is called the A_p constant of w .

In the case of Calderón–Zygmund operators, the A_p class turns out to be sufficient for boundedness, although not always necessary. In the limiting case when $p = 1$, Muckenhoupt did not prove a strong inequality but a weak one; namely, $M : L^1(w) \rightarrow L^{1,\infty}(w)$ if and only if $w \in A_1$, where the A_1 class is defined as follows.

Definition 1.4.2. *A weight w is in the A_1 class if*

$$(1.23) \quad [w]_{A_1} := \sup_Q \frac{1}{|Q|} \int_Q w(x) dx \left(\operatorname{ess\,sup}_Q w^{-1} \right) < \infty,$$

where the supremum is taken over all the cubes Q in \mathbb{R}^n and $[w]_{A_1}$ is called the A_1 constant of w .

Equivalently, w is in A_1 if

$$Mw(x) \leq [w]_{A_1} w(x), \text{ a.e. } x \in \mathbb{R}^n.$$

Since the A_p classes are increasing (i.e. $A_p \subset A_q$ if $p \leq q$) a natural way of defining the limiting class A_∞ is as the union of all A_p classes,

$$A_\infty := \bigcup_{p \geq 1} A_p.$$

However, there exist different characterizations of the A_∞ class of weights (see, for instance, [49, Thm. 9.3.3.]). Let us recall the following one, that we will use later on. We say that a weight $w \in A_\infty$ if there exist $0 < C, \varepsilon < \infty$ such that for all cubes Q and all measurable subsets A of Q we have

$$(1.24) \quad \frac{w(A)}{w(Q)} \leq C \left(\frac{|A|}{|Q|} \right)^\varepsilon,$$

where $w(E) = \int_E w(x)dx$ for any measurable set $E \subset \mathbb{R}^n$.

Similarly, although there are several equivalent ways of defining the A_∞ constant in the literature (see, for instance, [86]), we only define here two of them that will play an important role in this work. The first one is due to Hrusčev [56]:

$$[w]_{A_\infty}^H := \sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \exp \left(\frac{1}{|Q|} \int_Q \log w^{-1} \right),$$

where the supremum is taken over all the cubes Q in \mathbb{R}^n . We also consider the A_∞ constant that was implicitly considered by Fujii to provide a characterization of the A_∞ class in [45] and later rediscovered by Wilson in [112]. The Fujii-Wilson A_∞ constant is defined as follows

$$(1.25) \quad [w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q),$$

where the supremum is over all the cubes Q in \mathbb{R}^n . Observe that the Fujii-Wilson A_∞ constant is more suitable than the classical one due to Hrusčev since $[w]_{A_\infty} \leq c_n [w]_{A_\infty}^H$.

Next, we recall the reverse Hölder inequality with sharp constants for A_∞ weights proved in [61] that we will use in the sequel.

Theorem 1.4.1. *If $w \in A_\infty$, then*

$$(1.26) \quad \left(\frac{1}{|Q|} \int_Q w^{r(w)} \right)^{1/r(w)} \leq 2 \frac{1}{|Q|} \int_Q w,$$

where $r(w) = 1 + \frac{1}{\tau_n [w]_{A_\infty}}$ and $\tau_n = 2^{11+n}$. Furthermore, $[w]_{A_\infty} \simeq r'(w)$.

Let us note that an improvement of this result was shown in [63], which will be generalized to spaces of homogeneous type in Chapter 2.

Next, let us recall some useful properties of A_p weights. Since the weights of the form $M_r w$ will play an important role in the second chapter of this memory, we need to remind that these weights satisfy the so-called Coifman–Rochberg theorem.

Theorem 1.4.2. *Let $r > 1$ and a weight w . Then $(Mw)^{1/r} \in A_1$ and $[(Mw)^{1/r}]_{A_1} \leq c_n r'$.*

Besides, it is easy to see that if $w_1, w_2 \in A_1$, then $w = w_1 w_2^{1-p} \in A_p$ and $[w]_{A_p} \leq [w_1]_{A_1} [w_2]_{A_1}^{p-1}$. In general, it is possible to show that, conversely, every weight in A_p can be written in terms of two A_1 weights. This *factorization* theorem was proved by P. Jones in [67].

Theorem 1.4.3. *Let $1 < p < \infty$ and $w \in A_p$. Then, there exist weights $w_1, w_2 \in A_1$ such that $w = w_1 w_2^{1-p}$, and they verify*

$$[w_1]_{A_1} \leq c_n [w]_{A_p},$$

and

$$[w_2]_{A_1} \leq c_n [w]_{A_p}^{\frac{1}{p-1}}.$$

From the above inequalities it follows that

$$[w]_{A_p} \leq [w_1]_{A_1} [w_2]_{A_1}^{p-1} \leq c_n [w]_{A_p}^2.$$

The proof of this theorem can be done in several manners. One of these ways of proving P. Jones' factorization theorem is using the well-known Rubio de Francia's extrapolation algorithm that we detail below. One variation of this argument adapted to spaces of homogeneous type will be used to prove one of the main results in Chapter 2.

Theorem 1.4.4. *Let $1 < s < \infty$ and v be a weight. Let $h \in L^s(w)$. Then, there exists a sublinear non-negative operator R for which the following properties hold:*

1. $0 \leq h \leq R(h)$.
2. $\|R(h)\|_{L^s(v)} \leq 2\|h\|_{L^s(v)}$.
3. $R(h)v^{1/s} \in A_1$, and

$$[R(h)v^{1/s}]_{A_1} \leq cs'.$$

Regarding to the two-weight problem, the question of finding a condition on the weights u and w satisfying the strong estimate (1.21) was much more complicated. In [103], Sawyer characterized the two-weight inequality showing that $M : L^p(w) \rightarrow L^p(u)$ if and only if the pair (u, w) satisfies the following testing condition known as Sawyer's S_p condition

$$(1.27) \quad [u, w]_{S_p} = \sup_Q \left(\frac{\int_Q M(\chi_Q \sigma)^p u dx}{\sigma(Q)} \right)^{1/p} < \infty,$$

where σ is the conjugate weight of w , that is, $\sigma = w^{1-p'}$ and $1 < p < \infty$.

However, Sawyer's S_p condition is not easy to test since it contains the Hardy-Littlewood maximal function into itself. Motivated by this fact, in [61] T. Hytönen

and C. Pérez found a simpler condition that resembles in some sense the A_p condition for which the two-weight problem for M holds. More precisely, if $p > 1$ and w and σ are different weights

$$(1.28) \quad \|M(f\sigma)\|_{L^p(w)} \leq Cp'(B_p[w, \sigma])^{1/p} \|f\|_{L^p(\sigma)},$$

where

$$(1.29) \quad B_p[w, \sigma] := \sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q \sigma \right)^p \exp \left(\frac{1}{|Q|} \int_Q \log \sigma^{-1} \right),$$

is known as the B_p constant of the weights w and σ . In the particular case, when $\sigma = w^{1-p'}$ this constant clearly satisfies

$$[w]_{A_p} \leq B_p[w, \sigma] \leq [w]_{A_p} [\sigma]_{A_\infty}^H.$$

As mentioned in the introduction, in [61] it is also proved an improvement of Buckley's result. What is actually shown in that work is a mixed $A_p - A_\infty$ bound in the two-weight setting that can be particularized to obtain the mixed sharp bound (10). More precisely,

$$(1.30) \quad \|M(f\sigma)\|_{L^p(w)} \leq Cp'([w]_{A_p} [\sigma]_{A_\infty})^{1/p} \|f\|_{L^p(\sigma)}.$$

In both (1.29) and (1.30), C is a dimensional constant. It is worth mentioning that this result was improved in [97] and the novelty is that the reverse Hölder's property is completely avoided in this work.

1.5 BMO and John-Nirenberg inequality

Given a function $f \in L^1_{loc}(\mathbb{R}^n)$ and a cube Q , as we mentioned before, its sharp maximal function M^\sharp is defined by

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy \simeq \sup_{Q \ni x} \inf_c \frac{1}{|Q|} \int_Q |f(y) - c| dy,$$

where the supremum is taken over all cubes Q containing x . As each of these integrals measures the mean oscillation of f on the cube Q , we will say that f has a bounded mean oscillation if the function $M^\sharp f$ is bounded. The space of all this functions with bounded mean oscillation is denoted by BMO . More specifically,

$$BMO := \{f \in L^1_{loc}(\mathbb{R}^n) : M^\sharp f \in L^\infty\}.$$

To define a norm on BMO we write

$$\|f\|_{BMO} = \|M^\sharp f\|_{L^\infty}.$$

It is easy to see that this is not properly a norm since every function which is constant almost everywhere has zero oscillation. However, these are the only functions with this property, so that we can think of BMO as the quotient of the above space by the space of constant functions. Therefore, two functions which differ by a constant coincide as functions in BMO . This space equipped with the norm $\|\cdot\|_{BMO}$ is a Banach space. BMO can also be characterized in terms of Hardy spaces as C. Fefferman proved in [43]. One of the main properties of BMO functions is the well-known John-Nirenberg inequality that asserts that, in some sense, the maximum possible rate of growth for a BMO function is logarithmic. A beautiful proof of this result can be found, for instance, in [68].

Theorem 1.5.1. *There exist two positive dimensional constants $0 \leq \alpha < 1 < \beta$ such that for any function $b \in BMO$,*

$$(1.31) \quad \sup_Q \frac{1}{|Q|} \int_Q \exp\left(\frac{\alpha}{\|b\|_{BMO}} |b(y) - b_Q| dy\right) \leq \beta.$$

1.6 Local mean oscillation decomposition

In this section we will recall the notion of non-increasing rearrangement, median value and local mean oscillation. We also define some local maximal functions in order to remind the local mean oscillation decomposition proved by A.K. Lerner [75] and later improved by T. Hytönen in [57].

Given a measurable function f on \mathbb{R}^n , its non-increasing rearrangement is a non-increasing function f^* on $(0, \infty)$ which satisfies that it is equimeasurable with f , that is,

$$|\{t \in (0, \infty) : f^*(t) > \alpha\}| = M_f(\alpha), \text{ for any } \alpha > 0,$$

where M_f denotes the distribution function of f with respect to Lebesgue measure. We can define the non-increasing rearrangement of a measurable function f on \mathbb{R}^n as follows.

$$f^*(t) := \inf \{\alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| < t\}, \quad 0 < t < \infty.$$

Observe that the above definition of rearrangement that we will be using along this work, is defined as a left-continuous function. However, it is also possible to define the non-increasing rearrangement of a function to be a right-continuous function (see for instance [8, p. 39]). Here we will use the convention that $\inf \emptyset = \infty$. Therefore, if $|\{x \in \mathbb{R}^n : |f(x)| < \alpha\}| \geq t$ for all $\lambda \geq 0$, then $f^*(t) = \infty$.

Next, let us list some properties of rearrangements:

1. $M_f(f^*(t)) \leq t$.
2. $f^*(M_f(\lambda)) \leq \lambda$.
3. If $|f(x)| \leq |g(x)|$ then $f^*(t) \leq g^*(t)$.
4. $(f + g)^*(t_1 + t_2) \leq f^*(t_1) + g^*(t_2)$.
5. $((f + c)\chi_E)^*(t) \leq (f\chi_E)^*(t) + |c|$, for $|E| < \infty$.
6. $f^*(\infty) = 0$.
7. $(|f|^p)^*(t) = f^*(t)^p$, for any $p > 0$.
8. $\int_{\mathbb{R}^n} |f(x)|^p dx = \int_0^\infty f^*(t)^p dt$, $p > 0$.
9. $f^*(t) \leq \frac{1}{t} \|f\|_{L^1}$.

Given a measurable function f on \mathbb{R}^n and a cube Q , the local mean oscillation of f on Q is defined by

$$\omega_\lambda(f; Q) = \inf_{c \in \mathbb{R}} ((f - c)\chi_Q)^*(\lambda|Q|) \quad (0 < \lambda < 1).$$

By a median value of f over a cube Q we mean a possibly non-unique, real number $m_f(Q)$ such that

$$\max(|\{x \in Q : f(x) > m_f(Q)\}|, |\{x \in Q : f(x) < m_f(Q)\}|) \leq |Q|/2.$$

Observe that the set of all median values of a function f is either one point or the closed interval. In the latter case we will assume for definiteness that $m_f(Q)$ is the maximal median value. Observe that it follows from the definitions that

$$|\{x \in Q : f(x) \geq m_f(Q)\}| \geq |Q|/2,$$

which implies

$$(1.32) \quad |m_f(Q)| \leq (f\chi_Q)^*(|Q|/2).$$

Given a cube Q_0 , denote by $\mathcal{D}(Q_0)$ the set of all dyadic cubes with respect to Q_0 . The dyadic local sharp maximal function $m_{\lambda;Q_0}^{\#,d}f$ is defined by

$$m_{\lambda;Q_0}^{\#,d}f(x) = \sup_{x \in Q' \in \mathcal{D}(Q_0)} \omega_\lambda(f; Q').$$

The following theorem, known as local mean oscillation decomposition or Lerner's formula was proved in [76]; a similar version of this result can be found in [75].

Theorem 1.6.1. *Let f be a measurable function on \mathbb{R}^n and let Q_0 be a fixed cube. Then there exists a (possibly empty) sparse family of cubes $Q_j^k \in \mathcal{D}(Q_0)$ such that for a.e. $x \in Q_0$,*

$$(1.33) \quad |f(x) - m_f(Q_0)| \leq 4m_{\frac{1}{2^{n+2}};Q_0}^{\#,d}f(x) + 2 \sum_{k,j} \omega_{\frac{1}{2^{n+2}}}(f; Q_j^k)\chi_{Q_j^k}(x).$$

Very recently, Hytönen observed in [57] that the local mean oscillation formula (1.33) holds without the local sharp maximal function. Namely,

Theorem 1.6.2. *For any measurable function f on a cube $Q_0 \subset \mathbb{R}^n$. Then for a.e. $x \in Q_0$,*

$$(1.34) \quad |f(x) - m_f(Q_0)| \leq 2 \sum_{k,j} \omega_{\frac{1}{2^{n+2}}}(f; Q_j^k)\chi_{Q_j^k}(x),$$

where the family of cubes $\{Q_j^k\} \subset \mathcal{D}(Q_0)$ is sparse.

A_1 theory on spaces of homogeneous type

In this chapter we will be working in the more general setting of the spaces of homogeneous type that generalizes the Euclidean situation in \mathbb{R}^n with the Lebesgue measure. Our purpose is to extend to this context weak and strong sharp mixed inequalities for Calderón–Zygmund operators and their commutators with BMO functions. These mixed bounds, which are formed by at least two different A_p constants, are better since they are strictly smaller than the original one-constant bounds. We also generalize two well-known inequalities to the homogeneous setting: the sharp reverse Hölder inequality for A_∞ weights in dyadic cubes and a precise version of the John–Nirenberg inequality.

2.1 Basics on spaces of homogeneous type

In this section we introduce the definition of spaces of homogeneous type, which generalize the Euclidean situation of \mathbb{R}^n with the Lebesgue measure and we give some basic properties and related results. Examples of spaces of homogeneous type include C^∞ compact Riemannian manifolds, graphs of Lipschitz functions and Cantor sets with Hausdorff measure. These and more examples are described in [25] as well as some applications of these spaces can be found in [85, 115].

Definition 2.1.1. *an space of homogeneous type is an ordered triple (X, ρ, μ) where X is a set, ρ is a quasimetric, that is:*

1. $\rho(x, y) = 0$ if and only if $x = y$.
2. $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.
3. $\rho(x, z) \leq \kappa(\rho(x, y) + \rho(y, z))$, for all $x, y, z \in X$.

for some constant $\kappa > 0$ (quasimetric constant), and the positive measure μ is doubling, that is

$$0 < \mu(B(x_0, 2r)) \leq D_\mu \mu(B(x_0, r)) < \infty,$$

for some constant D_μ (doubling constant).

In the context of spaces of homogeneous type, for brevity we will say that a constant is *absolute* if it only depends on the triple (X, ρ, μ) . Particularly, κ and D_μ appearing in the above definition are absolute constants.

Fortunately, some constructions and results in classical harmonic analysis still exist in some form in spaces of homogeneous type, such as certain covering lemmas. We will use the Lebesgue differentiation theorem, very recently shown to hold in spaces of homogeneous type in [2, Lemma 2.3.] where the usual standard assumptions have been removed and whose proof is implicit in [106].

Lemma 2.1.1. *Given an space of homogeneous type (X, ρ, μ) , the Lebesgue differentiation theorem holds: for μ -almost every $x \in X$,*

$$\lim_{r \rightarrow 0} \frac{1}{\mu(B_\rho(x, r))} \int_{B_\rho(x, r)} |f(y) - f(x)| d\mu(y) = 0.$$

An important tool in the following will be the concept of a dyadic grid \mathcal{D} on an space of homogeneous type. The following result is due to Hytönen and Kairema [59] (see also Christ [25]).

Theorem 2.1.1. *There exists a family of sets $\mathcal{D} = \cup_{k \in \mathbb{Z}} D_k$, called a dyadic decomposition of X , constants $C < \infty$, $0 < \varepsilon$, $0 < \delta < 1$, and a corresponding family of points $\{x_c(Q)\}_{Q \in \mathcal{D}}$ such that:*

1. $X = \bigcup_{Q \in D_k} Q$, for all $k \in \mathbb{Z}$.
2. If $Q_1 \cap Q_2 \neq \emptyset$, then $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$.
3. For every $Q \in D_k$ there exists at least one child cube $\check{Q} \in D_{k-1}$ such that $\check{Q} \subseteq Q$.
4. For every $Q \in D_k$ there exists exactly one parent cube $\hat{Q} \in D_{k+1}$ such that $Q \subseteq \hat{Q}$.
5. If Q_2 is a child of Q_1 then $\mu(Q_2) \geq \varepsilon \mu(Q_1)$.
6. $B(x_c(Q), \delta^k) \subset Q \subset B(x_c(Q), C\delta^k)$.

We will refer to the last property as the *sandwich property*. Note that the sets $Q \in \mathcal{D}$ are referred to as dyadic cubes with center $x_c(Q)$ and sidelength δ^k , but it is important to emphasize that these are not cubes in any standard sense even if the underlying space is \mathbb{R}^n . For an exact characterization of the sets which can be dyadic

cubes we refer the reader to [60]. We also will need the dilations λQ , $\lambda > 1$, of a given dyadic cube Q . These will actually be balls containing Q : given a cube Q , we define

$$\lambda Q = B(x_c(Q), \lambda C \delta^k).$$

Now we recall an extension of Calderón–Zygmund decomposition to an space of homogeneous type. More general results that also hold for Orlicz norms and not only for L^1 averages are proved in [2, Thm. 2.7] and [2, Thm. 2.8.], respectively. Given a dyadic grid \mathcal{D} , denote by $M^{\mathcal{D}}$ the maximal function over cubes in \mathcal{D} .

Theorem 2.1.2. *Given an space of homogeneous type (X, ρ, μ) such that $\mu(X) = \infty$ and a dyadic grid \mathcal{D} , suppose f is a measurable function such that $\int_Q f(x) d\mu(x) \rightarrow 0$ as $\mu(Q) \rightarrow \infty$. Then, the following hold:*

1. For each $\lambda > 0$, there exists a collection $\{Q_j\} \subset \mathcal{D}$ that is pairwise disjoint, maximal with respect to inclusion and such that

$$\Omega_\lambda = \{x \in X : M^{\mathcal{D}} f(x) > \lambda\} = \bigcup_j Q_j.$$

Moreover, there exists a constant C_X such that for every j ,

$$\lambda < \frac{1}{\mu(Q)} \int_Q |f(x)| d\mu(x) \leq C_X \lambda.$$

2. Given $a > \frac{2}{\varepsilon}$, where ε is as in Theorem 2.1.1, for each $k \in \mathbb{Z}$ let $\{Q_j^k\}_j$ be the collection of maximal dyadic cubes in 1. with

$$\Omega_k = \{x \in X : M^{\mathcal{D}} f(x) > a^k\} = \bigcup_j Q_j^k.$$

Then, the set of cubes $\mathcal{S} = \{Q_j^k\}$ is sparse and $E(Q_j^k) = Q_j^k \setminus \Omega_{k+1}$.

If $\mu(X) < \infty$, then 1. holds provided that $\lambda > \int_X |f(x)| d\mu(x)$ and 2. holds for all k such that $a^k > \int_X |f(x)| d\mu(x)$.

Theorem 2.1.3. *Given an space of homogeneous type (X, ρ, μ) such that $\mu(X) = \infty$ and a dyadic grid \mathcal{D} , suppose f is a function such that $\int_Q |f| d\mu \rightarrow 0$ as $\mu(Q) \rightarrow \infty$. Then, for any $\lambda > 0$ there exists a family $\{Q_j\} \subset \mathcal{D}$ and functions b and g , such that:*

1. $f = b + g$.
2. $g = f \chi_{\Omega_\lambda^c} + \sum_j f \chi_{Q_j}$.

3. For μ -a.e. $x \in X$, $|g(x)| \leq C_X \lambda$.
4. $b = \sum_j b_j$, where $b_j = (f - f_{Q_j})\chi_{Q_j}$.
5. $\text{supp } b_j \subset Q_j$ and $\int_{Q_j} b_j(x) d\mu(x) = 0$.

If $\mu(X) < \infty$, then this decomposition still exists if we take $\lambda > \int_X |f(x)| d\mu(x)$.

Note that the definitions of the weights and A_p constants are similar to the ones defined in Chapter 1 substituting the Lebesgue measure by μ . Some further remarks regarding to A_p constants in spaces of homogeneous type are listed below.

We say that a weight is a nonnegative locally integrable function w on (X, μ) that takes values in $(0, \infty)$ almost everywhere. For any $1 < p < \infty$ we define the A_p constant of the weight w on the space of homogeneous type X as follows

$$(2.1) \quad [w]_{A_p} := \sup_Q \left(\frac{1}{\mu(Q)} \int_Q w(x) d\mu \right) \left(\frac{1}{\mu(Q)} \int_Q w(x)^{1-p'} d\mu \right)^{p-1}.$$

In the definition above we can take Q to be either in the family of dyadic cubes or balls, since the concept of a non-dyadic cube is not defined in spaces of homogeneous type. Note that the A_p constant is comparable when considering suprema over families of dyadic cubes or balls by using the sandwich property of dyadic cubes and the doubling property of the measure μ .

It is crucial to note that we will take this constant with respect to cubes (by which we always will mean dyadic cubes), because the A_∞ constant below is not comparable in the same way, even in the classical Euclidean case.

We define the Fujii–Wilson A_∞ constant in an space of homogeneous type as follows:

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q) d\mu.$$

While this A_∞ constant is comparable using dyadic cubes or balls, the constant of comparison depends on the measure w (which is doubling since $w \in A_\infty$). Hence to achieve sharp bounds in spaces of homogeneous type we cannot simply switch between these constants defined with respect to cubes or balls since we introduce a w -dependent factor. We will take up again this discussion in Section 2.2 where the sharpness of the reverse Hölder inequalities involving this A_∞ constant depend on the definition over cubes or balls.

Next we recall the definition of Calderón–Zygmund operators in spaces of homogeneous type that could be found, for instance, in [25].

Definition 2.1.2. We say that T is a Calderón–Zygmund operator if:

1. $T : L^2(X, \mu) \rightarrow L^2(X, \mu)$.
2. There exists a function $K : X \times X \setminus \{x = y\} \rightarrow \mathbb{R}$ such that

$$T(f)(x) = \int_X K(x, y)f(y)d\mu(y), \quad x \notin \text{supp}(f), \quad f \in C_c^\infty.$$

3. K is such that there exists $\eta > 0$ such that for all $x_0 \neq y \in X$ and $x \in X$ it satisfies the size condition:

$$(2.2) \quad |K(x_0, y)| \lesssim \frac{1}{\mu(B(x_0, \rho(x_0, y)))}$$

and the smoothness conditions for $\rho(x_0, x) \leq \eta\rho(x_0, y)$:

$$(2.3) \quad |K(x, y) - K(x_0, y)| \lesssim \left(\frac{\rho(x, x_0)}{\rho(x_0, y)}\right)^\eta \frac{1}{\mu(B(x_0, \rho(x_0, y)))},$$

and

$$(2.4) \quad |K(y, x) - K(y, x_0)| \lesssim \left(\frac{\rho(x, x_0)}{\rho(x_0, y)}\right)^\eta \frac{1}{\mu(B(x_0, \rho(x_0, y)))}.$$

These operators are of weak $(1, 1)$ type as proved in [33].

Theorem 2.1.4. Let T be a Calderón–Zygmund operator on an space of homogeneous type. Then T is bounded from L^1 to $L^{1, \infty}$.

Let us define sparse families of cubes in spaces of homogeneous type in order to state the boundedness properties of Calderón–Zygmund operators using the techniques originally due to Lerner [77]. Consider a dyadic grid $\mathcal{D} = \cup_k D_k$ as defined in Theorem 2.1.1. A sparse family $\mathcal{S} \subset \mathcal{D}$ is a collection of dyadic cubes for which there exists a collection of sets $\{E(Q) : Q \in \mathcal{S}\}$ such that the sets $E(Q)$ are pairwise disjoint, $E(Q) \subset Q$ and $\mu(Q) \leq 2\mu(E(Q))$. Given a sparse family $\mathcal{S} \subset \mathcal{D}$, we define a sparse operator as it was done in (1.9). We also introduce the decomposition of Lerner, proved in the homogeneous setting in [2].

Theorem 2.1.5. *For any Calderón-Zygmund operator T on an space of homogeneous type X , we have that*

$$(2.5) \quad \|Tf\|_Y \leq C \sup_{\mathcal{D}, S} \|\mathcal{A}_{\mathcal{D}, S} f\|_Y,$$

where \mathcal{D} is a dyadic grid, C only depends on the operator and the space X , and Y is any Banach function space.

Finally, let us introduce the concept of iterated commutator that we will use in the sequel.

Definition 2.1.3. *Given a Calderón-Zygmund operator T with kernel K and a function b in BMO , we define the k -th order commutator with b , for an integer $k \geq 0$, as follows*

$$[T, b]_k(f)(x) = \int_X (b(y) - b(x))^k K(x, y) f(y) d\mu(y).$$

In the particular case when $k = 1$, $[T, b]_1$ is the classical commutator and we will denote it by $[T, b]$.

Throughout this chapter, X will denote an space of homogeneous type equipped with a quasimetric ρ with quasimetric constant κ and a positive doubling measure μ with doubling constant D_μ .

2.2 Useful inequalities in spaces of homogeneous type

In this section we start proving some inequalities that will be fundamental in the proofs of the main theorems in this chapter. We refer to the so called reverse Hölder inequality and the John-Nirenberg inequality within the homogeneous setting.

2.2.1 Sharp reverse Hölder inequality for A_∞ weights

First, we present an adaption of the sharp reverse Hölder inequality for A_∞ weights in spaces of homogeneous type from the argument in [63]. Observe that in the same work there is a weak version of this inequality that we state below. They call this result a weak inequality since on the right hand side we have the dilation $2\kappa B$ of the ball B .

Lemma 2.2.1. *Let $w \in A_\infty$ and define*

$$r = r_w = 1 + \frac{1}{\tau[w]_{A_\infty}} = 1 + \frac{1}{6(32\kappa^2(4\kappa^2 + \kappa)^2)^{D_\mu} [w]_{A_\infty}}$$

2.2. USEFUL INEQUALITIES IN SPACES OF HOMOGENEOUS TYPE

where τ depends on κ , the quasimetric constant of X . Then

$$\left(\int_B w^r d\mu \right)^{1/r} \leq 2(4\kappa)^{D_\mu} \int_{2\kappa B} w d\mu,$$

for any ball $B \in X$.

However, this lemma is not sufficient for our purposes. The difficulty lies in the fact that the Fujii-Wilson A_∞ constant is comparable when it is defined with respect to cubes or balls, but the constant of comparison depends on the weight w . This provides a difficulty in converting between the constants, and since cubes are essential in the following lemmas, we need an appropriate reverse Hölder inequality for cubes. Here is the lemma that we use with respect to cubes.

Lemma 2.2.2. *Let $w \in A_\infty$ and let*

$$0 < r \leq \frac{1}{\tau[w]_{A_\infty} - 1} = \frac{1}{2D_X[w]_{A_\infty} - 1},$$

with $D_X = 1/\varepsilon$, where ε is the absolute constant appearing in the dyadic decomposition of X . Then

$$\int_Q w^{1+r} d\mu \leq 2 \left(\int_Q w d\mu \right)^{1+r},$$

for any cube $Q \subset X$.

The proof will use the following sublemma.

Lemma 2.2.3. *Let $w \in A_\infty$ and Q_0 a cube. Then for all*

$$0 < r \leq \frac{1}{2D_X[w]_{A_\infty} - 1}$$

we have

$$\int_{Q_0} (Mw)^{1+r} d\mu \leq 2[w]_{A_\infty} \left(\int_{Q_0} w d\mu \right)^{1+r}.$$

Proof of Lemma 2.2.3. Assume without loss of generality that $w = w\chi_{Q_0}$. Let $\Omega_\lambda = Q_0 \cap \{Mw > \lambda\}$. Then

$$\begin{aligned} \int_{Q_0} (Mw)^{1+r} d\mu &= \int_0^\infty r\lambda^{r-1} Mw(\Omega_\lambda) d\lambda \\ &= \int_0^{w_{Q_0}} r\lambda^{r-1} \int_{Q_0} Mw d\mu d\lambda + \int_{w_{Q_0}}^\infty r\lambda^{r-1} Mw(\Omega_\lambda) d\lambda. \end{aligned}$$

Now select a dyadic cube Q_j if it is maximal with respect to the following condition: $\lambda < w_{Q_j}$. Then $\Omega_\lambda = \cup_j Q_j$ where $\lambda < f_{Q_j} w \leq \frac{1}{\varepsilon} \lambda$ and ε is the absolute constant from Theorem 2.1.1. Hence we have

$$\int_{Q_0} (Mw)^{1+r} d\mu \leq w_{Q_0}^r [w]_{A_\infty} w(Q_0) + \int_{w_{Q_0}}^\infty r \lambda^{r-1} \sum_j \int_{Q_j} Mw d\mu d\lambda.$$

Now we can localize

$$Mw(x) = M(w\chi_{Q_j})(x)$$

by the maximality of the cubes Q_j for any $x \in Q_j$. Then,

$$\begin{aligned} \int_{Q_j} Mw d\mu &= \int_{Q_j} M(w\chi_{Q_j}) \leq [w]_{A_\infty} w(Q_j) \leq [w]_{A_\infty} w(\hat{Q}_j) \\ &= [w]_{A_\infty} w_{\hat{Q}_j} \mu(\hat{Q}_j) \leq [w]_{A_\infty} \lambda \frac{1}{\varepsilon} \mu(Q_j), \end{aligned}$$

where \hat{Q}_j is the parent of the cube Q_j and we have used the definition of A_∞ and the maximality and containment properties of the cubes. Call $\frac{1}{\varepsilon} = D$. Hence

$$\sum_j \int_{Q_j} Mw d\mu \leq \sum_j [w]_{A_\infty} \lambda D \mu(Q_j) \leq [w]_{A_\infty} \lambda D \mu(\Omega_\lambda)$$

so

$$\int_{Q_0} (Mw)^{1+r} \leq w_{Q_0}^r [w]_{A_\infty} w(Q_0) + r [w]_{A_\infty} D \int_{w_{Q_0}}^\infty \lambda^r \mu(\Omega_\lambda) d\lambda.$$

Dividing by $\mu(Q_0)$, we obtain

$$\int_{Q_0} (Mw)^{1+r} d\mu \leq w_{Q_0}^{1+r} [w]_{A_\infty} + \frac{rD[w]_{A_\infty}}{1+r} \int_{Q_0} (Mw)^{1+r} d\mu,$$

so by subtracting the last term on the right hand side from both sides of the equation, so to get the desired constant of 2 we must have that

$$1 - \frac{rD[w]_{A_\infty}}{1+r} \geq \frac{1}{2},$$

which after some calculation results in choosing $0 < r \leq \frac{1}{2D[w]_{A_\infty} - 1}$ as stated. \square

Next we move to the proof of Lemma 2.2.2.

Proof of Lemma 2.2.2. Without loss of generality, let $w = w\chi_{Q_0}$. Then

$$\int_{Q_0} w^{1+r} d\mu \leq \int_{Q_0} (Mw)^r w d\mu = \int_0^\infty r \lambda^{r-1} w(\Omega_\lambda) d\lambda,$$

2.2. USEFUL INEQUALITIES IN SPACES OF HOMOGENEOUS TYPE

where $\Omega_\lambda = Q_0 \cap \{Mw > \lambda\}$. Note that as in the previous lemma we can decompose $\Omega_\lambda = \cup_j Q_j$ where the Q_j are the Calderón-Zygmund cubes. Then splitting up the integral we get

$$\begin{aligned} \int_{Q_0} (Mw)^r w d\mu &= \int_0^{w_{Q_0}} r\lambda^{r-1} w(Q_0) d\lambda + \int_{w_{Q_0}}^\infty r\lambda^{r-1} w(\Omega_\lambda) d\lambda \\ &\leq w_{Q_0}^r w(Q_0) + \int_{w_{Q_0}}^\infty r\lambda^{r-1} \sum_j w(Q_j) d\lambda. \end{aligned}$$

Now by the decomposition, we have that $w_{Q_j} \leq D_X \lambda \mu(Q_j)$, where $D_X = \frac{1}{\varepsilon}$ since the decomposition is with respect to dyadic cubes, so we get

$$\begin{aligned} \int_{Q_0} (Mw)^r w d\mu &\leq w_{Q_0}^r w(Q_0) + rD_X \int_{w_{Q_0}}^\infty r\lambda^r \sum_j \mu(Q_j) d\lambda \\ &\leq w_{Q_0}^r w(Q_0) + rD_X \int_{w_{Q_0}}^\infty \lambda^r \mu(\Omega_\lambda) d\lambda \\ &\leq w_{Q_0}^r w(Q_0) + \frac{rD_X}{1+r} \int_{Q_0} (Mw)^{1+r}. \end{aligned}$$

Hence, dividing by $\mu(Q_0)$ and using Lemma 2.2.3, we arrive at

$$\begin{aligned} \int_{Q_0} w^{1+r} &\leq w_{Q_0}^{1+r} + \frac{rD_X 2[w]_{A_\infty}}{1+r} \left(\int_{Q_0} w \right)^{1+r} \\ &\leq \frac{rD_X 2[w]_{A_\infty} + 1+r}{1+r} \left(\int_{Q_0} w \right)^{1+r}. \end{aligned}$$

Therefore, choosing r in the mentioned range, we can make the constant on the right hand side less than or equal to 2. □

2.2.2 John-Nirenberg inequality and related lemmas

Here we prove a precise version of John-Nirenberg inequality in spaces of homogeneous type whose proof follows the same ideas as in the original and beautiful proof in [68, p. 31]. This result, which is interesting on its own, will allow us to prove two lemmas that are directly involved in the proof of the sharp bounds for commutators in Section 2.5.

Lemma 2.2.4 (John-Nirenberg inequality). *There are absolute constants $0 \leq \alpha_X < 1 < \beta_X$ such that*

$$(2.6) \quad \sup_Q \frac{1}{\mu(Q)} \int_Q \exp \frac{\alpha_X}{\|b\|_{BMO}} |b(y) - b_Q| d\mu(y) \leq \beta_X.$$

In fact, we can take $\alpha_X = \ln \sqrt[3]{2^\varepsilon}$, where $0 < \varepsilon < 1$ is an absolute constant.

Proof of Lemma 2.2.4. Suppose that b is bounded, so that the above supremum makes sense for all α . Then we will prove (2.6) with a bound independent of $\|b\|_\infty$.

