ATTRACTORS FOR A RANDOM EVOLUTION EQUATION WITH INFINITE MEMORY: THEORETICAL RESULTS

TOMÁS CARABALLO, MARÍA J. GARRIDO-ATIENZA, BJÖRN SCHMALFUSS,
AND JOSÉ VALERO

ABSTRACT. The long-time behavior of solutions (more precisely, the existence of random pullback attractors) for an integro-differential parabolic equation of diffusion type with memory terms, more particularly with terms containing both finite and infinite delays, as well as some kind of randomness, is analyzed in this paper. We imposed general assumptions not ensuring uniqueness of solutions, which implies that the theory of multivalued dynamical system has to be used. Furthermore, the emphasis is put on the existence of random pullback attractors by exploiting the techniques of the theory of multivalued nonautonomous/random dynamical systems.

1. INTRODUCTION

In our paper [5] we analyzed the long-time behavior of solutions (more precisely, the existence of pullback attractors) for an integro-differential parabolic equation of diffusion type with memory terms, expressed by convolution integrals involving infinite delays and by a forcing term with bounded delay, which represent the past history of one or more variables. In particular, we focused on the following non-autonomous reaction-diffusion equation with memory

$$(1) \quad \frac{\partial u}{\partial t} - \Delta u + \int_{-\infty}^{t} \gamma\left(t - s\right) \Delta u\left(x, s\right) ds + g\left(x, t, u\left(x, t\right)\right) = h\left(x, t, u\left(x, t - r\right)\right),$$

with Dirichlet boundary condition, where x belongs to a bounded domain $\mathcal{O} \subset \mathbb{R}^N$ with smooth boundary, $t \in \mathbb{R}$, the functions h and g satisfy suitable assumptions, and γ is given in a standard way as $\gamma(t) = -\gamma_0 e^{-d_0 t}$ with $d_0, \gamma_0, r > 0$. Instead of analyzing directly equation (1), we developed in [5] an abstract theory for a more general type of non-autonomous evolution equations which included as a particular case the previous one. These types of equations are important in many physical phenomena as it is assumed they are better described if one considers in the equations of the model some terms which take into account the past history of the system (see, e.g. [15] and the references therein). Although, in some situations, the contribution of the past history may not be so relevant to significantly affect the long time dynamics of the problem, in certain models, such as those describing high viscosity liquids at low temperatures, or the thermomechanical behavior of polymers (see [11], [15] and the references therein) the past history may play a nontrivial role.

However, it is also reasonable to think that certain models from the real world are more realistic if some random or stochastic terms are also considered in the formulation. In this respect, in the paper [2] the asymptotic behavior of a stochastic version of equation (1) (with an additive noise) and with conditions ensuring uniqueness of the Cauchy problem was analyzed. Namely, the problem

$$\frac{\partial u}{\partial t} - \Delta u + \int_{-\infty}^{t} \gamma(t-s)\,\Delta u(x,s)\,ds + g(x,u(x,t))$$
$$= h(x,u(x,t)) + \partial_t W(t,x),$$

(2)

is considered, where $W(t,\cdot)$ is an appropriate Hilbert valued Wiener process (see [2]) and the functions h and g are independent of time and delay. The technique used in [2] is based in the well-known Dafermos transformation (see [9]) and the existence of a random attractor is also proved.

In this paper we aim to consider a more general case in which the terms in the equations do not necessarily ensure uniqueness of solutions (and henceforth we will have a multivalued dynamical system), they may contain some kind of delay and randomness can be present as well. In fact, we are interested in studying a random problem which can come from a genuine stochastic model after having performed a conjugation procedure. In the cases of additive or multiplicative linear noise, the so-called Ornstein-Uhlenbeck (O-U) process plays a key role in this conjugation (see, e.g. [3] for more details). However, we will not focus on performing such transformation explicitly in this paper. Instead, we will consider a rather general abstract evolution equation containing delay (both bounded and unbounded) and some random terms, which will have the special structure provided when one uses the aforementioned change of variable by means of the (O-U) process. Furthermore, our objective is to tackle our problem by exploiting the techniques from the theory o multivalued non-autonomous/random dynamical systems without using the Dafermos transformation. We intend to apply the results of this work to a stochastic perturbation of equation (1) in a forthcoming paper.

It is worth mentioning that the problem in which we are interested in (see equation (5)) has a non-autonomous character, which means that for each fixed random parameter we can study a non-autonomous dynamical system and we can analyze the existence of a pullback attractor. However, when we are interested in the random problem, the significant concept to worry about is the concept of random pullback attractor. But this requires of more technicalities and the analysis of measurability properties which make a big difference between the current paper and [5].

The content of the paper is the following. In Section 2 we recall some basic concepts and results from the theory of pullback and random attractors in the set-valued or multivalued set-up. The description of the problem to be analyzed in the paper is included in Section 3, in particular, we consider a random evolution equation with delays with very general coefficients. The existence of solutions to our random problems as well as the existence of absorbing sets for the multivalued non-autonomous/random dynamical system generated by the problem is proved in Section 4, while the compactness of the dynamical system is proved in Section 5, ensuring the existence of a non-autonomous pullback attractor. But, to prove that this pullback attractor and the multivalued non-autonomous dynamical system are

random ones, it is necessary to prove extra measurability properties of both the dynamical system and the attractor. This is carried out in Section 6.

2. General theory of pullback and random attractors

We will recall the general theory of pullback attractors for set–valued non–autonomous and random dynamical systems.

A pair (Ω, θ) where $\theta = (\theta_t)_{t \in \mathbb{R}}$ is a flow on Ω, that is,

$$\theta : \mathbb{R} \times \Omega \to \Omega,$$

$$\theta_0 = \mathrm{id}_\Omega, \qquad \theta_{t+\tau} = \theta_t \circ \theta_\tau =: \theta_t \theta_\tau \quad \text{for } t, \tau \in \mathbb{R},$$

is called a non–autonomous perturbation.

Let $\mathcal{P} := (\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. On $\mathcal{P}$ we consider a measurable non–autonomous flow θ :

$$\theta : (\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}) \to (\Omega, \mathcal{F}).$$

In addition, $\mathbb{P}$ is supposed to be ergodic with respect to θ, which means that every θ_t-invariant set has measure zero or one for $t \in \mathbb{R}$. Hence $\mathbb{P}$ is invariant with respect to θ_t. The quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, which is the model for a noise, is called a metric dynamical system.

We also assume that Ω is a Polish space with the metric d_Ω and that $\mathcal{F}$ is the Borel σ-algebra of Ω.

If we replace in the definition of a metric dynamical system the probability space $\mathcal{P}$ by its completion $\mathcal{P}^c := (\Omega, \bar{\mathcal{F}}^\mathbb{P}, \bar{\mathbb{P}})$ the above measurability property is not true in general (see Arnold [1, Appendix A]). But for fixed $t \in \mathbb{R}$ we have that the mapping $\theta_t : (\Omega, \bar{\mathcal{F}}^\mathbb{P}) \to (\Omega, \bar{\mathcal{F}}^\mathbb{P})$ is measurable.

Let $X = (X, d_X)$ be a Polish space and let us denote by $P(X)$ $(P_f(X))$ the set of all non-empty (non-empty closed) subsets of the space X. Let $D : \omega \to D(\omega) \in P(X)$ be a multivalued mapping describing a family of sets parameterized by ω. Such family is said to be closed if $D(\omega) \in P_f(X)$ for any $\omega \in \Omega$.

Let $D : \omega \to D(\omega) \in P_f(X)$ be a multi–valued mapping in X over $\mathcal{P}$. This mapping is called a random set if

$$\omega \mapsto \inf_{y \in D(\omega)} d_X(x, y)$$

is a random variable for every $x \in X$. It is well known that a mapping is a random set if and only if for every open set O in X the inverse image $\{\omega : D(\omega) \cap O \neq \emptyset\}$ is measurable, i.e., it belongs to $\mathcal{F}$, see [14, Proposition 2.1.4].

Clearly, all this is also valid if we replace $\mathcal{P}$ by $\mathcal{P}^c$. It is also evident that if D is a random set with respect to $\mathcal{P}$, then it is also random with respect to $\mathcal{P}^c$.

We now introduce non–autonomous and random dynamical systems.

Definition 1. *A multi–valued map $G : \mathbb{R}^+ \times \Omega \times X \to P_f(X)$ is called a multi–valued non–autonomous dynamical system (MNDS) if:*

    i) *$G(0, \omega, \cdot) = \mathrm{id}_X$,*

    ii) *$G(t + \tau, \omega, x) \subset G(t, \theta_\tau \omega, G(\tau, \omega, x))$ (cocycle property)   for all $t, \tau \in \mathbb{R}^+, x \in X, \omega \in \Omega$.*

It is called a strict MNDS if, moreover, $G(t + \tau, \omega, x) = G(t, \theta_\tau \omega, G(\tau, \omega, x))$ for all $t, \tau \in \mathbb{R}^+, x \in X, \omega \in \Omega$.

An MNDS is called a multi–valued random dynamical system (MRDS) if the multi–valued mapping $(t, \omega, x) \to G(t, \omega, x)$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(X)$ measurable, i.e. $\{(t, \omega, x) : G(t, \omega, x) \cap O \neq \emptyset\} \in \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(X)$ for every open set O of X.

Our first aim is to formulate a general sufficient condition ensuring that an MNDS defines an MRDS. To do that, we introduce an adequate concept of continuity for multivalued mappings: we say that $G(t, \omega, \cdot)$ is upper–semicontinuous at x_0 if for every neighborhood $\mathcal{U}$ of the set $G(t, \omega, x_0)$ there exists $\delta > 0$ such that if $d_X(x_0, y) < \delta$ then $G(t, \omega, y) \subset \mathcal{U}$. In general $G(t, \omega, \cdot)$ is called upper–semicontinuous if it is upper–semicontinuous at every x_0 in X. It is not difficult to extend this definition and obtain upper–semicontinuity with respect to all variables.

Lemma 2. *If the mapping $(t, \omega, x) \mapsto G(t, \omega, x)$ is upper–semicontinuous, then it is measurable in the sense of Definition 1, see [4].*

In order to define the concept of attractor (both pullback and random) we need to recall some definitions.

A multivalued mapping $D : \omega \to D(\omega) \in P(X)$ is said to be negatively (resp. strictly) invariant for the MNDS G if $D(\theta_t \omega) \subset G(t, \omega, D(\omega))$ (resp. $=$), for $\omega \in \Omega$, $t \in \mathbb{R}^+$.

