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# ON PTÁK'S GENERALIZATION OF HANKEL OPERATORS

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*Abstract.* In 1997 Pták defined generalized Hankel operators as follows: Given two contractions $T_1 \in \mathcal{B}(\mathscr{H}_1)$ and $T_2 \in \mathcal{B}(\mathscr{H}_2)$, an operator $X\colon \mathscr{H}_1 \to \mathscr{H}_2$ is said to be a generalized Hankel operator if $T_2 X = X T_1^*$ and $X$ satisfies a boundedness condition that depends on the unitary parts of the minimal isometric dilations of $T_1$ and $T_2$. This approach, call it (P), contrasts with a previous one developed by Pták and Vrbová in 1988, call it (PV), based on the existence of a previously defined generalized Toeplitz operator. There seemed to be a strong but somewhat hidden connection between the theories (P) and (PV) and we clarify that connection by proving that (P) is more general than (PV), even strictly more general for some $T_1$ and $T_2$, and by studying when they coincide. Then we characterize the existence of Hankel operators, Hankel symbols and analytic Hankel symbols, solving in this way some open problems proposed by Pták.

*Keywords*: Toeplitz operators, Hankel operators, minimal isometric dilation

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## 1. Introduction

What objects should be called generalized Toeplitz or Hankel operators? Toeplitz and Hankel operators on the Hardy space $H^2$ are two of the most important and widely studied classes of operators on Hilbert spaces (see [6], [9], [10], [19], [21], [31], [32], [35], [43] or [46]). These classes are intimately related from its very definition: Every function $\varphi \in L^\infty(\mathbb{T})$ defines a Toeplitz operator $T_\varphi$ and a Hankel operator $H_\varphi$ by

$$\begin{aligned} T_\varphi\colon H^2 &\to H^2 \\ f &\to P_+(\varphi \cdot f) \end{aligned} \qquad \text{and} \qquad \begin{aligned} H_\varphi\colon H^2 &\to H^2_- \\ f &\to P_-(\varphi \cdot f) \end{aligned}$$

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where $P_+$ is the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2$ and $P_- = 1 - P_+$ is the orthogonal projection onto $H^2_-$. In the language of dilation theory, $T_\varphi$ and $H_\varphi$ are compressions of the multiplication (or Laurent) operator $M_\varphi$ induced by $\varphi$. This function $\varphi$ is called the symbol of $T_\varphi$ (it is unique) and a symbol of $H_\varphi$ (it is not unique; the kernel of the application $\varphi \to H_\varphi$ is $H^\infty$, whose functions are called analytic symbols).

Toeplitz and Hankel operators satisfy simple relations that characterize these classes. The characteristic relation for Hankel operators was given by Nehari in 1957 [31] and it tells us that a bounded linear map $X\colon H^2 \to H^2_-$ is a Hankel operator if and only if $XS = S_-^* X$ where $S$ is the unilateral forward shift in $H^2$ defined by $(Sf)(\zeta) := \zeta \cdot f(\zeta)$ and $S_-$ is the unilateral forward shift in $H^2_-$ defined by $(S_- f)(\zeta) := \overline{\zeta} \cdot f(\zeta)$. The characteristic relation for Toeplitz operators was given by Brown and Halmos in 1963 [10] and it tells us that a bounded linear map $X\colon H^2 \to H^2$ is a Toeplitz operator if and only if $X = S^* X S$.

It is no wonder that a number of possible generalizations of the notion of a Toeplitz or Hankel operator have been proposed in literature. To start with, one may substitute $L^2(\mathbb{T})$ by some complex $L^2(\mu)$ and $H^2$ by a suitable subspace of analytic functions (without the aim of being exhaustive, we refer the reader to, e.g., [2], [3], [4], [7], [13], [14], [25], [44], [45] or [46] and references therein). In this approach, the role of symbol is played by functions from $L^\infty(\mu)$. More generally, one may start with an algebra $A$ of continuous functions on a compact and Hausdorff topological space, and with a probability measure $\nu$ representing a character of $A$. Then $L^2(\nu)$ substitutes for $L^2(\mathbb{T})$ and the closure of $A$ in $L^2(\nu)$ substitutes for $H^2$. This leads to a generalization that preserves the harmonic flavour of the duality between $\mathbb{T}$ and $\mathbb{Z}$ (see [16] or [30] and the references in the latter paper). One more way: treat Hankel and Toeplitz operators as bilinear forms and generalize them by considering bilinear forms having a similar behaviour (as in [1], [12] or [29] where one can also find further references).

In this paper we want to concentrate on generalizations made by exploiting the properties of the characteristic relations. Namely, let $T_1$ and $T_2$ be two contractions defined, respectively, on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ and call $X\colon \mathscr{H}_1 \to \mathscr{H}_2$ a generalized Toeplitz operator if $X = T_2 X T_1^*$ and a generalized Hankel operator if $T_2 X = X T_1^*$. The purpose of this way of extending the classical notions of Toeplitz and Hankel operators is two-fold: on the one hand, to obtain new operators that share some of the properties that make these classes important and, on the other hand, to find out which of these interesting properties of the classical case depend only on the characteristic relations and not on the machinery of complex function theory that sometimes is used to prove them. For Toeplitz operators, this line of research was started, to the best of authors' knowledge, by Douglas [18] and Sz.-Nagy and Foiaş [41], [42]. In 1988 Pták and Vrbová [37] undertook the problem of obtaining an abstract analogue for Hankel operators, something that had been previously treated in the particular case when $T_1$ and $T_2$ are backward shifts of arbitrary multiplicity by Rosenblum [39], Page [33], Bonsall and Power [8] and Peller [34].

In this approach, an essential problem—already identified by Douglas, Sz.-Nagy and Foiaş in the Toeplitz case—is to find out what operators play the role of symbols. In the classical case, the symbols are multiplication operators induced by functions from $L^\infty(\mathbb{T})$ or, from another point of view, operators that commute with the unitary bilateral shift $V$ defined in $L^2(\mathbb{T})$ by $(Vf)(\zeta) := \zeta \cdot f(\zeta)$. The fact that $V$ and $V^*$ are the minimal isometric dilations of the backward shifts $S_-^*$ and $S^*$, respectively, suggests, as is the case, that operators intertwinning the minimal isometric dilations of $T_1$ and $T_2$ and the structure of these minimal isometric dilations (including lifting theorems) must be of the greatest importance for the problem at hand. Pták and Vrbová started by considering common symbols for Toeplitz and Hankel operators. Their research revealed a surprising fact: a certain boundedness condition must be satisfied by $X$ in order to ensure that the Hankel type intertwinning relation $T_2X = XT_1^*$ can be lifted. This boundedness condition, that will be described later, is trivially satisfied in the classical case and, more generally, also when $T_1$ and $T_2$ are backward shifts, so that one of the main virtues of Pták and Vrbová's paper [37] is to uncover its true meaning in the general approach. More recently, Pták [36] has proposed that it is more natural to consider different, rather than common, sets of symbols for Toeplitz and Hankel operators. These two approaches, let us call them (PV) and (P), yield exactly the same class of generalized Toeplitz operators but not the same class of generalized Hankel operators and Prof. Pták encouraged us to analyze and clarify the relationship between the theories (PV) and (P). The purpose of this paper is to show that, indeed, (P) is more natural and strictly more general than (PV), to study when these theories provide the same classes of generalized Hankel operators and to extend to the (P)-framework a number of results given in the (PV)-framework that depend on the notions of the Hankel operator or the analytic symbol. This research is continued in our paper [26] where we study finite-rank and compact generalized Hankel operators. We also refer the interested reader to our papers [27] and [28], which contains of a number of remarks, examples and results about invertibility and spectral properties of generalized Toeplitz operators.

Prof. V. Pták, who introduced us to this interesting topic and encouraged us to continue his line of research, died one year after this paper was completed. During almost fifteen years we had very close contact with him and his research group. His comments helped us to search in the right directions and to clarify some obscure points. We will miss very much the continuous flow of lively and interesting discussions that we had with him. We take this opportunity to thank, for their hospitality during our visits, to other members of the Mathematical Institute of the Academy of Sciences of the Czech Republic; notably M. Fiedler, V. Müller and, in *illo tempore*,

J. Fuka and P. Vrbová. We also thank V. Vasyunin (Steklov Mathematical Institute, St. Petersburg) for his helpful remarks and simplifications of some of our original proofs.

## 2. Terminology and Notation

Our terminology and notation will be mostly standard, e.g., given two Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, we denote by $\mathcal{B}(\mathscr{H},\mathscr{K})$ the set of all operators (:= bounded linear mappings) from $\mathscr{H}$ into $\mathscr{K}$ and we simply write $\mathcal{B}(\mathscr{H})$ if $\mathscr{K} = \mathscr{H}$. The range $\overline{X(\mathscr{H})}$ of an operator $X \in \mathcal{B}(\mathscr{H},\mathscr{K})$ will be denoted by $\operatorname{ran}(X)$ and its kernel by $\ker(X)$. However, and although we refer the reader to the excellent books by Böttcher and Silbermann [9], Douglas [19], Halmos [20], [21], Nikolskii [32], Young [43] and Sz.-Nagy and Foiaş [40], let us fix some of the (maybe not so standard) notation that will be used in the sequel.