Fix a cube Q_0 and a dyadic cube $Q \in \mathcal{D}(Q_0)$. Denote by \hat{Q} the parent of Q , namely, the unique element in $\mathcal{D}(Q_0)$ which contains Q and lies in the previous generation of cubes.

Then we can show that

$$(2.7) \quad |b_Q - b_{\hat{Q}}| \leq \frac{1}{\varepsilon} \|b\|_{BMO},$$

where $0 < \varepsilon < 1$ is an absolute constant as in [4] (see also [25] or [59] for further details). Indeed,

$$\begin{aligned} |b_Q - b_{\hat{Q}}| &\leq \frac{1}{\mu(Q)} \int_Q |b - b_{\hat{Q}}| \\ &\leq \frac{\mu(\hat{Q})}{\mu(Q)} \frac{1}{\mu(\hat{Q})} \int_{\hat{Q}} |b - b_{\hat{Q}}| \\ &\leq \frac{1}{\varepsilon} \|b\|_{BMO}. \end{aligned}$$

Next, consider the Calderón–Zygmund decomposition of $(b - b_{Q_0}) \chi_{Q_0}$ described in [2, Thm. 2.7.] for the level $2\|b\|_{BMO}$. Then there exists a collection of pairwise disjoint cubes $\{Q_i\} \subset \mathcal{D}$, maximal with respect to inclusion, satisfying

$$2\|b\|_{BMO} < \frac{1}{\mu(Q_i)} \int_{Q_i} |(b - b_{Q_0}) \chi_{Q_0}| < 2C_X \|b\|_{BMO}$$

and

$$|(b - b_{Q_0}) \chi_{Q_0}| < 2\|b\|_{BMO}, \text{ on } (\cup Q_i)^c.$$

Clearly, $Q_i \subset Q_0$ for each j , and

$$\mu(\cup Q_i) \leq \frac{\|(b - b_{Q_0}) \chi_{Q_0}\|_{L^1}}{2\|b\|_{BMO}} \leq \frac{\mu(Q_0)}{2}.$$

Since the cubes Q_i are maximal, we have that $(|b - b_{Q_0}|)_{\hat{Q}_i} \leq 2\|b\|_{BMO}$. Next, using the last inequality together with (2.7) we get

$$|b_{Q_i} - b_{Q_0}| \leq |b_{Q_i} - b_{\hat{Q}_i}| + |b_{\hat{Q}_i} - b_{Q_0}| \leq \left(\frac{1}{\varepsilon} + 2\right) \|b\|_{BMO}.$$

2.2. USEFUL INEQUALITIES IN SPACES OF HOMOGENEOUS TYPE

Denote $X(\alpha) = \sup_Q \frac{1}{\mu(Q)} \int_Q \exp \frac{\alpha}{\|b\|_{BMO}} |b - b_Q| d\mu(x)$, which is finite since we are assuming that b is bounded. From the properties of the cubes Q_i we arrive at

$$\begin{aligned} & \frac{1}{\mu(Q_0)} \int_{Q_0} \exp \left(\frac{\alpha}{\|b\|_{BMO}} |b - b_{Q_0}| \right) d\mu(x) \\ & \leq \frac{1}{\mu(Q_0)} \int_{Q_0 \setminus \cup Q_i} e^{2\alpha} d\mu(x) \\ & \quad + \sum_j \frac{\mu(Q_i)}{\mu(Q_0)} \frac{1}{\mu(Q_i)} \left(\int_{Q_i} \exp \left(\frac{\alpha}{\|b\|_{BMO}} |b - b_{Q_i}| d\mu(x) \right) e^{(\frac{1}{\varepsilon} + 2)\alpha} \right) \\ & \leq e^{2\alpha} + \frac{1}{2} e^{(\frac{1}{\varepsilon} + 2)\alpha} X(\alpha). \end{aligned}$$

Taking the supremum over all cubes Q_0 , we get the bound

$$X(\alpha) \left(1 - \frac{1}{2} e^{(\frac{1}{\varepsilon} + 2)\alpha} \right) \leq e^{2\alpha},$$

which implies that $X(\alpha) \leq C$, if α is small enough.

Since $0 < \varepsilon < 1$, if we impose that $\frac{1}{2} e^{(\frac{1}{\varepsilon} + 2)\alpha} < 1$, then $\alpha < \frac{\varepsilon \ln 2}{2\varepsilon + 1}$. Therefore we can choose an smaller parameter α , such as $\alpha_X = \ln \sqrt[3]{2\varepsilon}$. \square

Now we will prove two lemmas involving the A_2 and A_∞ constants of a particular weight that we will need in the following, extended from those in [63] that were motivated by [26].

Lemma 2.2.5. *There are absolute constants γ and c such that*

$$[we^{2\operatorname{Re}zb}]_{A_2} \leq c[w]_{A_2}$$

for all

$$|z| \leq \frac{\gamma}{\|b\|_{BMO}([w]_{A_\infty} + [\sigma]_{A_\infty})},$$

where $\gamma = \max\{C_1\alpha_X, C_2\alpha_X\}$ with C_1 and C_2 absolute constants.

Proof of Lemma 2.2.5. In the following we will omit the measure in the integrals for the sake of simplicity. We will use the sharp reverse Hölder inequality twice, first for $r = 1 + \frac{1}{\tau[w]_{A_\infty}}$ and then for $r = 1 + \frac{1}{\tau[\sigma]_{A_\infty}}$. With the sharp reverse Hölder inequality for the first choice of r , Hölder's inequality and the sharp John-Nirenberg inequality (2.6), we have

$$\int_Q we^{2\operatorname{Re}zb} \leq \left(\int_Q w^r \right)^{1/r} \left(\int_Q e^{r'2\operatorname{Re}z(b-b_Q)} \right)^{1/r'} e^{2\operatorname{Re}zb_Q}$$

$$\leq \left(2 \int_Q w \right) \cdot \beta_X \cdot e^{2\operatorname{Re}zb_Q},$$

for $|z| \leq \frac{C_1 \alpha_X}{\|b\|_{BMO}[w]_{A_\infty}}$. Note that the constant α_X comes from (2.6) and C_1 is an absolute constant from the sharp reverse Hölder inequality since by our choice of r , $r' = C_1[w]_{A_\infty}$. We can also get a similar bound as above for the second choice of $r = 1 + \frac{1}{\tau[\sigma]_{A_\infty}}$, giving us

$$\int_Q w^{-1} e^{-2\operatorname{Re}zb} \leq \left(2 \int_Q w^{-1} \right) \cdot \beta_X \cdot e^{-2\operatorname{Re}zb_Q}$$

for $|z| \leq \frac{C_2 \alpha_X}{\|b\|_{BMO}[\sigma]_{A_\infty}}$. Multiplying these two estimates and taking supremum, we finish the proof by showing that for all z as in the assumption

$$\left(\int_Q w e^{2\operatorname{Re}zb} \right) \left(\int_Q w^{-1} e^{-2\operatorname{Re}zb} \right) \leq 4\beta_X^2[w]_{A_2}.$$

□

We also have a similar lemma for the A_∞ weight constant.

Lemma 2.2.6. *There are absolute constants γ' and c such that*

$$[w e^{2\operatorname{Re}zb}]_{A_\infty} \leq c[w]_{A_\infty}$$

for all

$$|z| \leq \frac{\gamma'}{\|b\|_{BMO}[w]_{A_\infty}},$$

where we can take

$$\gamma' = \frac{\alpha_X}{4\tau}$$

being τ an absolute constant from Lemma 2.2.2.

Proof of Lemma 2.2.6. The proof follows in a similar way as in [61, Lemma 7.4.], substituting the appropriate constants from the sharp John Nirenberg inequality in Lemma 2.6 and the sharp reverse Hölder inequality in Lemma 2.2.2. □

2.3 Strong estimates for Calderón–Zygmund operators

The purpose of this section is proving a strong estimate for Calderón–Zygmund operators from which we can derive a mixed A_1 – A_∞ estimate that will be very useful in the sequel. The proof of the main result in this section will make use of the following inequality of Coifman–Fefferman type proved using sparse operators.

Proposition 2.3.1. *Let T be a Calderón–Zygmund operator and let $1 < p < \infty$. If $w \in A_p$ then*

$$(2.8) \quad \int_X |Tf(x)|w(x)d\mu(x) \leq C[w]_{A_p} \int_X Mf(x)w(x)d\mu(x),$$

where C is an absolute constant that also depends on T .

Before proving Proposition 2.3.1 we need to recall the following lemma that will allow us to obtain the precise constant in (2.8) and that can be found in [49, Ex. 9.2.5] as well as in [46, p. 388] in the context of two weights.

Lemma 2.3.1. *Let μ be a positive doubling measure and $1 < p < \infty$. If $w \in A_p$ then*

$$(2.9) \quad \left(\frac{\mu(A)}{\mu(Q)} \right)^p \leq [w]_{A_p} \frac{w(A)}{w(Q)},$$

where $A \subset Q$ is a μ -measurable set and Q is a cube.

Proof of Proposition 2.3.1. We have that

$$\int_X |Tf(x)|w(x)d\mu(x) \leq C_{X,T} \sup_{\mathcal{Q},S} \int_X \left| \sum_{Q \in S} \left(\int_Q f(x) \right) \chi_Q(x) \right| w(x)d\mu(x)$$

by (2.5). Using (2.9), we obtain that

$$\begin{aligned} \int_X |Tf(x)|w(x)d\mu(x) &\leq C_{X,T} \sup_{\mathcal{Q},S} \sum_{Q \in S} \left(\int_Q |f(x)| \right) w(Q) \\ &\leq C_{X,T} \sup_{\mathcal{Q},S} [w]_{A_p} \sum_{Q \in S} \left(\int_Q f(x) \right) w(E(Q)) \\ &\leq C_{X,T} [w]_{A_p} \sup_{\mathcal{Q},S} \sum_{Q \in S} \int_{E(Q)} Mf(x)w(x)d\mu(x), \end{aligned}$$

Finally, since the family $E(Q)$ is disjoint, we can bound the above by

$$\begin{aligned} \int_X |Tf(x)|w(x)d\mu(x) &\leq C[w]_{A_p} \sup_{\mathcal{Q},S} \int_X Mf(x)w(x)d\mu(x) \\ &\leq C[w]_{A_p} \int_X Mf(x)w(x)d\mu(x), \end{aligned}$$

where C is an absolute constant that depends also on T , proving (2.8) as wanted. \square

It is possible to improve Proposition 2.3.1 replacing the A_p constant of the weight by the smaller A_∞ constant as it is shown in [62]. To prove this result, we will need to use the following Carleson-type lemma whose proof in spaces of homogeneous type follows the same argument as in the original proof in [61, Thm. 4.5.] taking into account the bound for the dyadic maximal function $M_w^\mathcal{D}$ in spaces of homogeneous type (see [111, Thm. 14.11] or [59, p. 28]).

Lemma 2.3.2. (*Dyadic embedding Carleson Lemma*) *Suppose that the non-negative numbers a_Q satisfy*

$$\sum_{Q \subseteq R} a_Q \leq Aw(R), \quad R \in \mathcal{D}.$$

Then, for all $p \in [1, \infty)$ and $f \in L^p(w)$,

$$\begin{aligned} \left(\sum_{Q \in \mathcal{D}} a_Q \left(\frac{1}{w(Q)} \int_Q f w d\mu \right)^p \right)^{1/p} &\leq A^{1/p} \|M_w^\mathcal{D} f\|_{L^p(w)} \\ &\leq A^{1/p} p' \|f\|_{L^p(w)}. \end{aligned}$$

Here we denote

$$M_w^\mathcal{D} f(x) = \sup_{Q \ni x} \frac{1}{w(Q)} \int_Q |f| w d\mu,$$

where the supremum is all over the dyadic cubes $Q \in \mathcal{D}$ containing x .

Lemma 2.3.3. *Let T be a Calderón–Zygmund operator and $w \in A_\infty$. Then*

$$(2.10) \quad \int_X |Tf(x)| w(x) d\mu(x) \leq C[w]_{A_\infty} \int_X Mf(x) w(x) d\mu(x),$$

where C is an absolute constant that also depends on T .

Proof. As a consequence of Theorem 2.1.5 we only need to prove (2.10) for sparse operators. More precisely, we are going to prove that for any sparse family $\mathcal{S} \subset \mathcal{D}$ the following inequality holds

$$\|\mathcal{A}_{\mathcal{S}, \mathcal{D}} f\|_{L^1(w)} \leq 8[w]_{A_\infty} \|Mf\|_{L^1(w)}.$$

Note that for $f \geq 0$, the left hand side in (2.10) equals

$$\begin{aligned} \sum_{Q \in \mathcal{S}} \int_Q f(x) d\mu(x) w(Q) &\leq \sum_{Q \in \mathcal{S}} \inf_{z \in Q} Mf(z) w(Q) \\ &\leq \sum_{Q \in \mathcal{S}} \left(\frac{1}{w(Q)} \int_Q (Mf(x))^{1/2} w(x) d\mu(x) \right)^2 w(Q). \end{aligned}$$

By the Carleson embedding theorem applied to $g = (Mf)^{1/2}$, we have

$$\sum_{Q \in \mathcal{S}} \left(\frac{1}{w(Q)} \int_Q g(x)w(x)d\mu(x) \right)^2 w(Q) \leq 4A \|g\|_{L^2(w)}^2 = 4A \|Mf\|_{L^1(w)},$$

provided that the Carleson condition

$$(2.11) \quad \sum_{\substack{Q \in \mathcal{S} \\ Q \subseteq R}} w(Q) \leq Aw(R),$$

holds. To prove (2.11),

$$\begin{aligned} \sum_{\substack{Q \in \mathcal{S} \\ Q \subseteq R}} w(Q) &\leq \sum_{\substack{Q \in \mathcal{S} \\ Q \subseteq R}} \frac{w(Q)}{\mu(Q)} \mu(Q) \\ &\leq 2 \sum_{\substack{Q \in \mathcal{S} \\ Q \subseteq R}} \inf_{z \in Q} M(w\chi_R)(z) \mu(E(Q)) \\ &\leq 2 \int_R Mf(w\chi_R)(z) d\mu(z) \\ &\leq 2[w]_{A_\infty} w(R), \end{aligned}$$

where $A = 2[w]_{A_\infty}$. □

Before stating our main result in this section, we prove the following lemma which was originally in [78] and shortly after it was improved in [79] avoiding an extra $\log p$ factor.

Lemma 2.3.4. *Let w be any weight and let $1 \leq p, r < \infty$. Then there is a constant $C = C_{X,T}$ such that*

$$\|Tf\|_{L^p((M_r w)^{1-p})} \leq Cp \|Mf\|_{L^p((M_r w)^{1-p})}.$$

The proof of this lemma is based in a variation of the Rubio de Francia algorithm that could be found in [92].

Proof of Lemma 2.3.4. We want to prove

$$\left\| \frac{Tf}{M_r w} \right\|_{L^p(M_r w)} \leq Cp \left\| \frac{Mf}{M_r w} \right\|_{L^p(M_r w)}.$$

By duality we have

$$\left\| \frac{Tf(x)}{M_r w(x)} \right\|_{L^p(M_r w)} = \left| \int_X Tf(x)h(x)d\mu(x) \right| \leq \int_X |Tf(x)||h(x)|d\mu(x),$$

for some h such that $\|h\|_{L^{p'}(M_r w)} = 1$. By a variation of Rubio de Francia's algorithm adapted to spaces of homogeneous type (see Theorem 1.4.4; for a proof in \mathbb{R}^n , see [92, Lemma 4.4]) with $s = p'$ and $v = M_r w$ there exists an operator R such that

1. $0 \leq h \leq R(h)$.
2. $\|R(h)\|_{L^{p'}(M_r w)} \leq 2C_{X,p}\|h\|_{L^{p'}(M_r w)}$.
3. $[R(h)(M_r w)^{1/p'}]_{A_1} \leq C_X p$.

Let us recall two facts: First, if two weights $w_1, w_2 \in A_1$, then $w = w_1 w_2^{1-p} \in A_p$ and $[w]_{A_p} \leq [w_1]_{A_1} [w_2]_{A_1}^{p-1}$. Second, by the Coifman-Rochberg theorem in spaces of homogeneous type [34, Prop. 5.32], if $r > 1$ then $(Mf)^{1/r} \in A_1$ and $[(Mf)^{1/r}]_{A_1} \leq C_X r'$. Combining these facts and 3. we obtain

$$\begin{aligned} [R(h)]_{A_3} &= [R(h)(M_r w)^{1/p'} ((M_r w)^{1/2p'})^{-2}]_{A_3} \\ &\leq [R(h)(M_r w)^{1/p'}]_{A_1} [(M_r w)^{1/2p'}]_{A_1}^2 \\ &\leq C_X p ((2p'r)')^2 \leq C_X p, \end{aligned}$$

since $(2p'r)' < 2$.

Thus, by Proposition 2.3.1 and using 1. and 2., we obtain

$$\begin{aligned} \int_X |Tf(x)|h(x)d\mu(x) &\leq \int_X |Tf(x)|R(h)(x)d\mu(x) \\ &\leq C[R(h)]_{A_3} \int_X M(f)(x)R(h)(x)d\mu(x) \\ &= C[R(h)]_{A_3} \int_X M(f)(x)R(h)(x) \\ &\quad \times (M_r w(x))^{-1} M_r w(x) d\mu(x) \\ &\leq C[R(h)]_{A_3} \left\| \frac{M(f)}{M_r w} \right\|_{L^p(M_r w)} \|R(h)\|_{L^{p'}(M_r w)} \\ &\leq C[R(h)]_{A_3} \left\| \frac{M(f)}{M_r w} \right\|_{L^p(M_r w)} \|h\|_{L^{p'}(M_r w)} \\ &\leq C p \left\| \frac{M(f)}{M_r w} \right\|_{L^p(M_r w)}, \end{aligned}$$

and we are done. □

Our main result in this section is the following which is nothing more than an adaption from that in [79].

2.3. STRONG ESTIMATES FOR CALDERÓN–ZYGmund OPERATORS

Theorem 2.3.1. *Let T be a Calderón–Zygmund operator and let $1 < p < \infty$. Then for any weight w and $r > 1$,*

$$(2.12) \quad \|Tf\|_{L^p(w)} \leq C p p' (r')^{\frac{1}{p'}} \|f\|_{L^p(M_r w)},$$

where C is an absolute constant that also depends on T .

Proof of Theorem 2.3.1. First we are going to prove the following inequality

$$(2.13) \quad \|Tf\|_{L^p(w)} \leq C p p' (r')^{1 - \frac{1}{p'}} \|f\|_{L^p(M_r w)},$$

from which follows (2.12). Indeed, it is clear that

$$1 - \frac{1}{pr} = 1 - \frac{1}{p} + \frac{1}{p} - \frac{1}{pr} = \frac{1}{p'} - \frac{1}{p'r'},$$

and since $t^{1/t} \leq 2$ when $t \geq 1$, it follows that

$$(r')^{1 - \frac{1}{pr}} = (r')^{\frac{1}{p'} - \frac{1}{p'r'}} \leq 2^{-1/p} (r')^{\frac{1}{p'}}.$$

Next consider the dual estimate of (2.13), namely

$$\|T^* f\|_{L^{p'}((M_r w)^{1-p'})} \leq C p p' (r')^{1 - \frac{1}{p'}} \|f\|_{L^{p'}(w^{1-p'})},$$

where T^* is the adjoint operator of T . Then, since T is also a Calderón–Zygmund operator we are under assumptions of Lemma 2.3.4 for T^* , we get

$$\left\| \frac{T^* f}{M_r w} \right\|_{L^{p'}(M_r w)} \leq C p' \left\| \frac{M f}{M_r w} \right\|_{L^{p'}(M_r w)}.$$

Using Hölder's inequality with exponent pr we have

$$\begin{aligned} & \frac{1}{\mu(Q)} \int_Q f w^{-1/p} w^{1/p} d\mu \\ & \leq \left(\frac{1}{\mu(Q)} \int_Q w^r d\mu \right)^{1/pr} \left(\frac{1}{\mu(Q)} \int_Q (f w^{-1/p})^{(pr)'} d\mu(x) \right)^{1/(pr)'}, \end{aligned}$$

and hence,

$$M(f)^{p'} \leq (M_r w)^{\frac{p'}{p}} M((f w^{-1/p})^{(pr)'})^{p'/(pr)'}$$

From this and the unweighted maximal theorem in spaces of homogeneous type that can be easily obtained from the proof in [48], changing the dimensional constant for an absolute one, we obtain

$$\begin{aligned}
 \left(\int_X \frac{M(f)^{p'}}{(M_r w)^{p'-1}} d\mu \right)^{1/p'} &\leq \left(\int_X M((f w^{-1/p})^{(pr)'})^{p'/(pr)'} d\mu \right)^{1/p'} \\
 &\leq C \left(\frac{p'}{p' - (pr)'} \right)^{1/(pr)'} \left(\int_X f^{p'} w^{1-p'} d\mu \right)^{1/p'} \\
 &= C \left(\frac{p'}{p' - (pr)'} \right)^{1/(pr)'} \left\| \frac{f}{w} \right\|_{L^{p'}(w)} \\
 &= C \left(\frac{rp - 1}{r - 1} \right)^{1-1/pr} \left\| \frac{f}{w} \right\|_{L^{p'}(w)} \\
 &\leq Cp \left(\frac{r}{r - 1} \right)^{1-1/pr} \left\| \frac{f}{w} \right\|_{L^{p'}(w)},
 \end{aligned}$$

proving (2.13) and consequently (2.12). □

From Theorem 2.3.1 we obtain the following estimates as immediate corollaries.

Corollary 2.3.1. *Let T be a Calderón–Zygmund operator and let $1 < p < \infty$. Then if $w \in A_\infty$ we obtain*

$$(2.14) \quad \|Tf\|_{L^p(w)} \leq C p p' [w]_{A_\infty}^{1/p'} \|f\|_{L^p(Mw)},$$

and if $w \in A_1$,

$$(2.15) \quad \|Tf\|_{L^p(w)} \leq C p p' [w]_{A_\infty}^{1/p'} [w]_{A_1}^{1/p} \|f\|_{L^p(w)},$$

where C is an absolute constant that also depends on T .

Proof of Corollary 2.3.1. The proofs of (2.14) and (2.15) are immediate. In the first case, the estimate is derived by applying the sharp reverse Hölder inequality for weights in the A_∞ class (Lemma 2.2.2) to (2.12) and using the fact that $r' \approx [w]_{A_\infty}$. The latter is a direct consequence of (2.14) since $w \in A_1$. □

Let us observe that a natural extension of our earlier results for Calderón–Zygmund operators can be obtained from a generalization of an extrapolation theorem due

to Duoandikoetxea [39] to spaces of homogeneous type. This generalization will allow us to get sharp bounds involving the A_p weight constant through an initial A_{p_0} boundedness assumption.

Theorem 2.3.2. *Assume that for some family of pairs of nonnegative functions (f, g) , for some $p_0 \in [1, \infty)$, and for all $w \in A_{p_0}$ we have*

$$\left(\int_X g^{p_0} w \right)^{1/p_0} \leq CN([w]_{A_{p_0}}) \left(\int_X f^{p_0} w \right)^{1/p_0},$$

where N is an increasing function and the constant C does not depend on w . Then for all $1 < p < \infty$ and all $w \in A_p$ we have

$$\left(\int_X g^p w \right)^{1/p} \leq CK(w) \left(\int_X f^p w \right)^{1/p},$$

where

$$K(w) = \begin{cases} N([w]_{A_p} (2\|M\|_{L^p(w)})^{p_0-p}), & \text{if } p < p_0; \\ N([w]_{A_p}^{\frac{p_0-1}{p-1}} (2\|M\|_{L^{p'}(w^{1-p'})})^{\frac{p-p_0}{p-1}}), & \text{if } p > p_0. \end{cases}$$

In particular, $K(w) \leq C_1 N(C_2 [w]_{A_p}^{\max\{1, \frac{p_0-1}{p-1}\}})$, for $w \in A_p$ where C_2 is an absolute constant.

Note that the proof of the previous result follows from Duoandikoetxea's proof except for the fact that we have to replace the sharp bound for the Hardy–Littlewood maximal function by the corresponding one in Buckley's theorem in [59, Prop. 7.13.], so the constant C_2 in the proof now depends on p and X .

As application of the last result we obtain an estimate for $L^p(w)$ norms with A_q weights for $q < p$.

Corollary 2.3.2. *Let T be an operator such that*

$$(2.16) \quad \|Tf\|_{L^p(w)} \leq CN([w]_{A_1}) \|f\|_{L^p(w)}$$

for all weights $w \in A_1$ and all $1 < p < \infty$, with C independent of w . Then we have

$$(2.17) \quad \|Tf\|_{L^p(w)} \leq CN([w]_{A_q}) \|f\|_{L^p(w)}$$

for all $w \in A_q$ and $1 \leq q < p < \infty$, with C independent of w . In particular, (2.17) holds with $N(t) = t$ if T is a Calderón–Zygmund operator.

The proof of this result follows directly from the proof in [39]. We only have to take into account that (2.16) holds as a consequence of (2.14) since $[w]_{A_\infty} \leq [w]_{A_1}$.

2.4 Weak estimates for Calderón–Zygmund operators

Our main theorem in this section is the following endpoint estimate for Calderón–Zygmund operators within the context of spaces of homogeneous type that is obtained as a consequence of Theorem 2.3.1.

Theorem 2.4.1. *Let T be a Calderón–Zygmund operator. Then for any weight w and $r > 1$,*

$$(2.18) \quad \|Tf\|_{L^{1,\infty}(w)} \leq C \log(e + r') \|f\|_{L^1(M_r w)},$$

where C is an absolute constant that also depends on T .

We also get the following estimates as corollaries of the above result choosing r as the sharp exponent in the reverse Hölder inequality for weights in the A_∞ class in the setting of spaces of homogeneous type (see Lemma 2.2.2) and taking into account that $r' \approx [w]_{A_\infty}$.

Corollary 2.4.1. *Let T be a Calderón–Zygmund operator. Then*

1. *If $w \in A_\infty$*

$$\|Tf\|_{L^{1,\infty}(w)} \leq C \log(e + [w]_{A_\infty}) \|f\|_{L^1(M_r w)}.$$

2. *If $w \in A_1$*

$$\|Tf\|_{L^{1,\infty}(w)} \leq C [w]_{A_1} \log(e + [w]_{A_\infty}) \|f\|_{L^1(w)}.$$

In both cases C is an absolute constant that also depends on T .

Before proving Theorem 2.4.1 we need to establish a lemma which follows similar ideas of [46, Ch. 4, Lemma 3.3].

Lemma 2.4.1. *Let T be a Calderón–Zygmund operator. If w is a weight and $a \in L^1(w)$ supported in a cube Q with $\int_Q a(y) d\mu(y) = 0$. Then, if we set $\tilde{Q} = LQ$ for L such that $L > \frac{1}{\eta} > 0$, the following inequality holds*

$$(2.19) \quad \int_{X \setminus \tilde{Q}} |T(a)(x)| w(x) d\mu(x) \leq C \int_X |a(x)| M w(x) d\mu(x),$$

with C an absolute constant depending on the kernel K .

Proof. Fix $y_0 \in X$. By Theorem 2.1.1, we have that $Q \subset B(y_0, C\delta^k)$. Now making use of the cancellation property of a , we obtain

$$\begin{aligned} \int_{X \setminus \tilde{Q}} |T(a)(x)|w(x)d\mu(x) &= \int_{X \setminus \tilde{Q}} \left| \int_Q K(x, y)a(y)d\mu(y) \right| w(x)d\mu(x) \\ &\leq \int_Q \int_{X \setminus \tilde{Q}} |K(x, y) - K(x, y_0)|w(x)d\mu(x)|a(y)|d\mu(y) \\ &\leq \int_Q I(y)|a(y)|d\mu(y). \end{aligned}$$

Then we only need to prove that I is bounded by $CMw(y)$ where $C = C_{X,K}$ is an absolute constant depending on the kernel K . For every $y \in Q$, using the smoothness property of K in the second variable since $\rho(y, y_0) \leq \eta\rho(x, y_0)$, we obtain

$$\begin{aligned} I(y) &= \int_{X \setminus \tilde{Q}} |K(x, y) - K(x, y_0)|w(x)d\mu(x) \\ &= \int_{X \setminus \tilde{Q}} \left(\frac{\rho(y, y_0)}{\rho(x, y_0)} \right)^\eta \frac{1}{\mu(B(y_0, \rho(x, y_0)))} w(x)d\mu(x) \\ &= \sum_{l=1}^{\infty} \int_{2^l Q \setminus 2^{l-1} Q} \left(\frac{\rho(y, y_0)}{\rho(x, y_0)} \right)^\eta \frac{1}{\mu(B(y_0, \rho(x, y_0)))} w(x)d\mu(x) \\ &\leq \sum_{l=1}^{\infty} \int_{2^l Q} \frac{2^\eta}{2^{l\eta}} \frac{\mu(2^l Q)}{\mu(2^{l-1} Q)} \frac{1}{\mu(2^l Q)} w(x)d\mu(x) \\ &\leq D_{X,K} \sum_{l=1}^{\infty} \frac{1}{2^{l\eta}} \frac{1}{\mu(2^l Q)} \int_{2^l Q} w(x)d\mu(x) \\ &\leq D_{X,K} Mw(y). \end{aligned}$$

Above we have used the fact that $\rho(y, y_0) < C\delta^k$ and $\rho(x, y_0) > \frac{C\delta^k}{\eta}$. Then, we have shown that (2.19) holds. □

Next, we are ready to prove the main theorem in this section.

Proof of Theorem 2.4.1. The proof of Theorem 2.4.1 is based on several ingredients that we will mention as we need them. We follow the proof of [93, Thm. 1.6]. We claim that the following inequality holds: for any $1 < p, r < \infty$

$$\begin{aligned} (2.20) \quad \|Tf\|_{L^{1,\infty}(w)} &= \sup_{\lambda > 0} \lambda w(\{y \in X : |Tf(y)| > \lambda\}) \\ &\leq C(p')^p (r')^{p-1} \|f\|_{L^1(M_r w)}, \end{aligned}$$

and this claim implies (2.18). Indeed, it suffices to fix $r > 1$ and choose $p = 1 + \frac{1}{\log r'}$. We then obtain

$$\begin{aligned} \|Tf\|_{L^{1,\infty}(w)} &\leq C(p')^{p-1}(1 + \log r')(r')^{p-1}\|f\|_{L^1(w)} \\ &\leq C \log(e + r')\|f\|_{L^1(M_r w)}. \end{aligned}$$

since $p' = 1 + \log r'$, $(p')^{p-1} = (1 + \log(r'))^{1/\log(r')} \leq e$, $(r')^{p-1} = (r')^{1/\log(r')} = e$ and $1 + \log r' = \log(er') \leq 2 \log(e + r')$.

We now prove (2.20). By the classical Calderón–Zygmund decomposition of a function $f \in L^\infty$ with compact support at a level λ , we obtain a family of non-overlapping dyadic cubes $\{Q_j\}$ satisfying

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} |f(x)| d\mu(x) \leq C_X \lambda.$$

Let $\Omega = \cup_j Q_j$. Since Q_j is a dyadic cube in \mathcal{D} as in Theorem 2.1.1, using the sandwich property we have that there exists a ball $B_j = B(x_c(Q_j), C\delta^k)$ such that $Q_j \subset B_j$. Let us denote $\tilde{Q}_j = (2\kappa + 1)B_j$ and $\tilde{\Omega} = \cup_j \tilde{Q}_j$. Observe that the dilation \tilde{Q}_j of the cube Q_j is actually a ball with center x_{B_j} and radius $(2\kappa + 1)r(B_j)$. Note that for simplicity we have denoted the center of a ball B by x_B and its radius by $r(B)$. Using the notation $f_Q = \frac{1}{\mu(Q)} \int_Q f(x) d\mu(x)$, we write $f = g + b$ where $g(x) = \sum_j f_{Q_j} \chi_{Q_j}(x) + f(x) \chi_{\Omega^c}$ and $b = \sum_j b_j$ with $b_j(x) = (f(x) - f_{Q_j}) \chi_{Q_j}(x)$. We have

$$\begin{aligned} w(\{y \in X : |Tf(y)| > \lambda\}) &\leq w(\tilde{\Omega}) + w(\{y \in (\tilde{\Omega})^c : |Tg(y)| > \lambda/2\}) \\ &\quad + w(\{y \in (\tilde{\Omega})^c : |Tb(y)| > \lambda/2\}) \equiv I + II + III. \end{aligned}$$

Now

$$\begin{aligned} I = w(\tilde{\Omega}) &\leq C \sum_j \frac{w(\tilde{Q}_j)}{\mu(\tilde{Q}_j)} \mu(Q_j) \leq \frac{C}{\lambda} \sum_j \frac{w(\tilde{Q}_j)}{\mu(\tilde{Q}_j)} \int_{Q_j} |f(x)| d\mu(x) \\ &\leq \frac{C}{\lambda} \sum_j \int_{Q_j} |f(x)| M w(x) d\mu(x) \leq \frac{C}{\lambda} \int_X |f(x)| M w(x) d\mu(x) \\ &\leq \frac{C}{\lambda} \int_X |f(x)| M_r w(x) d\mu(x). \end{aligned}$$

The second term is estimated as follows using Chebyshev's inequality and (2.12). For

each $p > 1$ we get

$$\begin{aligned}
 II &= w(\{x \in (\tilde{\Omega})^c : |T(g)(x)| > \lambda/2\}) \\
 &\leq C(p')^p (r')^{p/p'} \frac{1}{\lambda^p} \int_X |g|^p M_r(w\chi_{(\tilde{\Omega})^c}) d\mu(x) \\
 &\leq C(p')^p (r')^{p/p'} \frac{1}{\lambda} \int_X |g| M_r(w\chi_{(\tilde{\Omega})^c}) d\mu(x) \\
 &\leq C(p')^p (r')^{p/p'} \frac{1}{\lambda} \left(\int_{X \setminus \Omega} |g| M_r(w\chi_{(\tilde{\Omega})^c}) d\mu(x) + \int_{\Omega} |g| M_r(w\chi_{(\tilde{\Omega})^c}) d\mu(x) \right) \\
 &= C(p')^p (r')^{p/p'} \frac{1}{\lambda} (II_1 + II_2).
 \end{aligned}$$

It is clear that

$$II_1 = \int_{X \setminus \Omega} |g| M_r(w\chi_{(\tilde{\Omega})^c}) d\mu(x) \leq \int_X |f| M_r w d\mu(x).$$

Next, we estimate II_2 as follows

$$\begin{aligned}
 II_2 &= \int_{\Omega} |g| M_r(w\chi_{(\tilde{\Omega})^c})(x) d\mu(x) \\
 &\leq \sum_j \int_{Q_j} |f_{Q_j}| M_r(w\chi_{(\tilde{\Omega})^c})(x) d\mu(x) \\
 &\leq \sum_j \int_{Q_j} \frac{1}{\mu(Q_j)} \int_{Q_j} |f(y)| d\mu(y) M_r(w\chi_{(\tilde{\Omega})^c})(x) d\mu(x) \\
 &\leq C \sum_j \frac{1}{\mu(Q_j)} \int_{Q_j} |f(y)| d\mu(y) \inf_{Q_j} M_r(w\chi_{(\tilde{\Omega})^c})(y) \mu(Q_j) \\
 &\leq C \sum_j \int_{Q_j} |f(y)| M_r w(y) d\mu(y) \\
 &\leq C \int_X |f(y)| M_r w(y) d\mu(y),
 \end{aligned}$$

where we have used that for any $r > 1$, non-negative function w with $M_r w(x) < \infty$ a.e., cube Q_j and $x \in Q_j$ we have

$$(2.21) \quad M_r(w\chi_{\tilde{\Omega}^c})(x) \lesssim \inf_{y \in Q_j} M_r(w\chi_{\tilde{Q}_j^c})(y).$$

The above inequality can be found, for instance, in [46, p. 159] for the Hardy–Littlewood maximal operator M in the classical setting. We give a short prove of (2.21) here for the sake of completeness. First note that since $\tilde{\Omega}^c \subset \tilde{Q}_j^c$, we only need to prove that, for any $y \in Q_j$, we have

$$(2.22) \quad M_r(w\chi_{\tilde{Q}_j^c})(x) \lesssim M_r(w\chi_{\tilde{Q}_j^c})(y).$$

Next, to estimate $M_r(w\chi_{\tilde{Q}_j^c})(x)$, we choose a ball P such that $x \in P$ and $P \cap \tilde{Q}_j^c \neq \emptyset$. Now if we take any $\tilde{x} \in P \cap \tilde{Q}_j^c$, we get that

$$\begin{aligned} r(\tilde{Q}_j) &= (2\kappa + 1)r(B_j) \leq \rho(x_{B_j}, \tilde{x}) \\ &\leq \kappa(\rho(x_{B_j}, x) + \rho(x, \tilde{x})) \\ &\leq \kappa r(B_j) + \kappa^2(\rho(x, x_P) + \rho(x_P, \tilde{x})) \\ &\leq \kappa r(B_j) + 2\kappa^2 r(P). \end{aligned}$$

Therefore, we have proved that

$$r(B_j) \leq \frac{2\kappa^2}{1 + \kappa} r(P) < 2\kappa r(P),$$

since $\kappa > 0$. Set $\beta = 2\kappa$. Now, following the argument in [44, p. 124], since we have that $B_j \cap P \neq \emptyset$ and $r(B_j) \leq \beta r(P)$ for a certain $\beta > 0$, we have that there exists a constant $c_\beta = c_\kappa > 0$ such that $B_j \subset c_\beta P := P'$.