Let $\mathcal{D}$ be a family of multivalued mappings $D : \omega \to D(\omega) \in P_f(X)$. We say that a multivalued mapping K is pullback $\mathcal{D}$-attracting if for every $D \in \mathcal{D}$,

$$\lim_{t \to +\infty} \mathrm{dist}_X(G(t, \theta_{-t} \omega, D(\theta_{-t} \omega)), K(\omega)) = 0, \text{ for all } \omega \in \Omega,$$

where by $\mathrm{dist}_X(A, B)$ we denote the Hausdorff semi-distance of two non-empty sets A, B:

$$\mathrm{dist}_X(A, B) = \sup_{x \in A} \inf_{y \in B} d_X(x, y).$$

The family $B : \omega \to B(\omega)$ is said to be pullback $\mathcal{D}$-absorbing if for every $D \in \mathcal{D}$ and $\omega \in \Omega$ there exists $\bar{t} = \bar{t}(\omega, D) > 0$ such that

$$G(t, \theta_{-t} \omega, D(\theta_{-t} \omega)) \subset B(\omega), \text{ for all } t \geq \bar{t}.$$

Throughout this work we always consider a particular system of sets as in [16]. Namely, let $\mathcal{D}$ be a set of multivalued mappings $D : \omega \to D(\omega) \in P_f(X)$ satisfying the *inclusion closed property*: suppose that $D \in \mathcal{D}$ and let $D' \in P_f(X)$ be a multivalued mapping such that $D'(\omega) \subset D(\omega)$ for $\omega \in \Omega$, then $D' \in \mathcal{D}$. The reason to consider such a system of sets is that then we will have a unique attractor in $\mathcal{D}$. An example of a family satisfying this property is the so-called familty of tempered sets. Let us consider the system $\mathcal{D}$ given by the multi–valued mappings $D : \omega \to D(\omega) \in P_f(X)$ with $D(\omega) \subset B_X(0, \varrho(\omega))$, the closed ball with center zero and radius $\varrho(\omega)$, which is supposed to have a subexponential growth (ϱ is tempered), i.e.

$$\lim_{t \to \pm\infty} \frac{\log^+ \varrho(\theta_t \omega)}{t} = 0 \quad \text{for } \omega \in \Omega.$$

$\mathcal{D}$ is called the family of subexponentially growing (or tempered) multi–functions. It is obvious that this family is inclusion closed.

Let us give the definition of global random pullback $\mathcal{D}$-attractor.

Definition 3. *The family $\mathcal{A} \in \mathcal{D}$ is said to be a global pullback $\mathcal{D}$-attractor for the MNDS G if it satisfies:*

    i) *$\mathcal{A}(\omega)$ is compact for any $\omega \in \Omega$;*
    ii) *$\mathcal{A}$ is pullback $\mathcal{D}$-attracting;*
    iii) *$\mathcal{A}$ is negatively invariant.*

$\mathcal{A}$ is said to be a strict global pullback $\mathcal{D}$-attractor if the invariance property in the third item is strict.

If, moreover, G is an MRDS and $\mathcal{A}$ is a random set with respect to $\mathcal{P}^c$, then $\mathcal{A}$ is called a random global pullback $\mathcal{D}$-attractor.

Now we can formulate the following theorem, proved in [4].

Theorem 4. *Suppose that the MNDS $G(t,\omega,\cdot)$ is upper–semicontinuous for $t \geq 0$ and $\omega \in \Omega$. Let $B \in \mathcal{D}$ be a multi–valued mapping such that the MNDS is $\mathcal{D}$-asymptotically compact with respect to B, i.e. for every sequence $t_n \to +\infty$ and $\omega \in \Omega$, we have that every sequence $y_n \in G(t_n, \theta_{-t_n}\omega, B(\theta_{-t_n}\omega))$ is relatively compact in X. In addition, suppose that B is pullback $\mathcal{D}$–absorbing. Then, the set $\mathcal{A}$ given by*

$$\text{(3)} \qquad \mathcal{A}(\omega) := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} G(t, \theta_{-t}\omega, B(\theta_{-t}\omega))}$$

is a pullback $\mathcal{D}$-attractor. Furthermore, $\mathcal{A}$ is the unique element from $\mathcal{D}$ with these properties. In addition, if G is a strict MNDS, then $\mathcal{A}$ is strictly invariant.

Moreover, assume that G is an MRDS, $\omega \to G(t,\omega,B(\omega))$ is a random set with respect to $\bar{\mathcal{F}}^{\mathbb{P}}$ (respectively $\mathcal{F}$) for $t \geq 0$, and also that $G(t,\omega,B(\omega))$ is closed for all $t \geq 0$ and $\omega \in \Omega$. Then the set $\mathcal{A}$ defined by (3) is a random set with respect to $\mathcal{P}^c$ (respectively $\mathcal{P}$).

3. Setting of the problem

Let $\mathcal{O} \subset \mathbb{R}^N$ be an open bounded set with C^∞–smooth boundary. Denote, as usual, $H = L^2(\mathcal{O})$, $V = H_0^1(\mathcal{O})$, that is, the closure of the sapce $C_0^\infty(\mathcal{O})$ with respect to the Sobolev space $W^{1,2}(\mathcal{O})$. The norms of these spaces are denoted by $\|\cdot\|$ and $\|\cdot\|_V$, respectively, and the scalar products by $(\cdot,\cdot)$ and $((\cdot,\cdot))$. We shall use $\langle \cdot, \cdot \rangle$ for the pairing between the spaces V' (the dual space of V) and V, and $\langle \cdot, \cdot \rangle_{q,p}$ for the pairing between $L^p(\mathcal{O})$ and $L^q(\mathcal{O})$, where $\frac{1}{p} + \frac{1}{q} = 1$ with $p \geq 2$.

Let A be a positive symmetric operator on H with compact inverse. Then the eigenfunctions of A, denoted by $\{w_i\}_{i \in \mathbb{N}}$, form a complete orthonormal system on H. By a bootstrap argument these elements are sufficiently smooth, in particular included in $L^p(\mathcal{O})$. The associated eigenvalues, denoted by $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$, have finite multiplicity. We consider the space

$$D(A^{\frac{1}{2}}) = \{u = \sum_{i=1}^\infty \hat{u}_i w_i \in H : \|u\|_{1/2}^2 = \sum_{i=1}^\infty |\hat{u}_i|^2 \lambda_i < \infty\},$$

where $\hat{u}_i = (u, w_i)$. In the sequel, A will be $-\Delta$, where Δ is the Laplacian operator endowed with homogeneous Dirichlet boundary conditions. Hence, $D(A^{\frac{1}{2}}) = V$ and $A : V \to V'$ is linear and continuous. Hence, for $u \in V$ it is well known that

$$\text{(4)} \qquad (Au, u) = \|u\|_V^2 = \|u\|_{1/2}^2.$$

Next we define the different phase spaces that will have an important role in our investigations. We consider the space $L^2(-\infty, r; V)$, $r \in \mathbb{R}$, of square integrable functions with values in V and with the measure $e^{\lambda_1 s} Leb$, where Leb is the standard Lebesgue's measure. The space $L^2(-\infty, r; V)$ is equipped with the following norm

$$\|\psi\|_{L^2(-\infty,r;V)}^2 = \int_{-\infty}^r e^{\lambda_1 s} \|\psi(s)\|_V^2 \, ds.$$

This Banach space is separable, see e.g. [13]. From now on, we denote $L_V^2 :=$ $L^2(-\infty, 0; V)$. Furthermore, for $h > 0$ we consider the space $C_h := C([-h, 0], H)$ of continuous functions on $[-h, 0]$ with values in H and with the supremum norm. Let us finally consider the space

$$\mathcal{H} = \{\psi \in L_V^2 \text{ such that } P_h \psi \in C_h\}^1,$$

where P_h is the operator of restriction to the interval $[-h, 0]$. $\mathcal{H}$ is a Banach space endowed with the norm $\|\psi\|_{\mathcal{H}} = \|\psi\|_{L_V^2} + \|\psi\|_{C_h}$. It can be proved easily that $\mathcal{H}$ is separable.

In this paper we are interested in studying the existence of solutions, as well as the longtime behavior of the following random delay system

$$(5) \qquad \begin{cases} \dfrac{du}{dt} + Au(t) = f(\theta_t\omega, u_t) + h(\theta_t\omega, u_t) - g(\theta_t\omega, u(t)), \text{ for } t \in [0, T], \\[2mm] u(t) = \psi(t), \text{ for } t \leq 0, \end{cases}$$

where $T > 0$, and the initial condition $\psi \in \mathcal{H}$. That u agrees with the initial condition ψ in the system (5) means that

$$(6) \qquad u(t) = \psi(t), \text{ for } t \in [-h, 0] \text{ but almost everywhere when } s < -h.$$

By $u_t(\cdot)$ we denote as usual the segment function on $\mathcal{H}$ defined by $u_t(s) = u(t+s)$, for $t \in \mathbb{R}^+$ and $s \in [-h, 0]$.

It is worth mentioning that equation (5) includes, as a particular case, the random evolution equation obtained from equation (2) if we perform a suitable change of variable involving an (O-U) process (see [2] for more details on this Hilbert valued (O-U)).

The assumptions on the different functions appearing in (5) are the following.

First, $g : \Omega \times L^p(\mathcal{O}) \to L^q(\mathcal{O})$, $q = p/(p-1)$, $h : \Omega \times C_h \to H$ and $f : \Omega \times L_V^2 \to V'$ are such that the mappings

$$(7) \qquad v \mapsto g(\omega, v), \ (\omega, \xi) \mapsto h(\omega, \xi), \ \zeta \mapsto f(\omega, \zeta)$$

are continuous in their respective spaces (ω is fixed for g and f), and for arbitrary fixed $v \in L^p(\mathcal{O})$, $\xi \in C_h$, $\zeta \in L_V^2$ we have that

$$(8) \qquad \omega \mapsto g(\omega, v), \ \omega \mapsto h(\omega, \xi), \ \omega \mapsto f(\omega, \zeta)$$

are measurable. Since L_V^2, C_h, $L^p(\mathcal{O})$, V' and H are separable, the functions

$$(\omega, v) \mapsto g(\omega, v), \ (\omega, \xi) \mapsto h(\omega, \xi), \ (\omega, \zeta) \mapsto f(\omega, \zeta)$$

are jointly measurable with respect to their arguments, see Castaing and Valadier [8], Chapter 3. We observe also that since Ω is separable, for the function h condition (7) implies (8).