We have already fixed $S$, $S_-$ and $V$ for the usual forward shifts in $H^2$, $H^2_-$ and $L^2(\mathbb{T})$. Let $T$ be a contraction defined on a Hilbert space $\mathscr{H}$. We denote by $U$ the minimal isometric dilation of $T$; this isometry $U$ is defined on a Hilbert space $\mathscr{K}$ and we will denote by $P(\mathscr{H})$ the orthogonal projection from $\mathscr{K}$ onto $\mathscr{H}$, by $\mathscr{H}^\perp$ the orthogonal complement of $\mathscr{H}$ in $\mathscr{K}$, and by $\mathscr{R}$ the unitary part of the Wold decomposition of $U$ (please note that if we had followed strictly the notation of [40], then we would have written $\mathscr{K}_+$ and $U_+$ instead of $\mathscr{K}$ and $U$). In what follows, we will mostly deal with two contractions $T_1$ and $T_2$ defined on respective Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, so we will write $\mathscr{K}_1$, $\mathscr{K}_2$, and so on; in particular, a very useful operator will be the co-isometry $\widetilde{T}_2 := (U_2|\mathscr{H}_2^\perp)^*$.

## 3. Theories (PV) and (P): Description and Comparison

We start by describing in some detail the approaches that we have called (PV) and (P); the proofs can be found in the corresponding papers [37] and [36]. The reader must be aware of the following fact: In [37] the operators are defined from $\mathscr{H}_2$ into $\mathscr{H}_1$ (from $\mathscr{K}_2$ into $\mathscr{K}_1$, etc.). To make the comparison between (PV) and (P) possible, we have rewritten the definitions and results of [37] interchanging the subindices 2 and 1.

**Theory (PV).** An operator $Y\colon \mathscr{K}_1 \to \mathscr{K}_2$ is said to be a symbol (with respect to $T_1$ and $T_2$) if $Y = U_2 Y U_1^*$. The set of all symbols is denoted by $\mathcal{S}_{\mathrm{PV}}(T_1, T_2)$. These symbols are operators defined essentially from $\mathscr{R}_1$ into $\mathscr{R}_2$ because every symbol $Y$ satisfies $Y = P(\mathscr{R}_2) Y P(\mathscr{R}_1)$. This is the reason why these spaces $\mathscr{R}_1$ and $\mathscr{R}_2$ are so important in the generalized theory of Toeplitz and Hankel operators.

This importance is hidden in the classical case because if $T_1 = T_2 = S^*$, then $\mathscr{R}_1 = \mathscr{K}_1 = \mathscr{R}_2 = \mathscr{K}_2 = L^2(\mathbb{T})$. Note that there are no non-zero symbols if either $\mathscr{R}_1$ or $\mathscr{R}_2$ is $\{0\}$—for instance if either $T_1$ or $T_2$ is a shift operator. Every symbol $Y$ defines two operators $T_Y \in \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ and $H_Y \in \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2^{\perp})$ by

$$T_Y := P(\mathscr{H}_2)Y\big|\mathscr{H}_1 \qquad \text{and} \qquad H_Y := P(\mathscr{H}_2^{\perp})Y\big|\mathscr{H}_1.$$

These operators satisfy the relations

$$T_Y = T_2 T_Y T_1^* \qquad \text{and} \qquad H_Y T_1^* = (U_2\big|\mathscr{H}_2^{\perp})^* H_Y.$$

The operator $H_Y$ also satisfies the following boundedness condition: there exists $c > 0$ such that for all $h_1 \in \mathscr{H}_1$ and all $h_2 \in \mathscr{H}_2^{\perp}$ the following inequality holds:

$$|\langle H_Y h_1, h_2\rangle| \leqslant c\|P(\mathscr{R}_1)h_1\|\,\|P(\mathscr{R}_2)h_2\|.$$

The minimum of the constants $c$ appearing above will be denoted by $\|H_Y\|_{\mathscr{R}}$. Operators satisfying this condition were called $\mathscr{R}$-bounded (with respect to $T_1$ and $T_2$) by Pták and Vrbová. This condition, as we will make precise immediately, plays an essential role in the searching of a symbol by lifting a Hankel-type intertwinning relation.

An operator $X \in \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ is said to be a Toeplitz operator (with respect to $T_1$ and $T_2$) if $X = T_2 X T_1^*$. The fundamental lifting theorem for Toeplitz operators (proved by different approaches in [18], [41], [42], [37] and [36]) says that they have a unique symbol: *An operator $X\colon \mathscr{H}_1 \to \mathscr{H}_2$ is a Toeplitz operator if and only if there is a unique symbol $Y$ such that $X = T_Y$, in which case $\|X\| = \|Y\|$.*

An operator $X \in \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2^{\perp})$ is said to be a Hankel operator (with respect to $T_1$ and $T_2$) if $X$ is $\mathscr{R}$-bounded and $(U_2\big|\mathscr{H}_2^{\perp})^* X = X T_1^*$. The fundamental lifting theorem for Hankel operators [37] says that they have (non-unique) symbols: *An operator $X\colon \mathscr{H}_1 \to \mathscr{H}_2^{\perp}$ is a Hankel operator if and only if there is a symbol $Y$ such that $X = H_Y$, in which case a symbol $Y$ can be found such that $\|X\|_{\mathscr{R}} = \|Y\|$.* Pták and Vrbová provided examples of non-$\mathscr{R}$-bounded operators satisfying the intertwinning relation $(U_2\big|\mathscr{H}_2^{\perp})^* X = X T_1^*$.

The classes of Toeplitz and Hankel operators defined in this (PV)-framework will be denoted, respectively, by $\mathcal{T}_{\mathrm{PV}}(T_1, T_2)$ and $\mathcal{H}_{\mathrm{PV}}(T_1, T_2)$.

**Theory (P).** Now Pták proposed to consider different, as the opposite to common, sets of symbols for Toeplitz and Hankel operators. The (P)-approach to Toeplitz operators is simply the same as (PV): the (P)-symbols for Toeplitz operators—now called Toeplitz symbols—are the same as the (PV)-symbols and the (P)-Toeplitz operators are defined exactly as the (PV)-Toeplitz operators; shortly,

$$\mathcal{T}_{\mathrm{PV}}(T_1, T_2) = \mathcal{T}_{\mathrm{P}}(T_1, T_2) \qquad \text{and} \qquad \mathcal{S}_{\mathrm{PV}}(T_1, T_2) = \mathcal{T}\mathcal{S}_{\mathrm{P}}(T_1, T_2).$$

For Hankel operators, the stress is put on the intertwinning relation by itself rather than in attaching them to a previously defined Toeplitz operator, as it was in the (PV)-approach. An operator $Z\colon \mathscr{K}_1 \to \mathscr{K}_2$ is said to be a Hankel symbol (with respect to $T_1$ and $T_2$) if $U_2 Z = Z U_1^*$. The set of all Hankel symbols is denoted by $\mathcal{HS}_{\mathrm{P}}(T_1, T_2)$. Hankel symbols are also essentially in $\mathcal{B}(\mathscr{R}_1, \mathscr{R}_2)$ because every Hankel symbol $Z$ satisfies $Z = P(\mathscr{R}_2) Z P(\mathscr{R}_1)$.

Every Hankel symbol $Z$ defines an operator $H_Z\colon \mathscr{H}_1 \to \mathscr{H}_2$ by $H_Z \coloneqq P(\mathscr{H}_2) Z\big|\mathscr{H}_1$. This operator $H_Z$ satisfies the intertwinning relation $T_2 H_Z = H_Z T_1^*$ and also a boundedness condition, which we will call (PV)-boundedness (with respect to $T_1$ and $T_2$), which is similar but not equivalent to $\mathscr{R}$-boundedness: there exists $c > 0$ such that for all $h_1 \in \mathscr{H}_1$ and all $h_2 \in \mathscr{H}_2$ the following inequality holds:

$$|\langle H_Z h_1, h_2 \rangle| \leqslant c \|P(\mathscr{R}_1) h_1\| \, \|P(\mathscr{R}_2) h_2\|.$$

The minimum of the constants $c$ appearing above will be denoted by $\|H_Z\|_{\mathrm{PV}}$; note that $\|Z\| \geqslant \|H_Z\|_{\mathrm{PV}}$. The reason why (PV)-boundedness is not the same as $\mathscr{R}$-boundedness is the following: $\mathscr{R}$-boundedness is defined for operators from $\mathscr{H}_1$ into $\mathscr{H}_2^{\perp}$, which is a space defined *a posteriori*, after considering $T_2$ and $\mathscr{K}_2$, whereas (PV)-boundedness is defined for operators from $\mathscr{H}_1$ into $\mathscr{H}_2$, which is a space given *a priori*. Moreover, since $\mathscr{R}_2$ is the unitary part of the minimal isometric dilation of the contraction $T_2$, the roles of $\mathscr{H}_2$ and $\mathscr{H}_2^{\perp}$ are not interchangeable with respect to these objects.

We say that an operator $X\colon \mathscr{H}_1 \to \mathscr{H}_2$ is a Hankel operator if $X$ is (PV)-bounded and $T_2 X = X T_1^*$. The key lifting theorem (given in [38] and [36]) says that Hankel operators have Hankel symbols: *An operator $X\colon \mathscr{H}_1 \to \mathscr{H}_2$ is a Hankel operator if and only if there is a Hankel symbol $Z$ such that $X = H_Z$, in which case a symbol $Z$ can be found such that $\|Z\| = \|X\|_{\mathrm{PV}}$.* The class of (P)-Hankel operators will be denoted by $\mathcal{H}_{\mathrm{P}}(T_1, T_2)$. Pták provided examples of non-(PV)-bounded operators satisfying the intertwinning relation $T_2 X = X T_1^*$.