Now, we can write the following

$$\frac{1}{\mu(P)} \int_P (w\chi_{\tilde{Q}_j^c})^r(z) d\mu(z) \lesssim \frac{1}{\mu(P')} \int_{P'} (w\chi_{\tilde{Q}_j^c})^r(z) d\mu(z),$$

where the last inequality is up to an absolute constant. Now, since $y \in Q_j \subset P'$, by definition of the maximal function, the right hand side is bounded by $M_r(w\chi_{\tilde{Q}_j^c})(y)$. Since P is an arbitrary ball containing x , again by the definition of the maximal function, taking the supremum over all balls containing x , the left hand side is equal to $M_r(w\chi_{\tilde{Q}_j^c})(x)$, and we obtain (2.21).

Now, combining estimates II_1 and II_2 , we obtain

$$w(\{x \in (\tilde{\Omega})^c : |T(g)(x)| > \lambda/2\}) \leq \frac{C}{\lambda} (p')^p (r')^{p-1} \|f\|_{L^1(M_r, w)}.$$

Next we estimate the term III , using the estimate in (2.19) and replacing w by $w\chi_{X \setminus \tilde{Q}_j}$ we have

$$\begin{aligned} III &= w(\{y \in X \setminus (\tilde{\Omega})^c : |T(b)(y)| > \frac{\lambda}{2}\}) \\ &\leq \frac{C}{\lambda} \int_{X \setminus \tilde{\Omega}} |b(y)| w(y) d\mu(y) \\ &\leq \frac{C}{\lambda} \sum_j \int_{X \setminus \tilde{Q}_j} |b_j(y)| w(y) d\mu(y) \\ &\leq \frac{C}{\lambda} \sum_j \int_X |b_j(y)| M(w\chi_{X \setminus \tilde{Q}_j})(y) d\mu(y) \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{C}{\lambda} \sum_j \int_{Q_j} |b(y)| M(w\chi_{X \setminus \tilde{Q}_j})(y) d\mu(y) \\
 &\leq \frac{C}{\lambda} \sum_j \left(\int_{Q_j} |f(y)| M w(y) d\mu(y) + \int_{Q_j} |g(y)| M(w\chi_{X \setminus \tilde{Q}_j})(y) d\mu(y) \right) \\
 &= \frac{C}{\lambda} (III_1 + III_2).
 \end{aligned}$$

To conclude the proof we only need to estimate III_2 . However

$$\begin{aligned}
 III_2 &= \sum_j \int_{Q_j} |f_{Q_j}| M(w\chi_{\tilde{Q}_j^c})(y) d\mu(y) \\
 &\leq \sum_j \int_{Q_j} \frac{1}{\mu(Q_j)} \int_{Q_j} |f(x)| d\mu(x) M(w\chi_{\tilde{Q}_j^c})(y) d\mu(y) \\
 &\leq \sum_j \int_{Q_j} |f(x)| d\mu(x) \inf_{Q_j} M(w\chi_{\tilde{Q}_j^c}) \\
 &\leq \sum_j \int_{Q_j} |f(x)| M(w\chi_{\tilde{Q}_j^c})(x) d\mu(x) \\
 &\leq C \int_X |f(x)| M w(x) d\mu(x),
 \end{aligned}$$

Combining the three estimates we have proved (2.20) and this concludes the proof of the theorem. □

2.5 Sharp estimates for commutators

In this section we prove a sharp $A_2 - A_\infty$ bound for commutators and the k -th iterate commutator of a Calderón–Zygmund operator using the precise version of the John–Nirenberg inequality in Section 2.2. It is worth mentioning that the optimality of the exponents in these results follow from the corresponding results in \mathbb{R}^n which were obtained by building specific examples of weights for each operator. However, a new approach to derive the optimality of the exponents without building explicit examples can be found in [84].

Firstly, we prove the following bound for a Calderón–Zygmund operator that will allow to get sharp $A_2 - A_\infty$ bounds for the commutators in spaces of homogeneous type adapting the corresponding result in [61].

Theorem 2.5.1. *Let T be a Calderón–Zygmund operator and $w \in A_2$. Then the following sharp weighted bound in an space of homogeneous type holds:*

$$\|T\|_{L^2(w)} \leq C[w]_{A_2}^{1/2}([w]_{A_\infty} + [\sigma]_{A_\infty})^{1/2}.$$

Note that it suffices to prove this result for sparse operators as it is done in [57, Sect. 2D] as a consequence of Theorem 2.1.5, which allows us to control a Calderón–Zygmund operator by a supremum of sparse operators in an space of homogeneous type. Observe that an equivalent result from that in [57] is proved in [107, Thm. 2.1.]. The latter works in a more general setting that also includes the case of spaces of homogeneous type. Because of that fact, only a brief sketch is given below.

Proof of Theorem 2.5.1. As stated in [57, Sect. 2D], Theorem 2.5.1 follows from verifying the following testing conditions:

1. $\|S_Q(\sigma \cdot \chi_Q)\|_{L^2(w)} \leq C_1 \|\chi_Q\|_{L^2(\sigma)}$
2. $\|S_Q(w \cdot \chi_Q)\|_{L^2(\sigma)} \leq C_2 \|\chi_Q\|_{L^2(w)}$

where

$$S_Q f = \sum_{\substack{L \in S \\ L \subseteq Q}} \left(\int_L f \right) \chi_L,$$

and S is a sparse family (this is a sparse operator). In fact, for any two weights w and σ (and, in particular for $\sigma = w^{1-p'}$), it follows

$$\sup_f \frac{\|S(f\sigma)\|_{L^p(w)}}{\|f\|_{L^p(\sigma)}} \simeq \sup_{Q \in \mathcal{D}} \frac{\|S_Q(\sigma)\|_{L^p(w)}}{\sigma(Q)^{1/p}} + \sup_{Q \in \mathcal{D}} \frac{\|S_Q(w)\|_{L^{p'}(\sigma)}}{w(Q)^{1/p'}}.$$

To verify the testing conditions, one simply follows the argument outlined in [57, Sect. 5A] to prove

$$\frac{\|S_Q(\sigma\chi_Q)\|_{L^2(w)}}{\sigma(Q)^{1/2}} \lesssim [w]_{A_2}^{1/2} [w]_{A_\infty}^{1/2},$$

and by symmetry,

$$\frac{\|S_Q(w\chi_Q)\|_{L^2(\sigma)}}{w(Q)^{1/2}} \lesssim [w]_{A_2}^{1/2} [\sigma]_{A_\infty}^{1/2}.$$

□

And finally as a corollary of the previous result and using a precise version of the John–Nirenberg inequality proved in Section 2.2, we prove the following generalized sharp weighted bound for the commutator and the k -th iterate commutator of a Calderón–Zygmund operator following the approach developed in [26].

Corollary 2.5.1. *Let T be a Calderón–Zygmund operator defined on an space of homogeneous type and $b \in BMO$. Then*

$$(2.23) \quad \|[T, b](f)\|_{L^2(w)} \leq C[w]_{A_2}^{1/2}([w]_{A_\infty} + [\sigma]_{A_\infty})^{3/2} \|b\|_{BMO} \|f\|_{L^2(w)}.$$

where C is an absolute constant. In general, for the k -th iterate commutator we get the following estimate

$$(2.24) \quad \|[T, b]_k(f)\|_{L^2(w)} \leq C[w]_{A_2}^{1/2}([w]_{A_\infty} + [\sigma]_{A_\infty})^{k+1/2} \|b\|_{BMO} \|f\|_{L^2(w)}.$$

Proof of Corollary 2.5.1. Firstly, we start proving (2.23). Let us “conjugate” the operator T as follows, that is, for any complex number z we define

$$(2.25) \quad T_z(f) = e^{zb}T(e^{-zb}f).$$

By using the Cauchy integral theorem, we get for appropriate functions,

$$[T, b](f) = \frac{d}{dz}T_z(f)|_{z=0} = \frac{-1}{2\pi i} \int_{|z|=\varepsilon} \frac{T_z(f)}{z^2} dz, \quad \varepsilon > 0,$$

Therefore we can write

$$\begin{aligned} \|[T, b](f)\|_{L^2(w)} &= \left\| (2\pi i)^{-1} \int_{|z|=\varepsilon} \frac{T(fe^{-zb})}{z^2} e^{zb} \right\|_{L^2(w)} \\ &\leq \frac{1}{C\varepsilon^2} \int_{|z|=\varepsilon} \left(\int_X |T(fe^{-zb})e^{zb}|^2 w d\mu(x) \right)^{1/2} |dz| \\ &= \frac{C}{\varepsilon} \|T(fe^{-zb})\|_{L^2(we^{2Rezb})} \\ &\leq \frac{C}{\varepsilon} [we^{2Rezb}]_{A_2}^{1/2} ([we^{2Rezb}]_{A_\infty} + [\sigma e^{2Rezb}]_{A_\infty})^{1/2} \\ &\quad \times \left(\int_X |fe^{-zb}|^2 we^{2Rezb} d\mu \right)^{1/2} \\ &= \frac{C}{\varepsilon} [we^{2Rezb}]_{A_2}^{1/2} ([we^{2Rezb}]_{A_\infty} + [\sigma e^{2Rezb}]_{A_\infty})^{1/2} \\ &\quad \times \left(\int_X |f|^2 w d\mu(x) \right)^{1/2}, \end{aligned}$$

where we have used the Minkowski inequality for integrals and the A_2 theorem for spaces of homogeneous type [4]. We also have

$$(2.26) \quad [we^{2Rezb}]_{A_2} \leq c[w]_{A_2}$$

and similarly for the A_∞ constants, for all $|z| \leq \frac{\delta}{\|b\|_{BMO}([w]_{A_\infty} + [\sigma]_{A_\infty})}$ where the δ is the minimum of the absolute constants from the corresponding lemmas.

All that remains is to bound

$$\frac{C}{\varepsilon} [w]_{A_2} \|f\|_{L^2(w)}.$$

Since $|z| = \varepsilon$ we are restricted to certain ε by (2.26), so we choose $\varepsilon = \frac{\delta}{\|b\|_{BMO}([w]_{A_\infty} + [\sigma]_{A_\infty})}$, so that

$$\frac{1}{\varepsilon} = \frac{1}{\delta} ([w]_{A_\infty} + [\sigma]_{A_\infty}) \|b\|_{BMO}$$

as wanted.

Putting everything together gives us the desired bound

$$\|[T, b](f)\|_{L^2(w)} \leq C [w]_{A_2}^{1/2} ([w]_{A_\infty} + [\sigma]_{A_\infty})^{3/2} \|b\|_{BMO} \|f\|_{L^2(w)}.$$

Finally, to prove the general estimate (2.24), we use again the Cauchy integral theorem to write the k -th commutator for appropriate functions as

$$[T, b]_k(f) = \frac{d^k}{dz^k} T_z(f)|_{z=0} = \frac{(-1)^k k!}{2\pi i} \int_{|z|=\varepsilon} \frac{T_z(f)}{z^{k+1}} dz, \quad \varepsilon > 0,$$

where T_z is defined as in (2.25). Then, following the computation for $[T, b]$ we can arrive at the desired bound for $[T, b]_k$.

□

Remark 2.5.1. *Corollary 2.5.1 can be proved under the weaker assumption that T is a linear operator that satisfies the sharp mixed $A_2 - A_\infty$ in spaces of homogeneous type.*

Weighted bounds for the multisublinear maximal operator

The aim of this chapter is to prove multilinear analogues of some results mentioned in the introduction of this dissertation in weighted Lebesgue spaces. Our main result in this chapter is a sharp mixed $A_{\vec{p}} - A_{\infty}$ bound for the multilinear maximal function that, as it happens in the linear setting, improves Buckley's one-constant bound. With respect to Buckley's estimate we only are able to give some partial results. We also prove a variety of results in the multiple two-weight setting using an adapted version of a Carleson-type lemma.

3.1 Some basics on multilinear theory of weights

Along this section we recall some basic notions and results related to the multilinear maximal function and the multiple theory of weights developed in [80].

Given $\vec{f} = (f_1, \dots, f_m)$, we define the multi(sub)linear maximal operator \mathcal{M} by

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i,$$

where the supremum is taken over all cubes containing x . The importance of this operator, that is smaller than the m -fold product of Hardy–Littlewood maximal functions, stems from the fact that it controls in several ways the class of multilinear Calderón–Zygmund operators as it was shown in [80].

However, this relationship becomes clearer when characterizing the weighted L^p spaces for which both operators are bounded. Let us first consider weights w_1, \dots, w_m and v and let us denote $\vec{w} = (w_1, \dots, w_m)$. Also let $1 < p_1, \dots, p_m < \infty$ and p be numbers such that $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ and denote $\vec{P} = (p_1, \dots, p_m)$.

The following theorem from [80] can be seen as a natural extension to the multilinear setting of Muckenhoupt's two-weight theorem.

Theorem 3.1.1. *Let $1 \leq p_j < \infty, j = 1, \dots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. Let ν and w_j be weights. Then the inequality*

$$(3.1) \quad \|\mathcal{M}(\vec{f})\|_{L^{p, \infty}(\nu)} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}$$

holds for any \vec{f} if and only if

$$(3.2) \quad \sup_Q \left(\frac{1}{|Q|} \int_Q \nu \right)^{1/p} \prod_{j=1}^m \left(\frac{1}{|Q|} \int_Q w_j^{1-p'_j} \right)^{1/p'_j} < \infty,$$

where $\left(\frac{1}{|Q|} \int_Q w_j^{1-p'_j} \right)^{1/p'_j}$ in the case $p_j = 1$ is understood as $(\inf_Q w_j)^{-1}$.

Let us remark here that if we denote

$$[v, \vec{w}]_{A_{\vec{p}}} := \sup_Q \left(\frac{1}{|Q|} \int_Q \nu \right) \prod_{j=1}^m \left(\frac{1}{|Q|} \int_Q w_j^{1-p'_j} \right)^{p/p'_j},$$

then the best constant appearing in (3.1) is comparable to $[v, \vec{w}]_{A_{\vec{p}}}^{1/p}$. Also observe that condition (3.2) combined with Lebesgue differentiation theorem also suggests the following way to define an analogue of the Muckenhoupt A_p classes in the multiple setting.

Definition 3.1.1. Let $1 \leq p_1, \dots, p_m < \infty$. Given $\vec{w} = (w_1, \dots, w_m)$, set

$$\nu_{\vec{w}} := \prod_{i=1}^m w_i^{p/p_i}.$$

We say that \vec{w} satisfies the $A_{\vec{p}}$ condition if

$$(3.3) \quad [\vec{w}]_{A_{\vec{p}}} := \sup_Q \left(\frac{1}{|Q|} \int_Q \nu_{\vec{w}} \right) \prod_{j=1}^m \left(\frac{1}{|Q|} \int_Q w_j^{1-p'_j} \right)^{p/p'_j} < \infty.$$

When $p_j = 1$, $\left(\frac{1}{|Q|} \int_Q w_j^{1-p'_j} \right)^{p/p'_j}$ is understood as $(\inf_Q w_j)^{-p}$.

As it is shown in [80], the multiple weight classes can be characterized in terms of the linear A_p classes. Observe that the following theorem also shows that as the index m increases, the $A_{\vec{p}}$ condition gets weaker.

Theorem 3.1.2. Let $\vec{w} = (w_1, \dots, w_m)$ and $1 \leq p_1, \dots, p_m < \infty$. Then $\vec{w} \in A_{\vec{p}}$ if and only if

$$(3.4) \quad \begin{cases} w_j^{1-p'_j} \in A_{mp'_j}, & j = 1, \dots, m \\ \nu_{\vec{w}} \in A_{mp}, \end{cases}$$

where the condition $w_j^{1-p'_j} \in A_{mp'_j}$ in the case $p_j = 1$ is understood as $w_j^{1/m} \in A_1$.

From (3.4) it is easy to see that when $p \geq 1$ we have

$$(3.5) \quad A_p \times \dots \times A_p \subsetneq A_{\min(p_1, \dots, p_m)} \times \dots \times A_{\min(p_1, \dots, p_m)} \subsetneq A_{p_1} \times \dots \times A_{p_m} \subsetneq A_{\vec{p}}.$$

Theorem 3.1.2 plays a fundamental role in the proof of the following characterization of the strong-type inequality for \mathcal{M} with one weight in [80]. Observe that the explicit (sharp) constants involved in it were not taken into account.

Theorem 3.1.3. *Let $1 < p_j < \infty, j = 1, \dots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. Then the inequality*

$$(3.6) \quad \|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}$$

holds for every \vec{f} if and only if \vec{w} satisfies the $A_{\vec{p}}$ condition.

3.2 One-weight estimates

Motivated by the previous results, in this section we prove the multilinear analogue of the mixed $A_p - A_\infty$ bound shown in [61] as well as some partial results related to Buckley's theorem in the multilinear setting in order to get sharp one-weight bounds for \mathcal{M} .

3.2.1 Sharp $A_{\vec{p}} - A_\infty$ multilinear estimate

Our main result in this section is the following $A_{\vec{p}} - A_\infty$ sharp bound for \mathcal{M} that, as it happens in the classical setting, improves Buckley's estimate.

Theorem 3.2.1. *Let $1 < p_i < \infty, i = 1, \dots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. Then the inequality*

$$(3.7) \quad \|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C_{n,m,\vec{p}} [w]_{A_{\vec{p}}}^{\frac{1}{p}} \prod_{i=1}^m ([\sigma_i]_{A_\infty})^{\frac{1}{p_i}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}$$

holds if $\vec{w} \in A_{\vec{p}}$, where $\sigma_i = w_i^{1-p'_i}$, $i = 1, \dots, m$. Furthermore the exponents are sharp in the sense that they cannot be replaced by smaller ones.

First, we will need to prove the following lemma.

Lemma 3.2.1. *For any non-negative integrable $f_i, i = 1, \dots, m$, there exist sparse families $\mathcal{S}_\alpha \in \mathcal{D}_\alpha$ such that for all $x \in \mathbb{R}^n$,*

$$\mathcal{M}(\vec{f})(x) \leq (2 \cdot 12^n)^m \sum_{\alpha=1}^{2^n} \mathcal{A}_{\mathcal{D}_\alpha, \mathcal{S}_\alpha}(\vec{f})(x),$$

where $\vec{f} = (f_1, \dots, f_m)$ and given a sparse family $\mathcal{S} = \{Q_j^k\}$ of cubes from a dyadic grid \mathcal{D} , the operator $\mathcal{A}_{\mathcal{D}, \mathcal{S}}$ is given by

$$\mathcal{A}_{\mathcal{D}, \mathcal{S}}(\vec{f}) = \sum_{j,k} \left(\prod_{i=1}^m (f_i)_{Q_j^k} \right) \chi_{Q_j^k}.$$

Proof of Lemma 3.2.1. First, by Proposition 1.2.1,

$$(3.8) \quad \mathcal{M}(\vec{f})(x) \leq 6^{mn} \sum_{\alpha=1}^{2^n} \mathcal{M}^{\mathcal{D}_\alpha}(\vec{f})(x),$$

where $\mathcal{M}^{\mathcal{D}_\alpha}$ denotes the multilinear maximal function defined with respect to \mathcal{D}_α . Consider $\mathcal{M}^d(\vec{f})$ taken with respect to the standard dyadic grid. We will use exactly the same argument as in the Calderón-Zygmund decomposition. For c_n which will be specified below and for $k \in \mathbb{Z}$ consider the sets

$$\Omega_k = \{x \in \mathbb{R}^n : \mathcal{M}^d(\vec{f})(x) > c_n^k\}.$$

Then we have that $\Omega_k = \cup_j Q_j^k$, where the cubes Q_j^k are pairwise disjoint with k fixed, and

$$c_n^k < \prod_{i=1}^m (f_i)_{Q_j^k} \leq 2^{mn} c_n^k.$$

From this and from Hölder's inequality,

$$\begin{aligned} |Q_j^k \cap \Omega_{k+1}| &= \sum_{Q_l^{k+1} \subset Q_j^k} |Q_l^{k+1}| \\ &< c_n^{-\frac{k+1}{m}} \sum_{Q_l^{k+1} \subset Q_j^k} \prod_{i=1}^m \left(\int_{Q_l^{k+1}} f_i \right)^{1/m} \\ &\leq c_n^{-\frac{k+1}{m}} \prod_{i=1}^m \left(\int_{Q_j^k} f_i \right)^{1/m} \leq 2^n c_n^{-1/m} |Q_j^k|. \end{aligned}$$

Hence, taking $c_n = 2^{m(n+1)}$, we obtain that the family $\{Q_j^k\}$ is sparse, and

$$\mathcal{M}^d(\vec{f})(x) \leq 2^{m(n+1)} \mathcal{A}_{\mathcal{D}, \mathcal{S}}(\vec{f})(x).$$

3.2. ONE-WEIGHT ESTIMATES

Applying the same argument to each $\mathcal{M}^{\mathcal{D}^\alpha}(\vec{f})$ and using (3.8), we get the statement of the lemma. \square

Next we proceed to the proof of Theorem 3.2.1.

Proof of Theorem 3.2.1. By (3.8), it suffices to prove the theorem for the dyadic maximal operators $\mathcal{M}^{\mathcal{D}^\alpha}$. Since the proof is independent of the particular dyadic grid, without loss of generality we consider \mathcal{M}^d taken with respect to the standard dyadic grid \mathcal{D} .

Let $a = 2^{m(n+1)}$. and $\Omega_k = \{x \in \mathbb{R}^n : \mathcal{M}^d(\vec{f})(x) > a^k\}$. We have seen in the proof of Lemma 3.2.1 that $\Omega_k = \cup_j Q_j^k$, where the family $\{Q_j^k\}$ is sparse and $a^k < \prod_{i=1}^m \frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i| \leq 2^{nm} a^k$. It follows that

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p \nu_{\vec{w}} dx &= \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} \mathcal{M}^d(\vec{f})^p \nu_{\vec{w}} dx \\ &\leq a^p \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i| dy_i \right)^p \nu_{\vec{w}}(Q_j^k) \\ &\leq a^p \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i| w_i^{\frac{1}{p_i}} w_i^{-\frac{1}{p_i}} dy_i \right)^p \nu_{\vec{w}}(Q_j^k) \\ &\leq a^p \sum_{k,j} \prod_{i=1}^m \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} dy_i \right)^{\frac{p}{\alpha_i}} \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} w_i^{-\frac{\alpha_i'}{p_i}} dy_i \right)^{\frac{p}{\alpha_i'}} \nu_{\vec{w}}(Q_j^k), \end{aligned}$$

where $\alpha_i = (p_i' r_i)'$ and r_i is the exponent in the sharp reverse Hölder inequality (1.26) for the weights σ_i which are in A_∞ for $i = 1, \dots, m$. Applying (1.26) for each σ_i , we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p \nu_{\vec{w}} dx &\leq a^p \sum_{k,j} \prod_{i=1}^m \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} dy_i \right)^{\frac{p}{\alpha_i}} \\ &\quad \times \left(2 \frac{1}{|Q_j^k|} \int_{Q_j^k} \sigma_i \right)^{\frac{p}{p_i'}} \nu_{\vec{w}}(Q_j^k) \\ &\leq C[\vec{w}]_{A_{\vec{P}}} \sum_{k,j} \prod_{i=1}^m \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} dy_i \right)^{\frac{p}{\alpha_i}} |Q_j^k|. \end{aligned}$$

Let E_j^k be the sets associated with the family $\{Q_j^k\}$. Using the properties of E_j^k and Hölder's inequality with the exponents p_i/p , we get

$$\begin{aligned}
 \int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p \nu_{\vec{w}} dx &\leq 2C[\vec{w}]_{A_{\vec{P}}} \sum_{k,j} \prod_{i=1}^m \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i(y_i)|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} dy_i \right)^{\frac{p}{\alpha_i}} |E_j^k| \\
 &\leq 2C[\vec{w}]_{A_{\vec{P}}} \sum_{k,j} \int_{E_j^k} \prod_{i=1}^m M \left(|f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} \right)^{\frac{p}{\alpha_i}} dx \\
 &\leq 2C[\vec{w}]_{A_{\vec{P}}} \int_{\mathbb{R}^n} \prod_{i=1}^m M \left(|f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} \right)^{\frac{p}{\alpha_i}} dx \\
 &\leq 2C[\vec{w}]_{A_{\vec{P}}} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} M \left(|f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} \right)^{\frac{p_i}{\alpha_i}} dx \right)^{\frac{p}{p_i}}.
 \end{aligned}$$

From this and by the boundedness of M ,

$$\begin{aligned}
 \int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p \nu_{\vec{w}} dx &\leq C[\vec{w}]_{A_{\vec{P}}} \prod_{i=1}^m ((p_i/\alpha_i)')^{\frac{p}{p_i}} \left\| |f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} \right\|_{L^{\frac{p_i}{\alpha_i}}(\mathbb{R}^n)}^{\frac{p}{p_i}} \\
 &\leq C[\vec{w}]_{A_{\vec{P}}} \prod_{i=1}^m (p_i' r_i')^{\frac{p}{p_i}} \|f_i\|_{L^{p_i}(w_i)}^p \\
 &\leq C[\vec{w}]_{A_{\vec{P}}} \prod_{i=1}^m ([\sigma_i]_{A_\infty})^{\frac{p}{p_i}} \|f_i\|_{L^{p_i}(w_i)}^p,
 \end{aligned}$$

where in next to last inequality we have used that $(p_i/\alpha_i)' \leq p_i' r_i'$ and in the last inequality we have used that $r_i' \approx [\sigma_i]_{A_\infty}$, for $i = 1, \dots, m$. This completes the proof of (3.7). \square

Let us show now the sharpness of the exponents in (3.7). Assume that $n = 1$ and $0 < \varepsilon < 1$. Let

$$w_i(x) = |x|^{(1-\varepsilon)(p_i-1)} \quad \text{and} \quad f_i(x) = x^{-1+\varepsilon} \chi_{(0,1)}(x), \quad i = 1, \dots, m.$$

On one hand, it is easy to check that $\nu_{\vec{w}} = |x|^{(1-\varepsilon)(pm-1)}$ and

$$(3.9) \quad [\vec{w}]_{A_{\vec{P}}} = [\nu_{\vec{w}}]_{A_{pm}} \approx (1/\varepsilon)^{mp-1}.$$

We also need to estimate $[\sigma_i]_{A_\infty}$, for $i = 1, \dots, m$. We have that

$$\sigma_i = w_i^{1-p_i'} = |x|^{\varepsilon-1} := \sigma.$$

Since σ is a power weight belonging to the A_1 class of weights, we obtain

$$(3.10) \quad [\sigma]_{A_\infty} \leq [\sigma]_{A_1} \approx \frac{1}{\varepsilon}.$$

Hence

$$(3.11) \quad \prod_{i=1}^m [\sigma]_{A_\infty}^{\frac{1}{p_i}} = [\sigma]_{A_\infty}^{\frac{1}{p}} = \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}.$$

Besides,

$$(3.12) \quad \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)} = (1/\varepsilon)^{1/p}.$$

On the other hand, we need to estimate $\|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})}$. First, let $f = x^{-1+\varepsilon}\chi_{(0,1)}(x)$ and observe that

$$\|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} = \|Mf\|_{L^{pm}(\nu_{\vec{w}})}^m$$

and if we pick $0 < x < 1$, we obtain

$$Mf(x) \geq \frac{1}{x} \int_0^x y^{-1+\varepsilon} dy = \frac{f(x)}{\varepsilon}.$$

Then the left-hand side of (3.7) can be bounded from below as follows:

$$(3.13) \quad \begin{aligned} \|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} &= \|Mf\|_{L^{pm}(\nu_{\vec{w}})}^m \geq \left(\frac{1}{\varepsilon}\right)^m \left(\int_{\mathbb{R}} f(x)^{mp} \nu_{\vec{w}}\right)^{\frac{m}{mp}} \\ &= \left(\frac{1}{\varepsilon}\right)^m \|f\|_{L^{pm}(\nu_{\vec{w}})}^m \\ &\approx \left(\frac{1}{\varepsilon}\right)^m \left(\frac{1}{\varepsilon}\right)^{1/p} \\ &\geq \left(\frac{1}{\varepsilon}\right)^{m+1/p}. \end{aligned}$$

since

$$\|f\|_{L^{mp}(\nu_{\vec{w}})}^{mp} \approx \frac{1}{\varepsilon},$$

and $\nu_{\vec{w}} \in A_{pm}$. By (3.9), (3.11) and (3.12) the right-hand side of (3.7) is at most $(1/\varepsilon)^{m+1/p}$. Since ε is arbitrary, this shows that the exponents $1/p$ and $1/p_i$ on the right-hand side of (3.7) cannot be replaced by smaller ones.

3.2.2 Buckley's multilinear theorem

Even though the result of Theorem 3.2.1 is a sharp and strictly smaller bound than the classical Buckley's estimate, it was also interesting to prove an extension of Buckley's result. However, contrary to the linear situation it seems that (3.7) cannot be used in order to derive a sharp multilinear version of Buckley's result. Perhaps, it is due to the fact that the right-hand side of (3.7) involves $m + 1$ suprema while the definition of $[\vec{w}]_{A_{\vec{P}}}$ involves only one supremum or else Lemma 3.2.2 below, where we show the relationship between the A_∞ constants of the σ_i weights and the $[\vec{w}]_{A_{\vec{P}}}$ constant, is not sharp. Anyway, we only could give some partial results expressed in the following theorem.

Theorem 3.2.2. *Let $1 < p_i < \infty, i = 1, \dots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. Denote by $\alpha = \alpha(p_1, \dots, p_m)$ the best possible power in*

$$(3.14) \quad \|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C_{n,m,p} [\vec{w}]_{A_{\vec{P}}}^\alpha \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.$$

Then we have the following results:

- (i) for all $1 < p_1, \dots, p_m < \infty$, $\frac{m}{mp-1} \leq \alpha \leq \frac{1}{p} \left(1 + \sum_{i=1}^m \frac{1}{p_i-1}\right)$;
- (ii) if $p_1 = p_2 = \dots = p_m = r > 1$, then $\alpha = \frac{m}{r-1}$.

In order to get an upper bound for α in part (ii) of Theorem 3.2.2, we shall need the following technical lemma. Its proof follows the same lines as the proof of [80, Th. 3.6].

Lemma 3.2.2. *Let $1 < p_j < \infty, j = 1, \dots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. If $\vec{w} \in A_{\vec{P}}$, then*

$$[\sigma_j]_{A_\infty} \leq C [\vec{w}]_{A_{\vec{P}}}^{p'_j/p}.$$

Proof. It was shown in [80, Th. 3.6] that if $\vec{w} \in A_{\vec{P}}$, then $\sigma_j \in A_{mp'_j}$. Our goal now is to check that

$$(3.15) \quad [\sigma_j]_{A_{mp'_j}} \leq [\vec{w}]_{A_{\vec{P}}}^{p'_j/p}.$$

Since $[\sigma_j]_{A_\infty} \leq C [\sigma_j]_{A_{mp'_j}}$, (3.15) would imply the statement of the lemma.

Fix $1 \leq j \leq m$, and define the numbers

$$q_j = p \left(m - 1 + \frac{1}{p_j} \right) \quad \text{and} \quad q_i = \frac{p_i}{p_i - 1} \frac{q_j}{p}, \quad i \neq j.$$

Since

$$\sum_{i=1}^m \frac{1}{q_i} = \frac{1}{m-1+1/p_j} \left(\frac{1}{p} + \sum_{i=1, i \neq j}^m (1-1/p_i) \right) = 1,$$

using Hölder inequality, we obtain

$$\begin{aligned} \int_Q w_j^{\frac{p}{p_j q_j}} &= \int_Q \left(\prod_{i=1}^m w_i^{\frac{p}{p_i q_j}} \right) \left(\prod_{i=1, i \neq j}^m w_i^{-\frac{p}{p_i q_j}} \right) \\ &\leq \left(\int_Q \prod_{i=1}^m w_i^{p/p_i} \right)^{1/q_j} \prod_{i=1, i \neq j}^m \left(\int_Q w_i^{-1/(p_i-1)} \right)^{1/q_i}. \end{aligned}$$

From this,

$$\begin{aligned} &\left(\int_Q w_j^{1-p'_j} \right) \left(\int_Q w_j^{\frac{p}{p_j q_j}} \right)^{\frac{q_j p_j}{p(p_j-1)}} \\ &\leq \left(\int_Q w_j^{1-p'_j} \right) \left[\left(\int_Q \prod_{i=1}^m w_i^{p/p_i} \right)^{1/q_j} \prod_{i=1, i \neq j}^m \left(\int_Q w_i^{1-p'_i} \right)^{1/q_i} \right]^{\frac{q_j p_j}{p(p_j-1)}} \\ &\leq \left(\int_Q w_j^{1-p'_j} \right) \left[\left(\int_Q \prod_{i=1}^m w_i^{p/p_i} \right) \prod_{i=1, i \neq j}^m \left(\int_Q w_i^{1-p'_i} \right)^{q_j/q_i} \right]^{\frac{p'_j}{p}}. \end{aligned}$$

Since

$$\frac{q_j}{q_i} = \frac{p \left(m-1 + \frac{1}{p_j} \right)}{\frac{p_i}{p_i-1} \frac{p \left(m-1 + \frac{1}{p_j} \right)}{p}} = \frac{p}{p'_i},$$

we obtain

$$\begin{aligned} &\left(\int_Q w_j^{1-p'_j} \right) \left(\int_Q w_j^{\frac{p}{p_j q_j}} \right)^{\frac{q_j p_j}{p(p_j-1)}} \\ &\leq \left(\int_Q w_j^{1-p'_j} \right) \left[\left(\int_Q \prod_{i=1}^m w_i^{p/p_i} \right) \prod_{i=1, i \neq j}^m \left(\int_Q w_i^{1-p'_i} \right)^{p/p'_i} \right]^{\frac{p'_j}{p}} \\ &\leq \left[\left(\int_Q \prod_{i=1}^m w_i^{p/p_i} \right) \prod_{i=1}^m \left(\int_Q w_i^{1-p'_i} \right)^{p/p'_i} \right]^{\frac{p'_j}{p}}. \end{aligned}$$

Therefore,

$$A_{mp'_j}(\sigma_j; Q) = \left(\frac{1}{|Q|} \int_Q w_j^{1-p'_j} \right) \left(\frac{1}{|Q|} \int_Q w_j^{\frac{p}{p_j q_j}} \right)^{\frac{q_j p_j}{p(p_j-1)}}$$

$$\begin{aligned} &\leq \left[\left(\frac{1}{|Q|} \int_Q \nu_{\vec{w}} \right) \prod_{i=1}^m \left(\frac{1}{|Q|} \int_Q w_i^{1-p'_i} \right)^{p/p'_i} \right]^{\frac{p'_j}{p}} \\ &= (A_{\vec{F}}(\vec{w}; Q))^{p'_j/p}, \end{aligned}$$

which proves (3.15). \square

Proof of Theorem 3.2.2. We start with part (i). Consider the example given after the proof of Theorem 3.2.1. Combining (3.9), (3.10), (3.12) and (3.13) with

$$\|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C[\vec{w}]_{A_{\vec{F}}}^\alpha \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)},$$

we obtain $m + 1/p \leq \alpha(mp - 1) + 1/p$ which yields $\alpha \geq \frac{m}{mp-1}$.