Moreover, assume the following inequalities

$$(9) \qquad \langle g(\omega, v), v \rangle_{q,p} \geq \eta \|v\|_{L^p(\mathcal{O})}^p - c_1(\omega), \text{ for } v \in L^p(\mathcal{O}),$$

$$(10) \qquad \|g(\omega, v)\|_{L^q(\mathcal{O})}^q \leq \nu \|v\|_{L^p(\mathcal{O})}^p + c_2(\omega), \text{ for } v \in L^p(\mathcal{O}),$$

$$(11) \qquad \|h(\omega, \xi)\| \leq c_3(\omega) + c_4(\omega) \|\xi\|_{C_h}, \text{ for } \xi \in C_h,$$

$$(12) \qquad 2\|f(\omega, \zeta)\|_{V'}^2 \leq c_5(\omega) + K\|\zeta\|_{L_V^2}^2, \text{ for } \zeta \in L_V^2,$$

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[1]$\mathcal{H}$ consists of *functions* of L_V^2 whose restriction to $[-h, 0]$ has a version in C_h.

where $\eta, \nu, K > 0$ and $c_i : \Omega \to \mathbb{R}^+$ are measurable with respect to $\mathcal{F}$. Also, the functions $t \mapsto c_1(\theta_t\omega)$, $t \mapsto c_3^2(\theta_t\omega)$ are assumed to be integrable on any finite interval and subexponentially growing (that is, tempered), whereas $t \mapsto c_2(\theta_t\omega)$, $t \mapsto c_5(\theta_t\omega)$ are integrable on any finite interval. For c_4 we suppose that $\mathbb{E}(c_4^2) < \infty$ (and then $t \mapsto c_4^2(\theta_t\omega)$ is integrable on any finite interval by the ergodic theorem), so that

$$\lim_{t \to \pm\infty} \frac{1}{t} \int_0^t c_4^2(\theta_s\omega)\,ds = \mathbb{E}(c_4^2)$$

on a $(\theta_t)_{t \in \mathbb{R}}$-invariant set of full measure.

On the other hand, and as a consequence of the Young inequality, we have

$$\|u\|_{L^p(\mathcal{O})}^p \geq \mu \|u\|^2 - C_\mu, \tag{13}$$

where $\mu > 0$ can be chosen arbitrarily and C_μ denotes a positive constant. In particular, we take μ such that

$$\mathbb{E}(c_4^2) < \frac{\eta\mu\lambda_1}{4e^{\lambda_1 h}}. \tag{14}$$

Note that from (12) it is straightforward that

$$2\int_r^t e^{\lambda_1 s}\|f(\theta_s\omega, u_s)\|_{V'}^2\,ds \leq \int_r^t e^{\lambda_1 s}c_5(\theta_s\omega)ds + K(t-r)\int_{-\infty}^t e^{\lambda_1 \tau}\|u(\tau)\|_V^2\,d\tau. \tag{15}$$

For f and $t > 0$ we also assume the following inequalities

$$2\int_0^t e^{\lambda_1 s}\|f(\theta_s\omega, u_s)\|_{V'}^2\,ds \leq \int_0^t e^{\lambda_1 s}c_6(\theta_s\omega)ds + \frac{d}{2}\int_{-\infty}^t e^{\lambda_1 s}\|u(s)\|_V^2\,ds, \tag{16}$$

$$2\int_0^t e^{\lambda_1 s}\|f(\theta_s\overline{\omega}, u_s) - f(\theta_s\omega, v_s)\|_{V'}^2\,ds \leq \int_0^t e^{\lambda_1 s}c_7(\theta_s\overline{\omega}, \theta_s\omega)ds$$
$$+ \frac{b}{2}\int_{-\infty}^t e^{\lambda_1 s}\|u(s) - v(s)\|_V^2\,ds, \tag{17}$$

for $u, v \in L^2(-\infty, t; V)$, $\overline{\omega}, \omega \in \Omega$, where $d, b < 1$, and $c_6 : \Omega \to \mathbb{R}^+$, $c_7 : \Omega \times \Omega \to \mathbb{R}^+$ are measurable with respect to $\mathcal{F}$ and $\mathcal{F} \otimes \mathcal{F}$, respectively, and the functions $t \mapsto c_6(\theta_t\omega)$, $t \mapsto c_7(\theta_t\overline{\omega}, \theta_t\omega)$ are integrable on any finite interval. Also, $t \mapsto c_6(\theta_t\omega)$ is subexponentially growing. Moreover, for any $t \in \mathbb{R}$, $\varepsilon > 0$ there exists $\delta > 0$ such that if $d_\Omega(\overline{\omega}, \omega) < \delta$, then

$$\int_0^t e^{\lambda_1 s}c_7(\theta_s\overline{\omega}, \theta_s\omega)ds < \varepsilon. \tag{18}$$

Additionally, we assume that the maps

$$\omega \mapsto \int_0^t c_i(\theta_s\omega)ds, \ i = 1, 3, 4, 5, \tag{19}$$

are continuous for any fixed $t \in \mathbb{R}$, and that for every $\omega_0 \in \Omega$, $t \in \mathbb{R}$ there exists a neighbourhood $\mathcal{U}$ of ω_0 and a constant $C(t, \omega_0)$ such that

$$\int_0^t c_j^2(\theta_s\omega)ds \leq C, \ \int_0^t c_i(\theta_s\omega)ds \leq C \text{ for any } \omega \in \mathcal{U}, \tag{20}$$

where $j = 3, 4$, $i = 1, 2, 5, 6$. It is obvious that for $i = 1, 5$ the property (19) implies condition (20).

Finally, let us assume that if $\omega^n \to \omega$, $u^n \to u$ in $L^2(0, T; H)$, $u^n \to u$ weakly in $L^p(0, T; L^p(\mathcal{O}))$ and $u^n \to u$ weakly in $L^2(-\infty, T; V)$ then

$$(21) \qquad f(\theta.\omega^n, u^n_.) \to f(\theta.\omega^n, u_.) \text{ weakly in } L^2(0, T; V'),$$

$$(22) \qquad g(\theta.\omega^n, u^n(\cdot)) \to g(\theta.\omega, u(\cdot)) \text{ weakly in } L^q(0, T; L^q(\mathcal{O})),$$

and

$$
\begin{aligned}
(23) \qquad &\lim_{n \to \infty} \inf \int_0^T e^{-\lambda_1(T-s)} \langle g(\theta_s \omega^n, u^n(s)), u^n(s) \rangle_{q,p} \, ds \\
&\geq \int_0^T e^{-\lambda_1(T-s)} \langle g(\theta_s \omega, u(s)), u(s) \rangle_{q,p} \, ds.
\end{aligned}
$$

We would like to point out that, comparing with the nonautonomous case [5], we have introduced the new conditions (18), (19) and (20). Also, assumptions (17), (21), (22) and (23) have been changed, as now the parameter ω is not fixed. These stronger conditions are needed in order to prove the measurability of the global pullback attractor, but not for checking its existence.

4. Existence of solutions and absorbing set

Our first aim is to prove the existence of weak solutions for the problem (5).

Definition 5. *Given $T > 0$, the function $u(t) = u(t, \omega, \psi) \in L^2(-\infty, T; V) \cap C([-h, T], H) \cap L^p(0, T; L^p(\mathcal{O}))$ is called a weak solution of (5) on $(0, T)$ with initial data $\psi \in \mathcal{H}$, if for arbitrary $v \in V \cap L^p(\mathcal{O})$,*

$$(24) \quad \frac{d}{dt}(u, v) + \ < Au, v \ > = \langle f(\theta_t \omega, u_t), v \rangle + (h(\theta_t \omega, u_t), v) - \langle g(\theta_t \omega, u(t)), v \rangle_{q,p},$$

and u agrees with the initial condition ψ according to (6).

From this definition and the above conditions, it follows that $\dfrac{du}{dt} \in L^2(0, T; V') + L^q(0, T; L^q(\mathcal{O}))$. Then, see [17], $\|u\| : [0, T] \to H$ is absolutely continuous and $\left\langle\!\!\left\langle \dfrac{du}{dt}, u \right\rangle\!\!\right\rangle = \dfrac{1}{2}\dfrac{d}{dt}\|u(t)\|^2$ for a.a. $t \in (0, T)$, where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the pairing between $V \cap L^p(\mathcal{O})$ and its conjugate space $V' + L^q(\mathcal{O})$. Therefore, (24) is equivalent to

$$(25) \quad u(t) + \int_0^t Au(s)ds = u(0) + \int_0^t \left(f(\theta_s\omega, u_s) + h(\theta_s\omega, u_s) - g(\theta_s\omega, u(s))\right) ds$$

in $V' + L^q(\mathcal{O})$, for $t \in [0, T]$.

In what follows we shall obtain some a priori estimates in the spaces C_h and L^2_V, that will be very useful when proving the existence of a weak solution by using Galerkin approximations, and the existence of an absorbing set as well.

Lemma 6. *Every weak solution of (5) satisfies, for $t \in [h, T]$, the estimate*

$$\|u_t\|_{C_h}^2 + \frac{1-d}{2} \int_{-t}^0 e^{\lambda_1 s} \|u_t(s)\|_V^2 ds$$

(26)
$$\leq e^{-\lambda_1(t-h) + e^{\lambda_1 h} \int_{-t}^0 \frac{4}{\eta\mu} c_4^2(\theta_{s+t}\omega) ds} \left(2\|\psi(0)\|^2 + d\|\psi\|_{L_V^2}^2 \right)$$

$$+ 2e^{\lambda_1 h} \int_{-t}^0 e^{\lambda_1 \tau + e^{\lambda_1 h} \int_\tau^0 \frac{4}{\eta\mu} c_4^2(\theta_{s+t}\omega) ds} c(\theta_{\tau+t}\omega) d\tau,$$

where

(27)
$$c(\omega) = \eta C_\mu + c_6(\omega) + 2c_1(\omega) + \frac{2}{\eta\mu} c_3^2(\omega).$$

Remark 7. *If we replace $\|\psi(0)\|^2$ by $\|\psi\|_{C_h}^2$ in the right-hand side of (26), then the inequality is true for all $t \in [0, T]$.*

Proof. Using (4), (9), (11) and (13) we derive the following energy equality

$$\frac{d\|u(t)\|^2}{dt} + \lambda_1 \|u(t)\|^2 + \|u(t)\|_V^2 + 2\eta \|u(t)\|_{L^p(\mathcal{O})}^p$$

(28)
$$\leq 2 < f(\theta_t\omega, u_t), u(t) > + 2c_1(\theta_t\omega) + 2 \left(c_3(\theta_t\omega) + c_4(\theta_t\omega) \|u_t\|_{C_h} \right) \|u(t)\|$$

$$\leq 2\|f(\theta_t\omega, u_t)\|_{V'}^2 + \frac{1}{2}\|u(t)\|_V^2 + 2c_1(\theta_t\omega) + \frac{2c_3^2(\theta_t\omega)}{\eta\mu} + \frac{2c_4^2(\theta_t\omega)}{\eta\mu} \|u_t\|_{C_h}^2$$

$$+ \eta \|u(t)\|_{L^p(\mathcal{O})}^p + \eta C_\mu.$$

Hence, by using Gronwall's lemma we obtain

$$\|u(t)\|^2 + \frac{1}{2} \int_0^t e^{-\lambda_1(t-s)} \|u(s)\|_V^2 ds \leq e^{-\lambda_1 t} \|\psi(0)\|^2 + 2 \int_0^t e^{-\lambda_1(t-s)} \|f(\theta_s\omega, u_s)\|_{V'}^2 ds$$

$$+ \int_0^t e^{-\lambda_1(t-s)} \left(2c_1(\theta_s\omega) + \frac{2c_3^2(\theta_s\omega)}{\eta\mu} + \frac{2c_4^2(\theta_s\omega)}{\eta\mu} \|u_s\|_{C_h}^2 + \eta C_\mu \right) ds.$$