The generalization proposed by Rosenblum [39] and Page [33] fits into this scheme: Let $T$ be the backward shift operator defined on a vector-valued Hardy space $\mathscr{H} = H^2(\mathscr{M})$ by the relation $(Tf)(\zeta) = \zeta^{-1}\big(f(\zeta) - f(0)\big)$. Then $U$ is the (unitary) bilateral shift defined on $\mathscr{K} = \mathscr{R} = L^2(\mathscr{M})$ by $(Uf)(\zeta) = \zeta^{-1} f(\zeta)$, hence (PV)-boundedness holds trivially and, therefore, the class of Hankel operators $\mathcal{H}_{\mathrm{P}}(T, T)$ coincides with the class defined by Rosenblum of all operators $X\colon H^2(\mathscr{M}) \to H^2(\mathscr{M})$ such that $TX = XT^*$. The set of Hankel symbols is then the set of all operators $Z\colon L^2(\mathscr{M}) \to L^2(\mathscr{M})$ such that $UZ = ZU^*$; if $\mathscr{M}$ is separable then this set can be identified with $L^\infty(\mathcal{B}(\mathscr{M}))$ via $Z \leftrightarrow JM_\varphi$ where $M_\varphi$ is the multiplication operator induced by a function $\varphi \in L^\infty(\mathcal{B}(\mathscr{M}))$ and $J$ is the unitary operator defined on $L^2(\mathscr{M})$ by $(Jf)(\zeta) = f(\zeta^{-1})$.

**Comparison between (PV) and (P).** We have already pointed out that the theories (PV) and (P) are the same for Toeplitz operators but not for Hankel operators; well, at least not formally. Theory (P) is apparently more general because the range space $\mathscr{H}_2$ of a (P)-Hankel operator is free, in the sense that it is not the orthogonal of some previously fixed space in the superspace where the minimal isometric dilation of $T_2$ lives. If we compare the definitions of a Hankel operator in both theories:

$$\begin{aligned} \text{(PV):}\quad & X\colon \mathscr{H}_1 \to \mathscr{H}_2^{\perp} \text{ is } \mathscr{R}\text{-bounded and } (U_2|\mathscr{H}_2^{\perp})^* X = XT_1^*, \\ \text{(P):}\quad & X\colon \mathscr{H}_1 \to \mathscr{H}_2 \text{ is (PV)-bounded and } T_2 X = XT_1^*, \end{aligned}$$

we can see that the connection between them is given by the co-isometry $\widetilde{T}_2$ defined by $\widetilde{T}_2 := (U_2|\mathscr{H}_2^{\perp})^*$ because every (PV)-Hankel operator satisfies the (P)-type intertwinning relation $\widetilde{T}_2 X = XT_1^*$. Then, what is the relation between the $\mathscr{R}$-boundedness with respect to $T_1$ and $T_2$ and the (PV)-boundedness with respect to $T_1$ and $\widetilde{T}_2$? We prove now that the former implies the latter.

**Theorem 1.** *Let $X$ be a* (PV)*-Hankel operator with respect to $T_1$ and $T_2$. Then $X$ is a* (P)*-Hankel operator with respect to $T_1$ and $\widetilde{T}_2$. In other words,*

$$\mathcal{H}_{\mathrm{PV}}(T_1, T_2) \subset \mathcal{H}_{\mathrm{P}}(T_1, \widetilde{T}_2).$$

This theorem tells us that for Hankel operators theory (P) is more general than theory (PV) because every (PV)-Hankel operator is also a (P)-Hankel operator. The converse is not true as we will see in Example 3 below.

P r o o f. $X$ is an operator in $\mathcal{B}(\mathscr{H}_1, \mathscr{H}_2^{\perp})$ and, as we know already, $X$ satisfies $\widetilde{T}_2 X = XT_1^*$, so we have to check that $X$ is (PV)-bounded with respect to $T_1$ and $\widetilde{T}_2$. Since $X$ is $\mathscr{R}$-bounded with respect to $T_1$ and $T_2$, it follows that there is a constant $c > 0$ such that

$$|\langle Xh_1 h_2\rangle| \leqslant c\|P(\mathscr{R}_1)h_1\|\,\|P(\mathscr{R}_2)h_2\| \qquad \text{for all } h_1 \in \mathscr{H}_1 \text{ and all } h_2 \in \mathscr{H}_2^{\perp}.$$

Now, $\widetilde{T}_2$ is a co-isometry defined on $\mathscr{H}_2^{\perp}$. Its minimal isometric dilation is a unitary operator $\widetilde{U}_2$ defined on some space $\widetilde{\mathscr{K}_2}$ that, therefore, coincides with the corresponding $\widetilde{\mathscr{R}}_2$ so that $\widetilde{\mathscr{R}}_2 \supset \mathscr{H}_2^{\perp}$. This implies that for all $h_1 \in \mathscr{H}_1$ and $h_2 \in \mathscr{H}_2^{\perp}$ we have

$$\begin{aligned} |\langle Xh_1, h_2\rangle| \leqslant c\|P(\mathscr{R}_1)h_1\|\,\|P(\mathscr{R}_2)h_2\| &\leqslant c\|P(\mathscr{R}_1)h_1\|\,\|h_2\| \\ &= c\|P(\mathscr{R}_1)h_1\|\,\|P(\widetilde{\mathscr{R}}_2)h_2\|, \end{aligned}$$

and this tells us that $X$ is (PV)-bounded with respect to $T_1$ and $\widetilde{T}_2$. This concludes the proof that $X \in \mathcal{H}_{\mathrm{P}}(T_1, \widetilde{T}_2)$. □

**Coincidence of (PV) and (P) for Hankel operators.** If $T_1 = T_2 = S^*$ on $H^2$ then $\widetilde{T}_2 = S_-^*$ and both $\mathcal{H}_{\mathrm{PV}}(T_1, T_2)$ and $\mathcal{H}_{\mathrm{P}}(T_1, \widetilde{T}_2)$ equal the set of all classical Hankel operators, and both $\mathcal{S}_{\mathrm{PV}}(T_1, T_2)$ and $\mathcal{HS}_{\mathrm{P}}(T_1, \widetilde{T}_2)$ equal the set of all multiplication operators induced by functions from $L^\infty(\mathbb{T})$. In order to figure out what assumptions would imply that $\mathcal{H}_{\mathrm{PV}}(T_1, T_2) = \mathcal{H}_{\mathrm{P}}(T_1, \widetilde{T}_2)$, let us make some heuristics about the corresponding symbols.

Every (PV)-symbol $Y \in \mathcal{S}_{\mathrm{PV}}(T_1, T_2)$ verifies $Y = U_2 Y U_1^*$. Every (P)-Hankel symbol $Z \in \mathcal{HS}_{\mathrm{P}}(T_1, \widetilde{T}_2)$ verifies $\widetilde{U}_2 Z = Z U_1^*$ where $\widetilde{U}_2$ is the minimal isometric dilation of $\widetilde{T}_2$. This dilation $\widetilde{U}_2$ is a unitary operator because $\widetilde{T}_2$ is a co-isometry, hence $Z = (\widetilde{U}_2)^* Z U_1^*$. In order to have that every (PV)-symbol $Y$ is a (P)-Hankel symbol, the relations $Y = U_2 Y U_1^*$ and $Y = (\widetilde{U}_2)^* Y U_1^*$ should be essentially the same. Now, on the one hand, $(\widetilde{U}_2)^*$ is a unitary operator defined on a space $\widetilde{\mathscr{K}_2}$ that contains $\mathscr{H}_2^\perp$. On the other hand, $U_2$ is unitary on $\mathscr{R}_2$, so both situations fit into the same frame if we require $\mathscr{H}_2^\perp$ to be contained in $\mathscr{R}_2$.

As we will prove, this condition $\mathscr{H}_2^\perp \subset \mathscr{R}_2$ is exactly what we need, but in its present form it is not adequate for our purposes. As it turned out after a number of fruitful discussions with V. Pták and V. Vasyunin, this condition can be reformulated in various different ways and leads to some unexpected facts. In particular, it implies that $T_2$ is a power partial isometry, i.e., that $T_2^n$ is a partial isometry for every $n \in \mathbb{N}$. Halmos and Wallen [22] proved that every power partial isometry is a direct sum whose summands are unitary operators, pure isometries, pure co-isometries and truncated shifts (or finite Jordan blocks in the language of matrices). As we are about to show in Lemma 1, due to V. Vasyunin, the inclusion $\mathscr{H}_2^\perp \subset \mathscr{R}_2$ holds if and only if $T_2$ is a power partial isometry without truncated shifts or, in other words, with no nilpotent part. Then we will give an adequate description of $\widetilde{U}_2$ and $\widetilde{\mathscr{K}_2}$ under the assumption $\mathscr{H}_2^\perp \subset \mathscr{R}_2$. Power partial isometries form a nice class of operators with some interesting applications. For more information about them we refer the reader to the papers [5], [11], [15] and [24].