Further, by Theorem 3.2.1 and Lemma 3.2.2,

$$\alpha \leq \frac{1}{p} + \sum_{i=1}^m \frac{1}{p_i} \frac{p'_i}{p} = \frac{1}{p} \left(1 + \sum_{i=1}^m \frac{1}{p_i - 1} \right).$$

This completes the proof of part (i).

Suppose now that $p_1 = p_2 = \dots = p_m = r$. Then $p = r/m$, $\nu_{\vec{w}} = \left(\prod_{j=1}^m w_j \right)^{1/m}$. Denote

$$A_{\vec{F}}(\vec{w}; Q) = \left(\frac{1}{|Q|} \int_Q \nu_{\vec{w}} \right) \prod_{i=1}^m \left(\frac{1}{|Q|} \int_Q \sigma_i \right)^{(r-1)/m},$$

where $\sigma_i = w_i^{1-r'}$. Set also

$$\mathcal{M}_{\vec{\sigma}}(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_Q |f_i|.$$

We will follow the method of the proof of Buckley's theorem given in [76]. By (3.8), without loss of generality we may assume that the maximal operators considered below are dyadic. We get

$$\prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i| = A_{\vec{F}}(\vec{w}; Q)^{\frac{m}{r-1}} \left(\frac{|Q|}{\nu_{\vec{w}}(Q)} \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_Q |f_i| \right)^{\frac{r-1}{m}} \right)^{\frac{m}{r-1}}.$$

Hence,

$$\mathcal{M}(\vec{f})(x) \leq [\vec{w}]_{A_{\vec{F}}}^{\frac{m}{r-1}} M_{\nu_{\vec{w}}}(\mathcal{M}_{\vec{\sigma}}(\vec{f})^{\frac{r-1}{m}} \nu_{\vec{w}}^{-1})(x)^{\frac{m}{r-1}}.$$

From this, using Hölder's inequality and the boundedness of the weighted dyadic maximal operator with the implicit constant independent of the weight, we obtain

$$\begin{aligned}
 \|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} &\leq [\vec{w}]_{A_{\vec{P}}}^{\frac{m}{r-1}} \|M_{\nu_{\vec{w}}}(\mathcal{M}_{\vec{\sigma}}(\vec{f})^{\frac{r-1}{m}} \nu_{\vec{w}}^{-1})\|_{L^{r'}(\nu_{\vec{w}})}^{\frac{m}{r-1}} \\
 &\leq C[\vec{w}]_{A_{\vec{P}}}^{\frac{m}{r-1}} \|\mathcal{M}_{\vec{\sigma}}(\vec{f})\|_{L^{r/m}(\nu_{\vec{w}}^{1-r'})} \\
 &\leq C[\vec{w}]_{A_{\vec{P}}}^{\frac{m}{r-1}} \prod_{i=1}^m \|M_{\sigma_i}(f_i \sigma_i^{-1})\|_{L^r(\sigma_i)} \\
 &\leq C[\vec{w}]_{A_{\vec{P}}}^{\frac{m}{r-1}} \prod_{i=1}^m \|f_i\|_{L^r(w_i)}.
 \end{aligned}$$

This proves that $\alpha \leq \frac{m}{r-1}$. But if $p_1 = p_2 = \dots = p_m = r$, then $\frac{m}{mp-1} = \frac{m}{r-1}$. Hence, using part (i), we get that $\alpha = \frac{m}{r-1}$. \square

It is worth observing that later on, using similar techniques as the ones that we use for proving Theorem 3.2.1, Li, Moen and Sun in [81] showed the following sharp version of Buckley's result.

Theorem 3.2.3. *Let $1 < p_1, \dots, p_m < \infty$, $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$ and $\vec{w} \in A_{\vec{P}}$. Then*

$$(3.16) \quad \|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C_{m,n,\vec{P}} [\vec{w}]_{A_{\vec{P}}}^{\max(\frac{p'_1}{p}, \dots, \frac{p'_m}{p})} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.$$

Moreover, the exponent in (3.16) is sharp.

3.3 Two-weight estimates

In this section we state and prove some two-weight estimates for the multilinear maximal function that are the generalization from the corresponding ones in the linear setting.

Throughout this section w_1, \dots, w_m and v will be weights and we will denote $\vec{w} = (w_1, \dots, w_m)$. Also let $1 < p_1, \dots, p_m < \infty$ and p be numbers such that $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ and denote $\vec{P} = (p_1, \dots, p_m)$.

3.3.1 A multilinear Carleson lemma

Firstly we state the main tool of this section. This lemma extends to the multilinear setting a nonstandard formulation of the (dyadic) Carleson embedding theorem proved in [61] and it will be very useful in the sequel.

Lemma 3.3.1. *Suppose that the nonnegative numbers $\{a_Q\}_Q$ satisfy*

$$(3.17) \quad \sum_{Q \subset R} a_Q \leq A \int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx, \quad \forall R \in \mathcal{D}$$

where σ_i are weights for $i = 1, \dots, m$. Then for all $1 < p_i < \infty$ and $p \in (1, \infty)$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ and for all $f_i \in L^{p_i}(\sigma_i)$,

$$(3.18) \quad \left(\sum_{Q \in \mathcal{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_Q f_i(y_i) \sigma_i(y_i) dy_i \right)^p \right)^{1/p} \leq A \| \mathcal{M}_{\vec{\sigma}}^d(\vec{f}) \|_{L^p(\nu_{\vec{\sigma}})}$$

$$\leq A \prod_{i=1}^m p'_i \| f_i \|_{L^{p_i}(\sigma_i)},$$

where $\mathcal{M}_{\vec{\sigma}}^d(\vec{f}) = \sup_{\substack{Q \ni x \\ Q \in \mathcal{D}}} \prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_Q |f_i(y_i)| \sigma_i(y_i) dy_i$.

Proof of Lemma 3.3.1. Let us see the sum

$$\sum_{Q \in \mathcal{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_Q f_i(y_i) \sigma_i(y_i) dy_i \right)^p$$

as an integral on a measure space $(\mathcal{D}, 2^{\mathcal{D}}, \mu)$ built over the set of dyadic cubes \mathcal{D} , assigning to each $Q \in \mathcal{D}$ the measure a_Q . Thus

$$\begin{aligned} & \sum_{Q \in \mathcal{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_Q f_i(y_i) \sigma_i(y_i) dy_i \right)^p = \\ & = \int_0^\infty p \lambda^{p-1} \mu \left\{ Q \in \mathcal{D} : \prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_Q f_i(y_i) \sigma_i(y_i) dy_i > \lambda \right\} \\ & =: \int_0^\infty p \lambda^{p-1} \mu(\mathcal{D}_\lambda) d\lambda. \end{aligned}$$

Let us denote by \mathcal{D}_λ^* the set of maximal dyadic cubes R with the property that $\prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_R f_i(y_i) \sigma_i(y_i) dy_i > \lambda$. Then the cubes $R \in \mathcal{D}_\lambda^*$ are disjoint and their union is equal to the set $\{\mathcal{M}_{\vec{\sigma}}^d(\vec{f}) > \lambda\}$. Thus

$$\mu(\mathcal{D}_\lambda) = \sum_{Q \in \mathcal{D}_\lambda} a_Q \leq \sum_{R \in \mathcal{D}_\lambda^*} \sum_{Q \subset R} a_Q$$

$$\begin{aligned}
 &\leq A \sum_{R \in \mathcal{D}_\lambda^*} \int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx \\
 &= A \int_{\{\mathcal{M}_\sigma^d(\vec{f}) > \lambda\}} \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx.
 \end{aligned}$$

Then we obtain

$$\begin{aligned}
 \sum_{Q \in \mathcal{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q)} \int_Q f_i(y_i) \sigma_i(y_i) dy_i \right)^p &\leq A \int_0^\infty p \lambda^{p-1} \int_{\{\mathcal{M}_\sigma^d(\vec{f}) > \lambda\}} \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx d\lambda \\
 &= A \int_{\mathbb{R}^n} \mathcal{M}_\sigma^d(\vec{f})^p \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx \\
 &\leq A \int_{\mathbb{R}^n} \prod_{i=1}^m ((M_{\sigma_i}^d(f_i))^{p_i} \sigma_i)^{\frac{p}{p_i}} dx \\
 &\leq A \prod_{i=1}^m \left(\int_{\mathbb{R}^n} (M_{\sigma_i}^d(f_i))^{p_i} \sigma_i dx \right)^{\frac{p}{p_i}} \\
 &\leq A \prod_{i=1}^m (p_i')^p \left(\int_{\mathbb{R}^n} |f_i|^{p_i} \sigma_i dx \right)^{\frac{p}{p_i}},
 \end{aligned}$$

where we have used that $\mathcal{M}_\sigma^d(\vec{f}) \leq \prod_{i=1}^m M_{\sigma_i}^d(f_i)$, Hölder's inequality and the boundedness properties of $M_{\sigma_i}^d(f_i)$ in $L^{p_i}(\sigma_i)$. \square

3.3.2 A multilinear Sawyer's theorem

Next we establish the following generalization of Sawyer's theorem for which it is necessary to define the Sawyer's condition in the multilinear setting.

Definition 3.3.1. *We say that the pair (v, \vec{w}) satisfies the $S_{\vec{P}}$ condition if*

$$[v, \vec{w}]_{S_{\vec{P}}} = \sup_Q \left(\int_Q \mathcal{M}(\sigma \vec{\chi}_Q)^p v dx \right)^{\frac{1}{p}} \left(\prod_{i=1}^m \sigma_i(Q)^{\frac{1}{p_i}} \right)^{-1} < \infty,$$

where $\sigma \vec{\chi}_Q = (\sigma_1 \chi_Q, \dots, \sigma_m \chi_Q)$ and $\sigma_i = w_i^{1-p_i'}$ for all $i = 1, \dots, m$ and all the suprema in the above definitions are taken over all cubes Q in \mathbb{R}^n .

Very recently it was shown in [82] a multilinear version of Sawyer's theorem using a kind of monotone property on the weights. The condition that we establish here is a sort of reverse Hölder inequality in the multilinear setting.

Definition 3.3.2. We say that the vector \vec{w} satisfies the $RH_{\vec{P}}$ condition if there exists a positive constant C such that

$$(3.19) \quad \prod_{i=1}^m \left(\int_Q \sigma_i dx \right)^{\frac{p}{p_i}} \leq C \int_Q \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx,$$

where $\sigma_i = w_i^{1-p'_i}$ for $i = 1, \dots, m$. We denote by $[\vec{w}]_{RH_{\vec{P}}}$ the smallest constant C in (3.19).

Observe that when $m = 1$ this reverse Hölder condition is superfluous and we recover the linear result of Moen in [87].

Theorem 3.3.1. Let $1 < p_i < \infty$, $i = 1, \dots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. Let v and w_i be weights. If we suppose that $\vec{w} \in RH_{\vec{P}}$ then there exists a positive constant C such that

$$(3.20) \quad \|\mathcal{M}(\vec{f}\vec{\sigma})\|_{L^p(v)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}, \quad f_i \in L^{p_i}(\sigma_i),$$

where $\sigma_i = w_i^{1-p'_i}$, if and only if $(v, \vec{w}) \in S_{\vec{P}}$. Moreover, if we denote the smallest constant C in (3.20) by $\|\mathcal{M}\|$, we obtain

$$(3.21) \quad [v, \vec{w}]_{S_{\vec{P}}} \lesssim \|\mathcal{M}\| \lesssim [v, \vec{w}]_{S_{\vec{P}}} [\vec{w}]_{RH_{\vec{P}}}^{1/p}.$$

Here we make some remarks related to the previous theorem.

Remark 3.3.1. In the particular case when $v = \nu_{\vec{w}}$, the following statements are equivalent:

1. $\vec{w} \in A_{\vec{P}}$.
2. $\sigma_i = w_i^{1-p'_i} \in A_{mp'_i}$, for $i = 1, \dots, m$ and $\nu_{\vec{w}} \in A_{mp}$.
3. $(\nu_{\vec{w}}, \vec{w}) \in S_{\vec{P}}$.
4. There exists a positive constant C such that

$$(3.22) \quad \|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}, \quad f_i \in L^{p_i}(w_i).$$

3.3. TWO-WEIGHT ESTIMATES

Indeed, the equivalence between 1., 2. and 4. was proved in [80, Th. 3.6, Th. 3.7]. It can be easily seen that in this particular case $[\nu_{\vec{w}}, \vec{w}]_{S_{\vec{P}}} \lesssim \|\mathcal{M}\|$ where $\|\mathcal{M}\|$ denotes the smallest constant in (3.22) and $[\vec{w}]_{A_{\vec{P}}} \lesssim [\nu_{\vec{w}}, \vec{w}]_{S_{\vec{P}}}^p$. Therefore we have that 4. implies 3. and 3. implies 1.. So we have obtained that all the statements are equivalent.

Additionally, following Theorem 3.2.1 we also have that $\|\mathcal{M}\| \lesssim [\vec{w}]_{A_{\vec{P}}}^{1/p} \prod_{i=1}^m [\sigma_i]_{\infty}^{\frac{1}{p_i}}$. So, we have obtained

$$(3.23) \quad [\vec{w}]_{A_{\vec{P}}}^{1/p} \lesssim [\nu_{\vec{w}}, \vec{w}]_{S_{\vec{P}}} \lesssim \|\mathcal{M}\| \lesssim [\vec{w}]_{A_{\vec{P}}}^{1/p} \prod_{i=1}^m [\sigma_i]_{\infty}^{\frac{1}{p_i}}.$$

Remark 3.3.2. As we have observed in the previous remark, $RH_{\vec{P}}$ condition is not necessary when $v = \nu_{\vec{w}}$ in Theorem 3.3.1. We are not sure if this condition can be removed in the general case.

Proof of Theorem 3.3.1. It is clear that (3.20) implies the $S_{\vec{P}}$ condition without using that $(v, \vec{w}) \in RH_{\vec{P}}$. Thus, it remains to prove that $(v, \vec{w}) \in S_{\vec{P}}$ implies (3.20) to complete the proof of the theorem.

As we did before it suffices to prove the theorem for the dyadic maximal operators $\mathcal{M}^{\mathcal{D}^\alpha}$. Since the proof is independent of the particular dyadic grid, without loss of generality we consider \mathcal{M}^d taken with respect to the standard dyadic grid \mathcal{D} . Next we proceed as in the proof of Lemma 3.2.1. Let $a = 2^{m(n+1)}$ and for $k \in \mathbb{Z}$ consider the following sets

$$\Omega_k = \{x \in \mathbb{R}^n : \mathcal{M}^d(\vec{f}\vec{\sigma}) > a^k\}.$$

Then we have that $\Omega_k = \cup_j Q_j^k$, where the cubes Q_j^k are pairwise disjoint with k fixed, and

$$a^k < \prod_{i=1}^m \frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i(y_i)| \sigma_i(y_i) dy_i \leq 2^{mn} a^k.$$

It follows that

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f}\vec{\sigma})^p v dx &= \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} \mathcal{M}^d(\vec{f}\vec{\sigma})^p v dx \\ &\leq a^p \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} a^{kp} v dx \end{aligned}$$

$$= a^p \sum_{k,j} a^{kp} v(E_j^k),$$

since $\Omega_k \setminus \Omega_{k+1} = \cup_j E_j^k$ where the sets E_j^k are the sets associated with the family $\{Q_j^k\}$. Then, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f}\vec{\sigma})^p v dx &\leq a^p \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i| \sigma_i dy_i \right)^p v(E_j^k) \\ &= a^p \sum_{k,j} v(E_j^k) \left(\prod_{i=1}^m \frac{\sigma_i(Q_j^k)}{|Q_j^k|} \right)^p \left(\prod_{i=1}^m \frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} |f_i| \sigma_i dy_i \right)^p \\ &= a^p \sum_{Q \in \mathcal{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} |f_i| \sigma_i dy_i \right)^p, \end{aligned}$$

where $a_Q = v(E(Q)) \left(\prod_{i=1}^m \frac{\sigma_i(Q)}{|Q|} \right)^p$, if $Q = Q_j^k$ for some (k, j) where $E(Q)$ denotes the corresponding set E_j^k associated to Q_j^k , and $a_Q = 0$ otherwise. If we apply the Carleson embedding to these a_Q , we will find the desired result provided that

$$\sum_{Q \subset R} a_Q \leq A \int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx, \quad R \in \mathcal{D}.$$

For $R \in \mathcal{D}$, we obtain

$$\begin{aligned} \sum_{Q \subset R} a_Q &= \sum_{Q_j^k \subset R} v(E_j^k) \left(\prod_{i=1}^m \frac{\sigma_i(Q_j^k)}{|Q_j^k|} \right)^p \\ &= \sum_{Q_j^k \subset R} \int_{E_j^k} \left(\prod_{i=1}^m \frac{\sigma_i(Q_j^k)}{|Q_j^k|} \right)^p v(x) dx \\ &\leq \sum_{Q_j^k \subset R} \int_{E_j^k} (\mathcal{M}(\vec{\sigma}\vec{\chi}_R))^p v(x) dx \\ &\leq [v, \vec{w}]_{S_{\vec{P}}}^p \prod_{i=1}^m \sigma_i(R)^{\frac{p}{p_i}} \\ &\leq [v, \vec{w}]_{S_{\vec{P}}}^p [\vec{w}]_{RH_{\vec{P}}} \int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx, \end{aligned}$$

where in the next to last inequality we have used the $S_{\vec{P}}$ condition and in the last inequality we have used the $RH_{\vec{P}}$ condition. Thus, by Lemma 3.3.1 we get the desired result and the proof is complete.

□

3.3.3 A multilinear B_p theorem

We are also able to prove an analogue of the B_p theorem (1.28) extended to the multilinear setting for which we need to define previously a multilinear $B_{\vec{p}}$ condition as follows.

Definition 3.3.3. We say that the pair (v, \vec{w}) satisfies the $B_{\vec{p}}$ condition if

$$[v, \vec{w}]_{B_{\vec{p}}} := \sup_Q \frac{v(Q)}{|Q|} \left(\prod_{i=1}^m \frac{w_i(Q)}{|Q|} \right)^p \exp \left(\frac{1}{|Q|} \int_Q \log \prod_{i=1}^m w_i^{-\frac{p}{p_i}} dx \right) < \infty.$$

Theorem 3.3.2. Let $1 < p_i < \infty$, $i = 1, \dots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. Let v and w_i be weights. Then

$$(3.24) \quad \|\mathcal{M}(\vec{f\sigma})\|_{L^p(v)} \lesssim [v, \vec{\sigma}]_{B_{\vec{p}}}^{1/p} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}, \quad f_i \in L^{p_i}(\sigma_i),$$

where $\sigma_i = w_i^{1-p'_i}$, $\vec{\sigma} = (\sigma_1, \dots, \sigma_m)$ and $\vec{f\sigma} = (f_1\sigma_1, \dots, f_m\sigma_m)$.

Proof of Theorem 3.3.2. To prove this result we proceed using the standard argument as before. We obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f\sigma})^p v dx &\leq a^p \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i| \sigma_i dy_i \right)^p v(Q_j^k) \\ &= a^p \sum_{k,j} v(Q_j^k) \left(\prod_{i=1}^m \frac{\sigma_i(Q_j^k)}{|Q_j^k|} \right)^p \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q_j^k)} \int_{Q_j^k} |f_i| \sigma_i dy_i \right)^p \\ &\leq a^p [v, \vec{\sigma}]_{B_{\vec{p}}} \sum_{k,j} |Q_j^k| \exp \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} \log \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx \right) \\ &\quad \times \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q_j^k)} \int_{Q_j^k} |f_i| \sigma_i dy_i \right)^p \end{aligned}$$

And it follows

$$\int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f\sigma})^p v dx \leq a^p [v, \vec{\sigma}]_{B_{\vec{p}}} \sum_{Q \in \mathcal{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q_j^k)} \int_{Q_j^k} |f_i| \sigma_i dy_i \right)^p,$$

where in next to last inequality we have used the $B_{\vec{p}}$ condition and

$$a_Q = |Q| \exp \left(\frac{1}{|Q|} \int_Q \log \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx \right),$$

if $Q = Q_j^k$ for some (k, j) and $a_Q = 0$, otherwise.

Next if we apply Carleson embedding to these a_Q , we obtain that (3.24) holds provided that

$$\sum_{Q \subset R} a_Q \leq A \int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx, \quad R \in \mathcal{D}.$$

For $R \in \mathcal{D}$, we have

$$\begin{aligned} \sum_{Q \subset R} a_Q &\leq \sum_{Q_j^k \subset R} |Q_j^k| \exp \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} \log \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx \right) \\ &\leq 2 \sum_{Q_j^k} |E_j^k| \exp \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} \log \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx \right) \\ &\leq 2 \sum_{Q_j^k \subset R} \int_{E_j^k} M_0 \left(\prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} \chi_R \right) dx \\ &\leq 2 \int_{\mathbb{R}^n} M_0 \left(\prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} \chi_R \right) dx \\ &\leq 2e \int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}}, \end{aligned}$$

where M_0 is the (dyadic) logarithmic maximal function described in [61, Lemma 2.1] and also discussed in [116]. Here we have used that M_0 is bounded from L^1 into itself, and this concludes the proof of (3.24). \square

3.3.4 A mixed $A_{\vec{p}} - W_{\vec{p}}^\infty$ theorem

We also prove a mixed $A_p - A_\infty$ bound for \mathcal{M} that extends the one in [61] to the multilinear setting, for which we need to define the corresponding generalization of the Fujii–Wilson A_∞ constant.

Definition 3.3.4. The vector of weights \vec{w} satisfies $W_{\vec{P}}^\infty$ condition if

$$[\vec{w}]_{W_{\vec{P}}^\infty} = \sup_Q \left(\int_Q \prod_{i=1}^m M(w_i \chi_Q)^{\frac{p}{p_i}} dx \right) \left(\int_Q \prod_{i=1}^m w_i^{\frac{p}{p_i}} dx \right)^{-1} < \infty.$$

The corresponding theorem is the following.

Theorem 3.3.3. Let $1 < p_i < \infty$, $i = 1, \dots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. Let v and w_i be weights. Then

$$(3.25) \quad \|\mathcal{M}(\vec{f}\vec{\sigma})\|_{L^p(v)} \lesssim ([v, \vec{w}]_{A_{\vec{P}}}, [\vec{\sigma}]_{W_{\vec{P}}^\infty})^{1/p} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}, \quad f_i \in L^{p_i}(\sigma_i),$$

where $\sigma_i = w_i^{1-p'_i}$, $\vec{\sigma} = (\sigma_1, \dots, \sigma_m)$ and $\vec{f}\vec{\sigma} = (f_1\sigma_1, \dots, f_m\sigma_m)$.

Theorem 3.3.3. Proceeding as we did in the previous theorems, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f}\vec{\sigma})^p v dx &\leq a^p \sum_{k,j} v(Q_j^k) \left(\prod_{i=1}^m \frac{\sigma_i(Q_j^k)}{|Q_j^k|} \right)^p \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q_j^k)} \int_{Q_j^k} |f_i| \sigma_i dy_i \right)^p \\ &\leq a^p [v, \vec{w}]_{A_{\vec{P}}} \sum_{Q \in \mathcal{D}} a_Q \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q_j^k)} \int_{Q_j^k} |f_i| \sigma_i dy_i \right)^p, \end{aligned}$$

where we have used the $A_{\vec{P}}$ condition and we have denoted by a_Q the following numbers

$$a_Q = \prod_{i=1}^m \sigma_i(Q)^{\frac{p}{p_i}},$$

if $Q = Q_j^k$ for some (j, k) , and $a_Q = 0$, otherwise. Therefore it suffices to check that (3.17) holds for every $R \in \mathcal{D}$. Indeed,

$$\begin{aligned} \sum_{Q \subset R} a_Q &= \sum_{Q_j^k \subset R} \prod_{i=1}^m \sigma_i(Q_j^k)^{\frac{p}{p_i}} = \sum_{Q_j^k \subset R} \prod_{i=1}^m \left(\frac{\sigma_i(Q_j^k)}{|Q_j^k|} \right)^{\frac{p}{p_i}} |Q_j^k| \\ &\leq 2 \sum_{Q_j^k \subset R} |E_j^k| \prod_{i=1}^m \left(\frac{\sigma_i(Q_j^k)}{|Q_j^k|} \right)^{\frac{p}{p_i}} \\ &\leq 2 \sum_{Q_j^k \subset R} \int_{E_j^k} \prod_{i=1}^m M(\sigma_i \chi_R)^{\frac{p}{p_i}} dx \end{aligned}$$

$$\begin{aligned} &\leq 2 \int_R \prod_{i=1}^m M(\sigma_i \chi_R)^{\frac{p}{p_i}} dx \\ &\leq 2[\vec{\sigma}]_{W_{\vec{P}}^\infty} \int_R \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}} dx, \end{aligned}$$

where E_j^k are the sets associated with the cubes Q_j^k and we have used that $\vec{\sigma} \in W_{\vec{P}}^\infty$. Therefore we have proved (3.25). \square

Observe that the relationship between the mixed bound below and (3.7) is not clear since, a priori, the multiple $W_{\vec{P}}^\infty$ constant appearing in the theorem above is not comparable to the product of linear A_∞ Fujii-Wilson constants in Theorem 3.2.1.

3.3.5 A sufficient condition for the two-weighted boundedness of \mathcal{M}

Finally, we give a sufficient condition for the “two-weighted” boundedness of \mathcal{M} with precise bounds generalizing the corresponding linear result from [95] and its multilinear counterpart in [88]. Let X be a Banach function space. Given a cube Q , define the X -average of f over Q and the maximal operator M_X by

$$\|f\|_{X,Q} = \|\tau_{\ell_Q}(f\chi_Q)\|_X, \quad M_X f(x) = \sup_{Q \ni x} \|f\|_{X,Q},$$

where ℓ_Q denotes the side length of Q and where $\tau_\delta f = f(\delta x)$, $\delta > 0$, $x \in \mathbb{R}^n$.

Theorem 3.3.4. *Let $1 < p_i < \infty$, $i = 1, \dots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. Let X_i be a Banach function space such that $M_{X_i'}$ is bounded on $L^{p_i}(\mathbb{R}^n)$. Let u and v_1, \dots, v_m be the weights satisfying*

$$K = \sup_Q \left(\frac{u(Q)}{|Q|} \right)^{\frac{1}{p}} \prod_{i=1}^m \|v_i^{-1/p_i}\|_{X_i,Q} < \infty.$$

Then

$$\|\mathcal{M}(\vec{f})\|_{L^p(u)} \leq C_{n,m} K \prod_{i=1}^m \|M_{X_i'}\|_{L^{p_i}(\mathbb{R}^n)} \|f_i\|_{L^{p_i}(v_i)}.$$

This result can be seen as a two weight version of (3.7) when considering function spaces X given by $X = L^{rp'}$ for $1 < p, r < \infty$ so that

$$\|M_{X'}\|_{L^p(\mathbb{R}^n)} = \|M_{(rp)'}\|_{L^p(\mathbb{R}^n)} \approx (r')^{1/p}.$$

3.3. TWO-WEIGHT ESTIMATES

Another interesting example is given when considering the Orlicz space space $X = L_B$ where B is a Young function for which $\|M_{X'}\|_{L^p(\mathbb{R}^n)} = \|M_{\bar{B}}\|_{L^p(\mathbb{R}^n)}$ is finite. In particular if $B(t) = t^{p'} (\log(e+t))^{p'-1+\delta}$, $\delta > 0, 1 < p < \infty$ it follows from [95] that

$$\|M_{X'}\|_{L^p(\mathbb{R}^n)} = \|M_{\bar{B}}\|_{L^p(\mathbb{R}^n)} \approx \left(\frac{1}{\delta}\right)^{1/p}.$$

Proof of Theorem 3.3.4. We start exactly as in the proof of Theorem 3.2.1. It suffices to prove the main result for \mathcal{M}^d . Let $\Omega_k = \{x \in \mathbb{R}^n : \mathcal{M}^d(\vec{f})(x) > a^k\} = \cup_j Q_j^k$, where $a = 2^{m(n+1)}$. Then

$$\int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p u dx \leq a^p \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i| v_i^{\frac{1}{p_i}} v_i^{-\frac{1}{p_i}} dy_i \right)^p u(Q_j^k).$$

By the generalized Hölder inequality ((1.7)),

$$\frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i| v_i^{\frac{1}{p_i}} v_i^{-\frac{1}{p_i}} dy_i \leq \|f_i v_i^{\frac{1}{p_i}}\|_{X'_i, Q_j^k} \|v_i^{-\frac{1}{p_i}}\|_{X_i, Q_j^k}.$$

Combining this with the previous estimate, using the properties of the sets E_j^k associated with $\{Q_j^k\}$, and applying Hölder's inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p u dx &\leq a^p \sum_{k,j} \left(\prod_{i=1}^m \|f_i v_i^{\frac{1}{p_i}}\|_{X'_i, Q_j^k} \|v_i^{-\frac{1}{p_i}}\|_{X_i, Q_j^k} \right)^p \frac{u(Q_j^k)}{|Q_j^k|} |Q_j^k| \\ &\leq 2a^p K^p \sum_{k,j} \left(\prod_{i=1}^m \|f_i v_i^{\frac{1}{p_i}}\|_{X'_i, Q_j^k} \right)^p |E_j^k| \leq 2a^p K^p \left\| \prod_{i=1}^m M_{X'_i}(f_i v_i^{1/p_i}) \right\|_{L^p}^p \\ &\leq 2a^p K^p \prod_{i=1}^m \|M_{X'_i}(f_i v_i^{1/p_i})\|_{L^{p_i}}^p \leq 2a^p K^p \prod_{i=1}^m \|M_{X'_i}\|_{L^{p_i}}^p \|f_i\|_{L^{p_i}(v_i)}^p, \end{aligned}$$

which completes the proof. \square

Weighted bounds for multilinear singular integral operators

This chapter is devoted to weighted bounds for multilinear singular integral operators. We first introduce a few facts related to the notion of multilinear Calderón–Zygmund operators as well as their boundedness properties in Lebesgue spaces. Next we prove a local mean oscillation estimate for multilinear Calderón–Zygmund operators. This result will play an important role in the proof of our main result in this chapter. Namely, we establish a control in norm from above of multilinear Calderón–Zygmund operators by a sort of multilinear sparse operators using an extension of the techniques of A. Lerner in [77] to the multilinear setting.

An analogous result is also obtained for a wider class of multilinear singular integral operators with non-smooth kernels. The main feature of this class of operators is that their kernels satisfy significantly weaker regularity assumptions than the standard Calderón–Zygmund kernels.

As an application of our main result, we will derive a multilinear analogue of the so-called A_2 theorem for both types of operators. Some remarks concerning the multilinear version of the A_p theorem as well as some open questions are also discussed at the end of this chapter.

4.1 Basics on multilinear Calderón–Zygmund operators

Multilinear Calderón–Zygmund theory can be traced back to the works of R. Coifman and Y. Meyer [31] in the seventies. Their work was oriented towards the study of certain singular integral operators, such as the commutator of Calderón. This theory, far from being a mere generalization of the linear theory, appears naturally in harmonic analysis. The boundedness results for the bilinear Hilbert transform obtained by M. Lacey and C. Thiele [72, 73], motivated the development of a systematic treatment of general multilinear Calderón–Zygmund operators. In this respect, the work of L. Grafakos and R. Torres [53] set the bases of the unweighted multilinear Calderón–Zygmund theory, whereas the corresponding weighted results connecting multilinear Calderón–Zygmund operators and the $A_{\vec{p}}$ class of weights were addressed by A. Lerner et al. in [80].

First, let us recall the definition of multilinear Calderón–Zygmund operator introduced in [53].

Definition 4.1.1. *Let T be a multilinear operator initially defined on the m -fold product of Schwartz spaces and taking values into the space of tempered distributions,*

$$T : S(\mathbb{R}^n) \times \cdots \times S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n).$$

We say that T is an m -linear Calderón–Zygmund operator if, for some $1 \leq q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q , where $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and if there exists a function K , defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$(4.1) \quad T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m,$$

for all $x \notin \bigcap_{j=1}^m \text{supp } f_j$,

$$(4.2) \quad |K(y_0, y_1, \dots, y_m)| \leq \frac{A}{\left(\sum_{k,l=0}^m |y_k - y_l| \right)^{mn}},$$

and

$$(4.3) \quad |K(y_0, \dots, y_j, \dots, y_m) - K(y_0, \dots, y'_j, \dots, y_m)| \leq \frac{A|y_j - y'_j|^\epsilon}{\left(\sum_{k,l=0}^m |y_k - y_l| \right)^{mn+\epsilon}},$$

for some $\epsilon > 0$ and all $0 \leq j \leq m$, whenever $|y_j - y'_j| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_j - y_k|$.

The following theorem proved in [53] summarizes the basic boundedness properties of multilinear Calderón–Zygmund operators in Lebesgue spaces.

Theorem 4.1.1. *Let T be a multilinear Calderón–Zygmund operator. Let p, p_j numbers satisfying $\frac{1}{m} \leq p < \infty$, $1 \leq p_j \leq \infty$, and $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$. Then, all the statements below are valid:*

- (i) *When all $p_j > 1$, then T can be extended to a bounded operator from $L^{p_1} \times \cdots \times L^{p_m}$ into L^p , where L^{p_k} should be replaced by L_c^∞ if some $p_k = \infty$.*
- (ii) *When some $p_j = 1$, then T can be extended to a bounded operator from $L^{p_1} \times \cdots \times L^{p_m}$ into $L^{p,\infty}$, where again L^{p_k} should be replaced by L_c^∞ if some $p_k = \infty$.*

(iii) When all $p_j = \infty$, then T can be extended to a bounded operator from the m -fold product $L_c^\infty \times \dots \times L_c^\infty$ into BMO .

Let us note that when all the indexes $p_j = 1$, this result is a generalization of the classical weak type $(1, 1)$ estimate for singular integral operators. Namely, the corresponding endpoint space to bound singular integral operators in the multilinear setting is now the m -fold product $L^1 \times \dots \times L^1$ and, by homogeneity, it is mapped into $L^{1/m, \infty}$, i.e.,

$$(4.4) \quad T : L^1(\mathbb{R}^n) \times \dots \times L^1(\mathbb{R}^n) \longrightarrow L^{1/m, \infty}(\mathbb{R}^n).$$

As we mentioned in Chapter 3, the multilinear maximal function controls multilinear Calderón–Zygmund operators. This relationship is reflected in the following estimate that can be found in [80, Thm 3.2.], which was motivated by [1].