By (16) it follows immediately that

$$2 \int_0^t e^{-\lambda_1(t-s)} \|f(\theta_s\omega, u_s)\|_{V'}^2 ds \leq \int_0^t e^{-\lambda_1(t-s)} c_6(\theta_s\omega) ds$$

$$+ \frac{d}{2} e^{-\lambda_1 t} \|\psi\|_{L_V^2}^2 + \frac{d}{2} \int_0^t e^{-\lambda_1(t-s)} \|u(s)\|_V^2 ds,$$

thus we have

$$\|u(t)\|^2 + \frac{1-d}{2} \int_0^t e^{-\lambda_1(t-s)} \|u(s)\|_V^2 ds$$

$$= \|u(t)\|^2 + \frac{1-d}{2} \int_{-t}^0 e^{\lambda_1 s} \|u_t(s)\|_V^2 ds$$

$$\leq e^{-\lambda_1 t} \|\psi(0)\|^2 + \frac{d}{2} e^{-\lambda_1 t} \|\psi\|_{L_V^2}^2 + \int_0^t e^{-\lambda_1(t-s)} \left(c(\theta_s\omega) + \frac{2c_4^2(\theta_s\omega)}{\eta\mu} \|u_s\|_{C_h}^2 \right) ds,$$

where $c(\omega)$ is given by (27). From this expression, for $t \geq h$ we obtain

$$\|u_t\|_{C_h}^2 \leq e^{-\lambda_1(t-h)} \left(\|\psi(0)\|^2 + \frac{d}{2} \|\psi\|_{L_V^2}^2 \right) + \int_0^t e^{\lambda_1 h} e^{-\lambda_1(t-s)} \left(c(\theta_s\omega) + \frac{2c_4^2(\theta_s\omega)}{\eta\mu} \|u_s\|_{C_h}^2 \right) ds,$$

and then, denoting for $v \in \mathcal{H}$

$$(29) \qquad \chi(v)(t) = \|v\|_{C_h}^2 + \frac{1-d}{2} \int_{-t}^0 e^{\lambda_1 s} \|v(s)\|_V^2 ds$$

we have

$\chi(u_t)(t)$

$$\leq e^{-\lambda_1(t-h)} \left(2\|\psi(0)\|^2 + d\|\psi\|_{L_V^2}^2 \right) + \int_0^t e^{\lambda_1 h} e^{-\lambda_1(t-s)} \left(2c(\theta_s\omega) + \frac{4c_4^2(\theta_s\omega)}{\eta\mu} \|u_s\|_{C_h}^2 \right) ds$$

$$\leq e^{-\lambda_1(t-h)} \left(2\|\psi(0)\|^2 + d\|\psi\|_{L_V^2}^2 \right) + \int_0^t e^{\lambda_1 h} e^{-\lambda_1(t-s)} \left(2c(\theta_s\omega) + \frac{4c_4^2(\theta_s\omega)}{\eta\mu} \chi(u_s)(s) \right) ds,$$

or equivalently,

$$e^{\lambda_1 t} \chi(u_t)(t) \leq e^{\lambda_1 h} \left(2\|\psi(0)\|^2 + d\|\psi\|_{L_V^2}^2 \right) + e^{\lambda_1 h} \int_0^t \left(e^{\lambda_1 s} 2c(\theta_s\omega) + \frac{4c_4^2(\theta_s\omega)}{\eta\mu} e^{\lambda_1 s} \chi(u_s)(s) \right) ds.$$

Then, applying again Gronwall's lemma we get

$$\chi(u_t)(t) \leq e^{-\lambda_1(t-h) + e^{\lambda_1 h} \int_0^t \frac{4}{\eta\mu} c_4^2(\theta_s\omega) ds} \left(2\|\psi(0)\|^2 + d\|\psi\|_{L_V^2}^2 \right)$$

$$+ 2e^{\lambda_1 h} \int_0^t e^{-\lambda_1(t-s) + e^{\lambda_1 h} \int_s^t \frac{4}{\eta\mu} c_4^2(\theta_\tau\omega) d\tau} c(\theta_s\omega) ds$$

$$\leq e^{-\lambda_1(t-h) + e^{\lambda_1 h} \int_{-t}^0 \frac{4}{\eta\mu} c_4^2(\theta_{s+t}\omega) ds} \left(2\|\psi(0)\|^2 + d\|\psi\|_{L_V^2}^2 \right)$$

$$+ 2e^{\lambda_1 h} \int_{-t}^0 e^{\lambda_1 \tau + e^{\lambda_1 h} \int_\tau^0 \frac{4}{\eta\mu} c_4^2(\theta_{s+t}\omega) ds} c(\theta_{\tau+t}\omega) d\tau.$$

$\square$

Lemma 8. *Let B be a bounded set of $\mathcal{H}$ and let $T > 0$, $\omega_0 \in \Omega$ be fixed. Then there exists a neighborhood $\mathcal{U}$ of ω_0 such that every weak solution $u(\cdot) = u(\cdot, \omega, \psi)$ of (5) with $\psi \in B$ and $\omega \in \mathcal{U}$ satisfies the inequality:*

$$(30) \qquad \|u(t)\|^2 \leq \|u(r)\|^2 + C \int_r^t \left(d(\theta_s\omega) + 1 \right) ds, \text{ for all } 0 \leq r \leq t \leq T,$$

where $d(\omega) = \sum_{i \in \{1,3,4,5\}} c_i(\omega)$ and $C = C(B, T, \omega_0) > 0$ is a constant.

Proof. First of all, note that thanks to (26) and the assumption (20) we obtain that there exists a neighborhood $\mathcal{U}$ of ω_0 such that $\|u_t\|_{C_h} \leq C_1 = C_1(B, T, \omega_0)$ for $t \in [0, T]$, $\omega \in \mathcal{U}$. Hence, arguing as in (28) we obtain

$$\frac{d\|u(t)\|^2}{dt} + \lambda_1 \|u(t)\|^2 + \|u(t)\|_V^2 + 2\eta \|u(t)\|_{L^p(\mathcal{O})}^p$$

$$\leq 2 < f(\theta_t\omega, u_t), u(t) > + 2c_1(\theta_t\omega) + 2C_1 \|h(\theta_t\omega, u_t)\|$$

$$\leq 2\|f(\theta_t\omega, u_t)\|_{V'}^2 + \frac{1}{2} \|u(t)\|_V^2 + 2c_1(\theta_t\omega) + 2C_1 \|h(\theta_t\omega, u_t)\|.$$

Once again, by Gronwall's lemma

$$\|u(t)\|^2 + \frac{1}{2} \int_r^t e^{-\lambda_1(t-s)} \|u(s)\|_V^2 ds \le e^{-\lambda_1(t-r)} \|u(r)\|^2$$

$$+ 2 \int_r^t e^{-\lambda_1(t-s)} \|f(\theta_s \omega, u_s)\|_{V'}^2 ds + 2 \int_r^t e^{-\lambda_1(t-s)} (c_1(\theta_s \omega) + C_1 \|h(\theta_s \omega, u_s)\|) ds.$$

On the other hand, by Lemma 6 and (20) we have that $\left(\int_{-\infty}^t e^{\lambda_1 s} \|u(s)\|_V^2 ds + 1 \right) \le C_2 = C_2(B, T, \omega_0)$ for $t \in [0, T]$, $\omega \in \mathcal{U}$. Therefore, in virtue of (11) and (15) we obtain

$$\|u(t)\|^2 + \frac{1}{2} \int_r^t e^{-\lambda_1(t-s)} \|u(s)\|_V^2 ds$$

$$\le e^{-\lambda_1(t-r)} \|u(r)\|^2 + \int_r^t e^{-\lambda_1(t-s)} c_5(\theta_s \omega) ds$$

$$+ KC_2(t-r) + 2 \int_r^t e^{-\lambda_1(t-s)} (c_1(\theta_s \omega) + C_1(c_3(\theta_s \omega) + C_1 c_4(\theta_s \omega))) ds$$

$$\le \|u(r)\|^2 + C \int_r^t \left(\sum_{i \in \{1,3,4,5\}} c_i(\theta_s \omega) + 1 \right) ds.$$

$\square$

For our further purposes we will also need the following technical lemma, whose proof can be found in [5].

Lemma 9. *Let $t \to J_n(t)$, $t \to J(t)$, $t \in [0, T]$, be continuous non-increasing functions such that $J_n(t) \to J(t)$ for a.a. t as $n \to \infty$. Then for all $t_0 \in (0, T]$ and any sequence $t_n \to t_0$ we have*

$$\limsup_{n \to \infty} J_n(t_n) \le J(t_0).$$

If, moreover, $J_n(0) \to J(0)$, then the result is true also for $t_0 = 0$.

Now we are ready to prove the existence of weak solutions by using the Galerkin method.

Theorem 10. *For every $\psi \in \mathcal{H}$ there exists at least one weak solution of (5).*

Proof. Recall that $\{w_j\}_{j=1}^\infty \subset C^\infty(\bar{\mathcal{O}})$ denotes the orthonormal basis consisting of the eigenfunctions of A. We denote by $V_n = [w_1, ..., w_n]$ the space spanned by $\{w_j\}_{j=1}^n$, and by $P_n : D(A^{m/2}) \to V_n$, $m \in \mathbb{N}$ the orthoprojector defined by $P_n u = \sum_{j=1}^n (u, w_j) w_j$. It is not difficult to check that V_n is complete in $H_0^1(\mathcal{O}) \cap L^p(\mathcal{O})$. For every fixed n we define

$$u^n(t) = \sum_{j=1}^n \gamma_j^n(t) w_j,$$

where the coefficients γ_j^n are required to satisfy the following system:

$$\frac{d}{dt} (u^n(t), w_j) + (Au^n(t), w_j) = (f(\theta_t \omega, u_t^n) + h(\theta_t \omega, u_t^n) - g(\theta_t \omega, u^n(t)), w_j),$$

(31) $$u^n(t) = P_n \psi(t), \text{ for } t \le 0,$$

for $1 \leq j \leq n$. Let $X_n = L^2(-\infty, 0; \mathbb{R}^n) \cap C([-h, 0]; \mathbb{R}^n)$. Consider the map $F_n : [0, T] \times X_n \times \Omega \to \mathbb{R}^n$ given by

$$F_n (t, \gamma, \omega) = P_n (-A\gamma + f(\theta_t\omega, \psi) + h(\theta_t\omega, \psi) - g(\theta_t\omega, \psi(0))),$$

where $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t)) \in X_n$. For the following we fix ω. It is easy to see that $\psi \in \mathcal{H}$ and $\gamma^m \to \gamma$ in X_n implies $\psi^m \to \psi$ in $\mathcal{H}$ and $\psi^m(0) \to \psi(0)$ in $L^p(\mathcal{O})$. Hence, the above continuity conditions (7) for the maps f, g, h imply that for a.a. t the map $\gamma \to F_n(t, \gamma, \omega)$ is continuous. Also, the measurability condition (8) together with the properties of θ_t give us the measurability of $t \to F_n(t, \gamma, \omega)$ for any γ. On the other hand, by (10)-(12) for every (t_0, γ_0) there is a neighborhood $\mathcal{U}_{t_0,\gamma_0,\omega}$ in $\mathbb{R} \times X_n$ and a Lebesgue integrable function $m(t)$ such that

$$\|F_n(t, \gamma, \omega)\|_{\mathbb{R}^n} \leq m(t) \text{ for all } (t, \gamma) \in \mathcal{U}_{t_0,\gamma_0,\omega}.$$

Hence, Theorem 1.1 in [13, p.36] implies the existence of at least one local solution for (31).