**Lemma 1.** *$\mathscr{H}^\perp \subset \mathscr{R}$ if and only if $T$ can be decomposed as $T = S_1 \oplus S_2^* \oplus V_3$, where $S_1$ and $S_2$ are forward shift operators and $V_3$ is unitary.*

Proof. The "if" implication is easy. To start with the "only if" implication, take $V_3$ as the unitary part of $T$ in the Langer-Wold decomposition for contractions [40, Theorem 3.2]. It remains to prove that if $T$ is completely non-unitary and $\mathscr{H}^\perp \subset \mathscr{R}$ then $T$ can be decomposed as $T_2 = S_1 \oplus S_2^*$. To see this, use the Sz.-Nagy and Foiaş's function model for the minimal isometric dilation of a completely non-unitary contraction [40], Chapter VI]: There exists two Hilbert spaces $\mathscr{D}$ and $\mathscr{D}_*$ (the defect spaces of $T$ and $T^*$), a contraction-valued analytic function $\Theta$ (the characteristic function of $T$) and a positive operator-valued function $\Delta$ ($\Delta = (I -$

$\Theta^*\Theta)^{1/2})$ such that

$$\begin{aligned}\mathscr{K} &= H^2(\mathscr{D}_*) \oplus \overline{\Delta L^2(\mathscr{D})},\\ \mathscr{H} &= \big[H^2(\mathscr{D}_*) \oplus \overline{\Delta L^2(\mathscr{D})}\big] \ominus \big[\Theta \oplus \Delta\big] H^2(\mathscr{D}),\\ \mathscr{H}^\perp &= \Theta H^2(\mathscr{D}) \oplus \Delta H^2(\mathscr{D}),\\ \mathscr{R} &= \{0\} \oplus \overline{\Delta L^2(\mathscr{D})}.\end{aligned}$$

Therefore, $\mathscr{H}^\perp \subset \mathscr{R}$ if and only if $\Theta = 0$. It follows that this happens if and only if $T$ decomposes as the direct sum of a forward unilateral shift operator plus a backward unilateral shift operator because if $\Theta = 0$, then $\Delta = I$, hence $\mathscr{H} = H^2(\mathscr{D}_*) \oplus H^2_-(\mathscr{D})$ and $T(f \oplus g) = zf \oplus \frac{f - f(0)}{z}$. In other words, $T$ it is the orthogonal sum of the forward shift of multiplicity $\dim(\mathscr{D}_*)$ and the backward shift of multiplicity $\dim(\mathscr{D})$. $\square$

**Lemma 2.** *Let $T \in \mathcal{B}(\mathscr{H})$ be a contraction such that $\mathscr{H}^\perp \subset \mathscr{R}$. Let $\mathscr{H}_{\mathrm{n}} \subset \mathscr{H}$ be the reducing subspace where the completely non-unitary part of the Langer-Wold decomposition of $T$ is defined. Then the minimal isometric dilation of $\widetilde{T} := (U\big|\mathscr{H}^\perp)^*$ is the unitary operator $U^*\big|\widetilde{\mathscr{K}}$, where $\widetilde{\mathscr{K}} = (\mathscr{H}_{\mathrm{n}} \cap \mathscr{R}) \oplus \mathscr{H}^\perp$.*

P r o o f. Use Lemma 1 to decompose $\mathscr{H}$ into three $T$-reducing subspaces $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_3$ such that $S_1 := T\big|\mathscr{H}_1$ is a forward shift operator, $S_2^* := T\big|\mathscr{H}_2$ is a backward shift operator and $V_3 := T\big|\mathscr{H}_3$ is unitary. It is clear that the minimal isometric dilation of $T$ can be written as

$$U = S_1 \oplus V_2^* \oplus V_3$$

where $V_2$ is a bilateral forward shift defined on some space $\mathscr{K}_2 \supset \mathscr{H}_2$. Hence we can write $\mathscr{K} = \mathscr{H}_1 \oplus \mathscr{K}_2 \oplus \mathscr{H}_3$ and, consequently,

$$\begin{aligned}\mathscr{R} &= \{0\} \oplus \mathscr{K}_2 \oplus \mathscr{H}_3,\\ \mathscr{H}^\perp &= \{0\} \oplus \mathscr{H}_2^\perp \oplus \{0\},\\ \mathscr{H}_{\mathrm{n}} &= \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \{0\}.\end{aligned}$$

Therefore, the co-isometry $\widetilde{T} = (U\big|\mathscr{H}^\perp)^*$ can be identified with $(V_2^*\big|\mathscr{H}_2^\perp)^*$ and this is a backward shift operator having, as is easy to check, the same multiplicity as $S_2^*$. It follows that the minimal isometric dilation of $\widetilde{T}$ is unitarily equivalent to $V_2$, so we may write

$$\widetilde{\mathscr{K}} = \{0\} \oplus \mathscr{K}_2 \oplus \{0\} \qquad \text{and} \qquad \widetilde{U} = 0 \oplus V_2 \oplus 0 = U^*\big|\widetilde{\mathscr{K}}.$$

Finally, note that

$$(\mathscr{R} \cap \mathscr{H}_{\mathrm{n}}) \oplus \mathscr{H}^\perp = \big[\{0\} \oplus \mathscr{H}_2 \oplus \{0\}\big] \oplus \big[\{0\} \oplus \mathscr{H}_2^\perp \oplus \{0\}\big] = \widetilde{\mathscr{K}}.$$

The fact that $\widetilde{\mathscr{K}}$ is $U^*$-reducing follows from the construction made. $\square$

**Theorem 2.** *If $\mathscr{H}_2^{\perp} \subset \mathscr{R}_2$ then*

$$\mathcal{S}_{\mathrm{PV}}(T_1,T_2) \supset \mathcal{HS}_{\mathrm{P}}(T_1,\widetilde{T}_2) \qquad \text{and} \qquad \mathcal{H}_{\mathrm{PV}}(T_1,T_2) = \mathcal{H}_{\mathrm{P}}(T_1,\widetilde{T}_2).$$

*Let $T_{2\mathrm{u}}$ be the unitary part of the Langer-Wold decomposition of $T_2$. Then the equality $\mathcal{S}_{\mathrm{PV}}(T_1,T_2) = \mathcal{HS}_{\mathrm{P}}(T_1,\widetilde{T}_2)$ holds if and only if either $\mathscr{R}_1 = \{0\}$, or $T_2$ is completely non-unitary, or the unitary operators $T_{2\mathrm{u}}$ and $U_1|\mathscr{R}_1$ are relatively singular.*

Proof. Let $Z \in \mathcal{HS}_{\mathrm{P}}(T_1,\widetilde{T}_2)$ be a (P)-Hankel symbol. Then Lemma 2 above tells us that $\widetilde{U}_2 = U_2^*|\widetilde{\mathscr{K}_2}$, where $\widetilde{\mathscr{K}_2} = (\mathscr{H}_{2\mathrm{n}} \cap \mathscr{R}_2) \oplus \mathscr{H}_2^{\perp}$, so that the intertwinning relation that $Z$ verifies is $(U_2^*|\widetilde{\mathscr{K}_2})Z = ZU_1^*$. Since we have $Z = P(\widetilde{\mathscr{K}_2})Z$, the intertwinning relation can be written as $U_2^*Z = ZU_1^*$. Note that $\widetilde{\mathscr{K}_2} \subset \mathscr{R}_2$, hence $Z = P(\widetilde{\mathscr{K}_2})Z = P(\mathscr{R}_2)Z$ takes its values in $\mathscr{R}_2 \subset \mathscr{K}_2$. Next, by using that $U_2^*$ is unitary in its reducing subspace $\mathscr{R}_2$, it follows that $Z = U_2ZU_1^*$. This proves that $Z \in \mathcal{S}_{\mathrm{PV}}(T_1,T_2)$.

Theorem 1 says that $\mathcal{H}_{\mathrm{PV}}(T_1,T_2) \subset \mathcal{H}_{\mathrm{P}}(T_1,\widetilde{T}_2)$. Take now $X \in \mathcal{H}_{\mathrm{P}}(T_1,\widetilde{T}_2)$ and let $Z \in \mathcal{HS}_{\mathrm{P}}(T_1,\widetilde{T}_2)$ be a (P)-Hankel symbol for $X$ so that $X = P(\mathscr{H}_2^{\perp})Z|\mathscr{H}_1$. As we have just proved, $Z \in \mathcal{S}_{\mathrm{PV}}(T_1,T_2)$. Therefore, $X \in \mathcal{H}_{\mathrm{PV}}(T_1,T_2)$.

To see when the equality $\mathcal{S}_{\mathrm{PV}}(T_1,T_2) = \mathcal{HS}_{\mathrm{P}}(T_1,\widetilde{T}_2)$ holds, we will use the decomposition given in Lemma 1 but written in matrix terms for the sake of convenience: There are three $T_2$-reducing subspaces $\mathscr{H}_{21}$, $\mathscr{H}_{22}$ and $\mathscr{H}_{23}$ such that $\mathscr{H}_2 = \mathscr{H}_{21} \oplus \mathscr{H}_{22} \oplus \mathscr{H}_{23}$ and, with respect to this decomposition, $T_2$ can be written as

$$T_2 = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_2^* & 0 \\ 0 & 0 & V_3 \end{pmatrix}$$

where $S_i \in \mathcal{B}(\mathscr{H}_{2i})$ $(i = 1,2)$ are forward shift operators and $V_3 \in \mathcal{B}(\mathscr{H}_{23}) = T_{2\mathrm{u}}$ is unitary. From the proof of Lemma 2 we know that

$$U_2 = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & V_2^* & 0 \\ 0 & 0 & V_3 \end{pmatrix} \qquad \text{and} \qquad \widetilde{U}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the bilateral backward shift operator $V_2^* \in \mathcal{B}(\mathscr{K}_{22})$ is the minimal isometric dilation of $S_2^*$. Now $Z \in \mathcal{HS}_{\mathrm{P}}(T_1,\widetilde{T}_2)$ if and only if $Z \in \mathcal{B}(\mathscr{K}_1,\widetilde{\mathscr{K}_2})$ and $\widetilde{U}_2Z = ZU_1^*$ or, equivalently, $Z = (\widetilde{U}_2)^*ZU_1^*$. In matrix terms, this means that $\mathcal{HS}_{\mathrm{P}}(T_1,\widetilde{T}_2)$ can be identified with the set

$$\left\{ Z\colon \mathscr{K}_1 \to \mathscr{K}_2\colon Z = \begin{pmatrix} 0 \\ Z_2 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ Z_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & V_2^* & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ Z_2 \\ 0 \end{pmatrix} U_1^* \right\}.$$