Theorem 4.1.2. *Let T be an m -linear Calderón–Zygmund operator and let $\delta > 0$ such that $\delta < 1/m$. Then for all \vec{f} in any product of $L^{q_j}(\mathbb{R}^n)$ spaces, with $1 \leq q_j < \infty$,*

$$(4.5) \quad M_\delta^\#(T(\vec{f}))(x) \leq CM(\vec{f})(x),$$

where $M^\#$ is the standard Fefferman–Stein sharp maximal function and $M_\delta^\#(f) = M^\#(|f|^\delta)^{1/\delta}$.

Finally, we recall two important results in [80] connecting the multilinear maximal function, Calderón–Zygmund operators and the $A_{\vec{P}}$ class of weights. The first one, which is a consequence of Theorem 4.1.2, can be viewed as a generalization of the Coifman–Fefferman theorem [28] to the multilinear case. The latter shows that the $A_{\vec{P}}$ classes are also the appropriate ones for the boundedness of multilinear Calderón–Zygmund operators.

Theorem 4.1.3. *Let T be a multilinear Calderón–Zygmund operator, let w be a weight in A_∞ and $p > 0$. There exists $C > 0$ (depending on $[w]_{A_\infty}^H$) so that the inequalities*

$$(4.6) \quad \|T(\vec{f})\|_{L^p(w)} \leq C\|\mathcal{M}(\vec{f})\|_{L^p(w)}$$

and

$$(4.7) \quad \|T(\vec{f})\|_{L^{p, \infty}(w)} \leq C\|\mathcal{M}(\vec{f})\|_{L^{p, \infty}(w)}$$

hold for all bounded functions \vec{f} with compact support.

Theorem 4.1.4. *Let T be a multilinear Calderón–Zygmund operator, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, and \vec{w} satisfy the $A_{\vec{p}}$ condition.*

(i) *If $1 < p_j < \infty$, $j = 1, \dots, m$, then*

$$(4.8) \quad \|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

(ii) *If $1 \leq p_j < \infty$, $j = 1, \dots, m$, and at least one of the $p_j = 1$, then*

$$(4.9) \quad \|T(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

4.2 Local mean oscillation estimate for multilinear Calderón–Zygmund operators

In this section we prove the following proposition that will be essential in the sequel and whose proof follows from the proof of Theorem 4.1.2.

Proposition 4.2.1. *Let T be a multilinear Calderón–Zygmund operator. For any cube $Q \subset \mathbb{R}^n$,*

$$(4.10) \quad \omega_\lambda(T(\vec{f}); Q) \leq c(T, \lambda, n) \sum_{l=0}^{\infty} \frac{1}{2^{l\epsilon}} \prod_{i=1}^m \left(\frac{1}{|2^l Q|} \int_{2^l Q} |f_i(y)| dy \right).$$

Proof of Proposition 4.2.1. Let $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i \chi_{Q^*}$, $i = 1, \dots, m$ and $Q^* = 2\sqrt{n}Q$. Then, we can write

$$\begin{aligned} \prod_{i=1}^m f_i(y_i) &= \prod_{i=1}^m (f_i^0(y_i) + f_i^\infty(y_i)) \\ &= \sum_{\alpha_1, \dots, \alpha_m \in \{0, \infty\}} f_1^{\alpha_1}(y_1) \dots f_m^{\alpha_m}(y_m) \\ &= \prod_{i=1}^m f_i^0 + \sum' f_1^{\alpha_1}(y_1) \dots f_m^{\alpha_m}(y_m), \end{aligned}$$

where each term in \sum' contains at least one $\alpha_i \neq 0$. Therefore

$$(4.11) \quad T(\vec{f})(z) = T(\vec{f}^0)(z) + \sum' T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(z),$$

and $\vec{f}^0 = (f_1^0, \dots, f_m^0)$. Next, by definition of local mean oscillation, we get

$$\begin{aligned} \omega_\lambda(T(\vec{f}); Q) &\leq ((T(\vec{f}) - c)\chi_Q)^*(\lambda|Q|) \\ &\leq \left((T(\vec{f}^0) + \sum' T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m}) - c)\chi_Q \right)^*(\lambda|Q|). \end{aligned}$$

We set now $c = \sum' T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(x)$ where x is the center of Q . Using the properties of rearrangements,

$$\begin{aligned} \omega_\lambda(T(\vec{f}); Q) &\leq (T(\vec{f}^0))^*(\lambda|Q|) \\ &\quad + \sum' \|T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(z) - T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(x)\|_{L^\infty(Q)}. \end{aligned}$$

For the first term we use the endpoint weak estimate for T in (4.4). Namely,

$$(T(\vec{f}^0))^*(\lambda|Q|) \leq \frac{C}{(\lambda|Q|)^m} \prod_{i=1}^m \int_{2\sqrt{n}Q} |f_i|.$$

In order to handle the second term we follow exactly the proof in [80, Thm 3.2], but taking into account that we only need the local estimate on a single cube. More precisely, if we consider the case when $\alpha_1 = \dots = \alpha_m = \infty$ and define $T(\vec{f}^\infty) = T(f_1^\infty, \dots, f_m^\infty)$, we obtain for any $x \in Q$

$$\begin{aligned} |T(\vec{f}^\infty)(z) - T(\vec{f}^\infty)(x)| &\leq C \int_{(\mathbb{R}^n \setminus 2\sqrt{n}Q)^m} \frac{|x-z|^\varepsilon}{(|z-y_1| + \dots + |z-y_m|)^{nm+\varepsilon}} \prod_{i=1}^m |f_i(y_i)| d\vec{y} \\ &\leq C \int_{(\mathbb{R}^n \setminus 2Q)^m} \frac{|x-z|^\varepsilon}{(|z-y_1| + \dots + |z-y_m|)^{nm+\varepsilon}} \prod_{i=1}^m |f_i(y_i)| d\vec{y} \\ &\leq C \sum_{l=1}^{\infty} \int_{(2^{l+1}Q)^m \setminus (2^lQ)^m} \frac{|x-z|^\varepsilon}{(|z-y_1| + \dots + |z-y_m|)^{nm+\varepsilon}} \prod_{i=1}^m |f_i(y_i)| d\vec{y} \\ &\leq C \sum_{l=1}^{\infty} \frac{|Q|^{\varepsilon/n}}{(2^l|Q|^{1/n})^{nm+\varepsilon}} \int_{(2^{l+1}Q)^m} \prod_{i=1}^m |f_i(y_i)| d\vec{y} \\ &\leq C \sum_{l=0}^{\infty} \frac{1}{2^{l\varepsilon}} \prod_{i=1}^m \frac{1}{|2^lQ|} \int_{2^lQ} |f_i(y_i)| d\vec{y}, \end{aligned}$$

where we have used the regularity of the kernel and that $|x-z| \simeq |Q|^{1/n}$ and $|z-y_i| \simeq |2^{k+1}Q|^{1/n} \simeq 2^k|Q|^{1/n}$. It remains to consider the rest of the terms in (4.11) such that $\alpha_{j_1} = \dots = \alpha_{j_k} = 0$ for some $\{j_1, \dots, j_k\} \subset \{1, \dots, m\}$ and $1 \leq k < m$. Using again the regularity of the kernel we get

$$\begin{aligned}
 & |T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(z) - T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(x)| \\
 & \leq C \prod_{j \in \{j_1, \dots, j_k\}} \int_{2\sqrt{n}Q} |f_j| dy_j \int_{(\mathbb{R}^n \setminus 2\sqrt{n}Q)^{m-k}} \frac{|x-z|^\varepsilon \prod_{j \notin \{j_1, \dots, j_k\}} |f_j| dy_j}{(|z-y_1| + \dots + |z-y_m|)^{nm+\varepsilon}} \\
 & \leq C \prod_{j \in \{j_1, \dots, j_k\}} \int_{2\sqrt{n}Q} |f_j| dy_j \sum_{l=1}^{\infty} \frac{|Q|^{\varepsilon/n}}{(2^l|Q|^{1/n})^{nm+\varepsilon}} \int_{(2^{l+1}Q)^{m-k}} \prod_{j \notin \{j_1, \dots, j_k\}} |f_j| dy_j \\
 & \leq C \sum_{l=1}^{\infty} \frac{|Q|^{\varepsilon/n}}{(2^l|Q|^{1/n})^{nm+\varepsilon}} \int_{(2^{l+1}Q)^m} \prod_{i=1}^m |f_i(y_i)| d\vec{y},
 \end{aligned}$$

and we arrive at the expression considered in the previous case. Therefore putting all together we obtain the desired result. \square

4.3 Control by multilinear sparse operators

In this section we generalize the result of A. Lerner in [76, 77] in which a Calderón-Zygmund operator is bounded from above by a supremum of sparse operators. We will need to introduce a natural multilinear extension of a dyadic sparse operator to the multilinear setting.

Definition 4.3.1. *Given a sparse family S over a dyadic grid \mathcal{D} , a multilinear sparse operator is an averaging operator over S of the following form*

$$\mathcal{A}_{\mathcal{D}, S}(\vec{f})(x) = \sum_{j, k} \left(\prod_{i=1}^m (f_i)_{Q_j^k} \right) \chi_{Q_j^k}(x).$$

Our main theorem in this section is the following.

Theorem 4.3.1. *Let T be a multilinear Calderón-Zygmund operator and let X be a Banach function space over \mathbb{R}^n equipped with Lebesgue measure. Then, for any appropriate \vec{f} ,*

$$\|T(\vec{f})\|_X \leq c_{T, m, n} \sup_{\mathcal{D}, S} \|\mathcal{A}_{\mathcal{D}, S}(|\vec{f}|)\|_X,$$

where the supremum is taken over arbitrary dyadic grids \mathcal{D} and sparse families $S \in \mathcal{D}$.

Next we proceed to the proof of Theorem 4.3.1.

4.3. CONTROL BY MULTILINEAR SPARSE OPERATORS

Proof of Theorem 4.3.1. Combining Proposition 4.2.1 and Theorem 1.6.2 with $Q_0 \in \mathcal{D}$, we get that there exists a sparse family $S = \{Q_j^k\} \in \mathcal{D}$ such that for a.e. $x \in Q_0$,

$$(4.12) \quad |T(\vec{f})(x) - m_{T(\vec{f})}(Q_0)| \leq C \sum_{l=0}^{\infty} \frac{1}{2^{l\epsilon}} \mathcal{T}_{S,l}(|\vec{f}|)(x),$$

where $c = c(n, T)$ and

$$\mathcal{T}_{S,l}(\vec{f})(x) = \sum_{j,k} \left(\prod_{i=1}^m (f_i)_{2^l Q_j^k} \right) \chi_{Q_j^k}(x).$$

By the weak type property of the m -linear Calderón-Zygmund operators (4.4), assuming, for instance, that each f_i is bounded and with compact support, we get $(T(\vec{f}))^*(+\infty) = 0$. Hence, it follows from (1.32) that $|m_{T(\vec{f})}(Q)| \rightarrow 0$ as $|Q| \rightarrow \infty$. Therefore, letting Q_0 to anyone of 2^n quadrants and using Fatou's lemma and (4.12), we obtain

$$\|T(\vec{f})\|_X \leq c(n, T) \sum_{l=0}^{\infty} \frac{1}{2^{l\epsilon}} \sup_{S \in \mathcal{D}} \|\mathcal{T}_{S,l}(\vec{f})\|_X.$$

Therefore, our goal here is to show that

$$(4.13) \quad \sup_{S \in \mathcal{D}} \|\mathcal{T}_{S,l}(\vec{f})\|_X \leq c(m, n) l \sup_{\mathcal{Q}, S} \|\mathcal{A}_{\mathcal{Q}, S} |\vec{f}|\|_X.$$

This estimate along with the previous ones would complete the proof. However we need to prove some intermediate results before we prove (4.13). Assume now that $f_i \geq 0$ and fix a sparse family $\mathcal{S} = \{Q_j^k\} \in \mathcal{D}$. Applying Proposition 1.2.1, we can decompose the cubes Q_j^k into 2^n disjoint families F_α such that for any $Q_j^k \in F_\alpha$ there exists a cube $P_{j,k}^{l,\alpha} \in \mathcal{D}_\alpha$ such that $2^l Q_j^k \subset P_{j,k}^{l,\alpha}$ and $\ell_{P_{j,k}^{l,\alpha}} \leq 6\ell_{2^l Q_j^k}$. Hence,

$$(4.14) \quad \mathcal{T}_{S,l}(\vec{f})(x) \leq 6^{nm} \sum_{\alpha=1}^{2^n} \sum_{j,k: Q_j^k \in F_\alpha} \left(\prod_{i=1}^m (f_i)_{P_{j,k}^{l,\alpha}} \right) \chi_{Q_j^k}(x),$$

where

$$\mathcal{T}_{l,\alpha}(\vec{f})(x) = \sum_{j,k} \left(\prod_{i=1}^m (f_i)_{P_{j,k}^{l,\alpha}} \right) \chi_{Q_j^k}(x).$$

We shall also need to define the following auxiliary operator

$$\mathcal{M}_{l,\alpha}(\vec{f}_{1,\dots,m-1}, g)(x) = \sum_{j,k} \left(\prod_{i=1}^{m-1} (f_i)_{P_{j,k}^{l,\alpha}} \right) \left(\frac{1}{|P_{j,k}^{l,\alpha}|} \int_{Q_j^k} g \right) \chi_{P_{j,k}^{l,\alpha}}(x),$$

which appears naturally in the following duality relation

$$(4.15) \quad \int_{\mathbb{R}^n} \mathcal{T}_{l,\alpha}(\vec{f})g \, dx = \int_{\mathbb{R}^n} \mathcal{M}_{l,\alpha}(\vec{f}_1, \dots, m-1, g) f_m \, dx.$$

Now we claim that $\mathcal{M}_{l,\alpha}$ satisfies that for any cube $Q \in \mathcal{D}_\alpha$,

$$(4.16) \quad \omega_\lambda(\mathcal{M}_{l,\alpha}(\vec{f}_1, \dots, m-1, g); Q) \leq c(\lambda, m, n) l g_Q \prod_{i=1}^{m-1} (f_i)_Q.$$

Indeed, let $Q \in \mathcal{D}_\alpha$ and let $x \in Q$. We have

$$\mathcal{M}_{l,\alpha}(\vec{f}_1, \dots, m-1, g)(x) = \sum_{k,j:P_{j,k}^{l,\alpha} \subset Q} + \sum_{k,j:Q \subseteq P_{j,k}^{l,\alpha}}.$$

The second sum is a constant (denote it by c) for $x \in Q$, while the first sum involves only the functions f_i which are supported in Q . We get the following simple estimate

$$(4.17) \quad |\mathcal{M}_{l,\alpha}(\vec{f}_1, \dots, m-1, g) - c|\chi_Q(x) \leq \prod_{i=1}^{m-1} M(f_i \chi_Q)(x) \mathcal{T}_l(g \chi_Q)(x),$$

where

$$\mathcal{T}_l g(x) = \sum_{j,k} \left(\frac{1}{|P_{j,k}^{l,\alpha}|} \int_{Q_{j,k}^k} g \right) \chi_{P_{j,k}^{l,\alpha}}(x).$$

It was proved in [77, Lemma 3.2] that $\|\mathcal{T}_l g\|_{L^{1,\infty}} \leq c(n)l\|g\|_{L^1}$. Using this estimate, the weak type (1, 1) of M , and reiterating the well known property of rearrangements, $(fg)^*(t) \leq f^*(t/2)g^*(t/2)$, $t > 0$, we get using (4.17)

$$\begin{aligned} \omega_\lambda(\mathcal{M}_{l,\alpha}(\vec{f}_1, \dots, m-1, g); Q) &\leq \left(\prod_{i=1}^{m-1} M(f_i \chi_Q) \mathcal{T}_l(g \chi_Q) \right)^*(\lambda|Q|) \\ &\leq \prod_{i=1}^{m-1} (M(f_i \chi_Q))^*(\lambda|Q|/2^i) (\mathcal{T}_l(g \chi_Q))^*(\lambda|Q|/2^{m-1}) \\ &\leq c(\lambda, m, n) l g_Q \prod_{i=1}^{m-1} (f_i)_Q, \end{aligned}$$

which completes the proof of (4.16).

Finally, let us proceed to show (4.13). By (4.14) it is enough to prove (4.13) with $\mathcal{T}_{l,\alpha}(\vec{f})$, for each $\alpha = 1, \dots, 2^n$, instead of $\mathcal{T}_{S,l}(\vec{f})$ on the left-hand side. By the standard limiting argument one can assume that the sum defining $\mathcal{T}_{l,\alpha}(\vec{f})$ is finite. Then the sum defining the corresponding operator $\mathcal{M}_{l,\alpha}(\vec{f}_1, \dots, m-1, g)$ in (4.15) will be

finite too. This means that the support of $\mathcal{M}_{l,\alpha}(\vec{f}_1, \dots, m-1, g)$ is compact. One can cover it by at most 2^n cubes $Q_\nu \in \mathcal{D}_\alpha$ such that

$$m_{\mathcal{M}_{l,\alpha}(\vec{f}_1, \dots, m-1, g)}(Q_\nu) = 0, \quad \nu = 1, \dots, 2^n.$$

Applying Theorem 1.6.2 along with (4.16), we get that there exists a sparse family $\mathcal{S}_\alpha \in \mathcal{D}_\alpha(Q_\nu)$ such that for a.e. $x \in Q_\nu$,

$$\mathcal{M}_{l,\alpha}(\vec{f}_1, \dots, m-1, g)(x) \leq c(m, n)l \sum_{Q_j^k \in \mathcal{S}_\alpha} \left(\prod_{i=1}^{m-1} (f_i)_{Q_j^k} \right) g_{Q_j^k} \chi_{Q_j^k}(x).$$

Hence, by Hölder's inequality,

$$\begin{aligned} \int_{Q_\nu} \mathcal{M}_{l,\alpha}(\vec{f}_1, \dots, m-1, g) f_m dx &\leq c(m, n)l \int_{\mathbb{R}^n} \mathcal{A}_{\mathcal{D}_\alpha, \mathcal{S}_\alpha}(\vec{f}) g dx \\ &\leq c(m, n)l \sup_{\mathcal{D}, \mathcal{S}} \|\mathcal{A}_{\mathcal{D}, \mathcal{S}}(\vec{f})\|_X \|g\|_{X'}. \end{aligned}$$

Summing up over Q_ν and using (4.15), we get

$$\int_{\mathbb{R}^n} \mathcal{T}_{l,\alpha}(\vec{f}) g dx \leq 2^n c(m, n)l \sup_{\mathcal{D}, \mathcal{S}} \|\mathcal{A}_{\mathcal{D}, \mathcal{S}}(\vec{f})\|_X \|g\|_{X'}.$$

By (1.5), taking here the supremum over $g \geq 0$ with $\|g\|_{X'} = 1$ gives (4.13) for $\mathcal{T}_{l,\alpha}(\vec{f})$, and therefore the proof is complete. \square

4.4 A_2 theorem for multilinear Calderón–Zygmund operators

As we mentioned before, one of the main results obtained in [80] is that if $\vec{w} \in A_{\vec{p}}$, then an analogue of (3.6) holds with a multilinear Calderón–Zygmund operator T instead of \mathcal{M} . Hence, it is natural to ask about the sharp dependence on $[\vec{w}]_{A_{\vec{p}}}$ in the corresponding inequality. As in the linear situation we want to apply Theorem 4.3.1 to $X = L^p(\nu_{\vec{w}})$. However, the exponent p is allowed to be smaller than one, that is, $1/m < p < \infty$. Therefore when $1/m < p < 1$, Theorem 4.3.1 cannot be applied since in this case $L^p(\nu_{\vec{w}})$ is not a Banach function space. In this section we will show the following particular case that can be seen as a multilinear analogue of the A_2 conjecture.

Theorem 4.4.1. *Let $T(\vec{f})$ be a multilinear Calderón–Zygmund operator. Assume that $p_1 = p_2 = \dots = p_m = m + 1$. Then*

$$(4.18) \quad \|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C_{T,m,n} [\vec{w}]_{A_{\vec{p}}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.$$

Proof of Theorem 4.4.1. First, we apply Theorem 4.3.1 with $X = L^p(\nu_{\vec{w}})$. Observe that with our particular choice of p_i , $p = \frac{m+1}{m}$. Fix $\mathcal{S} \in \mathcal{D}$ and assume that $f_i \geq 0$. By duality, we obtain

$$\|\mathcal{A}_{\mathcal{D}, \mathcal{S}}(\vec{f})\|_{L^p(\nu_{\vec{w}})} = \sup_{\|g\|_{L^{p'}(\nu_{\vec{w}}^{-1/(p-1)})} = 1} \sum_{j,k} \prod_{i=1}^m (f_i)_{Q_j^k} \int_{Q_j^k} g.$$

Observe again that by the choice of p_i we have $p/p'_i = 1$. Let us write for simplicity

$$A_{\vec{P}}(\vec{w}; Q) = \left(\frac{1}{|Q|} \int_Q \nu_{\vec{w}} \right) \prod_{i=1}^m \left(\frac{1}{|Q|} \int_Q \sigma_i \right).$$

Then, we have

$$\begin{aligned} & \sum_{j,k} \prod_{i=1}^m (f_i)_{Q_j^k} \int_{Q_j^k} g \\ &= \sum_{j,k} A_{\vec{P}}(\vec{w}; Q_j^k) \left(\prod_{i=1}^m \frac{1}{\sigma_i(Q_j^k)} \int_{Q_j^k} f_i \right) \left(\frac{1}{\nu_{\vec{w}}(Q_j^k)} \int_{Q_j^k} g \right) |Q_j^k| \\ &\leq 2[\vec{w}]_{A_{\vec{P}}} \sum_{j,k} \int_{E_j^k} \prod_{i=1}^m M_{\sigma_i}^{\mathcal{D}}(f_i \sigma_i^{-1}) M_{\nu_{\vec{w}}}^{\mathcal{D}}(g \nu_{\vec{w}}^{-1}) dx \\ &\leq 2[\vec{w}]_{A_{\vec{P}}} \int_{\mathbb{R}^n} \prod_{i=1}^m M_{\sigma_i}^{\mathcal{D}}(f_i \sigma_i^{-1}) M_{\nu_{\vec{w}}}^{\mathcal{D}}(g \nu_{\vec{w}}^{-1}) dx. \end{aligned}$$

Now applying Hölder's inequality we get

$$\begin{aligned} & \int_{\mathbb{R}^n} \prod_{i=1}^m M_{\sigma_i}^{\mathcal{D}}(f_i \sigma_i^{-1}) M_{\nu_{\vec{w}}}^{\mathcal{D}}(g \nu_{\vec{w}}^{-1}) dx \\ &\leq \left\| \prod_{i=1}^m M_{\sigma_i}^{\mathcal{D}}(f_i \sigma_i^{-1}) \right\|_{L^p(\nu_{\vec{w}}^{-(p-1)})} \|M_{\nu_{\vec{w}}}^{\mathcal{D}}(g \nu_{\vec{w}}^{-1})\|_{L^{p'}(\nu_{\vec{w}})}. \end{aligned}$$

First, applying again Hölder's inequality with exponents p_i/p in the first term on the right hand side and using that $p-1 = \frac{1}{p_i-1}$, we get

$$\left\| \prod_{i=1}^m M_{\sigma_i}^{\mathcal{D}}(f_i \sigma_i^{-1}) \right\|_{L^p(\nu_{\vec{w}}^{-(p-1)})} \leq \prod_{i=1}^m \|M_{\sigma_i}^{\mathcal{D}}(f_i \sigma_i^{-1})\|_{L^{p_i}(\sigma_i)} \leq c \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.$$

Finally, for the second term we have

$$\|M_{\nu_{\vec{w}}}^{\mathcal{D}}(g \nu_{\vec{w}}^{-1})\|_{L^{p'}(\nu_{\vec{w}})} \leq c \|g\|_{L^{p'}(\nu_{\vec{w}}^{-1/(p-1)})} = c,$$

and we are done. \square

4.5 Basics on multilinear generalized Calderón–Zygmund operators

In this section, we introduce the class of multilinear generalized Calderón–Zygmund operators also known as operators with non-smooth kernels in the literature (see [42, 41, 40, 51, 5]). The main feature of this class of operators is that their kernels satisfy regularity conditions that are significantly weaker than those of standard Calderón–Zygmund kernels. An important example of our model operator with non-smooth kernel is the m -th order commutator of Calderón.

$$\mathcal{C}_{m+1}(a_1, \dots, a_m, f)(x) = \int_{\mathbb{R}^n} \frac{\prod_{j=1}^m (A_j(x) - A_j(y))}{(x - y)^{m+1}} f(y) dy, \quad x \in \mathbb{R},$$

where $A'_j = a_j$.

The main motivation for studying this class of operators relies on the fact that there are operators whose kernels are not regular enough to fall under the scope of Calderón–Zygmund theory, but certain classes of such operators can be proved to be of weak type $(1, 1)$. Therefore, a natural question is whether one can weaken the smoothness condition imposed on Calderón–Zygmund operators so that they are still of weak type $(1, 1)$.

Regarding this problem, in [42] it was introduced a weaker condition on the kernel of such operators than the standard Hörmander condition. This condition was proved to be sufficient for these operators to be of weak type $(1, 1)$ by using a family of integral operators which plays the role of approximation to the identity. The study of these operators was motivated by the necessity of working in subsets of \mathbb{R}^n which do not possess any smoothness in their boundaries and, therefore, are not spaces of homogeneous type, which is the natural framework where Calderón–Zygmund theory is developed. Such measurable sets do appear naturally in PDES, i.e. Riesz transforms on Lie groups, elliptic operators.

In the last years, different authors [41, 40, 51] were interested in developing a systematic theory for this class of operators by using a weaker condition than the standard Hölder–Lipschitz smoothness condition for Calderón–Zygmund operators. This new condition, which is stronger than the one introduced by Duong and McIntosh in [42], also allows to develop a weighted theory for this class of operators.

In this section, we will introduce the notation and basic assumptions that we will need to define the class of generalized Calderón–Zygmund operators and some basic results concerning the boundedness properties for these operators. We refer the interested reader to [41, 40, 51] for a more detailed information on these issues.

We will work with a class of integral operators $\{A_t\}_{t>0}$ that plays the role of an approximation to the identity as in [42, 40, 41, 51]. We assume that such operators A_t are associated with kernels $a_t(x, y)$ in the sense that

$$(4.19) \quad A_t f(x) = \int_{\mathbb{R}^n} a_t(x, y) f(y) dy,$$

for every function $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and that the kernels $a_t(x, y)$ satisfy the following size conditions

$$(4.20) \quad |a_t(x, y)| \leq h_t(x, y) = t^{-\frac{n}{s}} h\left(\frac{|x-y|^s}{t}\right),$$

where s is a fixed constant and h is a positive, bounded, decreasing function satisfying

$$(4.21) \quad \lim_{r \rightarrow \infty} r^{n+\eta} h(r^s) = 0,$$

for some $\eta > 0$.

Also recall that a multilinear operator T has m formal transposes. The j th transpose $T^{*,j}$ of T is defined via

$$\langle T^{*,j}(f_1, \dots, f_m), g \rangle = \langle T(f_1, \dots, f_{j-1}, g, f_{j+1}, \dots, f_m), f_j \rangle,$$

for every $f_1, \dots, f_m, g \in \mathcal{S}(\mathbb{R}^n)$. It is easy to see that the kernel $K^{*,j}$ of $T^{*,j}$ is related to the kernel K of T via the identity

$$K^{*,j}(x, y_1, \dots, y_{j-1}, y_j, y_{j+1}, \dots, y_m) = K(y_j, y_1, \dots, y_{j-1}, x, y_{j+1}, \dots, y_m).$$

Observe that if T maps a product of Banach spaces $X_1 \times \dots \times X_m$ into another Banach space Y , then $T^{*,j}$ maps $X_1 \times \dots \times X_{j-1} \times Y^* \times X_{j+1} \times \dots \times X_m$ into X_j^* , where X_j^* and Y^* denote the dual of the spaces X_j and Y , respectively. Furthermore, the norms of T and T^* are equal.

In order to state our main assumptions in this section we will denote T by T^{*0} and K by K^{*0} when necessary for the sake of simplicity.

Next, let T be a multilinear operator associated with a kernel $K(x, y_1, \dots, y_m)$ in the sense of (4.1). The basic assumptions we will be working with from now on are the following. Throughout this section, we will assume the following size estimate on the kernel K ,

$$(4.22) \quad |K(x, y_1, \dots, y_m)| \leq \frac{A}{(|x-y_1| + \dots + |x-y_m|)^{mn}},$$

for some $A > 0$ and all (x, y_1, \dots, y_m) with $x \neq y_j$ for some j .

Assumption (H1). Assume that for each $i = 1, \dots, m$ there exist operators $\{A_t^{(i)}\}_{t>0}$ with kernels $a_t^{(i)}(x, y)$ that satisfy conditions (4.20) and (4.21) with constants s and η and that for every $j = 0, 1, \dots, m$, there exist kernels $K_t^{*,j,(i)}(x, y_1, \dots, y_m)$ such that

$$(4.23) \quad \begin{aligned} & \langle T^{*,j}(f_1, \dots, A_t^{(i)} f_i, \dots, f_m), g \rangle \\ &= \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n)^m} K_t^{*,j,(i)}(x, y_1, \dots, y_m) f_1(y_1) \dots f_m(y_m) g(x) d\vec{y} dx, \end{aligned}$$

for all f_1, \dots, f_m in $\mathcal{S}(\mathbb{R}^n)$ with $\bigcap_{k=1}^m \text{supp } f_k \cap \text{supp } g = \emptyset$. Also assume that there exist a function $\phi \in C(\mathbb{R})$ with $\text{supp } \phi \subset [-1, 1]$ and a constant $\varepsilon > 0$ so that for every $j = 0, 1, \dots, m$ and every $i = 1, 2, \dots, m$, we have

$$(4.24) \quad \begin{aligned} & |K^{*,j}(x, y_1, \dots, y_m) - K_t^{*,j,(i)}(x, y_1, \dots, y_m)| \\ & \leq \frac{A}{(|x - y_1| + \dots + |x - y_m|)^{mn}} \sum_{k=1, k \neq i}^m \phi\left(\frac{|y_i - y_k|}{t^{1/s}}\right) \\ & \quad + \frac{At^{\varepsilon/s}}{(|x - y_1| + \dots + |x - y_m|)^{mn+\varepsilon}}, \end{aligned}$$

whenever $2t^{1/s} \leq |x - y_i|$.

If T satisfies Assumption (H1) we will say that T is an m -linear operator with *generalized Calderón–Zygmund kernel* K . The collection of functions K satisfying (4.23) and (4.24) with parameters m, A, s, η and ε will be denoted by $m - GCZK(A, s, \eta, \varepsilon)$. We say that T is of class $m - GCZO(A, s, \eta, \varepsilon)$ if T has an associated kernel K in $m - GCZK(A, s, \eta, \varepsilon)$.

Remark 4.5.1. Observe that condition (4.3) is a stronger condition than assumption (H1). It is possible to show that, for suitably chosen $A_t^{(i)}$, condition (4.24) is a consequence of (4.3). For any $m > 0$, we can construct $a_t^{(i)}(x, y)$ with the following properties as in [42, Prop. 2] or [40, p. 2101]:

$$(4.25) \quad a_t^{(i)}(x, y) = 0, \quad \text{when } |x - y| \geq t^{1/s},$$

and

$$(4.26) \quad \int_{\mathbb{R}^n} a_t^{(i)}(x, y) dx = 1,$$

for all $y \in \mathbb{R}^n$, $t > 0$. For example, we can choose

$$a_t^{(i)}(x, y) = c_n t^{-n/s} \chi_{B(y, t^{1/s})}(x),$$

where c_n is a dimensional constant such that condition (4.26) is satisfied. Let us consider the operators $A_t^{(i)}$ associated with the kernels $a_t^{(i)}(x, y)$ in the sense of (4.19).

Next, let us prove that condition (4.3) implies that there exists a positive constant C and ε so that

$$|K^{*,j}(x, y_1, \dots, y_m) - K_t^{*,j,(i)}(x, y_1, \dots, y_m)| \leq \frac{At^{\varepsilon/s}}{(|x - y_1| + \dots + |x - y_m|)^{mn+\varepsilon}},$$

whenever $2t^{1/s} \leq |x - y_i|$.

In fact, we have that

$$\begin{aligned} & |K^{*,j}(x, y_1, \dots, y_i, \dots, y_m) - K_t^{*,j,(i)}(x, y_1, \dots, y_i, \dots, y_m)| \\ & \leq \left| K^{*,j}(x, y_1, \dots, y_i, \dots, y_m) - \int_{\mathbb{R}^n} K^{*,j}(x, y_1, \dots, z, \dots, y_m) a_t^{(i)}(z, y_i) dz \right| \\ & \leq \int_{|z - y_i| \leq t^{1/s}} |K^{*,j}(x, y_1, \dots, y_i, \dots, y_m) - K^{*,j}(x, y_1, \dots, z, \dots, y_m)| |a_t^{(i)}(z, y_i)| dz \\ & \leq \frac{At^{\varepsilon/s}}{(|x - y_1| + \dots + |x - y_m|)^{mn+\varepsilon}}, \end{aligned}$$

where we have used (4.26) and (4.3) together with the fact that $|z - y_i| \leq t^{1/s} \leq \frac{|x - y_i|}{2}$.

The following boundedness result proved in [41] holds for generalized Calderón–Zygmund operators.

Theorem 4.5.1. *Assume that T is a multilinear operator in $m - GCZO(A, s, \eta, \varepsilon)$. Let $1 < q_1, \dots, q_m, q < \infty$ be given numbers such that*

$$\frac{1}{q_1} + \dots + \frac{1}{q_m} = \frac{1}{q}.$$

Assume that T maps $L^{q_1}(\mathbb{R}^n) \times \dots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Let p, p_j be numbers satisfying $1/m \leq p < \infty$, $1 \leq p_j \leq \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. Then all the statements below hold:

1. *When all $p_j > 1$, then $T : L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$.*
2. *When some $p_j = 1$, then $T : L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n) \rightarrow L^{p,\infty}(\mathbb{R}^n)$.*

Furthermore, there exists a constant C_{n,m,p_j,q_j} such that the following estimate holds

$$(4.27) \quad \|T\|_{L^1 \times \dots \times L^1 \rightarrow L^{1/m}, \infty} \leq C_{n,m,p_j,q_j} (A + \|T\|_{L^{q_1} \times \dots \times L^{q_m} \rightarrow L^q}).$$

We also need further assumptions in order to prove our main result in the following section. In this respect, many different assumptions could be made in order to get weighted estimates for this class of operators with non-smooth kernels. We refer the reader to [41, 40, 5] and the references therein for further details.

Assumption (H2). Assume that there exist operators $\{B_t\}_{t>0}$ with kernels $b_t(x, y)$ that satisfy conditions (4.20) and (4.21) with constants s and η . Let

$$(4.28) \quad K_t^{(0)}(x, y_1, \dots, y_m) = \int_{\mathbb{R}} K(z, y_1, \dots, y_{m+1}) b_t(x, z) dz.$$

Also assume that the kernels $K_t^{(0)}(x, y_1, \dots, y_m)$ satisfy the following estimates: there exist a function $\phi \in \mathcal{C}(\mathbb{R})$ with $\text{supp } \phi \subset [-1, 1]$ and positive constants A and ε such that

$$|K_t^{(0)}(x, y_1, \dots, y_m)| \leq \frac{A}{(|x - y_1| + \dots + |x - y_m|)^{mn}}$$

whenever $2t^{1/s} \leq \min_{1 \leq j \leq m} |x - y_j|$, and

$$(4.29) \quad \begin{aligned} & |K(x, y_1, \dots, y_m) - K_t^{(0)}(x', y_1, \dots, y_m)| \\ & \leq \frac{A}{(|x - y_1| + \dots + |x - y_m|)^{mn}} \sum_k \phi\left(\frac{|x - y_k|}{t^{1/s}}\right) \\ & \quad + \frac{At^{\varepsilon/s}}{(|x - y_1| + \dots + |x - y_m|)^{mn+\varepsilon}}, \end{aligned}$$

whenever $2|x - x'| \leq t^{1/s}$ and $2t^{1/s} \leq \max_{1 \leq j \leq m} |x - y_j|$.