For the remainder of the proof, and in particular the compactness and the convergence argument, we refer the reader to [5]. □

We observe here that every weak solution of (5) can be extended to a globally defined one (i.e. for all $t \in \mathbb{R}$) simply by concatenating solutions. Let then $\mathcal{S}(\psi, \omega)$ be the set of all globally defined solutions to (5) corresponding to $\psi \in \mathcal{H}$ and $\omega \in \Omega$. We define the multivalued map $G : \mathbb{R}^+ \times \Omega \times \mathcal{H} \to P(\mathcal{H})$ as follows

$$G(t, \omega, \psi) = \{u_t : u(\cdot, \omega, \psi) \in \mathcal{S}(\psi, \omega)\}.$$

The next lemma can be proved in a similar way as in [7, Proposition 4] or [4, Lemma 5.1].

Lemma 11. *G satisfies the strict cocycle property* $G(t+\tau, \omega, \psi) = G(t, \theta_\tau\omega, G(\tau, \omega, \psi))$ *for all* $t, \tau \in \mathbb{R}^+$, $\psi \in \mathcal{H}$ *and* $\omega \in \Omega$.

Let us prove the existence of an absorbing set in the space $\mathcal{H}$. In what follows, we denote by $B_{\mathcal{H}}(0, R)$ a closed ball in $\mathcal{H}$ centered at 0 with radius R. We can assume without lost of generality that $c(\omega) > 0$ for all $\omega \in \Omega$.

In the following we need frequently (14).

Lemma 12. *Let* $\mathcal{D} = \{D(\omega)\}_{\omega \in \Omega}$ *be the family of tempered sets in* $\mathcal{H}$. *Then the ball in* $\mathcal{H}$ *given by* $B_{\mathcal{H}}(0, R(\omega))$ *is a* $\mathcal{D}$-*absorbing set, where*

$$(32) \qquad R^2(\omega) = \frac{8e^{\lambda_1 h}}{1-d} \int_{-\infty}^0 e^{\lambda_1 r + e^{\lambda_1 h} \int_r^0 \frac{4c_4^2(\theta_\tau\omega)}{\eta\mu} d\tau} c(\theta_r\omega) dr,$$

and the random variable $c(\omega)$ *has been defined in (27).*

Proof. First of all, note that for $u \in G(t, \omega, \psi)$

$$\frac{1-d}{2} \|u\|_{\mathcal{H}}^2 \leq \chi(u)(t) + \frac{1-d}{2} e^{-\lambda_1 t} \|\psi\|_{L_V^2}^2,$$

where the definition of χ was given in (29). Let $D \in \mathcal{D}$. Thanks to (26) we obtain

$$\sup_{u \in G(t, \theta_{-t}\omega, D(\theta_{-t}\omega))} \chi(u)(t) \le 2e^{-\lambda_1(t-h) + e^{\lambda_1 h} \int_{-t}^0 \frac{4}{\eta\mu} c_4^2(\theta_s\omega)ds} \sup_{\psi \in D(\theta_{-t}\omega)} \|\psi\|_{\mathcal{H}}^2$$

$$+ 2e^{\lambda_1 h} \int_{-\infty}^0 e^{\lambda_1 \tau + e^{\lambda_1 h} \int_\tau^0 \frac{4}{\eta\mu} c_4^2(\theta_s\omega)ds} c(\theta_\tau\omega) d\tau,$$

where the last integral exists due to condition (14) and the temperedness of $c(\omega)$. Hence

$$\sup_{u \in G(t, \theta_{-t}\omega, D(\theta_{-t}\omega))} \|u\|_{\mathcal{H}}^2 \le \frac{1}{2} R^2(\omega) + \left(\frac{4}{1-d} \left(e^{-\lambda_1(t-h) + e^{\lambda_1 h} \int_{-t}^0 \frac{4}{\eta\mu} c_4^2(\theta_s\omega)ds} \right) + e^{-\lambda_1 t} \right)$$

$$\times \sup_{\psi \in D(\theta_{-t}\omega)} \|\psi\|_{\mathcal{H}}^2,$$

and $R(\omega) > 0$. This estimate is true for any $t \ge 0$. Hence, by (14) for any $\omega \in \Omega$, $D \in \mathcal{D}$ there exists $\bar{t} = \bar{t}(D, \omega)$ such that for $t \ge \bar{t}$,

$$\sup_{u \in G(t, \theta_{-t}\omega, D(\theta_{-t}\omega))} \|u\|_{\mathcal{H}}^2 \le R^2(\omega).$$

$\square$

Lemma 13. *The absorbing ball given in Lemma 12 belongs to $\mathcal{D}$.*

For the proof we refer to Caraballo et al. [6], Lemma 4.6.

5. ASYMPTOTIC COMPACTNESS AND THE PULLBACK ATTRACTOR

Now, after an auxiliary lemma, we prove that the cocycle is pullback asymptotically compact.

Lemma 14. *Let $\psi^n \in B$, where B is bounded in $\mathcal{H}$, and $\psi^n \to \psi$ weakly in L_V^2, $\psi^n(0) \to \psi(0)$ weakly in H. Consider any sequence of weak solutions $u^n(\cdot) = u^n(\cdot, \omega^n, \psi^n)$ with initial data ψ^n corresponding to ω^n, where $\omega^n \to \omega$. Then there exists a subsequence u^{n_k} and a function u such that u^{n_k} converges to u in $C([\tau, T]; H)$ for all $0 < \tau < T$. Moreover, $u^{n_k} \to u$ weakly in $L^2(0, T; V)$ for all $T > 0$. Also, the have the following property:*

$$(33) \qquad \limsup_{n \to \infty} \|u_T^{n_k} - u_T\|_{L_V^2}^2 \le \frac{1}{1-b} e^{-\lambda_1 T} \limsup_{n \to \infty} \|\psi^n(r) - \psi(r)\|_{L_V^2}^2.$$

Hence, if $\psi^n \to \psi$ in L_V^2, then $u_T^{n_k} \to u_T$ in L_V^2 for any $T > 0$.
If, moreover, $\psi^n \to \psi$ in C_h, then $u^{n_k} \to u$ in $C([-h, T]; H)$, for all $T > 0$, and $u(\cdot) = u(\cdot, \omega, \psi)$ is a solution of (5) corresponding to the initial data ψ and the parameter ω.

Proof. First fix $0 < \tau < T$. In view of inequality (26) and assumptions (20), the sequence u^n is bounded in $L^\infty(0, T; H) \cap L^2(-\infty, T; V)$, and the norm $\|u_t^n\|_{C_h}$ is uniformly bounded in $[0, T]$. Further, using (12) and (20), from (28) we easily obtain that u^n is bounded in $L^p(0, T; L^p(\mathcal{O}))$, which implies, together with (10)-(12) and (20), that $f(\theta.\omega^n, u^n)$, $h(\theta.\omega^n, u^n)$ and $g(\theta.\omega^n, u^n(\cdot))$ are bounded in $L^2(0, T; V)$, $L^2(0, T; H)$ and $L^q(0, T; L^q(\mathcal{O}))$, respectively. Hence, the equality

$$\frac{du^n(t)}{dt} = -Au^n(t) + f(\theta_t\omega^n, u_t^n) + h(\theta_t\omega^n, u_t^n) - g(\theta_t\omega^n, u^n(t))$$

implies that $\dfrac{du^n}{dt}$ is bounded in $L^2\left(0,T;V'\right)+L^q\left(0,T;L^q\left(\mathcal{O}\right)\right)$. If we choose $r\geq max\{1,N\left(\frac{1}{q}-\frac{1}{2}\right)\}$, then $\dfrac{du^n}{dt}$ is bounded in $L^q\left(0,T;H^{-r}\left(\mathcal{O}\right)\right)$.

Hence, up to a subsequence, we have the convergences

(34)
$$u^n \to u \text{ weakly star in } L^\infty(0,T;H),$$
$$\frac{du^n}{dt} \to \frac{du}{dt} \text{ weakly in } L^q\left(0,T;H^{-r}\left(\mathcal{O}\right)\right),$$
$$u^n \to u \text{ strongly in } L^2\left(0,T;H\right),$$
$$u^n \to u \text{ weakly in } L^2\left(-\infty,T;V\right) \text{ and } L^p\left(0,T;L^p\left(\mathcal{O}\right)\right),$$
$$h(\theta.\omega^n,u^n_\cdot) \to \zeta_h \text{ weakly in } L^2\left(0,T;H\right).$$

In view of conditions (21)-(22), we have that $g\left(\theta.\omega^n,u^n(\cdot)\right)\to g\left(\theta.\omega,u(\cdot)\right)$ weakly in $L^q\left(0,T;L^q\left(\mathcal{O}\right)\right)$, and $f(\theta.\omega^n,u^n_\cdot)\to f(\theta.\omega,u_\cdot)$ weakly in $L^2\left(0,T;V'\right)$.

In a standard way (see the proof of Lemma 17 in [5]) we can check that $u_n\left(t_n\right)\to u\left(t_0\right)$ weakly in H for any sequence $t_n\to t_0$, $t_n,t_0\in[0,T]$. Hence, $u=\psi$ in $(-\infty,0)$ and $u(0)=\psi(0)$.