Analogously, $\mathcal{S}_{\mathrm{PV}}(T_1, T_2)$ can be described as

$$\left\{ Y\colon \mathscr{K}_1 \to \mathscr{K}_2 \colon Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \text{ and } \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & V_2^* & 0 \\ 0 & 0 & V_3 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} U_1^* \right\}.$$

Consequently, $\mathcal{HS}_{\mathrm{P}}(T_1, \widetilde{T}_2)$ equals $\mathcal{S}_{\mathrm{PV}}(T_1, T_2)$ if and only if both $Y_1$ and $Y_3$ are zero. Now, $Y_1 = P(\mathscr{H}_{21})Y$ is always zero because $Y_1 = S_1 Y_1 U_1^*$ so that $Y_1 \in \mathcal{S}_{\mathrm{PV}}(T_1, S_1)$, and this set reduces to $\{0\}$ because $S_1$ is a shift operator. On the other hand, $Y_3 = P(\mathscr{H}_{23})Y$ verifies $Y_3 = V_3 Y_3 U_1^*$, hence $Y_3 \in \mathcal{S}_{\mathrm{PV}}(T_1, V_3)$ and, according to [37, 4.2], this set reduces to $\{0\}$ if and only if either $\mathscr{H}_{23} = \mathscr{H}_{2\mathrm{u}}$ reduces to $\{0\}$, or $\mathscr{R}_1 = \{0\}$, or the unitary operators $U_1\big|\mathscr{R}_1$ and $V_3 = T_{2\mathrm{u}}$ are relatively singular. $\square$

**Example 1.** If $T_2$ is a backward shift operator on, say, $\mathscr{H}_2 = H^2(\mathscr{H})$ then $U_2$ is a unitary operator, a bilateral shift, on $\mathscr{K}_2 = \mathscr{R}_2 = L^2(\mathscr{H})$ and this space contains $\mathscr{H}_2^{\perp} = H^2_-(\mathscr{H})$. Then we have

$$\mathcal{HS}_{\mathrm{P}}(T_1, \widetilde{T}_2) = \mathcal{S}_{\mathrm{PV}}(T_1, T_2) \qquad \text{and} \qquad \mathcal{H}_{\mathrm{PV}}(T_1, T_2) = \mathcal{H}_{\mathrm{P}}(T_1, \widetilde{T}_2)$$

so that both theories coincide. If, moreover, we take $T_1 = T_2$ then (PV)-boundedness and $\mathscr{R}$-boundedness are both the same as the usual boundedness. This is what happens in the classical case: one takes $T_1 = T_2 = S^*$, hence $\widetilde{T}_2 = S_-^*$ and the equality $XS = S_-^* X$ characterizes Hankel operators in both (PV) and (P).

**Example 2.** Take $T_1 = S^*$ and $T_2 = V^*$ so that $T_2$ is unitary and $\mathscr{H}_2^{\perp} = \{0\}$. Then

$$Y \in \mathcal{S}_{\mathrm{PV}}(S^*, V^*) \Longleftrightarrow Y = V^* Y V \Longleftrightarrow YV = VY.$$

Therefore, the set of symbols $\mathcal{S}_{\mathrm{PV}}(S^*, V^*)$ coincides with the set of all multiplication operators induced by functions in $L^\infty(\mathbb{T})$. We also have $\widetilde{\mathscr{K}_2} = (\mathscr{H}_{2_{\mathrm{n}}} \cap \mathscr{R}_2) \oplus \mathscr{H}_2^{\perp} = \{0\}$, so that $\mathcal{HS}_{\mathrm{P}}(T_1, \widetilde{T}_2) = \{0\}$: this proves that the inclusion given in Theorem 2 can be strict. Note also that both classes of the (PV)- and the (P)-Hankel operators are trivial, although the class of the (PV)-symbols is not trivial; we will see in the next section that this cannot happen within the theory (P).

**Example 3.** Let $T_2$ be the zero operator in $\mathbb{C}$. It is well-known that $U_2 = S$ in $H^2$ so that $\mathscr{R}_2 = \{0\}$, $\mathcal{S}_{\mathrm{PV}}(T_1, T_2) = \{0\}$ and, consequently, $\mathscr{H}_2^{\perp} \not\subset \mathscr{R}_2$. On the other hand, $\widetilde{T}_2 = S^*$ in $H^2$, so that $\widetilde{U}_2 = V^*$ in $L^2$. Therefore, $Y \in \mathcal{HS}_{\mathrm{P}}(T_1, \widetilde{T}_2)$ if and only if $V^* Y = Y U_1^*$. Now, if we take the co-isometry $T_1 = S_-^*$ then $U_1 = V$ and we have that $Y \in \mathcal{HS}_{\mathrm{P}}(T_1, \widetilde{T}_2)$ if and only if $Y$ commutes with $V$. This tells us that $\mathcal{HS}_{\mathrm{P}}(T_1, \widetilde{T}_2)$ coincides with the set of all multiplication operators induced by functions in $L^\infty(\mathbb{T})$ and that $\mathcal{H}_{\mathrm{P}}(T_1, \widetilde{T}_2)$ is the set of all bounded operators $X\colon H^2_- \to H^2$ such that $S^* X = X S_-$, i.e., the adjoints of the classical Hankel operators. This example shows that the theory (P) can be strictly more general that the theory (PV).

**Remarks.** The theorems and examples above tell us that the theory (P) is more general than the theory (PV) and can be strictly more general. Even when both theories provide the same class of Hankel operators, under the hypothesis $\mathscr{H}_2^{\perp} \subset \mathscr{R}_2$, the set of symbols for the (P)-approach is smaller than the set of (PV)-symbols, so the former gives a more precise description.

The paper [37] contains a number of results that, as the notion of an analytic symbol, depend on the definition of a Hankel operator. In the next section we will show that most of these results hold also for the (P)-approach and that, in some sense, we obtain sharper conclusions; e.g., we cannot have, as it may happen in the (PV)-approach, that all non-zero Hankel symbols produce the zero operator as the unique Hankel operator.

Since, from now on, we will work only within the (P)-framework, we will suppress the corresponding subindex "P".

## 4. Analytic Symbols

In the classical case, the set of analytic symbols is $H^\infty$ and it can be viewed as the set of symbols such that the associated Toeplitz operator commutes with $S$ or also as the set of symbols such that the associated Hankel operator is zero. In our case we have to deal with two different sets of analytic symbols, namely, analytic Toeplitz symbols and analytic Hankel symbols.

**Definition.** A Toeplitz symbol $Y$ is said to be analytic if $P(\mathscr{H}_2^{\perp})Y\big|\mathscr{H}_1 = 0$; the set of all analytic Toeplitz symbols is denoted by $\mathcal{ATS}(T_1, T_2)$. Thus a Toeplitz symbol $Y$ is analytic when the Hankel operator $H_Y$ associated to $Y$ in the (PV)-approach is zero. A Toeplitz operator is said to be analytic if its symbol is analytic; the set of all analytic Toeplitz operators will be denoted by $\mathcal{AT}(T_1, T_2)$. The existence and a number of properties of analytic Toeplitz operators was already studied in [37].

A Hankel symbol $Z$ is said to be analytic if its Hankel operator $H_Z = P(\mathscr{H}_2)Z\big|\mathscr{H}_1$ is zero, and the set of all analytic Hankel symbols is denoted by $\mathcal{AHS}(T_1, T_2)$. Thus $\mathcal{AHS}(T_1, T_2)$ is the kernel of the linear mapping $Z \in \mathcal{HS}(T_1, T_2) \to H_Z \in \mathcal{H}(T_1, T_2)$.

We start by giving the (P)-version of Nehari's Theorem. It follows easily from the fundamental lifting theorem for (P)-Hankel operators quoted in the description of the theory (P) and from the fact that $\|Z\| \geqslant \|H_Z\|_{\mathrm{PV}}$.

**Theorem 3.** *Let $\mathscr{M}_1 \subset \mathscr{H}_1$ be a closed subspace such that*

$$\mathscr{H}_0 := \operatorname{lin}\{T_1^{*n}(\mathscr{M}_1);\ \ n = 0, 1, 2, \ldots\}$$

*is dense in $\mathscr{H}_1$. Let $X\colon \mathscr{H}_1 \to \mathscr{H}_2$ be an operator satisfying $T_2 X h_0 = X T_1^* h_0$ for all $h_0 \in \mathscr{H}_0$. Then the following assertions are equivalent:*

(a) $X$ *is (PV)-bounded.*

(b) $X$ *is a Hankel operator.*

*If $X$ satisfies* (a) *or* (b) *and $Z \in \mathcal{HS}(T_1,T_2)$ is a Hankel symbol for $X$ then we have $\|X\|_{\mathrm{PV}} = \operatorname{dist}(Z, \mathcal{AHS}(T_1,T_2))$ and this infimum is attained.*

We will now study when there exist non-trivial Hankel symbols, analytic Hankel symbols and Hankel operators. The existence of non-zero Hankel symbols is easily established.