Remark 4.5.2. Assumption (H2) can be proved to be weaker than condition (4.3) by choosing appropriate kernels $b_t(x, y)$ similarly as it was done in Remark 4.5.1 (see also [40, Prop. 2.3.]).

Example 4.5.1. As we mentioned at the beginning of this section, the m -th order commutator of Calderón falls under the scope of the generalized Calderón–Zygmund theory. It first appeared in the study of the Cauchy integral along Lipschitz curves and led to the first proof of the L^2 boundedness of the latter. Their boundedness properties were studied when $m = 1$ in [18, 19] and when $m = 1, 2$ in [30].

Recall that the m -th order commutator of Calderón is given by the following expression

$$\mathcal{C}_{m+1}(a_1, \dots, a_m, f)(x) = \int_{\mathbb{R}^n} \frac{\prod_{j=1}^m (A_j(x) - A_j(y))}{(x - y)^{m+1}} f(y) dy, \quad x \in \mathbb{R},$$

where $A'_j = a_j$. Define

$$e(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

Since $A'_j = a_j$, the multilinear operator $C_{m+1}(f, a_1, \dots, a_m)$ can be expressed as follows

$$\begin{aligned} & C_{m+1}(a_1, \dots, a_m, f)(x) \\ & := \int_{\mathbb{R}} K(x, y_1, \dots, y_{m+1}) a_1(y_1) \dots a_m(y_m) f(y_{m+1}) dy_1 \dots dy_{m+1}, \end{aligned}$$

where the kernel K is

$$K(x, y_1, \dots, y_{m+1}) = \frac{(-1)^{e(y_{m+1}-x)m}}{(x-y_{m+1})^{m+1}} \prod_{l=1}^m \chi_{(\min(x, y_{m+1}), \max(x, y_{m+1}))}(y_l).$$

The m -th order commutator of Calderón was shown to verify assumptions (H1) and (H2) in [41, 5], respectively. It was done by considering kernels $a_t(x, y)$ and $b_t(x, y)$ in the following form. Let $\phi \in C^\infty(\mathbb{R})$ be even, $0 \leq \phi \leq 1$, $\phi(0) = 1$ and $\text{supp } \phi \subset [-1, 1]$. Set $\Phi = \phi'$ and $\Phi_t(x) = t^{-1}\Phi(x/t)$. Define

$$\begin{aligned} a_t(x, y) &= \Phi_t(x-y)\chi_{(x, \infty)}(y), \\ b_t(x, y) &= \Phi_t(x-y)\chi_{(-\infty, x)}(y). \end{aligned}$$

Then the kernels $a_t(x, y)$ and $b_t(x, y)$ satisfy (4.20) and (4.21) with constants $s = \eta = 1$. If we consider the operators A_t and B_t associated with $a_t(x, y)$ and $b_t(x, y)$, respectively, as in (4.19) then assumptions (H1) and (H2) can be proved to hold (see [41, Thm. 4.1.] and [5, Prop. 5.1.]).

The multilinear maximal function also controls this class of multilinear operators with non-smooth kernels as it is shown in the following result proved in [5, Thm. 4.1.].

Theorem 4.5.2. *Let T be an operator in m -GCZO(A, s, η, ε) verifying assumption (H2) and let $0 < \delta < 1/m$. Also assume that there exist some $1 \leq q_1, \dots, q_m < \infty$ and $0 < q < \infty$ with $\frac{1}{q} = \frac{1}{q_1} + \dots + \frac{1}{q_m}$, such that T maps $L^{q_1} \times \dots \times L^{q_m}$ to L^q . Then for all \vec{f} in any product of $L^{p_j}(\mathbb{R}^n)$ spaces, with $1 \leq p_j < \infty$,*

$$(4.30) \quad M_\delta^\#(T(\vec{f}))(x) \leq CM(\vec{f})(x).$$

4.6 A_2 theorem for multilinear generalized Calderón–Zygmund operators

In this section we prove the analogous result to Proposition 4.2.1 for multilinear generalized Calderón–Zygmund operators.

Proposition 4.6.1. *Let T be a multilinear operator in m -GCZO(A, s, η, ε) verifying Assumption (H2). Let $1 < q_1, \dots, q_m, q < \infty$ be given numbers such that*

$$\frac{1}{q_1} + \dots + \frac{1}{q_m} = \frac{1}{q}.$$

Assume that T maps $L^{q_1}(\mathbb{R}^n) \times \dots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Then, for any cube $Q \subset \mathbb{R}^n$,

$$\omega_\lambda(T(f); Q) \leq c(T, n, \lambda) \sum_{l=0}^{\infty} \frac{1}{2^{l\varepsilon}} \prod_{i=1}^m \frac{1}{|2^l Q|} \int_{2^l Q} |f(y)| dy.$$

The proof of this result follows the same scheme as Proposition 4.2.1 with minor modifications (see also the proof in [5, Thm. 4.1.] for further details).

Proof. As we did before, set $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i \chi_{Q^*}$, $i = 1, \dots, m$ and $Q^* = 5\sqrt{n}Q$. Then, using (4.11), the definition of local mean oscillation, the properties of rearrangements and setting $c = \sum' T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(x_0)$ where x_0 is the center of Q , we obtain

$$\begin{aligned} \omega_\lambda(T(\vec{f}); Q) &\leq (T(\vec{f}^0))^*(\lambda|Q|) \\ &\quad + \sum' \|T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(x) - T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(x_0)\|_{L^\infty(Q)}. \end{aligned}$$

For the first term above we use the endpoint estimate for T in Theorem 4.5.1. Namely,

$$(T(\vec{f}^0))^*(\lambda|Q|) \leq \frac{C}{(\lambda|Q|)^m} \prod_{i=1}^m \int_{Q^*} |f_i|.$$

In order to handle the second term we proceed as before introducing slight modifications with respect to the proof of Proposition 4.2.1. More precisely, consider first the case when $\alpha_1 = \dots = \alpha_m = \infty$ and define $T(\vec{f}^\infty) = T(f_1^\infty, \dots, f_m^\infty)$, then for any $x \in Q$ and for $t > 0$ such that $t^{1/s} = \sqrt{n}|Q|^{1/n}$, we can write

$$\begin{aligned} |T(\vec{f}^\infty)(x) - T(\vec{f}^\infty)(x_0)| &\leq \int_{(\mathbb{R}^n \setminus Q^*)^m} |K(x, \vec{y}) - K(x_0, \vec{y})| \prod_{i=1}^m |f_i(y_i)| d\vec{y} \\ &\leq \int_{(\mathbb{R}^n \setminus Q^*)^m} |K(x, \vec{y}) - K_t^{(0)}(x, \vec{y})| \prod_{i=1}^m |f_i(y_i)| d\vec{y} \\ &\quad + \int_{(\mathbb{R}^n \setminus Q^*)^m} |K_t^{(0)}(x, \vec{y}) - K(x_0, \vec{y})| \prod_{i=1}^m |f_i(y_i)| d\vec{y} \\ &= I + II. \end{aligned}$$

Let us estimate I . Since $x \in Q$ and $y_j \in \mathbb{R}^n \setminus Q^*$, we get that

$$|y_j - x| > |x_0 - y_j| - |x - x_0| \geq 2\sqrt{n}|Q|^{1/n} = 2t^{1/s},$$

for all $j = 1, \dots, m$. By (4.29), we get

$$\begin{aligned}
 I &\leq \int_{(\mathbb{R}^n \setminus Q^*)^m} \frac{At^{\varepsilon/s}}{(|x - y_1| + \dots + |x - y_m|)^{mn+\varepsilon}} \prod_{i=1}^m |f_i(y_i)| d\vec{y} \\
 &\leq C \int_{(\mathbb{R}^n \setminus 2Q)^m} \frac{|Q|^{\varepsilon/n}}{(|x - y_1| + \dots + |x - y_m|)^{mn+\varepsilon}} \prod_{i=1}^m |f_i(y_i)| d\vec{y} \\
 &\leq C \sum_{l=1}^{\infty} \int_{(2^{l+1}Q)^m \setminus (2^lQ)^m} \frac{|Q|^{\varepsilon/n}}{(|x - y_1| + \dots + |x - y_m|)^{nm+\varepsilon}} \prod_{i=1}^m |f_i(y_i)| d\vec{y} \\
 &\leq C \sum_{l=1}^{\infty} \frac{|Q|^{\varepsilon/n}}{(2^l|Q|^{1/n})^{nm+\varepsilon}} \int_{(2^{l+1}Q)^m} \prod_{i=1}^m |f_i(y_i)| d\vec{y} \\
 &\leq C \sum_{l=0}^{\infty} \frac{1}{2^{l\varepsilon}} \prod_{i=1}^m \frac{1}{|2^lQ|} \int_{2^lQ} |f_i(y_i)| d\vec{y},
 \end{aligned}$$

where we have used that $\phi(\frac{|x-y_j|}{t^{1/s}}) = 0$ for all $j = 1, \dots, m$, since $|x - y_j| > 2t^{1/s}$ for all $j = 1, \dots, m$. Proceeding as before and applying (4.29) since $|x - x_0| \leq \frac{\sqrt{n}|Q|^{1/n}}{2} = t^{1/s}/2$, we also obtain the same bound for II . It remains to consider the rest of the terms in (4.11) such that $\alpha_{i_1} = \dots = \alpha_{i_k} = 0$ for some $\{i_1, \dots, i_k\} \subset \{1, \dots, m\}$ and $1 \leq k < m$. We can write

$$\begin{aligned}
 &|T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(x) - T(f_1^{\alpha_1}, \dots, f_m^{\alpha_m})(x_0)| \\
 &\leq \int_{(\mathbb{R}^n)^m} |K(x, \vec{y}) - K_t^{(0)}(x, \vec{y})| d\vec{y} \\
 &\quad + \int_{(\mathbb{R}^n)^m} |K_t^{(0)}(x, \vec{y}) - K(x_0, \vec{y})| d\vec{y} \\
 &= III + IV.
 \end{aligned}$$

For III , using again (4.29), we have

$$\begin{aligned}
 &\int_{(\mathbb{R}^n)^m} |K(x, \vec{y}) - K_t^{(0)}(x, \vec{y})| d\vec{y} \\
 &\leq C \prod_{i \in \{i_1, \dots, i_k\}} \int_{Q^*} |f_i| dy_i \int_{(\mathbb{R}^n \setminus Q^*)^{m-k}} \frac{t^{\varepsilon/s} \prod_{i \notin \{i_1, \dots, i_k\}} |f_i| dy_i}{(|x - y_1| + \dots + |x - y_m|)^{nm+\varepsilon}} \\
 &\quad + C \prod_{i \in \{i_1, \dots, i_k\}} \int_{Q^*} |f_i| dy_i \int_{(\mathbb{R}^n \setminus Q^*)^{m-k}} \frac{\prod_{i \notin \{i_1, \dots, i_k\}} |f_i| dy_i}{(|x - y_1| + \dots + |x - y_m|)^{nm}} \\
 &= III_1 + III_2,
 \end{aligned}$$

since $2t^{1/s} \leq \max_{1 \leq j \leq m} |x - y_j|$. Next, for III_1 we get

$$\begin{aligned}
 III_1 &\leq C \prod_{i \in \{i_1, \dots, i_k\}} \int_{Q^*} |f_i| dy_i \int_{(\mathbb{R}^n \setminus 2Q)^{m-k}} \frac{t^{\varepsilon/s} \prod_{i \notin \{i_1, \dots, i_k\}} |f_i| dy_i}{(|x - y_1| + \dots + |x - y_m|)^{nm+\varepsilon}} \\
 &\leq C \prod_{i \in \{i_1, \dots, i_k\}} \int_{Q^*} |f_i| dy_i \sum_{l=1}^{\infty} \frac{|Q|^{\varepsilon/n}}{(2^l |Q|^{1/n})^{nm+\varepsilon}} \int_{(2^{l+1}Q)^{m-k}} \prod_{i \notin \{i_1, \dots, i_k\}} |f_i| dy_i \\
 &\leq C \sum_{l=2}^{\infty} \frac{|Q|^{\varepsilon/n}}{(2^{l+1} |Q|^{1/n})^{nm+\varepsilon}} \int_{(2^{l+1}Q)^m} \prod_{i=1}^m |f_i| dy_i \\
 &\leq C \sum_{l=0}^{\infty} \frac{1}{2^{l\varepsilon}} \prod_{i=1}^m \frac{1}{|2^l Q|} \int_{2^l Q} |f_i(y_i)| d\vec{y},
 \end{aligned}$$

where we have used that $t^{1/s} \approx |Q|^{1/n}$. Observe that the estimate for III_2 is similar to the previous one but we have to take into account as before that for those $y_j \in (Q^*)^c$, $|x - y_j| \geq 2t^{1/s}$ and, therefore $\phi(\frac{|x-y_j|}{t^{1/s}}) = 0$. Otherwise, since $\phi \in \mathcal{C}(\mathbb{R}^n)$ with compact support, we can bound it by $\|\phi\|_{L^\infty}$. Therefore the dimensional constant C appearing above also depends on the operator T . By a similar argument, since $x_0 \in Q$, we can derive the same estimate for IV . Therefore putting all together we obtain the desired result. \square

Observe that as a consequence of Proposition 4.6.1 we can derive the following control in norm for generalized Calderón–Zygmund operators as it was done in Theorem 4.3.1 for multilinear Calderón–Zygmund operators.

Corollary 4.6.1. *Let T be an operator in m -GCZO(A, s, η, ε) verifying assumption (H2) and let X be a Banach function space over \mathbb{R}^n equipped with Lebesgue measure. Also assume that there exist some $1 \leq q_1, \dots, q_m < \infty$ and $0 < q < \infty$ with $\frac{1}{q} = \frac{1}{q_1} + \dots + \frac{1}{q_m}$, such that T maps $L^{q_1} \times \dots \times L^{q_m}$ to L^q . Then, for any appropriate \vec{f} ,*

$$\|T(\vec{f})\|_X \leq c_{T,m,n} \sup_{\mathcal{D}, \mathcal{S}} \|\mathcal{A}_{\mathcal{D}, \mathcal{S}}(|\vec{f}|)\|_X,$$

where the supremum is taken over arbitrary dyadic grids \mathcal{D} and sparse families $\mathcal{S} \in \mathcal{D}$.

Proof. The proof of this result is similar to the one for Calderón–Zygmund operators. Observe that combining Proposition 4.6.1 and Theorem 1.6.2 with $Q_0 \in \mathcal{D}$, we get that there exists a sparse family $\mathcal{S} = \{Q_j^k\} \subset \mathcal{D}$ such that for a.e. $x \in Q_0$,

$$|T(f) - m_{T(f)}(Q_0)| \leq C \sum_{l=0}^{\infty} \frac{1}{2^{l\varepsilon}} \mathcal{T}_{\mathcal{S}, l}(|f|)(x),$$

where $C = C(n, T)$ and

$$\mathcal{T}_{S,t}(f)(x) = \sum_{j,k} (f)_{2^j Q_j^k} \chi_{Q_j^k}.$$

Taking into account that from this point on the only fact that is used in the proof of Theorem 4.3.1 is that multilinear Calderón–Zygmund operators satisfy the endpoint estimate (4.4) (and generalized Calderón–Zygmund operators do satisfy this requirement, see (4.27)), we have that the proof follows the same scheme as the proof of Theorem 4.3.1 since it does not depend on the operator T we are working with. Hence, the result follows immediately. \square

As a consequence we can also derive the corresponding A_2 theorem for multilinear generalized Calderón–Zygmund operators.

Corollary 4.6.2. *Let T be an operator in m -GCZO(A, s, η, ε) verifying assumption (H2). Also assume that there exist some $1 \leq q_1, \dots, q_m < \infty$ and $0 < q < \infty$ with $\frac{1}{q} = \frac{1}{q_1} + \dots + \frac{1}{q_m}$, such that T maps $L^{q_1} \times \dots \times L^{q_m}$ to L^q . If $p_1 = p_2 = \dots = p_m = m+1$, then it holds*

$$(4.31) \quad \|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C_{T,m,n}[\vec{w}]_{A_{\vec{p}}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.$$

4.7 Further remarks and open questions

The results listed in this chapter led to several interesting questions and open problems that we summarized below:

1. As we already mentioned, Theorem 4.4.1 as well as Corollary 4.6.2 can be regarded as a multilinear “ A_2 theorems” and it is natural to ask how to extend them to all $1 < p_i < \infty$. Let us observe that in the linear setting the general bound for a Calderón–Zygmund operator can be obtained from the case $p = 2$ by the sharp version of the extrapolation theorem of Rubio de Francia obtained in [37] (see also [39] or [34] for simpler proofs). Hence, it would be desirable to obtain a multilinear version of this result. Having such analogue, inequality (4.18) probably would be a starting point to extrapolate from.
2. However, this result in the linear setting can be proved also without the use of extrapolation. Indeed, in the linear situation one can easily prove the sharp bound for a Calderón–Zygmund operator (8) with $\mathcal{A}_{\mathcal{Q},S}$ instead of T for all

$1 < p < \infty$ (as it was done in [34] for $p = 2$) and then apply that T is bounded by these sparse operators with $X = L^p(w)$. This kind of proof for $\mathcal{A}_{\mathcal{D},S}$ is very close in spirit to the proof of Buckley's inequality found in [74]. Thus, it was natural to ask whether it was possible to find a similar proof for a multilinear version of $\mathcal{A}_{\mathcal{D},S}$. This question was partially answered by Li, Moen and Sun in [81], where they proved the following theorem.

Theorem 4.7.1. *Suppose that $1 < p_1, \dots, p_m < \infty$ with $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p}$ and $\vec{w} \in A_{\vec{p}}$. Then*

$$\|\mathcal{A}_{\mathcal{D},S}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{p}}}^{\max(1, \frac{p'_1}{p}, \dots, \frac{p'_m}{p})} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.$$

Observe that when $1/m < p < 1$, we have that if we choose $X = L^p(\nu_{\vec{w}})$, then it is not a Banach function space and consequently, Theorem 4.3.1 does not hold. However, in [81] from Theorem 4.7.1 the authors derive the following result.

Theorem 4.7.2. *Let T be a multilinear Calderón–Zygmund operator, $\vec{P} = (p_1, \dots, p_m)$ with $1 < p_1, \dots, p_m < \infty$ and $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p} \leq 1$. Suppose that $\vec{w} = (w_1, \dots, w_m)$ with $\vec{w} \in A_{\vec{p}}$. Then*

$$(4.32) \quad \|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C_{n,m,\vec{P},T} [\vec{w}]_{A_{\vec{p}}}^{\max(1, \frac{p'_1}{p}, \dots, \frac{p'_m}{p})} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.$$

Moreover, the exponent on $[\vec{w}]_{A_{\vec{p}}}$ is the best possible.

Observe that, under the assumptions of Theorem 4.6.2 and as a consequence of Theorem 4.7.2, this multilinear A_p theorem extends to the more general setting of multilinear generalized Calderón–Zygmund operators.

3. From our last observation it is natural to ask if one can replace X by a quasi-Banach space in Theorem 4.3.1. A different proof must be found since the duality property of Banach spaces is the key point in this proof.

Compactness of commutators of bilinear singular integrals

Throughout this chapter we will study the smoothing effect of commutators of different classes of bilinear singular integrals. For the purposes of this dissertation, “smoothing” will mean the improvement of the boundedness to the stronger condition of compactness. This study is motivated by the work of A. Uchiyama [108], who proved that linear commutators of Calderón–Zygmund operators and pointwise multiplication with symbols in an appropriate subspace of BMO are compact. Therefore these commutators behave better than just being bounded, a result earlier proved by Coifman, Rochberg and Weiss in [32].

First, we will focus on the study of compactness of commutators of the class of bilinear operators $\{T_\alpha\}$ that extends the case of bilinear Calderón–Zygmund operators studied in [11]. We will also examine the case of the more singular family of bilinear fractional integrals that can be seen as fractional versions of the bilinear Hilbert transform. Finally, we study the compactness of commutators of bilinear Calderón–Zygmund operators and their iterates in weighted Lebesgue spaces determining the appropriate products of weighted Lebesgue spaces in which this property still holds.

5.1 Basics on compactness properties of bilinear operators

Along this section we will fix the notation as well as remember some notions and results related to compactness of bilinear operators. Even though the definition of compactness in the bilinear setting goes back to Calderón’s foundational article [17], here we will use two different definitions of compactness of a bilinear operator introduced in the work of Bényi and Torres [11].

Definition 5.1.1. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces and let $T : X \times Y \rightarrow Z$ be a bilinear operator. We say that T is:*

1. *Jointly compact (or simply compact) if the set $\{T(x, y) : \|x\|_X, \|y\|_Y \leq 1\}$ is precompact in Z .*
2. *Compact in the first variable if $T_y = T(\cdot, y) : X \rightarrow Z$ is compact for all $y \in Y$.*
3. *Compact in the second variable if $T_x = T(x, \cdot) : Y \rightarrow Z$ is compact for all $x \in X$.*

4. Separately compact if T is compact both in the first and second variable.

Let us make some remarks related to the last definitions.

Remark 5.1.1. Observe that, given three complete norm spaces X , Y and Z , if we denote by $B_{1,X}$ the closed unit ball in X , the definition of compactness specifically requires that if $\{(x_n, y_n)\} \subseteq B_{1,X} \times B_{1,Y}$, then the sequence $\{T(x_n, y_n)\}$ has a convergent subsequence in Z . Clearly, any compact bilinear operator T is continuous.

In general it is only true that separate compactness implies separate continuity. However, if we further consider one of the spaces X or Y to be Banach, the boundedness of T follows from separate compactness as well. For further details on these compactness properties we refer the reader to [11].

Throughout this chapter, the relevant space for the multiplicative symbols in our commutators will be a subspace of BMO , which we denote by CMO . We define CMO to be the closure of C_c^∞ in the BMO norm. It is convenient to mention that the notion CMO for this space is not uniformly used in the literature. Further details can be found in [11]; see the historical comments in Bourdaud [15]. We will only use the fact that, by definition, C_c^∞ is dense in CMO .

Let us recall the fact that if X, Y are normed spaces and Z is a Banach space, the collection of all compact bilinear operators $T : X \times Y \rightarrow Z$ is a closed subset of the collection of continuous bilinear operators. More precisely, it says that the limit of a sequence of bilinear compact operators is a compact operator. This result will be very useful along this chapter and can be found in [17] as well as in [11, Prop. 3].

Let T be a bilinear Calderón-Zygmund operator as defined in Chapter 4. For simplicity, we will assume that the kernel K which is defined away from the diagonal $x = y = z$, as well as ∇K satisfy the following decay conditions used in [53]. Namely,

$$(5.1) \quad |K(x, y, z)| \leq \frac{C}{(|x - y| + |x - z|)^{2n}},$$

and

$$(5.2) \quad |\nabla K(x, y, z)| \leq \frac{C}{(|x - y| + |x - z|)^{2n+1}}$$

where ∇ denotes the gradient in all possible variables and C is a positive constant. Recall that the gradient condition implies the standard smoothness condition (4.3) when $\varepsilon = 1$ as a consequence of the mean value theorem.

If $b \in BMO$, the bilinear commutators can be (formally) expressed in the form

$$\begin{aligned} [T, b]_1(f, g)(x) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y, z)(b(y) - b(x))f(y)g(z) dydz, \\ [T, b]_2(f, g)(x) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y, z)(b(z) - b(x))f(y)g(z) dydz. \end{aligned}$$

Furthermore, given $\vec{b} = (b_1, b_2) \in BMO \times BMO$, we define the iterated commutator as follows:

$$\begin{aligned} (5.3) \quad [T, \vec{b}] &= [[T, b_1]_1, b_2]_2 = [[T, b_2]_2, b_1]_1 \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y, z)(b_1(y) - b_1(x))(b_2(z) - b_2(x))f(y)g(z) dydz. \end{aligned}$$

In general, we can define $[T, \vec{b}]_{\vec{\alpha}}$ for any multi-index $\vec{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, formally as

$$[T, \vec{b}]_{\vec{\alpha}}(f, g)(x) = \iint (b_1(y) - b_1(x))^{\alpha_1} (b_2(z) - b_2(x))^{\alpha_2} K(x, y, z) f(y)g(z) dydz.$$

Let us recall the main result in [11], which extends the linear result of Uchiyama in [108] and confirms that the smoothing effect of commutators of such operators with CMO symbols is also present in the bilinear setting.

Theorem 5.1.1. *Let T be a bilinear Calderón–Zygmund operator. If $b \in CMO$, $1/p+1/q = 1/r$, $1 < p, q < \infty$ and $1 \leq r < \infty$, then, for $i = 1, 2$, $[T, b]_i : L^p \times L^q \rightarrow L^r$ is compact.*

The proof of the above result, as well as the main results proved in this chapter and other compactness results in the literature, make use of a characterization of precompactness in Lebesgue spaces, known as the Fréchet-Kolmogorov-Riesz theorem that can be found for example, in Yosida’s book [117, p. 275].

Theorem 5.1.2. *Let $1 \leq r < \infty$. A subset $\mathcal{K} \subseteq L^r$ is precompact if and only if the following three conditions are satisfied:*

- (a) \mathcal{K} is bounded in L^r ;
- (b) $\lim_{A \rightarrow \infty} \int_{|x| > A} |f(x)|^r dx = 0$ uniformly for $f \in \mathcal{K}$;
- (c) $\lim_{t \rightarrow 0} \|f(\cdot + t) - f\|_{L^r} = 0$ uniformly for $f \in \mathcal{K}$.

5.2 Compactness for commutators of the class $\{T_\alpha\}$

In this section we will show an extension of Theorem 5.1.1 that includes the commutators of the family $\{T_\alpha\}$ that is defined as follows.

Definition 5.2.1. Let $0 < \alpha < 2n$ and K_α be a kernel on \mathbb{R}^{3n} defined away from the diagonal $x = y = z$ that satisfies

$$(5.4) \quad |K_\alpha(x, y, z)| \lesssim \frac{1}{(|x - y| + |x - z|)^{2n-\alpha}}$$

and

$$(5.5) \quad |K_\alpha(x, y, z) - K_\alpha(x + h, y, z)| \lesssim \frac{|h|}{(|x - y| + |x - z|)^{2n-\alpha+1}},$$

with the analogous estimates in the y and z variables. We define the bilinear operator

$$(5.6) \quad T_\alpha(f, g)(x) = \int_{\mathbb{R}^{2n}} K_\alpha(x, y, z) f(y) g(z) dy dz$$

where f, g are bounded functions with compact support.

It is clear that when $\alpha = 0$, this class of operators correspond to the bilinear Calderón–Zygmund operators. An example of the above operator is the bilinear Riesz potential operators I_α , given by the kernel

$$K_\alpha(x, y, z) = \frac{1}{(|x - y| + |x - z|)^{2n-\alpha}}.$$

Let us observe that with respect to boundedness, the commutators of the family $\{T_\alpha\}$ behaves similarly as in the end-point case $\alpha = 0$.

Theorem 5.2.1. Let $0 < \alpha < 2n$, $1 < p, q < \infty$, $r \geq 1$, $\frac{\alpha}{n} < \frac{1}{p} + \frac{1}{q}$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\alpha}{n}$ and $b \in BMO$. The following estimates hold:

$$\begin{aligned} \|[T_\alpha, b]_1(f, g)\|_{L^r} &\lesssim \|b\|_{BMO} \|f\|_{L^p} \|g\|_{L^q}, \\ \|[T_\alpha, b]_2(f, g)\|_{L^r} &\lesssim \|b\|_{BMO} \|f\|_{L^p} \|g\|_{L^q}. \end{aligned}$$

A proof of the above result can be found in [23, 83] while the linear case goes back to Chanillo [20]. The corresponding results for the multilinear Calderón–Zygmund operators used in the proof of Theorem 5.1.1 can be found in [98, 105, 80].

Our goal in this section is to improve boundedness to compactness for this wide class of bilinear operators. In the linear setting it has been considered in different contexts, see for example [24, 109, 13, 22].

5.2. COMPACTNESS FOR COMMUTATORS OF THE CLASS $\{T_\alpha\}$

Observe that the boundedness result in Theorem 5.2.1 for the operators $[T_\alpha, b]_1$, $[T_\alpha, b]_2$ when $r > 1$, and $1/p + 1/q < 1$, can be obtained in an alternative way. Indeed, the kernel bound (5.4) implies that

$$|T_\alpha(f, g)(x)| \lesssim \int_{\mathbb{R}^{2n}} \frac{|f(y)||g(z)|}{(|x-y| + |x-z|)^{2n-\alpha}} dydz = I_\alpha(|f|, |g|)(x).$$

As shown by Moen [90], the operator I_α satisfies appropriate weighted estimates. Therefore, so does T_α , and we can use the ‘‘Cauchy integral trick’’. An exposition of this ‘‘trick’’ can be found in Section 5.3, which deals with the more singular versions BI_α of the operators T_α .

Our first main result in this section is an extension of Theorem 5.1.1 that encompasses the commutators of the family $\{T_\alpha\}_{0 < \alpha < 2n}$.

Theorem 5.2.2. *Let $0 < \alpha < 2n$, $1 < p, q < \infty$, $1 \leq r < \infty$, $\frac{\alpha}{n} < \frac{1}{p} + \frac{1}{q}$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\alpha}{n}$, and let $b \in CMO$. If T_α is the bilinear operator defined by (5.6) whose kernel K_α satisfies (5.4) and (5.5), then $[T_\alpha, b]_1, [T_\alpha, b]_2 : L^p \times L^q \rightarrow L^r$ are compact.*

Proof of Theorem 5.2.2. We will show the result for the commutator in the first variable, $[T_\alpha, b]_1$, since the proof for $[T_\alpha, b]_2$ is similar by symmetry. We will use the Fréchet-Kolmogorov-Riesz theorem that characterizes the pre-compactness of a set in L^r . By the form of the norm estimates in Theorem 5.2.1, density and the results about limits of compact bilinear operators in the operator norm that we already mentioned in Section 5.1, we may assume that $b \in C_c^\infty$.

Denote by B_{1, L^p} and B_{1, L^q} the unit balls in L^p and L^q , respectively and let $\mathcal{K} = [T_\alpha, b]_1(B_{1, L^p}, B_{1, L^q})$. Since $[T_\alpha, b]_1$ is a bounded operator (Theorem 5.2.1), it is clear that \mathcal{K} is a bounded set in L^r , thus fulfilling condition (a) in Theorem 5.1.2. Now we can proceed to prove condition (b) in Theorem 5.1.2. Let us introduce the following two indices:

$$\alpha_p = \alpha(1/p + 1/q)^{-1}1/p \quad \text{and} \quad \alpha_q = \alpha(1/p + 1/q)^{-1}1/q.$$

Clearly, $\alpha_p + \alpha_q = \alpha$. Since $1/p + 1/q - \alpha/n > 0$, there exist $s_p > p > 1$ and $s_q > q > 1$ such that

$$1/s_p = 1/p - \alpha_p/n \quad \text{and} \quad 1/s_q = 1/q - \alpha_q/n.$$

Now, since $p, q > 1$, we see that $n > \max(\alpha_p, \alpha_q)$. In particular, this yields

$$(|x-y| + |x-z|)^{2n-\alpha} = (|x-y| + |x-z|)^{(n-\alpha_p)+(n-\alpha_q)} \geq |x-y|^{n-\alpha_p} |x-z|^{n-\alpha_q}.$$

Pick now $R > 1$ large enough so that $R > 2 \max\{|x| : x \in \text{supp } b\}$. Using (5.4) we see that, for $|x| > R$, we have

$$\begin{aligned} |[T_\alpha, b]_1(f, g)(x)| &\lesssim \|b\|_{L^\infty} \int_{\mathbb{R}^n} \int_{y \in \text{supp } b} \frac{|f(y)||g(z)|}{(|x-y| + |x-z|)^{2n-\alpha}} dy dz \\ &\leq \|b\|_{L^\infty} \int_{y \in \text{supp } b} \int_{\mathbb{R}^n} \frac{|f(y)||g(z)|}{|x-y|^{n-\alpha_p} |x-z|^{n-\alpha_q}} dz dy \\ &\lesssim \frac{\|b\|_{L^\infty}}{|x|^{n-\alpha_p}} \int_{y \in \text{supp } b} |f(y)| \int_{\mathbb{R}^n} \frac{|g(z)|}{|x-z|^{n-\alpha_q}} dz dy \\ &\lesssim \frac{\|b\|_{L^\infty} I_{\alpha_q}(|g|)(x) \|f\|_{L^p} |\text{supp } b|^{1/p'}}{|x|^{n-\alpha_p}}. \end{aligned}$$

Here, we used I_α for the linear Riesz potential, $I_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$. Next, we observe that, since $s_p(n - \alpha_p) \geq np > n$, the function $|x|^{s_p(\alpha_p - n)}$ is integrable at infinity. Therefore, for a given $\varepsilon > 0$, we will be able to select an $R = R(\varepsilon)$ (but independent of f and g) such that

$$\left(\int_{|x| > R} |x|^{s_p(\alpha_p - n)} dx \right)^{1/s_p} \lesssim \varepsilon.$$

Notice now that the indices $s_p, s_q > 1$ satisfy $1/r = 1/s_p + 1/s_q$. Therefore, we can raise the previous pointwise estimate for $|[T_\alpha, b]_1(f, g)(x)|$ to the power r , integrate over $|x| > R$, use the Hölder's inequality and the $L^q \rightarrow L^{s_q}$ boundedness of I_{α_q} to get

$$\left(\int_{|x| > R} |[T_\alpha, b]_1(f, g)(x)|^r dx \right)^{1/r} \lesssim \varepsilon \|f\|_{L^p} \|I_{\alpha_q}(|g|)\|_{L^{s_q}} \lesssim \varepsilon \|f\|_{L^p} \|g\|_{L^q};$$

this, in turn, proves that condition (b) in Theorem 5.1.2 is satisfied.