Further, since $u:[0,T]\to H$ is continuous, we will obtain that $u^n\to u$ in $C\left([\tau,T],H\right)$ for any $0<\tau<T$ if we check that $u^n\left(t_n\right)\to u\left(t_0\right)$ strongly in H for any $t_n,t_0\in[\tau,T]$. As $\|u\left(t_0\right)\|\leq\liminf\|u^n\left(t_n\right)\|$, for this aim it is enough to obtain that

(35)
$$\limsup\|u^n\left(t_n\right)\|\leq\|u\left(t_0\right)\|.$$

In view of Lemma 8 there exists a neighborhood $\mathcal{U}$ of ω such that for any $\omega^n\in\mathcal{U}$ the continuous functions

$$J_n\left(t\right)=\|u^n(t)\|^2-C\int_0^t\left(d\left(\theta_s\omega^n\right)+1\right)ds,$$

are non-increasing in $[0,T]$. Passing to the limit we obtain that $u\left(\cdot\right)$ is a solution of the following problem:

$$\frac{du}{dt}+Au=f\left(\theta_t\omega,u_t\right)+\zeta_h-g\left(\theta_t\omega,u\right),$$
$$u(0)=\psi\left(0\right),\ u=\psi \text{ in } L^2_V.$$

We note that $h\left(\theta_s\omega^n,u^n_s\right)\to\zeta_h$ weakly in $L^2\left(0,T;H\right)$ and assumption (19) implies that ζ_h satisfies

$$\int_r^t\|\zeta_h(s)\|\,ds\leq\liminf_{n\to\infty}\int_r^t\|h(\theta_s\omega^n,u^n_s)\|\,ds$$
$$\leq\liminf_{n\to\infty}\int_r^t\left(c_3\left(\theta_s\omega^n\right)+C_1c_4\left(\theta_s\omega^n\right)\right)ds$$
$$=\int_r^t\left(c_3\left(\theta_s\omega\right)+C_1c_4\left(\theta_s\omega\right)\right)ds,$$

where we have used that $\|u^n_t\|_{C_h}\leq C_1=C_1(B,T,\omega)$ for all n and $t\in[0,T]$. Then repeating the same steps as in Lemma 8 we obtain that the continuous function

$$J\left(t\right)=\|u(t)\|^2-C\int_0^t\left(d\left(\theta_s\omega\right)+1\right)ds$$

is also non-increasing in $[0, T]$. Moreover, $u^n \to u$ strongly in $L^2(0, T; H)$ and (19) implies, passing to a subsequence, that $J_n(t) \to J(t)$ for a.a. $t \in (0, T)$. Therefore, using Lemma 9 we get (35).

Finally, we apply a diagonal argument to prove that the result is valid in an arbitrary interval $0 < \tau < T$.

If, moreover, $\psi^n \to \psi$ in C_h, then arguing as before one can check that $u^n \to u$ in $C([-h, T]; H)$. Hence, it follows from (7) that $h(\theta.\omega, u.) = \zeta_h$, and then u is a solution of (5) corresponding to the initial data ψ.

Finally, we shall obtain (33), which implies that if $\psi^n \to \psi$ in L_V^2, then $u_T^n \to u_T$ in L_V^2 for any $T > 0$. Indeed, the difference $v^n = u^n - u$ satisfies

$$\frac{d}{dt}\|v^n(t)\|^2 + \lambda_1 \|v^n(t)\|^2 + \|v^n(t)\|_V^2 \leq 2\|f(\theta_t\omega^n, u_t^n) - f(\theta_t\omega, u_t)\|_{V'}^2 + \frac{1}{2}\|v^n(t)\|_V^2$$
$$+ 2(h(\theta_t\omega^n, u_t^n) - h(\theta_t\omega, u_t), v^n(t))$$
$$- 2\langle g(\theta_t\omega^n, u^n(t)) - g(\theta_t\omega, u), v^n(t)\rangle_{q,p}.$$

Applying Gronwall's lemma and taking into account assumption (17), we deduce

(36)

$$\|v^n(T)\|^2 + \frac{1-b}{2}\int_0^T e^{-\lambda_1(T-s)}\|v^n(s)\|_V^2\, ds$$

$$\leq e^{-\lambda_1 T}\|v^n(0)\|^2 + \frac{b}{2}\int_{-\infty}^0 e^{-\lambda_1(T-s)}\|v^n(s)\|_V^2\, ds + \int_0^T e^{-\lambda_1(T-s)}c_7(\theta_s\omega^n, \theta_s\omega)ds$$

$$+ 2\int_0^T e^{-\lambda_1(T-s)}(h(\theta_s\omega^n, u_s^n) - h(\theta_s\omega, u_s), v^n(s))ds$$

$$- 2\int_0^T e^{-\lambda_1(T-s)}\langle g(\theta_s\omega^n, u^n(s)) - g(\theta_s\omega, u(s)), v^n(s)\rangle_{q,p}ds.$$

Using $\psi^n(0) \to \psi(0)$ in H, condition (18) and (34) we obtain that

$$\lim_{n\to\infty} e^{-\lambda_1 T}\|v^n(0)\|^2 = 0,$$

$$\lim_{n\to\infty}\int_0^T e^{-\lambda_1(T-s)}c_7(\theta_s\omega^n, \theta_s\omega)ds = 0,$$

$$\lim_{n\to\infty}\int_0^T e^{-\lambda_1(T-s)}(h(\theta_s\omega^n, u_s^n) - h(\theta_s\omega, u_s), v^n(s))ds = 0,$$

$$\lim_{n\to\infty}\int_0^T e^{-\lambda_1(T-s)}\langle g(\theta_s\omega, u(s)), v^n(s)\rangle_{q,p}ds = 0.$$

Further, by (11), (20), (22) and (23) we have

$$\limsup_{n\to\infty}\int_0^T e^{-\lambda_1(T-s)}\langle -g(\theta_s\omega^n, u^n(s)), v^n(s)\rangle_{q,p}ds$$

$$= \int_0^T e^{-\lambda_1(T-s)}\langle g(\theta_s\omega^n, u^n(s)), u(s)\rangle_{q,p}ds$$

$$- \liminf_{n\to\infty}\int_0^T e^{-\lambda_1(T-s)}\langle g(\theta_s\omega^n, u^n(s)), u^n(s)\rangle_{q,p}ds$$

$$\leq 0.$$

Finally, from (36) we derive

$$\limsup_{n\to\infty} \int_0^T e^{-\lambda_1(T-s)} \|v^n(s)\|_V^2 \, ds$$

$$\leq \frac{b}{1-b} \limsup_{n\to\infty} \int_{-\infty}^0 e^{-\lambda_1(T-s)} \|v^n(s)\|_V^2 \, ds$$

$$= \frac{b}{1-b} e^{-\lambda_1 T} \limsup_{n\to\infty} \int_{-\infty}^0 e^{\lambda_1 s} \|\psi^n(s) - \psi(s)\|_V^2 \, ds,$$

and since

$$\|v_T^n\|_{L_V^2}^2 = \int_{-T}^0 e^{\lambda_1 s} \|v_T^n(s)\|_V^2 \, ds + \int_{-\infty}^{-T} e^{\lambda_1 s} \|v_T^n(s)\|_V^2 \, ds$$

$$= \int_0^T e^{-\lambda_1(T-s)} \|v^n(s)\|_V^2 \, ds + e^{-\lambda_1 T} \int_{-\infty}^0 e^{\lambda_1 r} \|v^n(r)\|_V^2 \, dr$$

we finally arrive at

$$\limsup_{n\to\infty} \|v_T^n\|_{L_V^2}^2 \leq \frac{1}{1-b} e^{-\lambda_1 T} \limsup_{n\to\infty} \int_{-\infty}^0 e^{\lambda_1 r} \|\psi^n(r) - \psi(r)\|_V^2 \, dr.$$

$$\square$$

As a consequence of Lemmas 11 and 14 we obtain the following result.

Corollary 15. *The map G has compact values. The map $\psi \mapsto G(t, \omega, \psi)$ has closed graph in $\mathcal{H}$ and is upper semicontinuous for fixed $\omega \in \Omega$, $t \geq 0$. Hence, G is a strict MNDS.*

The next step is to prove the asymptotic compactness.

Lemma 16. *The MNDS G is pullback $\mathcal{D}$-asymptotically compact.*

Proof. Let $y_n \in G(t_n, \theta_{-t_n}\omega, D(\theta_{-t_n}\omega))$, where $D \in \mathcal{D}$ and $t_n \to +\infty$. Then we have to prove that the sequence y_n is precompact in $\mathcal{H}$. Observe that for $T > 0$,

$$G(t_n, \theta_{-t_n}\omega, D(\theta_{-t_n}\omega)) = G(T, \theta_{-T}\omega, G(t_n - T, \theta_{-t_n}\omega, D(\theta_{-t_n}\omega)))$$

$$= G(T, \theta_{-T}\omega, G(t_n - T, \theta_{-t_n+T}\theta_{-T}\omega, D(\theta_{-t_n+T}\theta_{-T}\omega)))$$

$$\subset G(T, \theta_{-T}\omega, B(\theta_{-T}\omega)),$$

if $t_n - T \geq \bar{t}(D, \theta_{-T}\omega)$, where B is the absorbing ball. Then $y_n \in G(T, \theta_{-T}\omega, \xi_n^T)$, where $\xi_n^T \in B(\theta_{-T}\omega)$. Let u^n be a sequence of solutions with initial condition ξ_n^T and $u_T^n = y_n$. Observe that $y_n(s-T) = \xi_n^T(s)$ for a.a. $s \leq 0$. Also, it is clear that u^n depend on T, but we omit this for simplicity of notation.

Since $B(\theta_{-T}\omega)$ is bounded in $\mathcal{H}$ we can assume (up to a subsequence) that $\xi_n^T \to \xi^T$ weakly in $L^2(-\infty, T; V)$. Also, since $y_n \in B(\omega)$ for n large enough, we have that $y_n \to y$ weakly in $L^2(-\infty, T; V)$, and $y(s-T) = \xi^T(s)$ for a.a. $s \leq 0$. Also, as the sequence $\xi_n^T(0)$ is bounded in H, we can suppose without loss of generality that $\xi_n^T(0) \to \xi^T(0)$ weakly in H.

From the proof of Lemma 14 it follows that u^n converges to some function u in the sense of (34). Also, it is clear from the above convergences that $u(s) = y(s-T)$,

for a.a. $s \leq T$, and then $u_T = y$ in $L^2(-\infty, 0; H)$. Lemma 14 implies, moreover, that

$$(37) \qquad u^n \to u \text{ in } C\left([\tau, T], H\right), \text{ for all } 0 < \tau < T.$$

Hence, if we take $T > h$, then we obtain that $y_n = u_T^n$ converges to $y = u_T$ in C_h, so $u_T = y$ in $\mathcal{H}$.

Finally, we need to prove that, up to a subsequence, $y_n \to y$ strongly in L_V^2. Using (33) and $\xi_n^T, \in B\left(\theta_{-T}\omega\right)$, $\left\|\xi^T\right\|_{L_V^2} \leq \liminf \left\|\xi_n^T\right\|_{L_V^2}$ we have

$$\limsup_{n \to \infty} \|y_n - y\|_{L_V^2} \leq \frac{1}{1-b} e^{-\lambda_1 T} \limsup_{n \to \infty} \int_{-\infty}^0 e^{\lambda_1 s} \left\|\xi_n^T(s) - \xi^T(s)\right\|_V^2 ds$$

$$\leq \frac{4}{1-b} R^2\left(\theta_{-T}\omega\right) e^{-\lambda_1 T},$$

where $R\left(\omega\right)$ is the radius of $B\left(\omega\right)$ given by (32).