**Theorem 4.** *$\mathcal{HS}(T_1,T_2)$ reduces to $\{0\}$ if and only if either $\mathscr{R}_1$ or $\mathscr{R}_2$ reduces to $\{0\}$, or the unitary operators $U_1^*\big|\mathscr{R}_1$ and $U_2\big|\mathscr{R}_2$ are relatively singular.*

P r o o f. If either $\mathscr{R}_1$ or $\mathscr{R}_2$ is trivial then we already know that $\mathcal{HS}(T_1,T_2)=\{0\}$. Assume, then, that neither $\mathscr{R}_1$ nor $\mathscr{R}_2$ reduces to $\{0\}$. In this case, given $Z \in \mathcal{HS}(T_1,T_2)$ we have $U_2Z = ZU_1^*$ and $Z = P(\mathscr{R}_2)ZP(\mathscr{R}_1)$. Therefore, the relation $U_2Z = ZU_1^*$ can be written as $(U_2\big|\mathscr{R}_2)(Z\big|\mathscr{R}_1) = (Z\big|\mathscr{R}_1)(U_1^*\big|\mathscr{R}_1)$ so that, by [18], $\mathcal{HS}(T_1,T_2) = \{0\}$ if and only if $U_1^*\big|\mathscr{R}_1$ and $U_2\big|\mathscr{R}_2$ are relatively singular. □

**Notation.** In what follows, we will make an intensive use of a co-isometry, denoted by $W$, associated to a contraction $T$. The co-isometry $W$ is defined on the space $\mathscr{P} = \overline{P(\mathscr{R})\mathscr{H}}$ by $W := (U^*\big|\mathscr{P})^*$. Note that if $T$ is itself a co-isometry then $\mathscr{P} = \mathscr{H}$ and $W = T$. There is a number of properties of this co-isometry that will be used and that the reader may find proved in [37] and [36]; in particular, the minimal isometric dilation of $W$ is the unitary operator $U\big|\mathscr{R}$ (see [37, 1.1], [36, 1.1 and 2.3] or [42]).

Pták's idea of using this co-isometry is essential in the two-step proofs given in [36]: first for the case when $T_1$ and $T_2$ are co-isometries and then for the general case by using the corresponding $W_1$ and $W_2$. For Hankel operators and Hankel symbols, the connection is given in the following lemma. Although some of the items in its statement are proved and used in [37, 1.4] and [36, 2.5], they never appear explicitly. For this reason, we find it convenient to give a complete proof.

**Lemma 3.** *For every Hankel operator $X \in \mathcal{H}(T_1,T_2)$ there is an operator $X_W \in \mathcal{B}(\mathscr{P}_1,\mathscr{P}_2)$ possessing the following properties:*

(1) *$Xh_1 = P(\mathscr{H}_2)X_WP(\mathscr{R}_1)h_1$ for every $h_1 \in \mathscr{H}_1$, and $X$ has finite rank if and only if $X_W$ has finite rank.*

(2) *$X_W = 0$ if and only if $X = 0$; in other words, the operator $X_W$ is the only one verifying the relation given above.*

(3) *$X_W$ is a Hankel operator with respect to $W_1$ and $W_2$.*

(4) *If $Y \in \mathcal{HS}(W_1,W_2)$ is a Hankel symbol for $X_W$ then $YP(\mathscr{R}_1)$ is a Hankel symbol for $X$.*

(5) *If* $Z \in \mathcal{HS}(T_1, T_2)$ *is a Hankel symbol for* $X$ *then the restriction* $Z\big|\mathscr{R}_1$ *is a Hankel symbol for* $X_W$.

(6) *The correspondences given in* (4) *and* (5) *map* $\mathcal{AHS}(T_1, T_2)$ *onto* $\mathcal{AHS}(W_1, W_2)$.

Proof. (1) Every $X \in \mathcal{H}(T_1, T_2)$ is (PV)-bounded, hence there exists $c > 0$ such that $|\langle Xh_1, h_2\rangle| \leqslant c\|P(\mathscr{R}_1)h_1\|\,\|P(\mathscr{R}_2)h_2\|$ for all $h_1 \in \mathscr{H}_1$ and $h_2 \in \mathscr{H}_2$. Take, on the one hand, $h_1 \in \mathscr{H}_1$. Then

$$\|Xh_1\|^2 = |\langle Xh_1, Xh_1\rangle| \leqslant c\|P(\mathscr{R}_1)h_1\|\,\|P(\mathscr{R}_2)Xh_1\| \leqslant c\|P(\mathscr{R}_1)h_1\|\,\|Xh_1\|$$

and, therefore, $\|Xh_1\| \leqslant c\|P(\mathscr{R}_1)h_1\|$ for all $h_1 \in \mathscr{H}_1$. The well-known Douglas' lemma [17] says that there is an operator $A\colon \mathscr{P}_1 \to \mathscr{H}_2$ such that $\|A\| \leqslant c$ and $AP(\mathscr{R}_1)\big|\mathscr{H}_1 = X$. Since $P(\mathscr{R}_1)\big|\mathscr{H}_1$ is dense in $\mathscr{P}_1$, it follows that $A$ has finite rank if and only if $X$ has finite rank. On the other hand, for every $h_2 \in \mathscr{H}_2$ we have

$$\begin{aligned}\|A^*h_2\| &= \sup\{|\langle P(\mathscr{R}_1)h_1, A^*h_2\rangle|\colon \|P(\mathscr{R}_1)h_1\| \leqslant 1\}\\ &= \sup\{|\langle AP(\mathscr{R}_1)h_1, h_2\rangle|\colon \|P(\mathscr{R}_1)h_1\| \leqslant 1\}\\ &= \sup\{|\langle Xh_1, h_2\rangle|\colon \|P(\mathscr{R}_1)h_1\| \leqslant 1\} \leqslant c\|P(\mathscr{R}_2)h_2\|.\end{aligned}$$

Consequently, $\|A^*h_2\| \leqslant c\|P(\mathscr{R}_2)h_2\|$ and, again by Douglas' lemma, it follows that there exists an operator $B\colon \mathscr{P}_2 \to \mathscr{P}_1$ such that $A^* = BP(\mathscr{R}_2)\big|\mathscr{H}_2$ and $\|B\| \leqslant c$. Again, $B$ has finite rank if $A^*$ has finite rank. If we define $X_W := B^*$ then $P(\mathscr{R}_2)X_W = X_W$ and we obtain

$$X = AP(\mathscr{R}_1)\big|\mathscr{H}_1 = (P(\mathscr{R}_2)\big|\mathscr{H}_2)^*X_WP(\mathscr{R}_1)\big|\mathscr{H}_1 = P(\mathscr{H}_2)X_WP(\mathscr{R}_1)\big|\mathscr{H}_1.$$

This proves (1).

(2) Assume that there is $X_0 \in \mathcal{B}(\mathscr{P}_1, \mathscr{P}_2)$ such that $P(\mathscr{H}_2)X_0P(\mathscr{R}_1)\big|\mathscr{H}_1 = 0$. Then $P(\mathscr{H}_2)X_0 = 0$ because $P(\mathscr{R}_1)\mathscr{H}_1$ is dense in $\mathscr{P}_1$. Since the range of $X_0$ is already contained in $\mathscr{P}_2$ it follows that $\operatorname{ran}(X_0) \subset \mathscr{P}_2 \cap \mathscr{H}_2^\perp$. Now, bearing in mind that $\mathscr{R}_2 = \mathscr{P}_2 \oplus (\mathscr{H}_2^\perp \cap \mathscr{R}_2)$ [36, 2.3], we have that $\mathscr{P}_2 \cap \mathscr{H}_2^\perp = \{0\}$ and, therefore, that $X_0 = 0$.

(3) Recall that the minimal isometric dilations of $W_1$ and $W_2$ are, respectively, $U_1\big|\mathscr{R}_1$ and $U_2\big|\mathscr{R}_2$. Using the fact that $X_W \in \mathcal{B}(\mathscr{P}_1, \mathscr{P}_2)$ verifies (1) and that $(U_1^*\big|\mathscr{P}_1)P(\mathscr{R}_1)\big|\mathscr{H}_1 = P(\mathscr{R}_1)T_1^*$ (see [36, 2.4]) we have

$$\begin{aligned}&(P(\mathscr{R}_2)\big|\mathscr{H}_2)^*X_W(U_1^*\big|\mathscr{P}_1)P(\mathscr{R}_1)\big|\mathscr{H}_1\\ &\qquad= (P(\mathscr{R}_2)\big|\mathscr{H}_2)^*X_WP(\mathscr{R}_1)T_1^*\\ &\qquad= P(\mathscr{H}_2)X_WP(\mathscr{R}_1)T_1^* && (\text{because } \operatorname{ran}(X_W) \subset \mathscr{R}_2)\\ &\qquad= XT_1^* = T_2X = T_2P(\mathscr{H}_2)X_WP(\mathscr{R}_1)\big|\mathscr{H}_1\\ &\qquad= T_2(P(\mathscr{R}_2)\big|\mathscr{H}_2)^*X_WP(\mathscr{R}_1)\big|\mathscr{H}_1 && (\text{because } \operatorname{ran}(X_W) \subset \mathscr{R}_2)\\ &\qquad= (P(\mathscr{R}_2)\big|\mathscr{H}_2)^*(U_2^*\big|\mathscr{P}_2)^*X_WP(\mathscr{R}_1)\big|\mathscr{H}_1 && (\text{by } [36, 2.4],\ \text{again}).\end{aligned}$$

This proves, by continuity, that on $\mathscr{P}_1$ we have the equality

$$(P(\mathscr{R}_2)\big|\mathscr{H}_2)^* X_W (U_1^*\big|\mathscr{P}_1) = (P(\mathscr{R}_2)\big|\mathscr{H}_2)^* (U_2^*\big|\mathscr{P}_2)^* X_W.$$

Now note that $(P(\mathscr{R}_2)\big|\mathscr{H}_2)^*$ is injective in $\mathscr{P}_2$ because

$$\ker\big[(P(\mathscr{R}_2)\big|\mathscr{H}_2)^*\big] = \mathscr{P}_2 \ominus \big[\operatorname{ran}(P(\mathscr{R}_2)\big|\mathscr{H}_2)\big] = \{0\},$$

therefore, $X_W(U_1^*\big|\mathscr{P}_1) = (U_2^*\big|\mathscr{P}_2)^* X_W$. This proves that $X_W$ verifies the characteristic Hankel intertwinning relation. Since $W_1$ and $W_2$ are co-isometries, it follows automatically that $X_W$ is (PV)-bounded with respect to $W_1$ and $W_2$. Hence $X_W \in \mathcal{H}(W_1, W_2)$.