Next, we will use the smoothness of b and that of the kernel K_α to show that condition (c) in Theorem 5.1.2 holds. Namely, we want to show that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^n} |[T_\alpha, b]_1(f, g)(x+t) - [T_\alpha, b]_1(f, g)(x)|^r dx = 0.$$

We use the following splitting from [11]:

$$[T_\alpha, b]_1(f, g)(x+t) - [T_\alpha, b]_1(f, g)(x) = A(x) + B(x) + C(x) + D(x),$$

where, for $\delta > 0$ to be chosen later, we have

$$A(x) = \iint_{|x-y| + |x-z| > \delta} (b(x+t) - b(x)) K_\alpha(x, y, z) f(y) g(z) dy dz$$

$$\begin{aligned}
 B(x) &= \iint_{|x-y|+|x-z|>\delta} (b(x+t) - b(y))(K_\alpha(x+t, y, z) - K_\alpha(x, y, z))f(y)g(z) dydz \\
 C(x) &= \iint_{|x-y|+|x-z|\leq\delta} (b(y) - b(x))K_\alpha(x, y, z)f(y)g(z) dydz \\
 D(x) &= \iint_{|x-y|+|x-z|\leq\delta} (b(x+t) - b(y))K_\alpha(x+t, y, z)f(y)g(z) dydz.
 \end{aligned}$$

The term A is easy to handle with the mean value theorem. Indeed,

$$|A(x)| \lesssim |t| \|\nabla b\|_{L^\infty} I_\alpha(|f|, |g|)(x).$$

Consequently, by the boundedness of I_α , we obtain

$$(5.7) \quad \|A\|_{L^r} \lesssim |t| \|\nabla b\|_{L^\infty} \|f\|_{L^p} \|g\|_{L^q}.$$

We now consider the terms B , C and D . Observe, that we actually obtain estimates for these terms that slightly improve the corresponding estimates for $\alpha = 0$ in [11]. Let us start with B .

$$\begin{aligned}
 |B(x)| &\leq \iint_{|x-y|+|x-z|>\delta} (b(x+t) - b(y))(K_\alpha(x+t, y, z) - K_\alpha(x, y, z))f(y)g(z) dydz \\
 &\leq 2\|b\|_{L^\infty} \iint_{|x-y|+|x-z|>\delta} |K_\alpha(x+t, y, z) - K_\alpha(x, y, z)| |f(y)| |g(z)| dydz \\
 &\lesssim |t| \|b\|_{L^\infty} \iint_{|x-y|+|x-z|>\delta} \frac{|f(y)||g(z)|}{(|x-y| + |x-z|)^{2n-\alpha+1}} dydz \\
 &\lesssim |t| \|b\|_{L^\infty} \iint_{\max(|x-y|, |x-z|) > \frac{\delta}{2}} \frac{|f(y)||g(z)|}{\max(|x-y|, |x-z|)^{2n-\alpha+1}} dydz \\
 &= |t| \|b\|_{L^\infty} \sum_{k=0}^{\infty} \iint_{2^{k-1}\delta < \max(|x-y|, |x-z|) \leq 2^k\delta} \frac{|f(y)||g(z)|}{\max(|x-y|, |x-z|)^{2n-\alpha+1}} dydz \\
 &\leq |t| \|b\|_{L^\infty} \sum_{k=0}^{\infty} \frac{1}{(2^k\delta)^{2n-\alpha+1}} \iint_{\max(|x-y|, |x-z|) \leq 2^k\delta} |f(y)||g(z)| dydz.
 \end{aligned}$$

Note now that

$$\{(y, z) \in \mathbb{R}^{2n} : \max(|x-y|, |x-z|) \leq 2^k\delta\} \subset B_{2^{k+1}\delta}(x) \times B_{2^{k+1}\delta}(x),$$

where $B_r(x)$ denotes the ball of radius r centered at x . Therefore, we can further estimate

$$|B(x)| \lesssim \frac{|t| \|b\|_{L^\infty}}{\delta} \sum_{k=0}^{\infty} \frac{|B_{2^k\delta}(x)|^{\frac{\alpha}{n}}}{2^k} \frac{1}{|B_{2^k\delta}(x)|} \int_{B_{2^k\delta}(x)} |f(y)| dy \frac{1}{|B_{2^k\delta}(x)|} \int_{B_{2^k\delta}(x)} |g(z)| dz$$

$$\lesssim |t| \|b\|_{L^\infty} \frac{1}{\delta} \mathcal{M}_\alpha(f, g)(x) \left(\sum_{k=0}^{\infty} 2^{-k} \right) = \frac{2|t| \|b\|_{L^\infty}}{\delta} \mathcal{M}_\alpha(f, g)(x),$$

where

$$\mathcal{M}_\alpha(f, g)(x) = \sup_{Q \ni x} |Q|^{\alpha/n} \left(\int_Q |f(y)| dy \right) \left(\int_Q |g(z)| dz \right).$$

Since the operator $\mathcal{M}_\alpha(f, g)$ is pointwise smaller than $I_\alpha(|f|, |g|)$, we get $\mathcal{M}_\alpha : L^p \times L^q \rightarrow L^r$. In turn, this yields

$$(5.8) \quad \|B\|_{L^r} \lesssim \frac{|t| \|b\|_{L^\infty}}{\delta} \|f\|_{L^p} \|g\|_{L^q}.$$

Let us now estimate the C term.

$$\begin{aligned} |C(x)| &\leq \iint_{|x-y|+|x-z|\leq\delta} |b(y) - b(x)| |K_\alpha(x, y, z)| |f(y)| |g(z)| dydz \\ &\lesssim \|\nabla b\|_{L^\infty} \sum_{k=0}^{\infty} \iint_{2^{-k-1}\delta < \max(|x-y|, |x-z|) \leq 2^{-k}\delta} \frac{|f(y)| |g(z)|}{\max(|x-y|, |x-z|)^{2n-\alpha-1}} dydz \\ &\lesssim \|\nabla b\|_{L^\infty} \sum_{k=0}^{\infty} \frac{2^{-k}\delta}{(2^{-k}\delta)^{2n-\alpha}} \iint_{\max(|x-y|, |x-z|) \leq 2^{-k}\delta} |f(y)| |g(z)| dydz \\ &\lesssim \delta \|\nabla b\|_{L^\infty} \mathcal{M}_\alpha(f, g)(x), \end{aligned}$$

where we have used a similar argument as before. From here, we get

$$(5.9) \quad \|C\|_{L^r} \lesssim \delta \|\nabla b\|_{L^\infty} \|f\|_{L^p} \|g\|_{L^q}.$$

For the last term D we have an identical estimate to the C term, except that x is now replaced by $x + t$. We have

$$\begin{aligned} |D(x)| &\leq \iint_{|x-y|+|x-z|\leq\delta} |b(x+t) - b(y)| |K_\alpha(x+t, y, z)| |f(y)| |g(z)| dydz \\ &\lesssim \|\nabla b\|_{L^\infty} \iint_{|x+t-y|+|x+t-z|\leq\delta+2|t|} \frac{|x+t-y| |f(y)| |g(z)|}{(|x+t-y| + |x+t-z|)^{2n-\alpha}} dydz \\ &\leq \|\nabla b\|_{L^\infty} \iint_{|x+t-y|+|x+t-z|\leq\delta+2|t|} \frac{|f(y)| |g(z)|}{(|x+t-y| + |x+t-z|)^{2n-\alpha-1}} dydz \\ &\lesssim (\delta + |t|) \|\nabla b\|_{L^\infty} \mathcal{M}_\alpha(f, g)(x+t). \end{aligned}$$

Thus, as above, we get

$$(5.10) \quad \|D\|_{L^r} \lesssim (\delta + |t|) \|\nabla b\|_{L^\infty} \|f\|_{L^p} \|g\|_{L^q}.$$

5.3. SEPARATE COMPACTNESS FOR COMMUTATORS OF THE CLASS $\{BI_\alpha\}$

Let $0 < \varepsilon < 1$ be given. For each $0 < |t| < \varepsilon^2$ we now select $\delta = |t|/\varepsilon$. Estimates (5.7), (5.8), (5.9) and (5.10) then prove

$$\|[T_\alpha, b]_1(f, g)(\cdot + t) - [T_\alpha, b]_1(f, g)(\cdot)\|_{L^r} \lesssim \varepsilon \|f\|_{L^p} \|g\|_{L^q},$$

that is, condition (c) in Theorem 5.1.2 holds. \square

Remark 5.2.1. *Iterated commutators can be considered as well. When dealing with operators as the ones in (5.3), for example*

$$[[T, b_1]_1, b_2]_2(f, g) = [T, b_1]_1(f, b_2g) - b_2[T, b_1]_1(f, g),$$

we know that for bilinear Calderón–Zygmund operators, the boundedness of such operators was studied in [105, 96], while for bilinear fractional integrals they were addressed in [83]. The compactness of iterated commutators is easier to prove by adapting the arguments in [11] to the family $\{T_\alpha\}$ as pointed out in that work.

5.3 Separate compactness for commutators of the class $\{BI_\alpha\}$

In this section we will study a more singular family of bilinear fractional integral operators,

$$(5.11) \quad BI_\alpha(f, g)(x) = \int_{\mathbb{R}^n} \frac{f(x-y)g(x+y)}{|y|^{n-\alpha}} dy.$$

These operators were first introduced by Grafakos in [47], and later studied by Grafakos and Kalton [50] and Kenig and Stein [69]. They can be seen as fractional versions of the bilinear Hilbert transform

$$BHT(f, g)(x) = p.v. \int_{\mathbb{R}} \frac{f(x-y)g(x+y)}{y} dy.$$

Next we recall an observation of Bernicot, Maldonado, Moen and Naibo in [12, Rmk. 2.1] regarding to the boundedness of these operators in certain weighted Lebesgue spaces that we will use later on. If $1 < s < r$ satisfy $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$ and $\frac{1}{s} - \frac{1}{r} = \frac{\alpha}{n}$. Then

$$(5.12) \quad BI_\alpha : L^p(w_1^p) \times L^q(w_2^q) \rightarrow L^r(w_1^r w_2^r)$$

where $w_1, w_2 \in A_{s,r}$, that is, for $i = 1, 2$,

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w_i^r dx \right) \left(\frac{1}{|Q|} \int_Q w_i^{-s'} dx \right)^{\frac{r}{s'}} < \infty.$$

For $i = 1, 2$ and $b \in BMO$, we define the commutators $[BI_\alpha, b]_i$ similarly to those of the operators T_α introduced in the previous section. Our goals in this section are twofold. First, we will prove that the commutators $[BI_\alpha, b]_i$, $i = 1, 2$, are bounded and then we will show that they are also compact. For the proof of the first result we will use what we call the ‘‘Cauchy integral trick’’.

Theorem 5.3.1. *Let $0 < \alpha < n$, $1 < p, q, r < \infty$, $\frac{1}{p} + \frac{1}{q} < 1$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\alpha}{n}$, and $b \in BMO$. Then, for $i = 1, 2$, we have*

$$\|[BI_\alpha, b]_i(f, g)\|_{L^r} \leq C \|b\|_{BMO} \|f\|_{L^p} \|g\|_{L^q}.$$

Proof of Theorem 5.3.1. As before, we only will work with the commutator in the first variable since the proof for the second variable is identical. We define $s > 1$ by $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. Without loss of generality, we may assume $f, g \in C_c^\infty(\mathbb{R}^n)$ and b is real valued. For $z \in \mathbb{C}$, consider the holomorphic function (in z)

$$T_z(f, g; \alpha) = e^{zb} BI_\alpha(e^{-zb} f, g),$$

and notice that by the Cauchy integral formula, for $\varepsilon > 0$,

$$[BI_\alpha, b]_1(f, g) = -\frac{d}{dz} T_z(f, g; \alpha) \Big|_{z=0} = -\frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{T_z(f, g; \alpha)}{z^2} dz.$$

Since $r > 1$, we can use Minkowski’s integral inequality (1.8) to obtain

$$\|[BI_\alpha, b]_1(f, g)\|_{L^r} \leq \frac{1}{2\pi\varepsilon^2} \int_{|z|=\varepsilon} \|T_z(f, g; \alpha)\|_{L^r} |dz|$$

and

$$\|T_z(f, g; \alpha)\|_{L^r}^r = \int_{\mathbb{R}^n} \left(|BI_\alpha(e^{-zb} f, g)| e^{(\operatorname{Re} z)b} \right)^r dx.$$

For $\varepsilon > 0$, $\varepsilon \lesssim \|b\|_{BMO}^{-1}$, and $|t| \leq \varepsilon$, by John-Nirenberg’s inequality (1.31), we have $e^{tb} \in A_{s,r}$. Therefore, by (5.12) with $w_1 = e^b$ and $w_2 = 1$, we have

$$\begin{aligned} \|T_z(f, g; \alpha)\|_{L^r} &= \left(\int_{\mathbb{R}^n} \left(|BI_\alpha(e^{-zb} f, g)| e^{(\operatorname{Re} z)b} \right)^r dx \right)^{1/r} \\ &\leq C \left(\int_{\mathbb{R}^n} (|e^{-zb} f| e^{(\operatorname{Re} z)b})^p dx \right)^{1/p} \left(\int_{\mathbb{R}^n} |g|^q dx \right)^{1/q} \\ &= C \|f\|_{L^p} \|g\|_{L^q}. \end{aligned}$$

The desired result follows from here. □

5.3. SEPARATE COMPACTNESS FOR COMMUTATORS OF THE CLASS $\{BI_\alpha\}$

Finally, we show the compactness of the commutator of BI_α .

Theorem 5.3.2. *Let $0 < \alpha < n$, $1 < p, q, r < \infty$, $\frac{1}{p} + \frac{1}{q} < 1$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{\alpha}{n}$, and $b \in CMO$. Then, $[BI_\alpha, b]_1, [BI_\alpha, b]_2 : L^p \times L^q \rightarrow L^r$ are separately compact.*

Proof of Theorem 5.3.2. We will work again with the commutator in the first variable. By a change of variables, this commutator can be rewritten as

$$[BI_\alpha, b]_1(f, g)(x) = \int_{\mathbb{R}^n} \frac{b(y) - b(x)}{|x - y|^{n-\alpha}} f(y)g(2x - y) dy.$$

We may assume that $b \in C_c^\infty(\mathbb{R}^n)$ and aim to prove that the conditions (a), (b) and (c) of Theorem 5.1.2 hold for the family of functions $[BI_\alpha, b]_1(f, g)$, where $g \in L^q$ is fixed and $f \in B_{1, L^p}$.

By Theorem 5.3.1, we already know that condition (a) is satisfied. Thus, we concentrate on proving (b) and (c). The estimates that yield (b) are reminiscent of the ones used in the proof of Theorem 5.2.2. Assume $R > 1$ is large enough so that $|x| \geq R$ implies $x \notin \text{supp } b$. Then

$$\begin{aligned} |[BI_\alpha, b]_1(f, g)(x)| &\leq \|b\|_{L^\infty} \int_{\text{supp } b} |x - y|^{\alpha-n} |f(y)g(2x - y)| dy \\ &\lesssim \|b\|_{L^\infty} |x|^{\alpha-n} \int_{\text{supp } b} |f(y)g(2x - y)| dy \\ &\leq \|b\|_{L^\infty} |x|^{\alpha-n} \left(\int_{\text{supp } b} |f(y)|^{q'} dy \right)^{1/q'} \|g\|_{L^q}. \end{aligned}$$

Let us write $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} < 1 = \frac{1}{q} + \frac{1}{q'}$. Then, $q' < p$. Now we have

$$\left(\int_{\text{supp } b} |f(y)|^{q'} dy \right)^{1/q'} \leq |\text{supp } b|^{\frac{1}{q'} - \frac{1}{p}} \|f\|_{L^p} = |\text{supp } b|^{\frac{1}{s'}} \|f\|_{L^p}.$$

Next, we raise to the power r and integrate with respect to x over the set $|x| > R$. Notice that, since $s > 1$, we have $\frac{1}{r} = \frac{1}{s} - \frac{\alpha}{n} < \frac{n-\alpha}{n} \Leftrightarrow r(n-\alpha) > n$. This allows us, for a given $\varepsilon > 0$, to control

$$\int_{|x| > R} |[BI_\alpha, b]_1(f, g)(x)|^r dx \lesssim \varepsilon^r$$

by taking $R = R(\varepsilon) > 0$ sufficiently large; which shows that, indeed, (b) is satisfied. We are left to show the continuity condition (c), that is,

$$\lim_{t \rightarrow 0} \|[BI_\alpha, b]_1(f, g)(\cdot + t) - [BI_\alpha, b]_1(f, g)\|_{L^r} = 0,$$

uniformly for $\|f\|_{L^p} \leq 1$ and $g \in L^q$ fixed. First, we lump our fixed function g into a general kernel

$$\begin{aligned} [BI_\alpha, b]_1(f, g)(x) &= \int_{\mathbb{R}^n} \frac{b(y) - b(x)}{|x - y|^{n-\alpha}} f(y) g(2x - y) dy \\ &= \int_{\mathbb{R}^n} (b(y) - b(x)) K_g(x, y) f(y) dy \end{aligned}$$

where

$$K_g(x, y) = \frac{g(2x - y)}{|x - y|^{n-\alpha}}.$$

Second, we split the commutator $[BI_\alpha, b]_1$ by following the decomposition used for $[T_\alpha, b]_1$. Namely, we write

$$[BI_\alpha, b]_1(f, g)(x + t) - [BI_\alpha, b]_1(f, g)(x) = A(x) + B(x) + C(x) + D(x),$$

where

$$\begin{aligned} A(x) &= \int_{|x-y|>\delta} (b(x+t) - b(x)) K_g(x, y) f(y) dy, \\ B(x) &= \int_{|x-y|>\delta} (b(x+t) - b(y)) (K_g(x+t, y) - K_g(x, y)) f(y) dy, \\ C(x) &= \int_{|x-y|\leq\delta} (b(y) - b(x)) K_g(x, y) f(y) dy, \\ D(x) &= \int_{|x-y|\leq\delta} (b(x+t) - b(y)) K_g(x+t, y) f(y) dy. \end{aligned}$$

We will now estimate each term in this decomposition. For A , the estimate is immediate. We clearly have $|A(x)| \leq |t| \|\nabla b\|_{L^\infty} BI_\alpha(|f|, |g|)(x)$. Since BI_α is bounded from $L^p \times L^q$ into L^r , we get $\|A\|_{L^r} \lesssim |t| \|f\|_{L^p} \|g\|_{L^q}$.

The estimate for the B term is the most delicate. For the sake of simplicity, we postpone it until the end of the proof.

We estimate C as follows

$$\begin{aligned} |C(x)| &\leq \int_{|x-y|\leq\delta} |b(y) - b(x)| \frac{|g(2x - y)|}{|x - y|^{n-\alpha}} |f(y)| dy \\ &\leq \|\nabla b\|_{L^\infty} \int_{|x-y|\leq\delta} |x - y| \frac{|g(2x - y)|}{|x - y|^{n-\alpha}} |f(y)| dy \\ &\leq \delta \|\nabla b\|_{L^\infty} BM_\alpha(f, g)(x), \end{aligned}$$

where BM_α is the associated bilinear fractional maximal operator,

$$BM_\alpha(f, g)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} |f(x - y)g(x + y)| dy.$$

The estimate for $D(x)$ is similar. Thus we have

$$|D(x)| \leq (\delta + |t|) \|\nabla b\|_{L^\infty} BM_\alpha(f, g)(x + t).$$

Since $BM_\alpha(f, g) \lesssim BI_\alpha(|f|, |g|)$, we have that BM_α is bounded from $L^p \times L^q$ into L^r . Thus, similarly to A , we get $\|C\|_{L^r} \lesssim \delta \|f\|_{L^p} \|g\|_{L^q}$ and $\|D\|_{L^r} \lesssim (\delta + |t|) \|f\|_{L^p} \|g\|_{L^q}$. Finally, we estimate B .

$$\begin{aligned} |B(x)| &\leq \int_{|x-y|>\delta} |b(x+t) - b(y)| \left| \frac{g(2x+2t-y)}{|x+t-y|^{n-\alpha}} - \frac{g(2x-y)}{|x-y|^{n-\alpha}} \right| |f(y)| dy \\ &\leq 2\|b\|_{L^\infty} \int_{|x-y|>\delta} \left| \frac{g(2x+2t-y)}{|x+t-y|^{n-\alpha}} - \frac{g(2x-y)}{|x-y|^{n-\alpha}} \right| |f(y)| dy \\ &\lesssim \int_{|x-y|>\delta} \left| \frac{1}{|x+t-y|^{n-\alpha}} - \frac{1}{|x-y|^{n-\alpha}} \right| |g(2x+2t-y)f(y)| dy \\ &\quad + \int_{|x-y|>\delta} \frac{|g(2x+2t-y) - g(2x-y)| |f(y)|}{|x-y|^{n-\alpha}} dy \\ &= E(x) + F(x). \end{aligned}$$

To estimate E , we note that

$$\left| \frac{1}{|x+t-y|^{n-\alpha}} - \frac{1}{|x-y|^{n-\alpha}} \right| \lesssim \frac{|t|}{|x-y|^{n-\alpha+1}},$$

which implies

$$\begin{aligned} E(x) &\lesssim |t| \int_{|x-y|>\delta} \frac{|g(2x+2t-y)f(y)|}{|x-y|^{n-\alpha+1}} dy \\ &\lesssim \frac{|t|}{\delta} BM_\alpha(f, \tau_{2t}g)(x). \end{aligned}$$

Here, τ_a is the shift operator $\tau_a g(x) = g(x+a)$. It follows from the boundedness of BM_α that

$$\|E\|_{L^r} \lesssim \frac{|t|}{\delta} \|f\|_{L^p} \|g\|_{L^q}.$$

For $F(x)$ we have

$$F(x) \lesssim BM_\alpha(f, \tau_{2t}g - g)(x),$$

so

$$\|F\|_{L^r} \lesssim \|f\|_{L^p} \|\tau_{2t}g - g\|_{L^q}.$$

Since $g \in L^q$, for a given $\varepsilon > 0$ we can find $\gamma = \gamma(\varepsilon, g) > 0$ such that $|t| < \gamma$ implies

$$\|\tau_{2t}g - g\|_{L^q} < \varepsilon.$$

Finally, by choosing $|t| < \varepsilon^2$ and $\delta = |t|/\varepsilon$ we get that

$$\|[BI_\alpha, b]_1(f, g)(\cdot + t) - [BI_\alpha, b]_1(f, g)\|_{L^r} \lesssim \varepsilon.$$

This shows that (c) holds, thus finishing our proof for the compactness in the first variable.

We now show that $[BI_\alpha, b]_1$ is compact in the second variable, that is, $[BI_\alpha, b]_1(f, \cdot) : L^q \rightarrow L^r$ is compact for a fixed $f \in L^p$. Conditions (a) and (b) of Theorem 5.1.2 follow from similar calculations to those performed above. Thus we will check condition (c) of Theorem 5.1.2. For $f \in L^p$ fixed and $g \in B_{1, L^q}$ we write

$$\begin{aligned} & [BI_\alpha, b]_1(f, g)(x+t) - [BI_\alpha, b]_1(f, g)(x) \\ &= \int_{\mathbb{R}^n} (b(2x+2t-y) - b(x+t)) \frac{f(2x+2t-y)g(y)}{|x+t-y|^{n-\alpha}} dy \\ &\quad - \int_{\mathbb{R}^n} (b(2x-y) - b(x)) \frac{f(2x-y)g(y)}{|x-y|^{n-\alpha}} dy \\ &= \int_{\mathbb{R}^n} (b(2x+2t-y) - b(x+t)) K_f(x+t, y) g(y) dy \\ &\quad - \int_{\mathbb{R}^n} (b(2x-y) - b(x)) K_f(x, y) g(y) dy, \end{aligned}$$

where this time we combine f with the kernel

$$K_f(x, y) = \frac{f(2x-y)}{|x-y|^{n-\alpha}}.$$

Before proceeding further, we make one reduction. Notice that

$$\begin{aligned} & [BI_\alpha, b]_1(f, g)(x+t) = \int_{\mathbb{R}^n} (b(2x+2t-y) - b(x+t)) K_f(x+t, y) g(y) dy \\ (5.13) \quad &= \int_{\mathbb{R}^n} (b(2x+2t-y) - b(2x-y)) K_f(x+t, y) g(y) dy \\ &\quad + \int_{\mathbb{R}^n} b(2x-y) - b(x+t) K_f(x+t, y) g(y) dy \end{aligned}$$

The first term in the sum (5.13) is bounded by

$$2\|\nabla b\|_{L^\infty} |t| BI_\alpha(|f|, |g|)(x+t)$$

and the L^r norm of this quantity will go to zero uniformly for $g \in B_{1, L^q}$ as t goes to 0. Thus it remains to estimate

$$\begin{aligned} & \int_{\mathbb{R}^n} (b(2x - y) - b(x + t))K_f(x + t, y)g(y) dy \\ & - \int_{\mathbb{R}^n} (b(2x - y) - b(x))K_f(x, y)g(y) dy = G(x) + H(x) + I(x) + J(x) \end{aligned}$$

where

$$\begin{aligned} G(x) &= \int_{|x-y|>\delta} (b(x) - b(x + t))K_f(x, y)g(y) dy, \\ H(x) &= \int_{|x-y|>\delta} (b(2x - y) - b(x + t))(K_f(x + t, y) - K_f(x, y))g(y) dy, \\ I(x) &= \int_{|x-y|\leq\delta} (b(x) - b(2x - y))K_f(x, y)g(y) dy, \\ J(x) &= \int_{|x-y|\leq\delta} (b(2x - y) - b(x + t))K_f(x + t, y)g(y) dy. \end{aligned}$$

The estimates for G, H, I , and J are handled similarly to the corresponding estimates for A, B, C , and D above, again, with H being the most complicated. For example the estimates for G, I , and J are as follows:

$$|G(x)| \leq |t| \|\nabla b\|_{L^\infty} BI_\alpha(f, g)(x),$$

$$|I(x)| \leq \delta \|\nabla b\|_{L^\infty} BM_\alpha(f, g)(x),$$

and

$$|J(x)| \leq (\delta + |t|) \|\nabla b\|_{L^\infty} BM_\alpha(f, g)(x + t).$$

Finally, for H we have

$$|H(x)| \lesssim \|b\|_{L^\infty} \left(\frac{|t|}{\delta} BM_\alpha(\tau_{2t}f, g)(x) + \frac{1}{\delta} BM_\alpha(\tau_{2t}f - f, g)(x) \right).$$

These estimates show that $[BI_\alpha, b]_1(f, g)$ is compact in the second variable as well, thus showing that it is separately compact. \square

Remark 5.3.1. *By inspecting the proof of Theorem 5.3.2 one can see that our method of proof only yields separate compactness. Indeed, the only non-uniform estimate concerns the very last terms, which we denote by F and H , where we use the fact that we can make the quantity $\|\tau_{2t}g - g\|_{L^q}$ (or $\|\tau_{2t}f - f\|_{L^p}$) small by taking t sufficiently small and, crucially, dependent on g (or f). In this case, a weaker smoothing property is obtained for the commutators of the more singular bilinear fractional integrals BI_α compared to the nicely behaved operators T_α . Therefore, the natural question that*

arises here is whether for $b \in CMO$ the commutators $[BI_\alpha, b]_i$, $i = 1, 2$ are jointly compact.

Besides, if we assume that $[BHT, b]_1 : L^p \times L^q \rightarrow L^r$, then using the same techniques as the ones used in this section, $[BHT, b]_1$ (and $[BHT, b]_2$) will be separately compact for $b \in CMO$. Thus, another natural question is the following: for $b \in BMO$, are the commutators $[BHT, b]_i$, $i = 1, 2$, bounded from $L^p \times L^q$ into L^r ?

5.4 Compactness of commutators of bilinear Calderón–Zygmund operators in weighted Lebesgue spaces

The purpose of this section is to show that the compactness of commutators of bilinear Calderón–Zygmund operators and their iterates carries over to the weighted setting when we consider the appropriate class of weights. Indeed, the following result is obtained for the commutator of a bilinear Calderón–Zygmund operator T .

Theorem 5.4.1. *Suppose $\vec{P} \in (1, \infty) \times (1, \infty)$, $p = \frac{p_1 p_2}{p_1 + p_2} > 1$, $b \in CMO$, and $\vec{w} \in A_p \times A_p$. Then $[T, b]_1$ and $[T, b]_2$ are compact operators from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^p(\nu_{\vec{w}})$.*

A similar result holds also for the iterated commutator defined as in (5.3).

Theorem 5.4.2. *Suppose $\vec{P} \in (1, \infty) \times (1, \infty)$, $p = \frac{p_1 p_2}{p_1 + p_2} > 1$, $\vec{b} \in CMO \times CMO$, and $\vec{w} \in A_p \times A_p$. Then $[T, \vec{b}]$ is a compact operator from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^p(\nu_{\vec{w}})$.*

To prove the above results we will need the following sufficient condition for precompactness in $L^r(w)$ obtained in [27] by adapting the arguments in [54] and avoiding the non-translation invariance of $L^r(w)$. Observe that for showing this version of the Fréchet-Kolmogorov-Riesz theorem the weight w must be assumed, essentially for the argument to work, to be in A_r .

Theorem 5.4.3. *Let $1 < r < \infty$ and $w \in A_r$ and let $\mathcal{K} \subset L^r(w)$. If*

- (i) \mathcal{K} is bounded in $L^r(w)$;
- (ii) $\lim_{A \rightarrow \infty} \int_{|x| > A} |f(x)|^r w(x) dx = 0$ uniformly for $f \in \mathcal{K}$;
- (iii) $\lim_{t \rightarrow 0} \|f(\cdot + t) - f\|_{L^r(w)} = 0$ uniformly for $f \in \mathcal{K}$;

then \mathcal{K} is precompact in $L^r(w)$.

Let us immediately note now that our choice for the class of vector weights in Theorems 5.4.1 and 5.4.2 is dictated by the previous compactness criterion. In both results we will need the weight $\nu_{\vec{w}} \in A_p$ to apply the above version of Fréchet-Kolmogorov-Riesz theorem. In general, if $\vec{w} \in A_{p_1} \times A_{p_2}$ or $\vec{w} \in A_{\vec{p}}$, the best class that $\nu_{\vec{w}}$ belongs to is A_{2p} , where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. However, if $\vec{w} \in A_p \times A_p$ then $\nu_{\vec{w}}$ is actually in A_p as a consequence of Hölder's inequality. We also point out there there exists examples with $\vec{w} \in A_{\vec{p}}$ and $\nu_{\vec{w}} \in A_p$, but $\vec{w} \notin A_p \times A_p$ (see Remark 5.4.2).

As pointed out in [27] in the linear setting the idea of considering truncated operators goes back to Krantz and Li [70]. We will follow the same approach here introducing the following truncated kernel:

$$K^\delta(x, y, z) = \begin{cases} K(x, y, z), & \max(|x - y|, |x - z|) > \delta \\ 0, & \max(|x - y|, |x - z|) \leq \delta. \end{cases}$$

We note that K^δ still obeys the size estimate $|K^\delta(x, y, z)| \leq \frac{C}{(|x-y|+|x-z|)^{2n}}$ uniformly in $\delta > 0$. Similarly define the operator $T^\delta(f, g)$ associated to K^δ .

First, we need the following lemma.

Lemma 5.4.1. *If $\vec{b} \in C_c^\infty \times C_c^\infty$, then*

$$|[T^\delta, \vec{b}]_\alpha(f, g)(x) - [T, \vec{b}]_\alpha(f, g)(x)| \lesssim \|\nabla b_1\|_{L^\infty}^{\alpha_1} \|\nabla b_2\|_{L^\infty}^{\alpha_2} \delta^{|\alpha|} \mathcal{M}(f, g)(x),$$

Consequently, if $\vec{w} \in A_{\vec{p}}$ we have

$$\lim_{\delta \rightarrow 0} \|[T^\delta, \vec{b}]_\alpha - [T, \vec{b}]_\alpha\|_{L^{p_1}(w_1) \times L^{p_2}(w_2) \rightarrow L^p(\nu_{\vec{w}})} = 0.$$

Proof of Lemma 5.4.1. We adapt the proof in [27, Lemma 7] for the linear situation of the result. Let us consider for simplicity the commutator in the first variable. The proof for the other commutator follow similarly. We have,

$$\begin{aligned}
 & |[T^\delta, b]_1(f, g)(x) - [T, b]_1(f, g)(x)| \\
 & \lesssim \iint_{\max(|x-y|, |x-z|) \leq \delta} |b(y) - b(x)| |K(x, y, z)| |f(y)g(z)| dy dz \\
 & \lesssim \|\nabla b\|_{L^\infty} \sum_{j=0}^{\infty} \iint_{2^{-j-1}\delta < \max(|x-y|, |x-z|) \leq 2^{-j}\delta} \frac{|f(y)||g(z)|}{(|x-y| + |x-z|)^{2n-1}} dy dz \\
 & \lesssim \|\nabla b\|_{L^\infty} \sum_{j=0}^{\infty} \iint_{2^{-j-1}\delta < \max(|x-y|, |x-z|) \leq 2^{-j}\delta} \frac{|f(y)||g(z)|}{\max(|x-y|, |x-z|)^{2n-1}} dy dz \\
 & \lesssim \|\nabla b\|_{L^\infty} \sum_{j=0}^{\infty} \frac{2^{-j}\delta}{(2^{-j}\delta)^{2n}} \iint_{\max(|x-y|, |x-z|) \leq 2^{-j}\delta} |f(y)||g(z)| dy dz \\
 & \lesssim \|\nabla b\|_{L^\infty} \delta \sum_{j=0}^{\infty} 2^{-j} \int_{|x-y| \leq 2^{-j}\delta} |f(y)| dy \int_{|x-z| \leq 2^{-j}\delta} |g(z)| dz \\
 & \lesssim \|\nabla b\|_{L^\infty} \delta \mathcal{M}(f, g)(x).
 \end{aligned}$$

and the rest of the result follows from the boundedness properties of the maximal function \mathcal{M} . \square

Lemma 5.4.1 states that $[T^\delta, \vec{b}]_\alpha$ converges in operator norm to $[T, b]_\alpha$ provided the functions in \vec{b} are smooth enough. Therefore, in order to prove that any of the commutators $[T, \vec{b}]_\alpha$ are compact it suffices to work with $[T^\delta, b]_\alpha$ for a fixed δ and our estimates may depend on δ . Notice that it is due to the fact that, as in the linear case, the limit in the operator norm of compact operators is compact. Moreover, since the bounds of the commutators with *BMO* functions are of the form

$$\|[T, \vec{b}]_\alpha(f, g)\|_{L^p(\nu_{\vec{b}})} \lesssim \|b_1\|_{BMO}^{\alpha_1} \|b_2\|_{BMO}^{\alpha_2} \|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)},$$

to show compactness when working with symbols in *CMO* we may also assume $\vec{b} \in C_c^\infty \times C_c^\infty$ by density and the estimates may depend on \vec{b} too.

Proof of Theorem 5.4.1. We will work with the commutator in the first variable since, by symmetry, the proof for the other commutator is identical. As already pointed out, we may fix $\delta > 0$ and assume $b \in C_c^\infty$. Suppose f, g belong to

$$B_{1, L^{p_1}(w_1)} \times B_{1, L^{p_2}(w_2)} = \{(f, g) : \|f\|_{L^{p_1}(w_1)}, \|g\|_{L^{p_2}(w_2)} \leq 1\},$$

with w_1 and w_2 in A_p . We need to show that the following conditions hold:

- (a) $[T^\delta, b]_1(B_{1,L^{p_1}(w_1)} \times B_{1,L^{p_2}(w_2)})$ is bounded in $L^p(\nu_{\vec{w}})$;
- (b) $\lim_{R \rightarrow \infty} \int_{|x| > R} |[T^\delta, b]_1(f, g)(x)|^p \nu_{\vec{w}} dx = 0$;
- (c) $\lim_{t \rightarrow 0} \|[T^\delta, b]_1(f, g)(\cdot + t) - [T^\delta, b]_1(f, g)\|_{L^p(\nu_{\vec{w}})} = 0$.

It is clear that the first condition (a) holds since

$$[T^\delta, b]_1 : L^{p_1}(w_1) \times L^{p_2}(w_2) \rightarrow L^p(\nu_{\vec{w}})$$

is bounded when $\vec{w} \in A_p \times A_p \subset A_{\vec{p}}$.