Using a diagonal argument one can prove that this inequality is true (passing to a subsequence) for every $T > 0$. Since $B \in \mathcal{D}$, for any $\varepsilon > 0$ there exists T_ε such that

$$\frac{4}{1-b} R^2\left(\theta_{-T}\omega\right) e^{-\lambda_1 T} < \varepsilon \text{ if } T \geq T_\varepsilon.$$

Thus, $\lim_{n \to +\infty} \|y_n - y\|_{L_V^2} = 0$. $\qquad \square$

As a consequence of Lemmas 11, 12, 16, Corollary 15 and Theorem 4 we can ensure the existence of a pullback attractor for our problem.

Theorem 17. *The MNDS generated by (5) posseses a pullback $\mathcal{D}$-attractor $\mathcal{A}$ in $\mathcal{H}$, which is strictly invariant.*

Remark 18. *This theorem is true without conditions (18), (19), (20) and writing assumptions (7), (17), (21), (22) and (23) just for ω fixed (see the non-autonomous case in [5]). In such a case Lemma 14 would be correct for ω fixed. These conditions are needed in the next section in order to prove that the pullback attractor is measurable.*

6. The random attractor

We need to prove now that the MNDS G is in fact an MRDS, and that the pullback $\mathcal{D}$-attractor is a random attractor.

Lemma 19. *The map $(t, \omega, \psi) \to G(t, \omega, \psi)$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(\mathcal{H})$ measurable.*

Proof. It is enough to check that $(t, \omega, \psi) \to G(t, \omega, \psi)$ is upper semicontinuous (see [4, Lemma 2.5]), and this is true if we prove that for any sequence $y^n \in G(t^n, \omega^n, \psi^n)$ such that $t_n \to t$, $\omega^n \to \omega$, $\psi^n \to \psi$, there exists a subsequence y^{n_k} converging in $\mathcal{H}$ to some $y \in G(t, \omega, \psi)$.

Take a sequence of solutions $u^n\left(\cdot\right) = u^n(\cdot, \omega^n, \psi^n)$ satisfying $y^n = u_{t^n}^n$. Let $t < T$. In view of Lemma 14 there exists a subsequence $u^{n_k}\left(\cdot\right)$ such that

$$(38) \qquad \begin{aligned} u^{n_k} &\to u \text{ in } C\left([-h, T]; H\right), \\ u^{n_k} &\to u \text{ in } L^2\left(-\infty, T; V\right), \end{aligned}$$

where $u(\cdot) = u\left(\cdot, \omega, \psi\right)$ is a solution with the initial data ψ.

Therefore,

$$y^n = u_{t^n}^n \to u_t = y \in G(t, \omega, \psi) \text{ in } C_h.$$

It remains to prove that $y^{n_k} \to y$ in L_V^2. Let $t^n = t + \tau^n < T$, where $\tau^n \to 0$. We observe that

$$\|y^{n_k} - y\|_{L_V^2}^2$$

$$= \int_{-\infty}^0 e^{\lambda_1 s} \|u_t^{n_k}(\tau^{n_k} + s) - u_t(s)\|_V^2 \, ds$$

$$\leq 2 \int_{-\infty}^0 e^{\lambda_1 s} \|u_t^{n_k}(\tau^{n_k} + s) - u_t(\tau^{n_k} + s)\|_V^2 \, ds$$

$$+ 2 \int_{-\infty}^0 e^{\lambda_1 s} \|u_t(\tau^{n_k} + s) - u_t(s)\|_V^2 \, ds.$$

By (38) it is straightforward to check that

$$\int_{-\infty}^0 e^{\lambda_1 s} \|u_t^{n_k}(\tau^{n_k} + s) - u_t(\tau^{n_k} + s)\|_V^2 \, ds \to 0.$$

On the other hand, if $v \in L^2(-\infty, T; V)$, then $v_\tau(\cdot) = v(\cdot + \tau) \to v(\cdot)$ in $L^2(-\infty, T - \varepsilon; V)$ as $\tau \to 0$ for any $\varepsilon > 0$, see [12]. Hence, choosing $\varepsilon > 0$ such that $t < T - \varepsilon$ we obtain

$$\int_{-\infty}^0 e^{\lambda_1 s} \|u_t(\tau^{n_k} + s) - u_t(s)\|_V^2 \, ds = \int_{-\infty}^t e^{-\lambda_1(t-s)} \|u(r + \tau^{n_k}) - u(r)\|_V^2 \, ds \to 0,$$

which concludes the proof. $\qquad\square$

We will define now a new pullback $\mathcal{D}$-absorbing family $K(\omega)$ given by

$$K(\omega) = \left\{ \begin{array}{c} y : \exists \omega^n, \ y^n \in G(2h, \theta_{-2h}\omega^n, B(\theta_{-2h}\omega^n)) \text{ such that} \\ \omega^n \to \omega, \ y^n \to y \text{ in } \mathcal{H} \end{array} \right\}.$$

We note that

$$G(2h, \theta_{-2h}\omega, B(\theta_{-2h}\omega)) \subset K(\omega) \text{ for all } \omega \in \Omega,$$

and then for any $\omega \in \Omega$, $D \in \mathcal{D}$, there exists $T(\omega, D)$ such that

$$G(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset G(2h, \theta_{-2h}\omega, G(t - 2h, \theta_{-t}\omega, D(\theta_{-t}\omega)))$$

$$\subset G(2h, \theta_{-2h}\omega, B(\theta_{-2h}\omega)) \subset K(\omega),$$

if $t \geq T$. Therefore, the family $K(\omega)$ is pullback $\mathcal{D}$-absorbing.

Lemma 20. *Assume that the radius $R(\omega)$ of the absorbing family $B(\omega)$ defined in Lemma 12 is locally bounded, that is, for any $\omega_0 \in \Omega$ there exists a neighborhood $\mathcal{U}$ of ω_0 and a constant $C(\omega_0)$ such that*

$$(39) \qquad\qquad R(\omega) \leq C \ \forall \omega \in \mathcal{U}.$$

Then for any fixed $t \geq 0$ the map $\omega \mapsto G(t, \omega, K(\omega))$ has closed values and is a random set with respect to $\bar{\mathcal{F}}^{\mathbb{P}}$.

Proof. We shall prove that $\omega \mapsto G(t, \omega, K(\omega))$ has closed graph. We consider the sequence $y^n \in G(t, \omega^n, K(\omega^n))$, where $\omega^n \to \omega$, $y^n \to y$ in $\mathcal{H}$. Then there are solutions $u^n(\cdot, \omega^n, x^n)$ such that $y^n = u_t^n$ and $x^n \in K(\omega^n)$.

Since $x^n(s) = y^n(s - t)$ for a.a. $s \leq 0$ and $y^n \to y$ in L_V^2, we deduce that $x^n \to x$ in L_V^2, where $x(s) = y(s - t)$ for a.a. $s \leq 0$.

We will check further that, up to a subsequence, $x^n \to x$ in C_h. In view of the definition of the set $K(\omega)$ one can choose $\overline{\omega}^n$, $\overline{x}^n \in G(2h, \theta_{-2h}\overline{\omega}^n, B(\theta_{-2h}\overline{\omega}^n))$ such that

$$
(40) \qquad d_\Omega(\overline{\omega}^n, \omega^n) < \frac{1}{n}, \ \|\overline{x}^n - x^n\|_{\mathcal{H}} < \frac{1}{n}.
$$

There exist $\psi^n \in B(\theta_{-2h}\overline{\omega}^n)$ and solutions $v^n(\cdot) = v^n(\cdot, \overline{\omega}^n, \psi^n)$ such that $\overline{x}^n = v_{2h}^n$. In view of condition (39), ψ^n is bounded in $\mathcal{H}$. Thus, passing to a subsequence we have that $\psi^n \to \psi$ weakly in L_V^2 and $\psi^n(0) \to \psi(0)$ weakly in H. Lemma 14 implies then that $v^n \to v$ in $C([\tau, 2h]; H)$ for any $0 < \tau < 2h$. Hence, $\overline{x}^n = v_{2h}^n \to v_{2h}$ in C_h. This, together with (40), gives

$$
x^n \to v_{2h} = x \text{ in } C_h,
$$

and therefore

$$
x^n \to x \text{ in } \mathcal{H}.
$$

Moreover, from (40) we have $\overline{x}^n \to x$ in $\mathcal{H}$, $\overline{\omega}^n \to \omega$ as well, so by the definition of $K(\omega)$ it is obvious that $x \in K(\omega)$.

Now, using again Lemma 14 we obtain the existence of a solution $u(\cdot) = u(\cdot, \omega, x)$ such that $y = u_t$, thus $y \in G(t, \omega, K(\omega))$, proving that the graph is closed. This fact immediately implies that the map $\omega \mapsto G(t, \omega, K(\omega))$ has closed values and also that $Gr(G(t, \cdot, K(\cdot)) \in \bar{\mathcal{F}}^{\mathbb{P}} \otimes \mathcal{B}(\mathcal{H})$.

Finally, applying well known results (see [8, Chapter III]) it follows that $\omega \mapsto G(t, \omega, K(\omega))$ is a random set with respect to $\bar{\mathcal{F}}^{\mathbb{P}}$. $\qquad \square$

Let us establish now some sufficient conditions ensuring that (39) is fulfilled.

Lemma 21. *Let the functions c_1, c_3, c_6 satisfiy the following assumption: for any $\omega_0 \in \Omega$ there exist a neighborhood $\mathcal{U}_1$ of ω_0, $0 < \delta(\omega_0) < \lambda_1 - e^{\lambda_1 h} \frac{4}{\eta\mu} \mathbb{E}(c_4^2)$ and $C(\omega_0)$, $t_0(\omega_0) > 0$ such that*

$$
(41) \qquad c_i(\theta_t\omega) \le C(\omega_0)e^{\delta(\omega_0)|t|}, \ i = 1, 6,
$$

$$
(42) \qquad c_3^2(\theta_t\omega) \le C(\omega_0)e^{\delta(\omega_0)|t|},
$$

for all $\omega \in \mathcal{U}_1$, $|t| \ge t_0$. Moreover, for any $\omega_0 \in \Omega$ there exists a neighborhood $\mathcal{U}_2$ such that for any $\gamma > 0$ there exists $r_0(\gamma, \omega_0) < 0$ such that

$$
(43) \qquad (-\mathbb{E}(c_4^2) + \gamma)r \le \int_r^0 c_4^2(\theta_\tau\omega)d\tau \le (-\mathbb{E}(c_4^2) - \gamma)r,
$$

for all $\omega \in \mathcal{U}_1$ and $r \le r_0$.
Then (39) holds.