(4) Let $Y \in \mathcal{HS}(W_1, W_2)$ be a Hankel symbol for $X_W$, that means $Y \in \mathcal{B}(\mathscr{R}_1, \mathscr{R}_2)$ verifies $(U_2\big|\mathscr{R}_2)Y = Y(U_1\big|\mathscr{R}_1)^*$ and $P(\mathscr{P}_2)Y\big|\mathscr{P}_1 = X_W$. Take $Z := YP(\mathscr{R}_1)$, then

$$P(\mathscr{H}_2)Z\big|\mathscr{H}_1 = P(\mathscr{H}_2)YP(\mathscr{R}_1)\big|\mathscr{H}_1 = P(\mathscr{H}_2)P(\mathscr{P}_2)YP(\mathscr{R}_1)\big|\mathscr{H}_1$$

because $P(\mathscr{H}_2)P(\mathscr{P}_2) = P(\mathscr{H}_2)P(\mathscr{R}_2)$ [36, 2.3]. Therefore,

$$P(\mathscr{H}_2)Z\big|\mathscr{H}_1 = P(\mathscr{H}_2)X_W P(\mathscr{R}_1)\big|\mathscr{H}_1 = X.$$

Moreover,

$$U_2 Z = U_2 Y P(\mathscr{R}_1) = (U_2\big|\mathscr{R}_2)YP(\mathscr{R}_1) = Y(U_1^*\big|\mathscr{R}_1)P(\mathscr{R}_1) = YP(\mathscr{R}_1)U_1^* = ZU_1^*$$

so that $Z$ is a Hankel symbol for $X$.

(5) Let $Z \in \mathcal{HS}(T_1, T_2)$ be a Hankel symbol for $X$. Then, as we have already pointed out, $Z = P(\mathscr{R}_2)ZP(\mathscr{R}_1)$. This implies that the intertwinning relation $U_2 Z = ZU_1^*$ can be written as $(U_2\big|\mathscr{R}_2)(Z\big|\mathscr{R}_1) = (Z\big|\mathscr{R}_1)(U_1\big|\mathscr{R}_1)^*$ and it follows that $Y := Z\big|\mathscr{R}_1$ is in $\mathcal{HS}(W_1, W_2)$. Let $H_Y = P(\mathscr{P}_2)Y\big|\mathscr{P}_1$ be the Hankel operator associated to $Y$. Using again the relation $P(\mathscr{H}_2)P(\mathscr{P}_2) = P(\mathscr{H}_2)P(\mathscr{R}_2)$ [36, 2.3], we obtain that $H_Y$ verifies

$$\begin{aligned} P(\mathscr{H}_2)H_Y P(\mathscr{R}_1)\big|\mathscr{H}_1 &= P(\mathscr{H}_2)P(\mathscr{P}_2)YP(\mathscr{R}_1)\big|\mathscr{H}_1 \\ &= P(\mathscr{H}_2)P(\mathscr{R}_2)ZP(\mathscr{R}_1)\big|\mathscr{H}_1 = P(\mathscr{H}_2)Z\big|\mathscr{H}_1 = X. \end{aligned}$$

Therefore, the uniqueness proved in (2) tells us that $H_Y = X_W$.

(6) Follows from (4), (5) and (2). □

**Lemma 4.** *If $Z \in \mathcal{AHS}(T_1, T_2)$ is an analytic Hankel symbol then $X := Z|\mathscr{H}_1$ is an analytic Toeplitz operator with respect to $T_1$ and $\widetilde{T}_2$.*

Proof. Since $Z$ is analytic we have that $P(\mathscr{H}_2)Z|\mathscr{H}_1 = 0$, hence $X = P(\mathscr{H}_2^\perp)Z|\mathscr{H}_1$ so that $X \in \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2^\perp)$. Let us prove first that $X \in \mathcal{T}(T_1, \widetilde{T}_2)$, i.e., $X = \widetilde{T}_2 X T_1^*$. Since, by the well-known properties of the minimal isometric dilation [40], for $i = 1, 2$ the space $\mathscr{H}_i$ is $U_i^*$-invariant and $T_i^* = U_i^*|\mathscr{H}_i$, we have $P(\mathscr{H}_i^\perp)U_i^* P(\mathscr{H}_i) = 0$ and this implies

$$\begin{aligned}
\widetilde{T}_2 X T_1^* &= (U_2|\mathscr{H}_2^\perp)^*(P(\mathscr{H}_2^\perp)Z|\mathscr{H}_1)(U_1^*|\mathscr{H}_1) = (U_2|\mathscr{H}_2^\perp)^* P(\mathscr{H}_2^\perp)ZU_1^*|\mathscr{H}_1 \\
&= (U_2|\mathscr{H}_2^\perp)^* P(\mathscr{H}_2^\perp)U_2 Z|\mathscr{H}_1 = P(\mathscr{H}_2^\perp)U_2^* P(\mathscr{H}_2^\perp)U_2 Z|\mathscr{H}_1 \\
&= P(\mathscr{H}_2^\perp)U_2^*\big[P(\mathscr{H}_2^\perp) + P(\mathscr{H}_2)\big]U_2 Z|\mathscr{H}_1 = P(\mathscr{H}_2^\perp)U_2^* U_2 Z|\mathscr{H}_1 \\
&= P(\mathscr{H}_2^\perp)Z|\mathscr{H}_1 = X,
\end{aligned}$$

as was required. Now, to prove that $X$ is analytic it suffices to show, according to [37, 4.3], that $(\widetilde{T}_2)^* X = X T_1^*$. Since $Z$ is analytic, we have $P(\mathscr{H}_2)Z|\mathscr{H}_1 = 0$ and, therefore,

$$(\widetilde{T}_2)^* X = (U_2|\mathscr{H}_2^\perp)X = U_2 P(\mathscr{H}_2^\perp)Z|\mathscr{H}_1 = U_2 Z|\mathscr{H}_1.$$

Using this and the relation $U_2 Z = Z U_1^*$, we have

$$(\widetilde{T}_2)^* X = U_2 Z|\mathscr{H}_1 = Z U_1^*|\mathscr{H}_1 = P(\mathscr{H}_2^\perp)Z U_1^*|\mathscr{H}_1 = X T_1^*,$$

which completes the proof. □

**Theorem 5.** *Let $\mathscr{N}_1$ and $\mathscr{N}_2$ be, respectively, the wandering subspaces of the shift parts of the Wold decompositions of $U_1^*|\mathscr{P}_1$ and $U_2^*|\mathscr{P}_2$. Then $\mathcal{AHS}(T_1, T_2) \neq \{0\}$ if and only if neither $\mathscr{N}_1$ nor $\mathscr{N}_2$ reduces to $\{0\}$.*

Proof. We will make the proof in two steps: we assume first that both $T_1$ and $T_2$ are co-isometries and this will be used afterwards to complete the proof in the general case.

So assume that $T_1$ and $T_2$ are co-isometries. Then $\mathscr{R}_i = \mathscr{K}_i$, $\mathscr{P}_i = \mathscr{H}_i$ for $i = 1, 2$. Note also that, in this case, $U_i^*|\mathscr{P}_i = T_i^*$ for $i = 1, 2$. We start by proving that $\mathcal{AHS}(T_1, T_2) = \{0\}$ if and only if $\mathcal{ATS}(T_1, \widetilde{T}_2) = \{0\}$ where, as before, $\widetilde{T}_2 = (U_2|\mathscr{H}_2^\perp)^*$.

($\Leftarrow$) Take $Z \in \mathcal{AHS}(T_1, T_2)$, then Lemma 4 tells us that $X := P(\mathscr{H}_2^\perp)Z|\mathscr{H}_1 = Z|\mathscr{H}_1$ is in $\mathcal{AT}(T_1, \widetilde{T}_2)$. Assume that $\mathcal{ATS}(T_1, \widetilde{T}_2) = \{0\}$. Then $\mathcal{AT}(T_1, \widetilde{T}_2) = \{0\}$, because the correspondence between the Toeplitz operators and the Toeplitz symbols is 1-1, and hence $Z\mathscr{H}_1 = 0$. Use that $U_1$ is an isometry to write $Z = U_2 Z U_1$ and, inductively, $Z = U_2^n Z U_1^n$ for all $n \in \mathbb{N}$. Then $Z(\mathscr{H}_1) = U_2^n Z U_1^n(\mathscr{H}_1) = \{0\}$ and, consequently, $ZU_1^n(\mathscr{H}_1) = \{0\}$ for all $n \in \mathbb{N}$. Now, by using that $\{U_1^n \mathscr{H}_1 \colon n = 0, 1, 2, \ldots\}$ is dense in $\mathscr{K}_1$, we deduce that $Z\mathscr{K}_1 = \{0\}$; that is, $Z = 0$.