We now show that the second condition (b) holds. It is worth pointing out that for our calculations to work, we need the more restrictive assumption $\nu_{\vec{w}} \in A_p$ which holds since $\vec{w} \in A_p \times A_p$. Let A be large enough so that $\text{supp } b \subset B_A(0)$ and let $R \geq \max(1, 2A)$. Then for $|x| > R$ we have

$$\begin{aligned} |[T^\delta, b]_1(f, g)(x)| &\leq \|b\|_{L^\infty} \int_{\text{supp } b} \int_{\mathbb{R}^n} \frac{|f(y)||g(z)|}{(|x-y| + |x-z|)^{2n}} dy dz \\ &\lesssim \|b\|_{L^\infty} \int_{\text{supp } b} |f(y)| \int_{\mathbb{R}^n} \frac{|g(z)|}{(|x| + |x-z|)^{2n}} dy dz \\ &\lesssim \|b\|_{L^\infty} \|f\|_{L^{p_1}(w_1)} \sigma_1(B_A(0))^{1/p'_1} \int_{\mathbb{R}^n} \frac{|g(z)|}{(|x| + |x-z|)^{2n}} dz \\ &\lesssim \frac{1}{|x|^n} \|b\|_{L^\infty} \|f\|_{L^{p_1}(w_1)} \sigma_1(B_A(0))^{1/p'_1} \int_{\mathbb{R}^n} \frac{|g(z)|}{(|x| + |x-z|)^n} dz, \end{aligned}$$

where in the second inequality we have used $|x-y| \geq |x| - |y| \geq \frac{|x|}{2}$ since $|y| < A \leq \frac{R}{2} \leq \frac{|x|}{2}$. Here σ_i , $i = 1, 2$, denotes as usual the conjugate weight of w_i , that is, $\sigma_i = w_i^{1-p'_i}$.

Let us now work with the factor

$$\int_{\mathbb{R}^n} \frac{|g(z)|}{(|x| + |x-z|)^n} dz.$$

We split this integral into a local and a global part. For the local part, since $|x| \leq 1$ we have

$$\int_{|z| \leq 1} \frac{|g(z)|}{(|x| + |x-z|)^n} dz \leq \int_{|z| \leq 1} |g(z)| dz \leq \|g\|_{L^{p_2}(w_2)} \sigma_2(B_1(0))^{1/p'_2}.$$

For the global part we notice that $|z| \leq |z-x| + |x|$, and hence

$$\int_{|z| \geq 1} \frac{|g(z)|}{(|x| + |x-z|)^n} dz \leq \int_{|z| \geq 1} \frac{|g(z)|}{|z|^n} dz \leq \|g\|_{L^{p_2}(w_2)} \left(\int_{|z| \geq 1} \frac{\sigma_2(z)}{|z|^{np'_2}} dz \right)^{1/p'_2}.$$

Since $w_2 \in A_p \subset A_{p_2}$, we have $\sigma_2 \in A_{p'_2}$, and then

$$(5.14) \quad \int_{|z| \geq 1} \frac{\sigma_2(z)}{|z|^{np'_2}} dz < \infty.$$

Estimate (5.14) can be found, for example, in [46, p. 412]. Combining everything, for $|x| > R$ we have

$$\begin{aligned} & |[T^\delta, b]_1(f, g)(x)| \\ & \lesssim \frac{\|b\|_{L^\infty}}{|x|^n} \sigma_1(B_A(0))^{1/p'_1} \left((\sigma_2(B_1(0)))^{1/p'_2} + \left(\int_{|z| \geq 1} \frac{\sigma_2(z)}{|z|^{np'_2}} dz \right)^{1/p'_2} \right) \end{aligned}$$

Raising both sides of the inequality to the power p and integrating over $|x| > R$ we have

$$\int_{|x| > R} |[T^\delta, b]_1(f, g)(x)|^p \nu_{\bar{w}} dx \lesssim_{b, \bar{P}, \bar{w}} \int_{|x| > R} \frac{\nu_{\bar{w}}(x)}{|x|^{np}} dx \rightarrow 0, \quad R \rightarrow \infty,$$

where we used again the fact that for $v \in A_r$, $r > 1$,

$$\int_{|x| > R} \frac{v(x)}{|x|^{nr}} dx < \infty.$$

We now show the uniform equicontinuity estimate (c). Note that

$$\begin{aligned} & [T^\delta, b]_1(f, g)(x+t) - [T^\delta, b]_1(f, g)(x) \\ & = \iint_{\mathbb{R}^{2n}} (b(y) - b(x+t)) K^\delta(x+t, y, z) f(y) g(z) dy dz \\ & \quad - \iint_{\mathbb{R}^{2n}} (b(y) - b(x)) K^\delta(x, y, z) f(y) g(z) dy dz \\ & = (b(x) - b(x+t)) \iint_{\mathbb{R}^{2n}} K^\delta(x, y, z) f(y) g(z) dy dz \\ & \quad + \iint_{\mathbb{R}^{2n}} (b(y) - b(x+t)) (K^\delta(x+t, y, z) - K^\delta(x, y, z)) f(y) g(z) dy dz \\ & = I_1(t, x) + I_2(t, x). \end{aligned}$$

To deal with I_1 we observe first that

$$|I_1(t, x)| \lesssim |t| \|\nabla b\|_{L^\infty} T_*(f, g)(x),$$

where

$$T_*(f, g)(x) = \sup_{\delta > 0} \left| \iint_{\max(|x-y|, |x-z|) \geq \delta} K(x, y, z) f(y) g(z) dy dz \right|.$$

By the pointwise estimate [52, (2.1)], for all $\eta > 0$

$$(5.15) \quad T_*(f, g)(x) \lesssim_\eta (M(|T(f, g)|^\eta)(x))^{1/\eta} + Mf(x)Mg(x),$$

where M is the Hardy-Littlewood maximal function. From (5.15) with $\eta = 1$ in our current situation it easily follows that for $\vec{w} \in A_p \times A_p$

$$\|I_1(t, x)\|_{L^p(\nu_{\vec{w}})} \lesssim |t| \|\nabla b\|_{BMO}.$$

To estimate I_2 , we split the region of integration into three subregions following the linear case in [27]. We will assume $|t| < \delta/2$ and let

$$E(t, x) = \{(y, z) : \max\{|x - y|, |x - z|\} > \delta, \max\{|x + t - y|, |x + t - z|\} > \delta\}$$

$$F(t, x) = \{(y, z) : \max\{|x - y|, |x - z|\} > \delta, \max\{|x + t - y|, |x + t - z|\} \leq \delta\}$$

$$G(t, x) = \{(y, z) : \max\{|x - y|, |x - z|\} \leq \delta, \max\{|x + t - y|, |x + t - z|\} > \delta\}.$$

Our goal is to prove that in these three regions $\|I_2\|_{L^p(\nu_{\vec{w}})}$ goes to zero uniformly as t goes to zero. For the integral over $E(t, x)$ we simply use the kernel bounds on K since

$$K^\delta(x + t, y, z) - K^\delta(x, y, z) = K(x + t, y, z) - K(x, y, z)$$

for $(y, z) \in E(t, x)$ and $|t| < \delta/2$. We get

$$\begin{aligned} & \left| \iint_{E(t, x)} (b(y) - b(x + t))(K^\delta(x + t, y, z) - K^\delta(x, y, z))f(y)g(z) dydz \right| \\ & \lesssim \|b\|_{L^\infty} |t| \iint_{\max\{|x-y|, |x-z|\} > \delta} \frac{|f(y)||g(z)|}{(|x-y| + |x-z|)^{2n+1}} dydz \\ & \lesssim \|b\|_{L^\infty} |t| \sum_{j=0}^{\infty} \iint_{2^{j-1}\delta < \max(|x-y|, |x-z|) \leq 2^j\delta} \frac{|f(y)||g(z)|}{(|x-y| + |x-z|)^{2n+1}} dydz \\ & \lesssim \frac{\|b\|_{L^\infty} |t|}{\delta} \mathcal{M}(f, g)(x). \end{aligned}$$

The integrals of $F(t, x)$ and $G(t, x)$ are more complicated. Notice that

$$\begin{aligned} & \iint_{F(t, x)} (b(y) - b(x + t))(K^\delta(x + t, y, z) - K^\delta(x, y, z))f(y)g(z) dydz \\ & = \iint_{F(t, x)} (b(x + t) - b(y))K(x, y, z)f(y)g(z) dydz. \end{aligned}$$

First let us show that for $|x|$ large, the above integral is zero. Let $R_0 \geq 1$ be large enough so that $\text{supp } b \subset B_{R_0}(0) = B_0$. We claim that if $|x| > 3R_0$ then

$$\iint_{F(t, x)} (b(x + t) - b(y))K(x, y, z)f(y)g(z) dydz = 0.$$

Indeed, if $|x| > 3R_0$ then

$$|x + t| \geq |x| - |t| > 3R_0 - R_0 = 2R_0 \Rightarrow b(x + t) = 0.$$

Hence

$$\begin{aligned} \iint_{F(t,x)} (b(x+t) - b(y))K(x,y,z)f(y)g(z) dydz \\ = - \iint_{F(t,x) \cap (\text{supp } b \times \mathbb{R}^n)} b(y)K(x,y,z)f(y)g(z) dydz. \end{aligned}$$

Furthermore, if $|x| \geq 3R_0$, $|t| < \frac{\delta}{2} \leq \delta \leq 1 \leq R_0$ and $(y, z) \in F(t, x) \cap (\text{supp } b \times \mathbb{R}^n) \subset F(t, x) \cap (B_0 \times \mathbb{R}^n)$. Then, on one hand $|y| < R_0$ but on the other hand

$$|y| = |x + t - (x + t - y)| \geq |x + t| - |x + t - y| \geq |x| - |t| - \delta \geq R_0.$$

Hence

$$F(t, x) \cap (\text{supp } b \times \mathbb{R}^n) \subset F(t, x) \cap (B_0 \times \mathbb{R}^n) = \emptyset,$$

and we may assume $x \in 3B_0$. Then we have

$$\begin{aligned} & \left| \iint_{F(t,x)} (b(x+t) - b(y))K(x,y,z)f(y)g(z) dydz \right| \\ & \leq 2\|b\|_{L^\infty} \iint_{F(t,x)} \frac{|f(y)||g(z)|}{(|x-y| + |x-z|)^{2n}} dydz \\ & \leq \frac{2\|b\|_{L^\infty}}{\delta^{2n}} \iint_{F(t,x)} |f(y)||g(z)| dydz. \end{aligned}$$

We are now going to use that the set $F(t, x)$ satisfies $|F(t, x)| \rightarrow 0$ uniformly in x as $t \rightarrow 0$. However, we need to split the integrals and $F(t, x)$ is not a product set. But $F(t, x)$ is contained in the union of two product sets:

$$F(t, x) \subset F^*(t, x) \cup F_*(t, x),$$

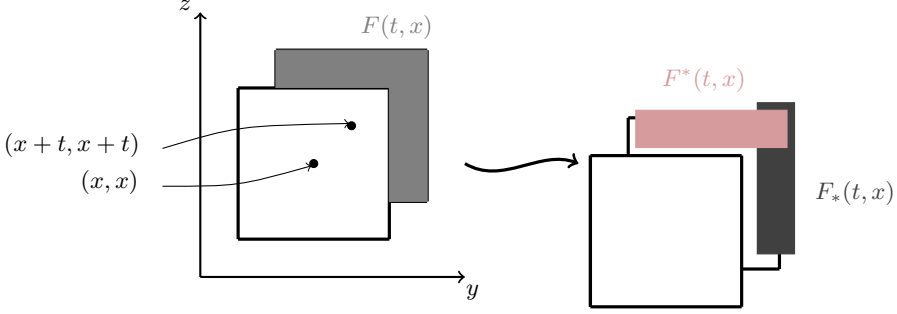
where

$$\begin{aligned} F_*(t, x) &= \{(y, z) : |x - y| > \delta, |x + t - y| \leq \delta, |x + t - z| \leq \delta\} \\ &= \{y : |x - y| > \delta, |x + t - y| \leq \delta\} \times B_\delta(x + t) \end{aligned}$$

and

$$\begin{aligned} F^*(t, x) &= \{(y, z) : |x - z| > \delta, |x + t - z| \leq \delta, |x + t - y| \leq \delta\} \\ &= B_\delta(x + t) \times \{z : |x - z| > \delta, |x + t - z| \leq \delta\}, \end{aligned}$$

as shown in the following figure:



Next define the set

$$(5.16) \quad S_t(x) = \{u \in \mathbb{R}^n : |x - u| > \delta, |x + t - u| \leq \delta\}.$$

Furthermore notice that since $x \in 3B_0$ we have $B_\delta(x + t) \subset 5B_0$, hence

$$F(t, x) \subset F^*(t, x) \cup F_*(t, x) \subset (S_t(x) \times 5B_0) \cup (5B_0 \times S_t(x)).$$

Therefore we obtain

$$\begin{aligned} & \iint_{F(t, x)} |f(y)| |g(z)| \, dydz \\ & \leq \iint_{S_t(x) \times 5B_0} |f(y)| |g(z)| \, dydz + \iint_{5B_0 \times S_t(x)} |f(y)| |g(z)| \, dydz. \end{aligned}$$

For the first term we have

$$\begin{aligned} \iint_{S_t(x) \times 5B_0} |f(y)| |g(z)| \, dydz & \leq \|f\|_{L^{p_1}(w_1)} \sigma_1(S_t(x))^{1/p_1'} \|g\|_{L^{p_2}(w_2)} \sigma_2(5B_0)^{1/p_2'} \\ & \leq \sigma_1(S_t(x))^{1/p_1'} \sigma_2(5B_0)^{1/p_2'}, \end{aligned}$$

and similarly for the second term

$$\iint_{5B_0 \times S_t(x)} |f(y)| |g(z)| \, dydz \leq \sigma_1(5B_0)^{1/p_1'} \sigma_2(S_t(x))^{1/p_2'}.$$

Next observe that the weight σ_1 belongs to $A_{2p_1'} \subset A_\infty$ and thus there exists $\varepsilon_1 > 0$ such that for any ball, B , and $S \subset B$

$$\frac{\sigma_1(S)}{\sigma_1(B)} \lesssim \left(\frac{|S|}{|B|} \right)^{\varepsilon_1}.$$

Since $x \in 3B_0$, $|t| < R_0$ we have $S_t(x) \subset 5B_0$. Indeed,

$$|u| = |x + t - (x + t - u)| \leq |x| + |t| + |x + t - u| < 3R_0 + R_0 + R_0 = 5R_0.$$

Thus,

$$\sigma_1(S_t(x)) \leq \left(\frac{|S_t(x)|}{|5B_0|} \right)^{\varepsilon_1} \sigma_1(5B_0),$$

and similarly, since $\sigma_2 \in A_{2p'_2}$

$$\sigma_2(S_t(x)) \leq \left(\frac{|S_t(x)|}{|5B_0|} \right)^{\varepsilon_2} \sigma_2(5B_0),$$

for some $\varepsilon_2 > 0$. Taking into account the definition of the sets $S_t(x)$ in (5.16) and the fact that Lebesgue measure is translation invariant, for any x we have

$$|S_t(x)| = |S_t(0)| = |\{u \in \mathbb{R}^n : |u| > \delta, |u - t| \leq \delta\}|.$$

Therefore, as $t \rightarrow 0$ we have that $|S_t(0)| \rightarrow 0$. Combining everything we obtain the following estimate

$$\begin{aligned} & \left(\int_{\mathbb{R}^n} \left| \iint_{F(t,x)} (b(x+t) - b(y))K(x,y,z)f(y)g(z) dydz \right|^p \nu_{\bar{w}} \right)^{1/p} \\ &= \left(\int_{3B_0} \left| \iint_{F(t,x)} (b(x+t) - b(y))K(x,y,z)f(y)g(z) dydz \right|^p \nu_{\bar{w}} \right)^{1/p} \\ &\leq \frac{2\|b\|_{L^\infty}}{\delta^{2n}} \nu_{\bar{w}}(3B_0)^{\frac{1}{p}} \sigma_1(5B_0)^{\frac{1}{p_1}} \sigma_2(5B_0)^{\frac{1}{p_2}} \left[\left(\frac{|S_t(0)|}{|5B_0|} \right)^{\frac{\varepsilon_1}{p_1}} + \left(\frac{|S_t(0)|}{|5B_0|} \right)^{\frac{\varepsilon_2}{p_2}} \right] \end{aligned}$$

We see that the last term goes to zero as $t \rightarrow 0$. The last term involving the integration region $G(t,x)$ is similar so we simply sketch the details. Again we may assume that $|x| < 3R_0$ where $R_0 \geq 1$ is the radius of a ball that contains the support of b . Then,

$$\begin{aligned} & \left| \iint_{G(t,x)} (b(y) - b(x+t))(K^\delta(x+t,y,z) - K^\delta(x,y,z))f(y)g(z) dydz \right| \\ &= \left| \iint_{G(t,x)} (b(y) - b(x+t))K(x+t,y,z)f(y)g(z) dydz \right| \\ &\leq \frac{2\|b\|_{L^\infty}}{\delta^{2n}} \iint_{G(t,x)} |f(y)||g(z)| dydz. \end{aligned}$$

From this point on, we handle the estimates the same way as we did for the $F(t,x)$ integral: while $G(t,x)$ is not a product set, it is contained in the union of two product sets whose Lebesgue measures go to zero uniformly in x as $t \rightarrow 0$. \square

Next, we concentrate now on the compactness of the iterated commutator. We will show that $[T^\delta, \vec{b}]$ satisfies the corresponding conditions (a), (b) and (c) of Theorem 5.4.3. The proof is similar to that of Theorem 5.4.1, but it is worth pointing out that

for the iterated commutator, these conditions hold under the weakest assumption on the class of weights, that is, $\vec{w} \in A_{\vec{p}}$. We indicate the needed modifications in the proof below.

Proof of Theorem 5.4.2. As before, we may assume $\vec{b} \in C_c^\infty \times C_c^\infty$, fix $\delta > 0$ and study $[T^\delta, \vec{b}]$. Suppose again f, g belong to

$$B_{1, L^{p_1}(w_1)} \times B_{1, L^{p_2}(w_2)} = \{(f, g) : \|f\|_{L^{p_1}(w_1)}, \|g\|_{L^{p_2}(w_2)} \leq 1\},$$

with w_1 and w_2 in A_p . Once again, condition (a) in Theorem 5.4.3 holds since $[T^\delta, \vec{b}]$ is bounded from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^p(\nu_{\vec{w}})$ when $\vec{w} \in A_{\vec{p}}$. Next, we show that condition (b) holds. Let A be large enough so that $\text{supp } b_1 \cup \text{supp } b_2 \subset B_A(0)$ and let $R \geq \max(2A, 1)$. Then, for $|x| \geq R$, we have

$$\begin{aligned} |[T^\delta, \vec{b}](f, g)(x)| &\lesssim \|b_1\|_{L^\infty} \|b_2\|_{L^\infty} \int_{\text{supp } b_1} \int_{\text{supp } b_2} \frac{|f(y)||g(z)|}{(|x-y| + |x-z|)^{2n}} dy dz \\ &\lesssim \frac{1}{|x|^{2n}} \|b_1\|_{L^\infty} \|b_2\|_{L^\infty} \int_{\text{supp } b_1} |f(y)| dy \int_{\text{supp } b_2} |g(z)| dz \\ &\lesssim \frac{1}{|x|^{2n}} \|b_1\|_{L^\infty} \|b_2\|_{L^\infty} \|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)} \sigma_1(\text{supp } b_1)^{1/p_1'} \sigma_2(\text{supp } b_2)^{1/p_2'}. \end{aligned}$$

We can raise the previous pointwise estimate to the power p and integrate over $|x| > R$ to get

$$\begin{aligned} &\int_{|x| > R} |[T^\delta, \vec{b}](f, g)(x)|^p \nu_{\vec{w}}(x) dx \\ &\leq \left(\|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)} \sigma_1(\text{supp } b_1)^{1/p_1'} \sigma_2(\text{supp } b_2)^{1/p_2'} \right)^p \int_{|x| > R} \frac{\nu_{\vec{w}}(x)}{|x|^{2np}} dx, \end{aligned}$$

which tends to zero as $R \rightarrow \infty$ since $\nu_{\vec{w}} \in A_{2p}$ (see (3.4)), and gives (b). To show that condition (c) also holds, we write

$$\begin{aligned} &|[T^\delta, \vec{b}](f, g)(x) - [T^\delta, \vec{b}](f, g)(x+t)| = \\ &\left| \iint_{\mathbb{R}^{2n}} (b_1(y) - b_1(x))(b_2(z) - b_2(x)) K^\delta(x, y, z) f(y) g(z) dy dz + \right. \\ &\left. \iint_{\mathbb{R}^{2n}} (b_1(y) - b_1(x+t))(b_2(z) - b_2(x+t)) K^\delta(x+t, y, z) f(y) g(z) dy dz \right| \\ &\leq |I_1(x, t)| + |I_2(x, t)|, \end{aligned}$$

where

$$I_1(x, t) = (b_1(x+t) - b_1(x)) \iint_{\mathbb{R}^{2n}} (b_2(z) - b_2(x)) K^\delta(x, y, z) f(y) g(z) dy dz$$

and

$$I_2(x, t) = \iint_{\mathbb{R}^{2n}} (K^\delta(x, y, z)(b_2(z) - b_2(x)) - K^\delta(x + t, y, z)(b_2(z) - b_2(x + t))) \\ \times (b_1(y) - b_1(x + t))f(y)g(z)dydz.$$

The pointwise estimate of $I_1(x, t)$ can be obtained as in the proof of Theorem 5.4.1:

$$|I_1(x, t)| \lesssim |t| \|\nabla b_1\|_{L^\infty} (T^*(f, b_2 g)(x) + \|b_2\|_{L^\infty} T^*(f, g)(x))$$

Thus, as $|t| \rightarrow 0$,

$$\|I_1\|_{L^p(\nu_{\bar{w}})} \lesssim \frac{|t|}{\delta} \|\nabla b_1\|_{L^\infty} \|b_2\|_{L^\infty} \|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)} \rightarrow 0.$$

Now, we split I_2 in two other integrals, that is,

$$I_2(x, t) = \iint_{\mathbb{R}^{2n}} (K^\delta(x, y, z) - K^\delta(x + t, y, z))(b_2(z) - b_2(x + t)) \times \\ \times (b_1(y) - b_1(x + t))f(y)g(z)dydz \\ + (b_2(x + t) - b_2(x)) \iint_{\mathbb{R}^{2n}} (b_1(y) - b_1(x + t))K^\delta(x, y, z)f(y)g(z)dydz \\ := I_{21}(x, t) + I_{22}(x, t).$$

We will now estimate each term in this decomposition. The estimate for the I_{21} term is the most delicate and we will postpone it until the end of the proof. We estimate I_{22} as follows:

$$|I_{22}(x, t)| \leq \\ |t| \|\nabla b_2\|_{L^\infty} \left| \iint_{\max(|x-y|, |x-z|) \geq \delta} (b_1(y) - b_1(x + t))K(x, y, z)f(y)g(z)dydz \right| \\ \leq |t| \|\nabla b_2\|_{L^\infty} (T^*(b_1 f, g)(x) + \|b_1\|_{L^\infty} T^*(f, g)(x)).$$

Therefore, as $|t| \rightarrow 0$,

$$\|I_{22}\|_{L^p(\nu_{\bar{w}})} \lesssim |t| \|\nabla b_2\|_{L^\infty} \|b_1\|_{L^\infty} \|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)} \rightarrow 0.$$

For the term I_{21} , we further split the integral into the three regions $E(x, t)$, $F(x, t)$ and $G(x, t)$ that we defined in the proof of Theorem 5.4.1. We will denote them as E , F and G , respectively, to simplify the notation. We note immediately that $I_{21}(x, t) = 0$ for $(y, z) \in \mathbb{R}^{2n} \setminus (E \cup F \cup G)$. Now, for the integral over E , we proceed in the same

way as we did for the corresponding part in the proof of Theorem 5.4.1 using the regularity on the kernel to get

$$\begin{aligned} |I_{21,E}(x,t)| &\lesssim |t| \|b_1\|_{L^\infty} \|b_2\|_{L^\infty} \iint_E \frac{|f(y)||g(z)|}{(|x-y|+|x-z|)^{2n+1}} dydz \\ &\lesssim \frac{|t|}{\delta} \|b_1\|_{L^\infty} \|b_2\|_{L^\infty} \mathcal{M}(f,g)(x). \end{aligned}$$

Therefore we have that, as $|t| \rightarrow 0$,

$$\|I_{21,E}\|_{L^p(\nu_{\vec{w}})} \lesssim \frac{|t|}{\delta} \|b_1\|_{L^\infty} \|b_2\|_{L^\infty} \|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)} \longrightarrow 0.$$

The integrals over the sets F and G are symmetric, so we only sketch the estimate for F . It is easy to see that these integrals can be reduced to the ones that we have already estimated in Theorem 5.4.1 for the same regions. Indeed,

$$\begin{aligned} |I_{21,F}(x,t)| &\lesssim \|b_1\|_{L^\infty} \|b_2\|_{L^\infty} \iint_F \frac{1}{(|x-y|+|x-z|)^{2n}} |f(y)||g(z)| dydz \\ &\lesssim \frac{\|b_1\|_{L^\infty} \|b_2\|_{L^\infty}}{\delta^{2n}} \iint_F |f(y)||g(z)| dydz. \end{aligned}$$

From this point on we can proceed as we did in the estimate of the commutator $[T, b]_1$ and we get that

$$\|I_{21,F}\|_{L^p(\nu_{\vec{w}})} \longrightarrow 0 \text{ as } |t| \rightarrow 0,$$

proving that $[T^\delta, \vec{b}]$ satisfies conditions (a), (b) and (c) when $\vec{w} \in A_{\vec{p}}$. In particular, $[\vec{b}, T^\delta] : L^{p_1}(w_1) \times L^{p_2}(w_2) \rightarrow L^p(\nu_{\vec{w}})$ is jointly compact when $\vec{w} \in A_p \times A_p$. \square

Finally, we make some further remarks regarding the results proved in this section.

Remark 5.4.1. *Let us observe that it is also possible to prove Theorems 5.4.1 and 5.4.2 by introducing the following smooth truncations. This approach could also be used to simplify some of the computations in the linear and multilinear setting. For the complete proof using this approach we refer the reader to [9].*

Let $\varphi = \varphi(x, y, z)$ be a non-negative function in $C_c^\infty(\mathbb{R}^{3n})$,

$$\text{supp } \varphi \subset \{(x, y, z) : \max(|x|, |y|, |z|) < 1\}$$

and such that

$$\int_{\mathbb{R}^{3n}} \varphi(u) du = 1.$$

For $\delta > 0$ let $\chi^\delta = \chi^\delta(x, y, z)$ be the characteristic function of the set

$$\{(x, y, z) : \max(|x - y|, |x - z|) \geq \frac{3\delta}{2}\},$$

and let

$$\psi^\delta = \varphi_\delta * \chi^\delta,$$

where

$$\varphi_\delta(x, y, z) = (\delta/4)^{-3n} \varphi(4x/\delta, 4y/\delta, 4z/\delta).$$

Clearly we have that $\psi^\delta \in C^\infty$,

$$\text{supp } \psi^\delta \subset \{(x, y, z) : \max(|x - y|, |x - z|) \geq \delta\},$$

$\psi^\delta(x, y, z) = 1$ if $\max(|x - y|, |x - z|) > 2\delta$, and $\|\psi^\delta\|_{L^\infty} \leq 1$. Moreover, $\nabla\psi^\delta$ is not zero only if $\max(|x - y|, |x - z|) \approx \delta$ and $\|\nabla\psi^\delta\|_{L^\infty} \lesssim 1/\delta$. Given a kernel K associated to a Calderón–Zygmund operator T , we define the truncated kernel

$$K^\delta(x, y, z) = \psi^\delta(x, y, z)K(x, y, z).$$

It follows that K^δ satisfies the same size and regularity estimates of K , (4.2) and (4.3), with a constant C independent of δ . As before, we let $T^\delta(f, g)$ be the operator defined pointwise by K^δ as in (4.1), now for all $x \in \mathbb{R}^n$.

Remark 5.4.2. Our results on bilinear commutators highlight one more time the fact that the higher the order of the commutator with CMO symbols, the less singular the operators are. In this chapter this is reflected in the less restrictive class of weights needed to achieve the estimates (a), (b) and (c) in Fréchet–Riesz–Kolmogorov theorem. Indeed, in Theorem 5.4.1, the assumption $A_p \times A_p$ on the weight is needed both to check condition (b) and to guarantee that the target weight falls in the right class. However, to obtain bilinear compactness in Theorem 5.4.2 we require the $A_p \times A_p$ assumption about the vector weight only because the sufficient condition from [27] on $L^p(\nu_{\vec{w}})$ precompactness requires $\nu_{\vec{w}} \in A_p$. As already mentioned, this last condition fails if \vec{w} is only assumed to belong to $A_{\vec{p}}$. Actually, our techniques can be used to obtain a more general theorem by assuming that $\vec{w} \in A_{\vec{p}}$ and $\nu_{\vec{w}} \in A_p$ instead of $\vec{w} \in A_p \times A_p$.

Theorem 5.4.4. Suppose $\vec{P} \in (1, \infty) \times (1, \infty)$, $p = \frac{p_1 p_2}{p_1 + p_2} > 1$, $b \in \text{CMO}$, and $\vec{w} \in A_{\vec{p}}$ with $\nu_{\vec{w}} \in A_p$. Then $[T, b]_1$ and $[T, b]_2$ are compact operators from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^p(\nu_{\vec{w}})$.

Theorem 5.4.5. *Suppose $\vec{P} \in (1, \infty) \times (1, \infty)$, $p = \frac{p_1 p_2}{p_1 + p_2} > 1$, $\vec{b} \in CMO \times CMO$, and $\vec{w} \in A_{\vec{P}}$ with $\nu_{\vec{w}} \in A_p$. Then $[T, \vec{b}]$ is a compact operator from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^p(\nu_{\vec{w}})$.*

As mentioned at the beginning of this section,

$$\vec{w} \in A_p \times A_p \Rightarrow \vec{w} \in A_{\vec{P}} \text{ and } \nu_{\vec{w}} \in A_p.$$

To see that the assumption $\vec{w} \in A_{\vec{P}}$ and $\nu_{\vec{w}} \in A_p$ is indeed weaker, consider the example $w_1(x) = |x|^{-\alpha}$ where $1 < \alpha < \frac{p_1}{p} = 1 + \frac{p_1}{p_2}$ and $w_2(x) = 1$ on \mathbb{R} . Then $\sigma_1(x) = |x|^{\alpha(p'_1 - 1)}$ belongs to $A_{2p'_1}$ since

$$\alpha < 1 + \frac{p_1}{p_2} < 1 + p_1 = \frac{2p'_1 - 1}{p'_1 - 1}.$$

Moreover, $\nu_{\vec{w}}(x) = |x|^{-\alpha \frac{p}{p_1}}$ belongs to $A_1(\subset A_p)$ since $\alpha \frac{p}{p_1} < 1$. However, the weight w_1 does not belong to any A_p class since it is not locally integrable. This vector weight also provides a new example of the properness of the containment $A_{p_1} \times A_{p_2} \subsetneq A_{\vec{P}}$ from [80, Sect. 7].

Remark 5.4.3. *It is natural to ask whether the sufficient condition about $L^p(w)$ precompactness in [27] may be extended to include weights $w \in A_q$ with $q > p$. A simple modification of the argument in [117, p. 275] gives the following result in this setting:*

Let $1 < r < \infty$, $w \in A_\infty$, and $\mathcal{K} \subset L^r(w)$. If

(a) \mathcal{K} is bounded in $L^r(w)$;

(b) $\lim_{A \rightarrow \infty} \int_{|x| > A} |f(x)|^r w dx = 0$ uniformly for $f \in \mathcal{K}$;

(c) $\|f(\cdot + t_1) - f(\cdot + t_2)\|_{L^r(w)} \rightarrow 0$ uniformly for $f \in \mathcal{K}$ as $|t_1 - t_2| \rightarrow 0$;

then \mathcal{K} is precompact.

This is different than the sufficient condition we employed in the proofs of our main theorems, specifically in the third assumption about equicontinuity. Note that, in general, the non-translation invariance of the measure deems our last condition strictly stronger than the corresponding one in [27]. Unfortunately, the arguments we used to prove Theorem 5.4.2 do not seem to hold anymore in this setting.

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List of symbols and notation

$A \lesssim B$	$A \leq CB$ for some numerical constant $C > 0$
$A \simeq B$	$A \lesssim B$ and $B \lesssim A$
C_A	numerical constant that depends on A
\mathbb{R}	the set of real numbers
\mathbb{C}	the set of complex numbers
\mathbb{R}^n	the n -dimensional Euclidean space
X'	the associate space of a Banach function space X
$B_{R,X}(x_0)$	the ball of X with center x_0 and radius R
$Q(x, r)$	the cube with center x and side length r
\hat{Q}	parent of a cube Q
\check{Q}	child of a cube Q
ℓ_Q	the side length of a cube Q
dx	Lebesgue measure
μ	non-negative measure
w	weight
\vec{w}	(w_1, \dots, w_m)
\vec{P}	(p_1, \dots, p_m)
$\nu_{\vec{w}}$	$\prod_{i=1}^m w_i^{\frac{p_i}{p}}$
$ E $	the Lebesgue measure of the set $E \subset \mathbb{R}^n$
$w(E)$	the w -measure of the set $E \subset \mathbb{R}^n$
$f_Q f = f_Q$	the average of a function f over a set Q
\vec{f}	(f_1, \dots, f_m)
$ \vec{f} $	(f_1 , \dots, f_m)

f^*	the non-increasing rearrangement of a function f
$\omega_\lambda(f; Q)$	the local mean oscillation of a function f over a cube Q
$m_f(Q)$	the median value of a function f over a cube Q
$m_{\lambda, Q}^{\sharp, d}$	the local sharp maximal function
$C^\infty(\mathbb{R}^n)$	the space of smooth functions from \mathbb{R}^n to \mathbb{C}
$C_c^\infty(\mathbb{R}^n)$	the space of functions in $C^\infty(\mathbb{R}^n)$ with compact support
$\partial_j^m f$	the m -th partial derivative of $f(x_1, \dots, x_n)$ with respect to x_j
$\partial^\beta f$	$\partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} f$
$S(\mathbb{R}^n)$	$\{f \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} x^\alpha \partial^\beta f(x) < \infty \quad \forall \alpha, \beta\}$
$L^p(X, \mu)$	the Lebesgue space over the measure space (X, μ)
L^p	the Lebesgue space over the measure space (\mathbb{R}^n, dx)
$L^{p, \infty}(X, \mu)$	the weak Lebesgue space over the measure space (X, μ)
$L^{p, \infty}$	the weak Lebesgue space over the measure space (\mathbb{R}^n, dx)
BMO	the space of bounded mean oscillation functions
CMO	the closure of $C_c^\infty(\mathbb{R}^n)$ in BMO
χ_E	the characteristic function of the set $E \in \mathbb{R}^n$
\mathcal{D}	the standard dyadic grid in \mathbb{R}^n
\mathscr{D}	a general dyadic grid in an space of homogeneous type X
$S = \{Q_j^k\}$	a sparse family of cubes
E_j^k	a pairwise disjoint family of sets associated to S (Sect. 1.2)
$\mathcal{A}_{S, \mathscr{D}}$	dyadic sparse operator on a sparse family $S \subset \mathscr{D}$
M	the Hardy-Littlewood maximal operator
M^\sharp	the Fefferman–Stein sharp maximal function
M_δ^\sharp	$(M^\sharp(f ^\delta))^{1/\delta}$
M_α	the fractional maximal operator

\mathcal{M}	the multilinear maximal operator
\mathcal{M}^d	the dyadic multilinear maximal operator w.r.t. \mathcal{D}
I_α	the fractional integral operator
T_α	the fractional bilinear integral operator
BI_α	the bilinear fractional singular integral operator
H	the Hilbert transform
BHT	the bilinear Hilbert transform
$[T, b](\cdot)$	$T(b\cdot) - bT(\cdot)$
$[T, b]_k$	iterated commutator of a linear Calderón–Zygmund operator
$[T, \vec{b}]_{\vec{\alpha}}$	iterated commutator of a multilinear Calderón–Zygmund operator