Remark 22. *Assumption (43) means that the ergodic property $\lim_{t \to -\infty} \frac{1}{t} \int_0^t c_4^2(\theta_\tau\omega)d\tau = \mathbb{E}(c_4^2)$ is satisfied uniformly in a neighborhood of each $\omega_0 \in \Omega$.*

Proof. For an arbitrary $\omega_0 \in \Omega$ let $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$. Take $\gamma \in (0, \frac{\lambda_1 \eta\mu}{4e^{\lambda_1 h}} - \mathbb{E}(c_4^2) - \frac{\delta(\omega_0)\eta\mu}{4e^{\lambda_1 h}})$ and denote $\delta = \lambda_1 - e^{\lambda_1 h} \frac{4}{\eta\mu} \mathbb{E}(c_4^2) - e^{\lambda_1 h} \frac{4}{\eta\mu} \gamma > \delta(\omega_0) > 0$. Let $\bar{t}(\omega_0, \gamma) = $

$\max\{t_0(\omega_0), -r_0(\omega_0, \gamma)\}$. Conditions (41)-(43) imply

$$\frac{8e^{\lambda_1 h}}{1-d} \int_{-\infty}^{-\bar{t}} e^{\lambda_1 r + e^{\lambda_1 h} \int_r^0 \frac{4c_4^2(\theta_\tau \omega)}{\eta\mu} d\tau} c(\theta_r \omega) dr$$

$$\leq \frac{8e^{\lambda_1 h}}{1-d} \int_{-\infty}^{-\bar{t}} e^{(\lambda_1 - e^{\lambda_1 h} \frac{4}{\eta\mu} \mathbb{E}(c_4^2) - e^{\lambda_1 h} \frac{4}{\eta\mu} \gamma) r} c(\theta_r \omega) dr$$

$$\leq \frac{8e^{\lambda_1 h}}{1-d} \int_{-\infty}^{-\bar{t}} e^{\delta r} (\eta C_\mu + (3 + \frac{2}{\eta\mu}) C(\omega_0) e^{-\delta(\omega_0) r}) dr \leq D_1(\omega_0),$$

for any $\omega \in \mathcal{U}$. On the other hand, from condition (20) we obtain easily that

$$\frac{8e^{\lambda_1 h}}{1-d} \int_{-\bar{t}}^0 e^{\lambda_1 r + e^{\lambda_1 h} \int_r^0 \frac{4c_4^2(\theta_\tau \omega)}{\eta\mu} d\tau} c(\theta_r \omega) dr \leq D_2(\omega_0),$$

for any $\omega \in \mathcal{U}$. Hence, assumption (39) follows. $\qquad\square$

In the sequel we strengthen conditions (41)-(42) and condition (20) (for the function $c_4(\omega)$) to ensure that the family $K(\omega)$ is tempered.

Lemma 23. *Assume that (43) is satisfied and that for any $\omega_0 \in \Omega$ and $k > 0$ there exist a neighborhood $\mathcal{U}$ of ω_0 and $C(\omega_0, k)$, $\bar{t}(\omega_0, k) > 0$ such that*

$$(44) \qquad\qquad c_i(\theta_t \omega) \leq C(\omega_0, k) e^{k|t|}, \ i = 1, 6,$$

$$(45) \qquad\qquad c_3^2(\theta_t \omega) \leq C(\omega_0, k) e^{k|t|},$$

for all $\omega \in \mathcal{U}$, $|t| \geq \bar{t}$. Also, assume that for every $\omega_0 \in \Omega$, $t_0 \in \mathbb{R}$ there exists a neighbourhood $\mathcal{U}$ of ω_0 and a constant $D(t_0, \omega_0)$ such that

$$(46) \qquad\qquad \int_t^{t+t_0} c_4^2(\theta_s \omega) ds \leq D, \ \ for \ any \ \omega \in \mathcal{U}, \ t \in \mathbb{R}.$$

Then $K \in \mathcal{D}$.

Proof. First, we ought to establish that the absorbing family $B(\omega)$ is locally uniformly tempered. We shall prove that for any ω_0 there exists a neighborhood $\mathcal{U}_0$ of ω_0 such that for any $\delta > 0$,

$$(47) \qquad\qquad \lim_{t \to +\infty} \sup_{\omega \in \mathcal{U}} e^{-\delta t} R^2(\theta_{-t}\omega) = 0.$$

Using assumption (43), the proof is quite similar to that in Lemma 4.6 from Caraballo et al. [6], so we omit it.

Further, we need to prove that for any $\delta > 0$,

$$(48) \qquad\qquad \lim_{t \to +\infty} e^{-\delta t} \sup_{y \in K(\theta_{-t}\omega)} \|y\|_{\mathcal{H}}^2 = 0.$$

For any $y \in K(\theta_{-t}\omega)$ there exist $y^n \in G(2h, \theta_{-2h-t}\omega^n, z^n)$, $z^n \in B(\theta_{-2h-t}\omega^n)$, $\omega^n \in \Omega$, such that $\omega^n \to \omega$, $y^n \to y$ in $\mathcal{H}$. From (26) we have

$$\|y^n\|_{C_h}^2 + \frac{1-d}{2} \int_{-2h}^0 e^{\lambda_1 s} \|y^n(s)\|_V^2 ds$$

$$\leq e^{-\lambda_1 h + e^{\lambda_1 h} \int_{-2h}^0 \frac{4}{\eta\mu} c_4^2(\theta_{s-t}\omega^n) ds} (2 \|z^n(0)\|^2 + d \|z^n\|_{L_V^2}^2)$$

$$+ 2e^{\lambda_1 h} \int_{-2h}^0 e^{\lambda_1 \tau + e^{\lambda_1 h} \int_\tau^0 \frac{4}{\eta\mu} c_4^2(\theta_{s-t}\omega^n) ds} c(\theta_{\tau-t}\omega^n) d\tau.$$

We observe that (46), (44)-(45) imply that for any $k > 0$ there exist a neighborhood $\mathcal{U}_1$ of ω and constants $D(\omega)$, $C(\omega, k)$, $\bar{t}(\omega, k) > 0$ such that

$$e^{-\lambda_1 h + e^{\lambda_1 h} \int_{-2h}^{0} \frac{4}{\eta\mu} c_4^2(\theta_{s-t}\omega^n)ds}$$

$$= e^{-\lambda_1 h + e^{\lambda_1 h} \int_{-2h-t}^{-t} \frac{4}{\eta\mu} c_4^2(\theta_s \omega^n)ds} \leq e^{-\lambda_1 h + e^{\lambda_1 h} \frac{4}{\eta\mu} D(\omega)}$$

and

$$\int_{-2h}^{0} e^{\lambda_1 \tau + e^{\lambda_1 h} \int_{\tau}^{0} \frac{4}{\eta\mu} c_4^2(\theta_{s-t}\omega^n)ds} c(\theta_{\tau-t}\omega^n)d\tau$$

$$= \int_{-2h}^{0} e^{\lambda_1 \tau + e^{\lambda_1 h} \int_{\tau-t}^{-t} \frac{4}{\eta\mu} c_4^2(\theta_s \omega^n)ds} c(\theta_{\tau-t}\omega^n)d\tau$$

$$\leq \int_{-2h}^{0} e^{\lambda_1 \tau + e^{\lambda_1 h} \frac{4}{\eta\mu} D(\omega)} (\eta C_\mu + (3 + \frac{2}{\eta\mu})C(\omega, k)e^{-k\tau}e^{kt})d\tau,$$

whenever ω^n belongs to the neighborhood $\mathcal{U}_1$ and $t \geq \bar{t}$. Therefore, there exist constants $D_1(\omega), D_2(\omega), D_3(\omega, k) > 0$ such that

$$\|y^n\|_{\mathcal{H}}^2 = \|y^n\|_{C_h}^2 + \int_{-2h}^{0} e^{\lambda_1 s} \|y^n(s)\|_V^2 \, ds + \int_{-\infty}^{-2h} e^{\lambda_1 s} \|y^n(s)\|_V^2 \, ds$$

$$= \|y^n\|_{C_h}^2 + \int_{-2h}^{0} e^{\lambda_1 s} \|y^n(s)\|_V^2 \, ds + e^{-2h\lambda_1} \int_{-\infty}^{0} e^{\lambda_1 s} \|z^n(s)\|_V^2 \, ds$$

$$\leq D_1(\omega) \|z^n\|_{\mathcal{H}}^2 + D_2(\omega) + D_3(\omega, k)e^{kt}$$

if $\omega^n \in \mathcal{U}_1$ and $t \geq \bar{t}$. Now, for any $\delta > 0$, choosing $k < \delta$ and using (47) we obtain that for any $\varepsilon > 0$ there exists $T(\varepsilon)$ such that

$$e^{-\delta t} \|y^n\|_{\mathcal{H}}^2 \leq e^{-\delta t}(D_1(\omega)R^2(\theta_{-2h-t}\omega^n) + D_2(\omega)) + D_3(\omega, k)e^{-(\delta-k)t} \leq \varepsilon$$

if $\omega^n \in \mathcal{U}_1 \cap \mathcal{U}_0$ and $t \geq T$.
Passing to the limit as $n \to \infty$

$$e^{-\delta t} \|y\|_{\mathcal{H}}^2 \leq \varepsilon \ \forall t \geq T(\varepsilon), \ y \in K(\theta_{-t}\omega).$$

Then, (48) follows.
The case where $t \to -\infty$ is treated similarly. Hence, $K \in \mathcal{D}$. $\qquad \square$

We are ready to prove that the pullback $\mathcal{D}$-attractor $\mathcal{A}$ defined in Theorem 17 is a random pullback $\mathcal{D}$-attractor.

Theorem 24. *Assume that conditions (43)-(46) are satisfied. Then $\mathcal{A}$ is a random $\mathcal{D}$-attractor.*

Proof. From Lemmas 23, 19 we know that the family of absorbing sets K belongs to $\mathcal{D}$ and that the map $(t, \omega, \psi) \to G(t, \omega, \psi)$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(\mathcal{H})$ measurable. Also, from Lemmas 20, 21 we have that for any fixed $t \geq 0$ the map $\omega \mapsto G(t, \omega, K(\omega))$ has closed values and is a random set with respect to $\bar{\mathcal{F}}^{\mathbb{P}}$.
Hence, Theorem 4 implies that

$$\mathcal{A}(\omega) := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} G(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}$$

and this set is random with respect to $\mathcal{P}^c$. $\qquad \square$

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E-mail address, Tomás Caraballo: `caraball@us.es`
E-mail address, María J. Garrido-Atienza: `mgarrido@us.es`
E-mail address, Björn Schmalfuß: `bjoern.schmalfuss@uni-jena.de`
E-mail address, Jose Valero: `jvalero@umh.es`

(T. Caraballo) Dpto. Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, 41080-Sevilla, Spain

(María J. Garrido-Atienza) Dpto. Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, 41080-Sevilla, Spain

(B. Schmalfuß) Institut für Mathematik, Institut für Stochastik, Ernst Abbe Platz 2, 07737, Jena,Germany

(J. Valero) Centro de Investigación Operativa, Universidad Miguel Hernández, Avda. de la Universidad, s/n, 03202-Elche, Spain