($\Rightarrow$) Since $T_2$ is a co-isometry we have that $\mathscr{H}_2^\perp \subset \mathscr{R}_2$. Now we may use this condition to prove, by using an argument similar to the one that was made to prove the first part of Theorem 2, that $\mathcal{TS}(T_1, \widetilde{T}_2) \subset \mathcal{HS}(T_1, T_2)$. Let us see that this inclusion implies the inclusion $\mathcal{ATS}(T_1, \widetilde{T}_2) \subset \mathcal{AHS}(T_1, T_2)$: Every $Y \in \mathcal{ATS}(T_1, \widetilde{T}_2)$ is, on the one hand, in $\mathcal{TS}(T_1, \widetilde{T}_2)$ so that $Y$ is also in $\mathcal{HS}(T_1, T_2)$; on the other hand, such $Y$ verifies $Y\mathscr{H}_1 \subset \mathscr{H}_2^\perp$ because $\mathscr{H}_2^\perp$ is the space where $\widetilde{T}_2$ is defined; therefore, $P(\mathscr{H}_2)Y\big|H_1 = 0$. This proves that $Y \in \mathcal{AHS}(T_1, T_2)$ and we may conclude that if $\mathcal{AHS}(T_1, T_2)$ reduces to $\{0\}$ then so does $\mathcal{ATS}(T_1, \widetilde{T}_2)$.

This proves that $\mathcal{AHS}(T_1, T_2) = \{0\}$ if and only if $\mathcal{ATS}(T_1, \widetilde{T}_2) = \{0\}$. Since, again by the well-known properties of the minimal isometric dilation, $(\widetilde{T}_2)^* = U_2\big|\mathscr{H}_2^\perp$ is completely non-unitary, we may use Pták and Vrbová's characterization of the existence of non-trivial analytic Toeplitz symbols [37, 4.6] to deduce that $\mathcal{ATS}(T_1, \widetilde{T}_2)$ does not reduce to $\{0\}$ if and only if both $\mathscr{N}_1$ and $\widetilde{\mathscr{N}_2}$ are non-trivial, where $\mathscr{N}_1$ is as defined in the statement and $\widetilde{\mathscr{N}_2}$ is the wandering subspace of the shift part of the Wold decomposition of $(\widetilde{U}_2)^*\big|(\mathscr{H}_2^\perp \cap \widetilde{\mathscr{R}}_2)$. Since $\widetilde{T}_2$ is a co-isometry, we already know that $\widetilde{\mathscr{R}}_2 = \widetilde{\mathscr{K}_2}$, hence $(\mathscr{H}_2^\perp \cap \widetilde{\mathscr{R}}_2) = \mathscr{H}_2^\perp$. On the other hand, we also know that $(\widetilde{T}_2)^* = (\widetilde{U}_2)^*\big|\mathscr{H}_2^\perp$ by the well-known properties of the minimal isometric dilation. It follows from these facts that $\widetilde{\mathscr{N}_2}$ is, indeed, the wandering subspace of the shift part of the Wold decomposition of $(\widetilde{T}_2)^* = U_2\big|\mathscr{H}_2^\perp$ or, in the notation of [42], that $\widetilde{\mathscr{N}_2} = \mathscr{L}_2 = \overline{(U_2 - T_2)\mathscr{H}_2}$. It is also well-known that $\mathscr{L}_2$ is a Hilbert space of the same dimension as $\mathscr{N}_2$, the wandering subspace of the shift part of the Wold decomposition of $T_2^*$, hence $\dim(\widetilde{\mathscr{N}_2}) = \dim(\mathscr{N}_2)$. Summarizing: $\mathcal{AHS}(T_1, T_2)$ does not reduce to $\{0\}$ if and only if both $\mathscr{N}_1$ and $\mathscr{N}_2$ are non-trivial, and this completes the proof in the case when $T_1$ and $T_2$ are co-isometries.

Assume now that $T_1$ and $T_2$ are arbitrary contractions and consider the co-isometries $W_1 := (U_1^*\big|\mathscr{P}_1)^*$ and $W_2 := (U_2^*\big|\mathscr{P}_2)^*$. Item (6) of Lemma 3 tells us that

$$\{Z\big|\mathscr{R}_1 \colon Z \in \mathcal{AHS}(T_1, T_2)\} = \mathcal{AHS}(W_1, W_2).$$

Therefore, $\mathcal{AHS}(T_1, T_2)$ is non-trivial if and only if $\mathcal{AHS}(W_1, W_2)$ is non-trivial. According to what we have already proved, we know that it is non-trivial if and only if $\mathscr{N}_1$ and $\mathscr{N}_2$ do not reduce to $\{0\}$, where $\mathscr{N}_i$ is the wandering subspace of the shift part of the Wold decomposition of $W_i^* = U_i^*\big|\mathscr{P}_i$ for $i = 1, 2$. □

**Theorem 6.** $\mathcal{H}(T_1, T_2) = \{0\}$ *if and only if* $\mathcal{HS}(T_1, T_2) = \{0\}$.

P r o o f. Clearly, $\mathcal{HS}(T_1, T_2) = \{0\}$ implies $\mathcal{H}(T_1, T_2) = \{0\}$. Assume, then, that $\mathcal{HS}(T_1, T_2)$ is not trivial. Theorem 4 tells us that neither $\mathscr{R}_1$ nor $\mathscr{R}_2$ is trivial and that the unitary operators $U_1^*\big|\mathscr{R}_1$ and $U_2\big|\mathscr{R}_2$ are not relatively singular. We have to see that these conditions imply that $\mathcal{H}(T_1,T_2)$ is non-trivial. By using Lemma 3, it is clear that we may restrict ourselves to the case when $T_1$ and $T_2$ are non-zero co-isometries. We will distinguish two subcases.

(1) Assume that the completely non-unitary parts $\mathscr{H}_{1\mathrm{n}}$ and $\mathscr{H}_{2\mathrm{n}}$ of the Wold decompositions of, respectively, $T_1^*$ and $T_2^*$ are both non-trivial. For $i=1,2$, we know that $T_i^*\big|\mathscr{H}_{i\mathrm{n}}$ is a unilateral forward shift with the wandering subspace $\mathscr{E}_i := \ker T_i$ satisfying $\mathscr{H}_{i\mathrm{n}} = \bigoplus\limits_{k\geqslant 0} T_i^{*k}\mathscr{E}_i$. Fix unit elements $e_1 \in \mathscr{E}_1$ and $e_2 \in \mathscr{E}_2$ and define the non-zero operator $X\colon \mathscr{H}_1 \to \mathscr{H}_2$ by $Xh_1 := \langle h_1, e_1\rangle\, e_2$ for $h_1 \in \mathscr{H}_1$. Let us see that $X$ is a Hankel operator. Since $e_1 \in \ker T_1$ and $e_2 \in \ker T_2$, for $h_1 \in \mathscr{H}_1$ we have, on the one hand, $XT_1^*h_1 = \langle T_1^*h_1, e_1\rangle\, e_2 = \langle h_1, T_1e_1\rangle\, e_2 = 0$ and, on the other hand, $T_2Xh_1 = \langle h_1, e_1\rangle\, T_2e_2 = 0$. Therefore, $T_2X = XT_1^* = 0$. Since we are dealing with co-isometries, (PV)-boundedness is automatically granted, hence $X \in \mathcal{H}(T_1,T_2)$. (More generally, it can be proved that any operator $X$ of the form $\sum\limits_{j=0}^{n} \big\langle h_1, T_1^{*n-j}e_1\big\rangle\, T_2^{*j}e_2$ is a Hankel operator with respect to $T_1$ and $T_2$.)

(2) Assume now that, say, $\mathscr{H}_{1\mathrm{n}}$ is trivial (the reasoning is similar if $\mathscr{H}_{2\mathrm{n}}$ is trivial). Then $T_1$ is unitary and $U_1^*\big|\mathscr{P}_1 = T_1^*$ is also unitary. Therefore, the completely non-unitary part of the Wold decomposition of $U_1^*\big|\mathscr{P}_1$ is trivial and, by Theorem 5, $\mathcal{AHS}(T_1,T_2) = \{0\}$. This means that the correspondence between Hankel symbols and Hankel operators is 1-1 in this case and, since by our hypothesis $\mathcal{HS}(T_1,T_2)$ is non-trivial, it follows that $\mathcal{H}(T_1,T_2)$ is non-trivial. □

**Corollary.** *The equality $\mathcal{HS}(T_1,T_2) = \mathcal{AHS}(T_1,T_2)$ holds only if both sets reduce to $\{0\}$.*

**Example 4.** Of course, the trivial example $T_1 = T_2 =$ identity on a Hilbert space shows that one may have $\mathcal{AHS}(T_1,T_2) = \{0\}$ and $\mathcal{HS}(T_1,T_2) = \mathcal{H}(T_1,T_2) \neq \{0\}$. A little less trivial example is given by $T_1 = V$ and $T_2 = V^*$ on $L^2(\mathbb{T})$. Then $\mathcal{AHS}(T_1,T_2)$ reduces to $\{0\}$ but, obviously, the identity on $L^2(\mathbb{T})$ is a Hankel operator with respect to $T_1$ and $T_2$; as a matter of fact, the set of all Hankel operators coincides with the set of the multiplication operators induced by functions from $L^\infty(\mathbb{T})$. In this case each Hankel operator coincides with its own unique symbol.

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